

INSTRUCTOR'S RESOURCE GUIDE AND SOLUTIONS MANUAL

ELKA M. BLOCK

FRANK PURCELL

CALCULUS WITH APPLICATIONS *and* CALCULUS WITH APPLICATIONS, BRIEF VERSION TENTH EDITIONS

Margaret L. Lial

American River College

Raymond N. Greenwell

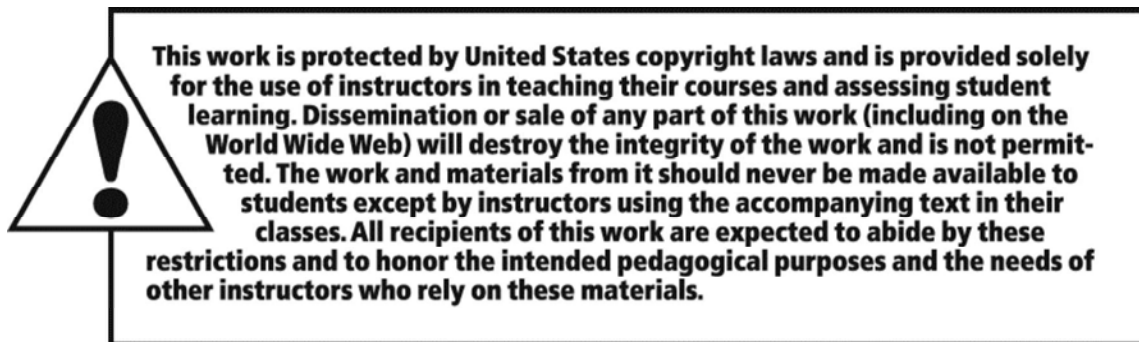
Hofstra University

Nathan P. Ritchey

Youngstown State University

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PREFACE

This book provides several resources for instructors using *Calculus with Applications*, Tenth Edition, by Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey.

- Hints for teaching *Calculus with Applications* are provided as a resource for faculty.
- One open-response form and one multiple-choice form of a pretest are provided. These tests are an aid to instructors in identifying students who may need assistance.
- One open-response form and one multiple-choice form of a final examination are provided.
- Solutions for nearly all of the exercises in the textbook are included. Solutions are usually not provided for exercises with open-response answers.

The following people have made valuable contributions to the production of this *Instructor's Resource Guide and Solutions Manual*: LaurelTech Integrated Publishing Services, editors; Judy Martinez and Sheri Minkner, typists; and Joe Vetere, Senior Author Support/Technology Specialist. Revision for the Tenth Edition was carried out by Twin Prime Editorial.

HINTS FOR TEACHING CALCULUS WITH APPLICATIONS

Algebra Reference

Some instructors obtain best results by going through this chapter carefully at the beginning of the semester. Others find it better to refer to it as needed throughout the course. Use whichever method works best for your students. As in the previous edition, we refer to the chapter as a “Reference” rather than a “Review,” and the regular page numbers don’t begin until Chapter 1. We hope this will make your students less anxious if you don’t cover this material.

Chapter 1

Instructors sometimes go to either of two extremes in this chapter and the next. Some feel that their students have already covered enough precalculus in high school or in previous courses, and consequently begin with Chapter 3. Unfortunately, if they are wrong, their students may do poorly. Other instructors spend at least half a semester on Chapters 1 and 2 and the algebra reference chapter, and subsequently have little time for calculus. Such a course should not be labeled as calculus. We recommend trying to strike a balance, which may still not make all your students happy. A few may complain that the review of algebra, functions, and graphs is too quick; such students should be sent to a more basic course. Those students who are familiar with this material may become lazy and develop habits that will hurt them later in the course. You may wish to assign a few challenging exercises to keep these students on their toes.

Chapter 1 of *Calculus with Applications* is identical to Chapter 1 of *Finite Mathematics* and may be skipped by students who have already taken a course using that text. In this edition, we have streamlined the chapter from four to three sections, allowing instructors to reach the calculus material more quickly.

Section 1.1

This section and the next may seem fairly basic to students who covered linear functions in high school. Nevertheless, many students who have graphed hundreds of lines in their lifetime still lack a thorough understanding of slope, which hampers their understanding of the derivative. Such students could benefit from doing dozens of exercises similar to 39–42.

Perpendicular lines are not used in future chapters and could be skipped if you are in a hurry.

Section 1.2

Much recent research has been devoted to students’ misunderstandings of the function concept. Such misunderstandings are among the major impediments to learning calculus. One way to help students is to study a simple class of functions first, as we do in this section. In this edition, even more of the general material on functions, including domain and range, is postponed until Chapter 2.

Supply and demand provides the students’ first experience with a mathematical model. Spend time developing both the economics and the mathematics involved.

Stress that for cost, revenue, and profit functions, x represents the number of units. For supply and demand functions, we use the economists’ notation of q to represent the number of units.

Emphasize the difference between the profit earned on 100 units sold as opposed to the number of units that must be sold to produce a profit of \$100.

Section 1.3

The statistical functions on a calculator can greatly simplify these calculations, allowing more time for discussion and further examples. In this edition, we use “parallel presentation” to allow the instructor a choice on the extent technology is used. This section may be skipped if you are in a hurry, but your students can benefit from the realistic model and the additional work with equations of lines.

Section 2.1

After learning about linear functions in the previous chapter, students now learn about functions in general. This concept is critical for success in calculus. Unless sufficient time is devoted to this section, the results will become apparent later when students don't understand the derivative. One device that helps students distinguish $f(x + h)$ from $f(x) + h$ is to use a box in place of the letter x , as we do in this section after Example 4.

Section 2.2

This section combines the topics of quadratic functions and translation and reflection, with a minimal amount of material on completing the square. Our experience is that students graph quadratics most easily by first finding the y -intercept, then finding the x -intercepts when they exist (using factoring or the quadratic formula), and finding the vertex last by locating the point midway between the x -intercepts or, if the quadratic formula was used, by letting $x = -b/(2a)$.

Quadratics are among a small group of functions that can be analyzed completely with ease, so they are used throughout the text. On the other hand, the advent of graphing calculators has made ease of graphing less important, so we rely on quadratics less than in previous editions.

Some instructors pressed for time may choose to skip translations and reflections. But we have found that students who understand that the graph of $f(x) = 5 - \sqrt{4 - x}$ is essentially the same as the graph of $f(x) = \sqrt{x}$, just shifted and reflected, will have an easier time when using the derivative to graph functions. Since students are familiar with very few classes of functions at this point, it helps to work with functions defined solely by their graphs, such as Exercises 31–34.

Exercises 39–46 cover stretching and shrinking of graphs in the vertical and horizontal directions. Covering these exercises carefully will not only give students a better grasp of functions, but will help them later to interpret the chain rule.

Section 2.3

Graphing calculators have made point plotting of functions less important than before. Plotting points by hand should not be entirely neglected, however, because a small amount is helpful when using the derivative to graph functions.

The two main goals of this section are to have an understanding of what an n -th degree polynomial looks like, and to be able to find the asymptotes of a rational function. Students who master these ideas will be better prepared for the chapter on curve sketching.

Exercise 62 is the first of several in this chapter asking students to find what type of function best fits a set of data. (See also Section 2.4, Exercises 45, 54 and 55, and Review Exercise 111.) The class can easily get bogged down in these exercises, particularly if you decide to explore the regression features in a calculator such as the TI-84 Plus. But there is a powerful payoff in terms of mastery of functions for the student who succeeds at these exercises.

Section 2.4

Some instructors may prefer at this point to continue with Chapter 3 and to postpone discussion of the exponential and logarithmic functions until later. The overwhelming preference of instructors we surveyed, however, was to cover exponential and logarithmic functions early and then to use these functions throughout the rest of the course. Instructors who wish to postpone this material will also need to omit for now those examples and exercises in Sections 3.1–4.3 that refer to exponential and logarithmic functions.

Students typically have no problem with $f(x) = 2^x$, but the number e often remains a mystery. Like π , the number e is a transcendental number, but students have had years of schooling to get used to π . Have your students approximate e with a calculator, as the textbook does before the definition of e . Notice how we use compound interest to help students get a handle on this number.

Section 2.5

Logarithms are a very difficult topic for many students. It's easy to say that a logarithm is just an exponent, but the fact that it is the exponent to which one must raise the base to get the number whose logarithm we are calculating is a rather obtuse concept. Therefore, spend lots of time going over examples that can be done without a calculator, such as $\log_2 8$. Students will also tend to come up with many incorrect pseudoproperties of logarithms, similar in form to the properties of logarithms given in this section. Take as much time and patience as necessary in gently correcting the many errors students inevitably will make at first.

Even after receiving a thorough treatment of logarithms, some students will still be stumped when solving a problem such as Example 8. Some of these students can get the correct answer using trial-and-error. The instructor should take consolation in the fact that at least such a student understands exponentials better than the one who uses logarithms incorrectly to solve Example 8 and comes up with the nonsensical answer $t = -7.51$ without questioning whether this makes sense. Be sure to teach your students to question the reasonableness of their answers; this will help them catch their errors.

Section 2.6

This section gives students much needed practice with exponentials and logarithms, and the applications keep students interested and motivated. Instructors should keep this in mind and not worry about having students memorize formulas. We have removed the formulas for present value in this edition, having decided that it's better for students to just solve the compound amount formula for P . This reduces by two the number of formulas that students need to remember.

There is a summary of graphs of basic functions in the end-of-chapter review.

Section 3.1

This is the first section on calculus, and perhaps the most important, since limits are what really distinguish calculus from algebra. Students will have the best understanding of limits if they have studied them graphically (as in Exercises 5–12), numerically (as in Exercises 15–20), and analytically (as in Exercises 31–52). The graphing calculator is a powerful tool for studying limits. Notice in Example 12 (c) and (d) that we have modified the method of finding limits at infinity by dividing by the highest power of x in the denominator, which avoid the problem of division by 0.

Section 3.2

The section on continuity should be straightforward if students have mastered limits from the previous section.

Section 3.3

This section introduces the derivative, even though that term doesn't appear until the next section. In a class full of business and social science majors, an instructor may wish to place less emphasis on velocity, an approach more suited to physics majors. But we have found velocity to be the manifestation of the derivative that is most intuitive to all students, regardless of their major.

Instructors in a hurry can skip the material on estimating the instantaneous rate of change from a set of data, but it helps solidify students' understanding of the derivative by giving them one more point of view.

Section 3.4

Students who have learned differentiation formulas in high school usually want (and deserve) some explanation of why they need to learn to take derivatives using the definition. You might try explaining to your students that getting the right formula is not the only goal; graphing calculators can give derivatives numerically. The most important thing for students to learn is the concept of the derivative, which they don't learn if they only memorize differentiation formulas.

Zooming in on a function with a graphing calculator until the graph appears to be a straight line gives students a very concrete image of what the derivative means.

After students have learned the differentiation formulas, they may forget about the definition of derivative. We have found that if we want them to use the definition on a test, it is important to say so clearly and emphatically, or they will simply use the shortcut formulas.

Section 3.5

One way to get students to focus on the concept of the derivative, rather than the mechanics, is to emphasize graphical differentiation. We have therefore devoted an entire section to this topic. Graphical differentiation is difficult for many students because there are no formulas to rely on. One must thoroughly understand what's going on to do anything. On the other hand, we have seen students who are weak in algebra but who possess a good intuitive grasp of geometry find this topic quite simple.

Section 4.1

Students tend to learn these differentiation formulas fairly quickly. These and the formulas in the next few sections are included in a summary at the end of the chapter.

Section 4.2

The product and quotient rules are more difficult for students to keep straight than those of the previous section. People seem to remember these rules better if they use an incantation such as "The first times the derivative of the second, plus the second times the derivative of the first." Some instructors have argued that this formulation of the product rule doesn't generalize well to products of three or more functions, but that's not important at this level. Some instructors allow their students to bring cards with formulas to the tests. This does not eliminate the need for students to understand the use of the formulas, but it does eliminate the anxiety students may have about forgetting a key formula under the pressure of an exam.

Section 4.3

No matter how many times an instructor cries out to his or her students, "Remember the chain rule!", many will still forget this rule at some time later in the course. But if a few more students remember the rule because the instructor reminds them so often, such reminders are worthwhile.

Section 4.4 and 4.5

In going through these sections, you may need to frequently refer to the rules of differentiation in the previous sections. You may also need to review the last three sections of Chapter 2.

Section 5.1 and 5.2

If students have understood Chapter 3, then the connection between the derivative, increasing and decreasing functions, and relative extrema should be obvious, and these sections should go quickly and smoothly.

Section 5.3

Students often confuse concave downward and upward with increasing and decreasing; carefully go over Figure 31 or the equivalent with your class.

Section 5.4

Graphing calculators have made curve sketching techniques less essential, but curve sketching is still one of the best ways to unify the various concepts introduced in this and the previous two chapters. Students should use graphing calculators to check their work.

Because this section is the culmination of many ideas, students often find it difficult and start to forget things they previously knew. For example, a student might state that a function is increasing on an interval and then draw it decreasing. The best solution seems to be lots of practice with immediate help and feedback from the instructor.

Students sometimes stumble over this topic because they use the rules for differentiation incorrectly, or because they make mistakes in algebra when simplifying. Exercises such as 35–39 are excellent for testing whether students really grasp the concepts.

Section 6.1

This section should not be conceptually difficult, but students need constant reminders to check the endpoints of an interval when finding the absolute maxima and minima.

Section 6.2

This section is one of the high points of the course. Some of the best applications of calculus involve maxima and minima. Notice that some exercises have a maximum or minimum at the endpoint of an interval, so students cannot ignore checking endpoints.

Almost everyone finds this material difficult because most people are not skilled at word problems. Remind your students that if they ever wonder whether mathematics is of any use, this section will show them.

Why are word problems so difficult? One theory is that word problems require the use of two different modes of thinking, which students are using simultaneously for the first time. People use words in daily life without difficulty, but when they study mathematics, they often turn off that part of their brain and begin thinking in a very formal, mechanical way. In word problems, both modes of thinking must be active. If and when the NCTM Standards become widely accepted in the schools, children will get more practice at an early age in such ways of thinking. Meanwhile, the steps for solving applied problems given in this book might make the process a little more straightforward, and hence achievable by the average student.

Section 6.3

This section continues the ideas of the previous one. The point of studying economic lot size should not be to apply Equation (3), but to learn how to apply calculus to solve such problems. We therefore urge you to cover Exercises 15–18, in which we vary the assumptions, so Equation (3) does not necessarily apply. In this edition, we have changed the presentation to be consistent with that of business texts.

The material on elasticity can be skipped, but it is an important application that should interest students who have studied even a little economics.

Section 6.4

There are two main reasons for covering implicit differentiation. First, it reinforces the chain rule. Second, it is needed for doing related rate problems. If you skip related rates due to lack of time, it is not essential to cover implicit differentiation, either.

Section 6.5

Related rate applications are less important than applied extrema problems, but they use some of the same skills in setting up word problems, and for that reason are worth covering. The best application exercises are under the heading “Physical Sciences,” because those are the exercises in which no formula is given to the student; the student must construct a formula from the words. The geometrical formulas needed are kept to a minimum: the Pythagorean theorem, the area of a circle, the volume of a sphere, the volume of a cone, and the volume of a cylinder with a triangular cross section. Some instructors allow their students to use a card with such formulas on the exam. These formulas are summarized in a table in the back of the book.

Section 6.6

Differentials may be skipped by instructors in a hurry; you need not fear that this omission will hamper your students in the chapter on integration. The differentials used there are not the same as those used here, and the required techniques are easily picked up when integration by substitution is covered. The exception is for instructors who intend to cover Section 10.3 on Euler’s Method, since differentials are used to motivate and derive that method.

As in the previous edition, our presentation of differentials emphasizes linear approximation, a topic of considerable importance in mathematics.

Section 7.1

Students sometimes start to get differentiation and antidifferentiation confused when they reach this section. Some believe the antiderivative of x^{-2} is $(-1/3)x^{-3}$; after all, if n is -2 , isn’t $n + 1 = -3$? Carefully clarify this point.

Section 7.2

The main difficulty here is teaching students what to choose as u . The advice given before Example 3 should be helpful.

Section 7.3

Some instructors who are pressed for time go lightly over the topic of the area as the limit of the sum of rectangular areas. This is possible, but care should be taken that students don't lose track of what the definite integral represents. Also, a light treatment here lessens the excitement of the Fundamental Theorem of Calculus.

We have continued the trend of the previous edition in covering three ways of approximating a definite integral with rectangles: the right endpoint, the left endpoint, and the midpoint. The trapezoidal rule is briefly introduced here as the average of the left sum and the right sum.

Section 7.4

The Fundamental Theorem of Calculus should be one of the high points of the course. Make a big deal about how the theorem unifies these two separate topics of area as a limit of sums and the antiderivative.

When using substitution on a definite integral, the text recommends changing the limits and the variable of integration. (See Example 4 and the Caution which follows the example.) Some instructors prefer instead to have their students solve an indefinite integral, and then to evaluate the integral using the limits on x . One advantage of this method is that students don't have to remember to change the limits. This method also has two disadvantages. The first is that it takes slightly longer, since it requires changing the integral to u and then back to x . Second, it prevents students at this stage from solving problems such as $\int_0^{1/2} x\sqrt{1-16x^4} dx$, which can be solved using the substitution $u = 4x^2$ and the fact that the integral $\int_0^1 \sqrt{1-u^2} du$ represents the area of a quarter circle. This is one section in which we deliberately did not use more than one method of presentation, because this would inevitably lead to confusion in the minds of some students.

Section 7.5

This section gives more motivation to the topic of integration. Consumers' and producers' surplus are important, realistic applications. We have downplayed sketching the curves that bound the area under consideration. Such sketches take time and are not necessary in solving these problems. But they clarify what is happening and make it possible to avoid memorizing formulas. Using a graphing calculator to sketch the curves can be helpful.

Section 7.6

The ubiquity of computers and graphing calculators has made numerical integration more important. Rather than computing a definite integral by an integration technique, one can just as easily enter the function into a calculator and press the integration key. Point out to students that this is valuable when the function to be integrated is complicated. On the other hand, using the antiderivative makes it easier to see the effect of changing the upper limit, say, from 2 to 3, or for working with the definite integral when one or both limits are parameters, such as a and b , rather than numbers.

Simpson's rule is the most accurate of the simple integration formulas. To achieve greater accuracy, a more complicated method must be used. This is why, unlike the trapezoidal rule, Simpson's rule is actually used by mathematicians and engineers.

You may wish to give your students the programs for the trapezoidal rule and Simpson's rule in *The Graphing Calculator Manual* that is available with the text.

Section 8.1

Students usually find column integration simpler than traditional integration by parts. We show both methods to give instructors a choice, and also to emphasize that the two methods are equivalent. Column integration makes organizing the details simpler, but is no more mechanical than the traditional method, as some have mistakenly claimed. At Hofstra University, students even use this method when neither the instructor nor the book discuss it. They find out about it from other students, and so it has become an underground method. Some instructors feel that students will lose any theoretical understanding of what they are doing if they use this method. Our experience is that almost no students at this level have a theoretical understanding of what integration by parts is about, but the better students can at least master the mechanics. With column integration, almost all of the students master the mechanics.

Section 8.2 and 8.3

These two sections give more applications of integration. Coverage of either section is optional.

Section 8.4

Improper integration is not really an application of integration, but it makes further applications of integration possible.

Many mathematicians use shorthand notation such as the following: $\int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = 0 - (-1) = 1$.

For students at this level, it may be best to avoid the shorthand notation.

Section 9.1

The major difficulty students have with this section, and indeed with this entire chapter, is that they cannot visualize surfaces in 3-dimensional space, even though they live there. Fortunately, such visualization is not really necessary for doing the exercises in this chapter. A student who wants to explore what various surfaces look like can use any of the commercial or public domain computer programs available.

Section 9.2

Students who have mastered the differentiation techniques should have no difficulty with this section.

Section 9.3

This section corresponds to the section on applied extrema problems in the chapter on applications of the derivative, but with less emphasis on word problems. In Exercise 30, we revisit the topic of the least squares line, first covered in Section 1.3.

Section 9.4

Lagrange multipliers are an important application of calculus to economics. At some colleges, the business school is very insistent that the mathematics department cover this material.

Section 9.5

This section corresponds to the section on differentials in the chapter on applications of the derivative.

Section 9.6

Students who have trouble visualizing surfaces in 3-dimensional space are sometimes bothered by double integrals over variable regions. Instructors should assure such students that all they need to do is draw a good sketch of the region in the xy -plane, and not try to draw the volume in three dimensions.

Section 10.1

Differential equations of the form $dy/dx = f(x)$ are treated lightly in this section because they were already covered in the chapter on integration, although the terminology and notation were different then. Remind students that solving such differential equations is the same as antidifferentiation. The rest of the section is on separable differential equations. Students sometimes have trouble with this section because they have forgotten how to find an antiderivative, particularly one requiring substitution.

Section 10.2

If you get this far, your students have covered most of the techniques for solving first-order differential equations. You can find further techniques in differential equations texts, but most first-order equations that come up in real applications are separable, linear, or not solvable by any exact method.

Section 10.3

This is one of the most calculator/computer-intensive sections of the book. In practice, more accurate methods than Euler's method are almost always used, but Euler's method introduces students to a way of solving problems that would otherwise be beyond their grasp. You may wish to give your students the program for Euler's method in *The Graphing Calculator Manual* that is available with the text.

Section 10.4

This is a fun section of assorted applications, showing students that the techniques they have learned were not in vain. You can pick and choose those applications of greatest interest to yourself and your students. You can also supplement

the text with applications from other sources, such as those published by the Consortium for Mathematics and Its Applications Project (COMAP).

Chapter 11

Probability is one of the best applications of calculus around. In fact, statistics instructors sometimes feel the temptation to start discussing the definite integral even when their students know no calculus. This chapter is just a brief introduction, but it covers some of the most important concepts, such as mean, variance, standard deviation, expected value, and probability as the area under the curve. The third section covers three of the most important continuous probability distributions: uniform, exponential, and normal.

Chapter 12

Except for geometric series, most of the material in this chapter is of greater interest to the professional mathematician or engineer than to the student in business, management, economics, life sciences or social sciences, so many instructors may choose to skip this chapter.

Many students confuse sequences with series, and often have trouble with the various tests for convergence. To avoid these sources of confusion, we emphasize summation as a unifying concept, whether of a few terms of a sequence or of an infinite series. Except for convergence of geometric series and a few words on the interval of convergence, we devote little coverage to the topic of convergence. We believe this approach is appropriate for a course at this level.

Section 12.1

Geometric sequences are the most important type of sequences for applications. Even instructors who wish to skip most of this chapter may want to cover the first two sections.

Section 12.2

This material is similar to the material in the Mathematics of Finance chapter in our textbook *Finite Mathematics*. The section shows how geometric sequences are critical for an understanding of annuities, mortgages, and amortization. Don't be put off by the strange notation for the amount or the present value of an annuity; this notation is not at all strange in the world of finance.

Section 12.3

Taylor polynomials are introduced here as an approximation method, with no hint of infinite series.

Section 12.4

The emphasis here is on geometric series, the most important and simplest type of infinite series to understand.

Section 12.5

Some students find Taylor series a strange and abstract concept. To help make this concept more concrete, cover Example 4 on the normal curve, as well as the derivation of the rule of 70 and the rule of 72, introduced without proof in Chapter 2.

Section 12.6

Students with a "zero" feature on their calculator may lack motivation to learn Newton's method unless you can interest them in how one might develop techniques for finding a zero. Newton's method is not, however, the method typically used by calculators; calculator manufacturers are usually reluctant to discuss the actual algorithms.

Section 12.7

We have no applications for this sections, but students and instructors who enjoy symbol manipulation may still find this section satisfying.

Chapter 13

This chapter is a brief introduction to trigonometry and its uses in calculus. Students who need a more thorough treatment of this subject would be better served by a calculus book designed for mathematics majors. The presentation is brief, with a limited number of examples. As a result, students may find some of these exercises challenging. Therefore, tread carefully through this chapter.

PRETESTS AND ANSWERS

Pretest, Form A

1. Evaluate the expression $-x^2 + 3y - z^3$ when $x = -2$, $y = 3$, and $z = -1$.

1. _____

Perform the indicated operations.

2. $(2x^2 - 3x + 7) - (5x^2 - 8x - 9)$

2. _____

3. $(y - 4)^2$

3. _____

4. $(2x - 1)(x^2 + 3x - 4)$

4. _____

5. $(5a + 2b)(5a - 2b)$

5. _____

Factor each polynomial completely.

6. $3x^2y^3 - 6x^3y^2 + 15x^2y^2$

6. _____

7. $2ac - 3ad + 8bc - 12bd$

7. _____

Solve each equation.

8. $6(x - 1) - 4(2x + 8) = 2 - 5(x + 2)$

8. _____

9. $\frac{1}{4}x - 7 = \frac{2x}{3} + \frac{1}{2}$

9. _____

10. $\frac{x - 1}{5} = \frac{3 - 2x}{4}$

10. _____

11. $(x - 2)(x + 1) = 4$

11. _____

12. Solve the inequality $6x - 8(x + 2) \leq 5 - (x + 14)$.
Write your answer in interval notation.

12. _____

13. Solve the equation $2x + 4y = ay - bx$ for y .

13. _____

14. Solve the equation $\frac{5}{a} + \frac{c}{2b} = \frac{7}{3a}$ for a .

14. _____

15. Solve the following system of equations.

$$\begin{aligned} 3x - 2y &= 12 \\ -4x + 5y &= -23 \end{aligned}$$

15. _____

Perform the indicated operation. Write answers in lowest terms.

16. $\frac{18a^2b^2}{27xy^3} \cdot \frac{10xy^2}{8ab^2}$

16. _____

17. $\frac{2x + 8}{3y - 15} \div \frac{x^2 + 7x + 12}{y^2 - 10y + 25}$

17. _____

18. $\frac{4}{x^2 - 1} + \frac{2}{x^2 + 3x + 2}$

18. _____

Simplify each expression. Write answers with only positive exponents.

19. $(2x^2y^{-3})^{-3}$

19. _____

20. $\frac{6x^{-2}y^5}{15x^4y^{-2}}$

20. _____

Simplify each radical. Assume all variables represent nonnegative real numbers.

21. $\sqrt{72x^3y^4}$

21. _____

22. $\sqrt[3]{72x^3y^4}$

22. _____

Write an equation in the form $ax + by = c$ for each line.

23. The line through the points with coordinates $(-3, -14)$ and $(5, 2)$

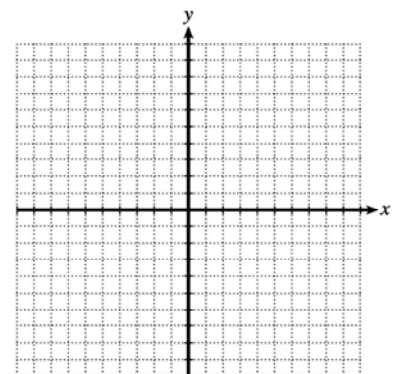
23. _____

24. The line through the points with coordinates $(3, -7)$ and $(3, 5)$

24. _____

25. Graph the solution of the inequality $2x - 3y \leq 12$.

25. _____



Pretest, Form B

Choose the best answer.

1. Evaluate the expression $\frac{y^2 - z}{-x^2 + 6}$ when $x = -2$, $y = 4$, and $z = -8$. 1. _____
- (a) $\frac{12}{5}$ (b) 4 (c) 12 (d) $\frac{4}{5}$

Perform the indicated operations.

2. $(x^2 - 3x + 7) + (4x^3 - 6x^2 - 5x + 4)$ 2. _____
- (a) $4x^3 - 6x^4 - 8x^2 + 11$ (b) $4x^3 - 5x^2 - 8x + 11$
 (c) $4x^3 - 5x^2 - 2x + 11$ (d) $4x^3 - 7x^2 - 8x + 11$
3. $(5x - 4w)^2$ 3. _____
- (a) $25x^2 - 16w^2$ (b) $25x^2 + 16w^2$
 (c) $25x^2 - 40wx + 16w^2$ (d) $25x^2 - 20wx + 16w^2$
4. $(x - 3)(x^2 + x - 2)$ 4. _____
- (a) $x^3 - 2x^2 - 5x + 6$ (b) $x^3 - 3x^2 - 5x + 6$
 (c) $x^3 - 3x^2 + 6x + 6$ (d) $x^3 - 2x^2 + 5x + 6$
5. $(4x + 9)(4x - 9)$ 5. _____
- (a) $16x^2 + 72x - 81y^2$ (b) $16x^2 - 72xy - 81y^2$
 (c) $16x^2 - 81y^2$ (d) $16x^2 - 72xy + 81y^2$

Factor each polynomial completely.

6. $14a^2b^3 - 7a^2b^2 + 35ab^4$ 6. _____
- (a) $7ab(2ab^2 - ab + 5b^3)$ (b) $a^2b^3(14 - 7b + 35b^2)$
 (c) $7a^3b(2b^2 - 7b + 5b^3)$ (d) $7ab^2(2ab - a + 5b^2)$
7. $6xt + 8x - 3yt - 4y$ 7. _____
- (a) $(2x - y)(3t + 4)$ (b) $(2x + y)(3t - 4)$
 (c) $(2x - y)(3t - 4)$ (d) $(2x + y)(3t + 4)$

Solve each equation.

8. $4m - (7m - 6) = -m$ 8. _____
 (a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) -3 (d) 3
9. $-\frac{5}{3} + \frac{2}{r-1} = \frac{1}{6}$ 9. _____
 (a) $-\frac{1}{3}$ (b) $\frac{23}{11}$ (c) 3 (d) -6
10. $\frac{3y+2}{5} = \frac{7-y}{4}$ 10. _____
 (a) $\frac{43}{7}$ (b) $\frac{27}{7}$ (c) $\frac{27}{17}$ (d) $\frac{43}{17}$
11. $2z^2 + z = 28$ 11. _____
 (a) $-4, 3$ (b) $4, -3$
 (c) $-4, \frac{7}{2}$ (d) $4, -\frac{7}{2}$
12. Solve the inequality $-6y + 2 \geq 4y - 7$.
 Write your answer in interval notation. 12. _____
 (a) $\left(-\infty, -\frac{9}{10}\right]$ (b) $\left[\frac{1}{2}, \infty\right)$
 (c) $\left[\frac{9}{10}, \infty\right)$ (d) $\left(-\infty, \frac{9}{10}\right]$
13. Solve the equation $x = \frac{4(y-z)}{3k}$ for y . 13. _____
 (a) $y = \frac{4}{3kx+4z}$ (b) $y = \frac{4}{kx-z}$
 (c) $y = -\frac{3kxz}{4}$ (d) $y = \frac{3x+4z}{4}$
14. Solve the equation $\frac{4}{x} + \frac{a}{y} = \frac{2}{3}$ for x . 14. _____
 (a) $x = \frac{2xy-12y}{3s}$ (b) $x = \frac{12y}{2y-3a}$
 (c) $x = \frac{12y+3ax}{2y}$ (d) $x = \frac{3a-2y}{12y}$

15. Solve the following system of equations and then determine the value of $x + y$ for the solution.

$$3x + 2y = 0$$

$$x - 5y = 17$$

- (a) -1 (b) 1 (c) 2 (d) -3

15. _____

Perform the indicated operation. Write answer in lowest terms.

16. $\frac{6yz^3}{30a^2b^4} \cdot \frac{15a^3b^3}{12y^2z^3}$

(a) $\frac{3a^2}{12aby}$

(b) $\frac{4y^3z^6}{25a^5b^7}$

(c) $\frac{a}{4by}$

(d) $\frac{b}{4ay}$

16. _____

17. $\frac{x^2 - y^2}{4a^5b^7} \div \frac{x^2 + 3xy + 2y^2}{12a^3b^{12}}$

(a) $\frac{3b^5}{2a^2}$

(b) $\frac{3b^5(x+y)}{a^2(x-2y)}$

(c) $\frac{3b^5(x-y)}{a^2(x+2y)}$

(d) $\frac{3b^3(x+y)}{(x-2y)}$

17. _____

18. $\frac{1}{2x} - \frac{2}{3y} + \frac{5}{6xy}$

(a) $\frac{2}{3xy}$

(b) $\frac{3y - 4x + 5}{6xy}$

(c) $3y - 4x + 5$

(d) $\frac{4x - 3y + 5}{6xy}$

18. _____

Simplify each expression. Write answers with only positive exponents.

19. $(3a^{-2}b^3)^{-4}$

(a) $-\frac{12}{a^6b}$

(b) $-\frac{12a^8}{b^{12}}$

(c) $\frac{a^8}{12b^{12}}$

(d) $\frac{a^8}{81b^{12}}$

19. _____

20. $\left(\frac{r^2m^{-1}}{r^3m^2} \right)^{-2}$

(a) r^2m^2

(b) $\frac{1}{r^2m^2}$

(c) r^2m^6

(d) $\frac{1}{r^2m^6}$

20. _____

Simplify each radical. Assume all variables represent nonnegative real numbers.

21. $\sqrt{108a^4b^3}$

(a) $3a^2b\sqrt{12b}$

(b) $6a^2b\sqrt{3b}$

(c) $6a^2b\sqrt{3a}$

(d) $54a^2b\sqrt{b}$

21. _____

22. $\sqrt[3]{108a^4b^3}$

(a) $3ab\sqrt[3]{4a}$

(b) $6a^2b\sqrt[3]{3b}$

(c) $36ab\sqrt[3]{a}$

(d) $9a^2b\sqrt[3]{12a}$

22. _____

Write an equation in the form $ax + by = c$ for each line.

23. The line through the points with coordinates $(1, -1)$ and $(-1, -2)$

(a) $x - 2y = 3$

(b) $x + 2y = 3$

(c) $x + 2y = -3$

(d) $x - 2y = -3$

23. _____

24. The line through the points with coordinates $(-5, -2)$ and $(-3, -2)$

(a) $x + y = -2$

(b) $x = -2$

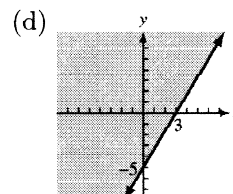
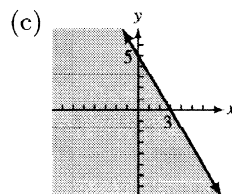
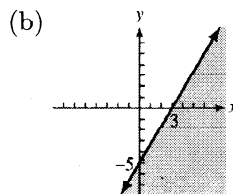
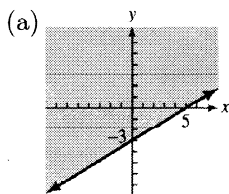
(c) $x - y = -2$

(d) $y = -2$

24. _____

25. Graph the solution of the inequality $5x - 3y \geq 15$.

25. _____



ANSWERS TO PRETESTS

PRETEST, FORM A

1. 6

11. $-2, 3$

20. $\frac{2y^7}{5x^6}$

2. $-3x^2 + 5x + 16$

12. $[-7, \infty)$

21. $6xy^2\sqrt{2x}$

3. $y^2 - 8y + 16$

13. $y = \frac{(b+2)x}{a-4}$

22. $2xy\sqrt[3]{9y}$

4. $2x^3 + 5x^2 - 11x + 4$

14. $a = -\frac{16b}{3c}$

23. $2x - y = 8$

5. $25a^2 - 4b^2$

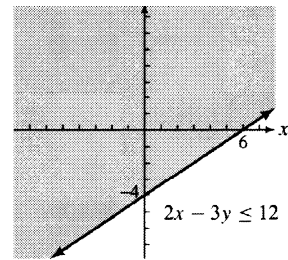
15. $(2, -3)$

24. $x = 3$

6. $3x^2y^2(y - 2x + 5)$

16. $\frac{5a}{6y}$

25.



PRETEST, FORM B

1. (c)

6. (d)

11. (c)

16. (c)

21. (b)

2. (b)

7. (a)

12. (d)

17. (c)

22. (a)

3. (c)

8. (d)

13. (d)

18. (b)

23. (a)

4. (a)

9. (b)

14. (b)

19. (d)

24. (d)

5. (c)

10. (c)

15. (a)

20. (c)

25. (b)

FINAL EXAMINATIONS AND ANSWERS

Final Examination, Form A

1. Find the coordinates of the vertex of the graph of $f(x) = -2x^2 + 30x + 45$.

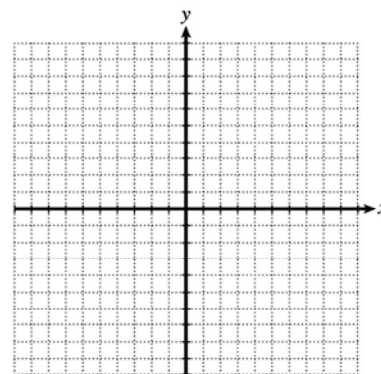
1. _____

2. Write the equation $2^a = d$ using logarithms.

2. _____

3. Graph the function $y = \log_2(x - 3) + 4$.

3. _____



4. Evaluate $\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}$.

4. _____

5. Find all values of x at which $g(x) = \frac{x-2}{x-5}$ is not continuous.

5. _____

6. Find the average rate of change of $y = \sqrt{2x+1}$ between $x = 4$ and $x = 12$.

6. _____

7. Find the derivative of $y = 7x^2e^{3x}$.

7. _____

8. Find the derivative of $y = \frac{\ln(3x)}{x+1}$.

8. _____

9. Find the instantaneous rate of change of $s(t) = 3t^2 - \frac{8}{t}$ at $t = 2$.

9. _____

10. Find an equation of the tangent line to the graph of $y = (x^2 + 3x + 3)^8$ at the point $(-1, 1)$.

10. _____

11. Find $h'(2)$ if $h(x) = \frac{6}{\sqrt{4x+17}}$.

11. _____

12. Find the largest open intervals where f is increasing or decreasing if $f(x) = 2x^3 - 24x^2 + 72x$.

12. _____

13. Find the locations and values of all relative extrema of g if $g(x) = \frac{x^2 + 5x + 3}{x - 1}$. 13. _____
14. For the function $f(x) = x^4 - 4x^3 - 5$, find
 (a) all intervals where f is concave upward; 14. (a) _____
 (b) all intervals where f is concave downward; (b) _____
 (c) all points of inflection. (c) _____
15. Find the fourth derivative of $h(x) = xe^x$. 15. _____
16. Find the locations and values of all absolute extrema of $f(x) = x^3 - 6x^2$ on the interval $[-1, 2]$. 16. _____
17. For a particular commodity, its price per unit in dollars is given by $P(x) = 120 - \frac{x^2}{10}$, where x , measured in thousands, is the number of units sold. This function is valid on the interval $[0, 34]$. How many units must be sold to maximize revenue? 17. _____
18. Find $\frac{dy}{dx}$ if $3\sqrt{x} - 5y^3 = 7xy$. 18. _____
19. If $xy = x - 9$, find $\frac{dy}{dt}$ if $\frac{dx}{dt} = 12$, $x = -3$, and $y = 4$. 19. _____
20. If $y = -3(2 + x^2)^3$, find dy . 20. _____
21. Using differentials, approximate the volume of coating on a sphere of radius 5 inches, if the coating is 0.03 inch thick. 21. _____
22. Find $\int (3x^2 - 7x + 2) dx$. 22. _____
23. Find $\int \frac{8}{x+9} dx$. 23. _____
24. Evaluate $\int_0^2 x\sqrt{3x^2 + 4} dx$. 24. _____
25. Find the area of the region between the x -axis and the graph of $f(x) = xe^{x^2}$ on the interval $[0, 2]$. 25. _____

26. Find the area of the region enclosed by the graphs of $f(x) = x^2 - 4$ and $g(x) = 1 - x^2$. 26. _____
27. Find $\int x^2 e^x dx$. 27. _____
28. Evaluate $\int_1^{e^2} 3x \ln x dx$. 28. _____
29. Find the average value of $f(x) = x(x^2 + 1)^3$ over the interval $[0, 2]$. 29. _____
30. Find the volume of the solid of revolution formed by rotating the region bounded by the graphs of $f(x) = \sqrt{x + 4}$, $y = 0$, and $x = 12$ about the x -axis. 30. _____
31. Determine whether the improper integral $\int_{-\infty}^{-1} x^{-2} dx$ converges or diverges. If it converges, find its value. 31. _____
32. If $f(x, y) = 3x^2 - 4xy + y^3$, find $f_{xy}(1, -3)$. 32. _____
33. For $f(x, y) = 2x^2 - 2xy + 2y^2 + 4x + 4y + 5$, find all of the following:
(a) relative maxima; 33. (a) _____
(b) relative minima; (b) _____
(c) saddle points. (c) _____
34. Maximize $f(x, y) = x^2 y$, subject to the constraint $y + 4x = 84$. 34. _____
35. Find dz , given $z = 3x^2 - 4xy + y^2$, where $x = 0$, $y = 2$, $dx = 0.02$, and $dy = 0.01$. 35. _____
36. Evaluate $\int_1^2 \int_0^3 e^{2x} dy dx$. 36. _____
37. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1 + e^x}{2y}$. 37. _____
38. Find the particular solution of the differential equation $\frac{dy}{dx} = 3x^2 - 7x + 2$; $y = 4$ when $x = 0$. 38. _____

39. The marginal cost function for a particular commodity is $\frac{dy}{dx} = 8x - 9x^2$. The fixed cost is \$20. Find the cost function. 39. _____
40. The probability density function of a random variable is defined by $f(x) = \frac{1}{4}$ for x in the interval $[12, 16]$. Find $P(X \leq 14)$. 40. _____
41. For the probability density function $f(x) = 3x^{-4}$ on $[1, \infty)$, find
(a) the expected value; 41. (a) _____
(b) the variance; (b) _____
(c) the standard deviation. (c) _____
42. The average height of a particular type of tree is 20 feet, with a standard deviation of 2 feet. Assuming a normal distribution, what is the probability that a tree of this kind will be shorter than 17 feet? 42. _____
43. For the function $f(x) = e^{-3x}$, find
(a) the Taylor polynomial of degree 4 at $x = 0$; 43. (a) _____
(b) an approximation, rounded to four decimal places, of $e^{-0.03}$ using the Taylor polynomial from part (a). (b) _____
44. Find the sum of the convergent geometric series $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \cdots$. 44. _____
45. Use Newton's method to find a solution to the nearest hundredth of $3x^3 - 2x^2 + x - 1 = 0$ in the interval $[0, 1]$. 45. _____
46. Use l'Hospital's rule to find $\lim_{x \rightarrow \infty} \frac{3e^{2x} - 3}{x}$. 46. _____
47. Find the derivative of $f(x) = \tan(x^2 + 1)$. 47. _____
48. Find the derivative of $y = x^3 \sin^2 x$. 48. _____
49. Find $\int \sec^2(3x + 1) dx$. 49. _____
50. Evaluate $\int_0^{\pi/6} \sin^3 x \cos x dx$. 50. _____

Final Examination, Form B

Choose the best answer.

1. Find the coordinates of the vertex of the graph of $f(x) = 2x^2 + 10x - 17$. 1. _____
 (a) $\left(-\frac{5}{2}, -\frac{59}{2}\right)$ (b) $\left(\frac{5}{2}, -\frac{59}{2}\right)$ (c) $(-5, -42)$ (d) $(5, -42)$
2. Write the equation $m^P = q$ using logarithms. 2. _____
 (a) $\log_m p = q$ (b) $\log_q p = m$
 (c) $\log_p q = m$ (d) $\log_m q = p$
3. Solve $\left(\frac{1}{e}\right)^{2x} = 14$. (Round to nearest thousandth.) 3. _____
 (a) -1.320 (b) 2.639 (c) -2.639 (d) 1.320
4. Evaluate $\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$. 4. _____
 (a) 0 (b) 3 (c) 6 (d) The limit does not exist.
5. Find all values of x at which $f(x) = \frac{x^2 - 5x + 6}{x^2 - 7x + 12}$ is not continuous. 5. _____
 (a) 3 (b) 4 (c) 3 and 4
 (d) The function is continuous everywhere.
6. Find the average rate of change of $y = x^2 + x$ between $x = 4$ 6. _____
 (a) 9 (b) 10 (c) 17 (d) 25
7. Find the derivative of $y = -3x^3e^{2x}$. 7. _____
 (a) $-27x^2e^{2x} - 6x^3e^{2x}$ (b) $-18x^2e^{3x}$
 (c) $-9x^2e^{2x} - 6x^3e^{2x}$ (d) $-6x^2e^{2x} - 6x^3e^{2x}$
8. Find the derivative of $y = \frac{x^2}{\ln 3x}$. 8. _____
 (a) $\frac{2x^2}{(\ln 3x)^2}$ (b) $\frac{6x \ln 3x - x}{3(\ln 3x)^2}$
 (c) $\frac{2x \ln 3x - x}{(\ln 3x)^2}$ (d) $\frac{6x \ln 3x + x}{3(\ln 3x)^2}$

9. Find the instantaneous rate of change of $s(t) = \frac{t^2}{2} - \frac{2}{t^2}$ at $t = 2$. 9. _____
- (a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$
10. Find an equation of the tangent line to the curve $f(x) = -\frac{1}{x+2}$ at the point $(2, -\frac{1}{4})$. 10. _____
- (a) $x - 16y = 6$ (b) $x - 4y = 3$
- (c) $f'(x) = \frac{1}{16}$ (d) $y = \frac{1}{(x+2)^2}$
11. Find $h'(3)$ if $h(x) = -\frac{400}{\sqrt{3x+7}}$. 11. _____
- (a) 100 (b) $\frac{25}{8}$ (c) $\frac{75}{8}$ (d) 200
12. Find the largest open interval(s) over which the function $g(x) = x^3 - 12x^2 + 36x + 1$ is increasing. 12. _____
- (a) $(2, 6)$ (b) $(-\infty, 2)$
- (c) $(-\infty, 6)$ (d) $(-\infty, 2)$ and $(6, \infty)$
13. Find the location and value of all relative extrema for the function $f(x) = \frac{x^2 + 5x + 3}{x - 1}$. 13. _____
- (a) Relative minimum of 1 at -2 ; relative maximum of 13 at 4
- (b) Relative maximum of 1 at -2 ; relative minimum of 13 at 4
- (c) Relative minimum of 1 at 4; relative minimum of 13 at -2
- (d) No relative extrema
14. Find the coordinates of all points of inflection of the function $g(x) = x^3 - 6x^2$. 14. _____
- (a) $(2, -16)$ (b) $(2, -12)$ (c) $(2, 0)$ (d) $(-2, 16)$
15. Find the third derivative of $f(x) = \sqrt{2x+1}$. 15. _____
- (a) $\frac{3}{8}(2x+1)^{-5/2}$ (b) $3(2x+1)^{-5/2}$
- (c) $-(2x+1)^{-3/2}$ (d) $-\frac{1}{4}(2x+1)^{-3/2}$

16. Find the absolute minimum of $f(x) = x^4 - 4x^3 - 5$ on the interval $[-1, 2]$. 16. _____
- (a) -30 (b) 0 (c) -21 (d) No absolute minimum

17. Botanists, Inc., a consulting firm, monitors the monthly growth of an unusual plant. They determine that the growth (in inches) is given by

$$g(x) = 4x - x^2,$$

where x represents the average daily number of ounces of water the plant receives. Find the maximum monthly growth of the plant.

17. _____

- (a) 2 inches (b) 8 inches
(c) 4 inches (d) 6 inches

18. Find $\frac{dy}{dx}$, given $4x^2 - 7 = 3\sqrt{y} + \frac{2}{x^3}$. 18. _____

- (a) $\frac{4\sqrt{y}(4x^5 - 3)}{3x^4}$ (b) $\frac{4\sqrt{y}(4x^5 + 3)}{3x^4}$
(c) $\frac{4\sqrt{y}(4x^3 - 3)}{3x^2}$ (d) $\frac{4\sqrt{y}(4x^3 + 3)}{3x^2}$

19. If $xy = y + 6$, find $\frac{dy}{dt}$ if $\frac{dx}{dt} = -3$, $x = 4$, and $y = 2$. 19. _____

- (a) 2 (b) 0 (c) -2 (d) -4

20. Evaluate dy , given $y = 45 - 2x - 3x^2$, with $x = 3$ and $\Delta x = 0.02$. 20. _____

- (a) -0.4 (b) 0.4 (c) 0.32 (d) -0.32

21. Using differentials, approximate the volume of coating on a cube with 3-cm sides, if the coating is 0.02 cm thick. 21. _____

- (a) 0.54 cm^3 (b) 27.54 cm^3
(c) 26.46 cm^3 (d) 81.54 cm^3

22. Find $\int (4x^2 + 3x - 5) dx$. 22. _____

- (a) $\frac{4}{3}x^3 + \frac{3}{2}x^2 - 5 + C$ (b) $\frac{4}{3}x^3 + \frac{3}{2}x^2 - 5x$
(c) $\frac{3}{4}x^3 + \frac{3}{2}x^2 - 5 + C$ (d) $8x + 3$

23. Find $\int \frac{-3}{(2x+5)} dx$.

23. _____

- (a) $-\frac{3}{2} \ln |2x + 5| + C$ (b) $-3 \ln |2x + 5| + C$
 (c) $-\frac{1}{6} \ln |2x + 5| + C$ (d) $\frac{1}{2} \ln |2x + 5| + C$

24. Evaluate $\int_0^1 2x\sqrt{4x^2 + 5} dx$.

24. _____

- (a) $\frac{3}{8}$ (b) $\frac{27 - 5\sqrt{5}}{6}$
 (c) $\frac{27 - 5\sqrt{5}}{8}$ (d) $\frac{27 - 5\sqrt{5}}{4}$

25. Find the area of the region between the x -axis and the graph of $f(x) = \sqrt{x+1}$ on the interval $[0, 3]$.

25. _____

- (a) $\frac{14}{3}$ (b) 7 (c) $\frac{1}{4}$ (d) $\frac{21}{2}$

26. Find the area of the region enclosed by the graphs of $f(x) = 9 - x^2$ and $g(x) = x^2 - 9$.

26. _____

- (a) -36 (b) 36 (c) 0 (d) 72

27. Find $\int 3x^2 e^{2x} dx$.

27. _____

- (a) $3e^{x^2} + C$ (b) $\frac{3}{2}x^2 e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$
 (c) $x^3 e^{2x} + C$ (d) $3x^2 e^{2x} - 6xe^{2x} + 6e^{2x} + C$

28. Evaluate $\int_1^4 9x^2 \ln x dx$.

28. _____

- (a) $192 \ln 4 - 45$ (b) $192 \ln 4 - 65$
 (c) $192 \ln 4 - 51$ (d) $192 \ln 4 - 63$

29. Find the average value of the function $f(x) = \sqrt{3x+1}$ over the interval $[0, 8]$.

29. _____

- (a) $\frac{31}{3}$ (b) $\frac{248}{9}$ (c) $\frac{31}{9}$ (d) $\frac{248}{3}$

30. Find the volume of the solid of revolution formed by rotating the region bounded by $f(x) = \sqrt{x-1}$, $y = 0$, and $x = 10$ about the x -axis. 30. _____
- (a) 40.5 (b) $\frac{81\pi}{2}$ (c) 18π (d) 40π
31. Evaluate $\int_{-\infty}^{-2} \frac{1}{x^3} dx$. 31. _____
- (a) $-\frac{1}{8}$ (b) Diverges (c) $\frac{1}{64}$ (d) $-\frac{1}{64}$
32. Given $z = f(x, y) = x^2 - 3xy + 2y^3$, find $f_{yx}(0, -2)$. 32. _____
- (a) -16 (b) 2 (c) -24 (d) -3
33. Which of the following applies to the function $f(x, y) = 2x^2 + 8y^3 - 12xy + 7$? 33. _____
- (a) f has a relative maximum at $(\frac{9}{2}, \frac{3}{2})$
(b) f has a relative minimum at $(0, 0)$
(c) f has a relative minimum at $(\frac{9}{2}, \frac{3}{2})$
(d) f has a saddle point at $(\frac{9}{2}, \frac{3}{2})$
34. Maximize $f(x, y) = x^2y$, subject to the constraint $y + 4x = 84$. 34. _____
- (a) 0 (b) 14 (c) 5488 (d) 3642
35. Find dz , given $z = \sqrt{x^2 + y^2}$, with $x = 3$, $y = 4$, $dx = -0.01$, and $dy = 0.02$. 35. _____
- (a) 0.022 (b) 0.01 (c) 0.001 (d) 0.05
36. Evaluate $\int_0^2 \int_2^4 x^2 y \, dy \, dx$. 36. _____
- (a) $\frac{112}{3}$ (b) 16 (c) 32 (d) 12

37. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{e^{2x} - 4}{y^2}$. 37. _____
- (a) $y = \left(\frac{3}{2}e^{2x} - 12x + C \right)^{1/3}$ (b) $y = \frac{1}{2}e^{2x} - 4x + C$
- (c) $y = \left(\frac{1}{2}e^{2x} - 4x + C \right)^{1/3}$ (d) $y^2 = e^{2x} - 4$
38. Find the particular solution of the differential equation $\frac{dy}{dx} = 7 - 2x + 3x^2$; $y = -2$ when $x = 0$. 38. _____
- (a) $y = 7x - x^2 + x^3 + C$ (b) $y = 7x - x^2 + x^3 - 2$
- (c) $y = -x^2 + x^3 + C$ (d) $y = 7x - x^2 + x^3 + 2 + C$
39. The marginal cost function for a particular commodity is $\frac{dy}{dx} = 10x - 7x^2$. The fixed cost is \$200. Find the cost function. 39. _____
- (a) $y = 200 + 5x^2 - 14x$ (b) $y = 5x^2 - 14x$
- (c) $y = 5x^2 - \frac{7x^3}{3} + 200$ (d) $y = 10 - 14x$
40. The probability density function of a random variable is defined by $f(x) = \frac{1}{5}$ for $[20, 25]$. Find $P(X \leq 22)$. 40. _____
- (a) 0.2 (b) 0.8 (c) 0.4 (d) 0.6
41. Find the standard deviation for the probability density function $f(x) = 3x^{-4}$ on $[1, \infty)$. 41. _____
- (a) 1.5 (b) 0.866 (c) 0.75 (d) 0.67
42. The average height of a particular type of tree is 20 feet, with a standard deviation of 2 feet. Assuming a normal distribution, what is the probability that a tree of this kind will be taller than 17 feet? 42. _____
- (a) 0.0668 (b) 0.9332 (c) 0.8023 (d) 0.1977
43. Find the monthly house payment necessary to amortize a \$150,000 loan at 7.2% for 30 years. 43. _____
- (a) \$1018.18 (b) \$975.34 (c) \$1112.87 (d) \$5478.44

44. Find the sum of the convergent geometric series $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$. 44. _____
- (a) 2 (b) 8 (c) 4 (d) $\frac{8}{3}$
45. The n th term of the Taylor series expansion of $\ln(1 + \frac{x}{2})$ is given by which of the following? 45. _____
- (a) $\frac{(-1)^n x^{n+1}}{(n+1) \cdot 2^{n+1}}$ (b) $\frac{x^{n+1}}{n+1}$ (c) $\frac{x^{n+1}}{(n+1) \cdot 2^{n+1}}$ (d) $\frac{(-1)^n x^{n+1}}{n+1}$
46. Evaluate $\lim_{x \rightarrow 0} \frac{-2e^{4x} + 2}{x}$. 46. _____
- (a) -2 (b) -8 (c) 0 (d) The limit does not exist.
47. Find the derivative of $f(x) = \tan \sqrt{2x+1}$. 47. _____
- (a) $\frac{\sec^2 \sqrt{2x+1}}{\sqrt{2x+1}}$ (b) $\frac{\sec^2 \sqrt{2x+1}}{2}$
- (c) $x \sec^2 \sqrt{2x+1}$ (d) $2x \sec^2 \sqrt{2x+1}$
48. Find the derivative of $y = x^2 \cos^3 x$. 48. _____
- (a) $x \cos^2 x (2 \cos x + 3x)$ (b) $x \cos^2 x (2 \cos x - 3x \sin x)$
- (c) $x \cos^2 x (2 \cos x - 3x)$ (d) $x \cos^2 x (2 \cos x + 3x \sin x)$
49. Find $\int \tan(2x) \sec^2(2x) dx$. 49. _____
- (a) $\frac{1}{4} \tan^2(2x) + C$ (b) $\tan^2(2x) + C$
- (c) $\frac{1}{4} \sec(2x) + C$ (d) $\sec(2x) + C$
50. Evaluate $\int_0^{\pi/2} (10 - 10 \sin^2 x \cos x) dx$. 50. _____
- (a) $\frac{20}{3}$ (b) $5\pi - 10$
- (c) $\frac{30\pi - 10}{3}$ (d) $\frac{15\pi - 10}{3}$

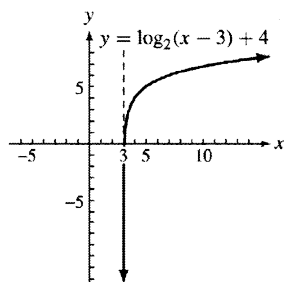
ANSWERS TO FINAL EXAMINATIONS

FINAL EXAMINATION, FORM A

1. $(7.5, 157.5)$

2. $\log_2 d = a$

3.



4. 4

5. 5

6. $\frac{1}{4}$

7. $y' = 7xe^{3x}(2 + 3x)$

8. $y' = \frac{x+1-x \ln(3x)}{x(x+1)^2}$

9. 14

10. $y = 8x + 9$

11. $-\frac{12}{125}$

12. Increasing on $(-\infty, 2)$ and $(6, \infty)$; decreasing on $(2, 6)$ 13. Relative maximum of 1 at -2 ; relative minimum of 13 at 414. (a) $(-\infty, 0)$ and $(2, \infty)$ (b) $(0, 2)$
(c) $(0, -5)$ and $(2, -21)$

15. $(x + 4)e^x$

16. Absolute maximum of 0 at 0; absolute minimum of -16 at 2

17. 20,000

18. $\frac{dy}{dx} = \frac{3 - 14x^{1/2}y}{14x^{3/2} + 30x^{1/2}y^2}$

19. 12

20. $dy = -18x(2 + x^2)^2 dx$

21. $3\pi \text{ in.}^3$

22. $x^3 - \frac{7}{2}x^2 + 2x + C$

23. $\frac{1}{8} \ln |x + 9| + C$

24. $\frac{56}{9}$

25. $\frac{1}{2}(e^4 - 1)$

26. $\frac{10\sqrt{10}}{3} \approx 10.54$

27. $e^x(x^2 - 2x + 2) + C$

28. $\frac{3}{4}(3e^4 + 1) \approx 123.60$

29. 39

30. 120π

31. Converges; 1
32. -4
33. (a) None (b) -3 at $(-2, -2)$ (c) None
34. 5488 when $x = 14$ and $y = 28$
35. -0.12
36. $\frac{3}{2}e^4 - \frac{3}{2}e^2 \approx 70.81$
37. $y^2 = x + e^x + C$
38. $y = x^3 - \frac{7}{2}x^2 + 2x + 4$
39. $y = 4x^2 - 3x^3 + 20$
40. 0.5
41. (a) $\frac{3}{2}$ (b) 0.75 (c) 0.866
42. 0.0668
43. (a) $1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8}$
(b) 0.9704
44. $\frac{15}{2}$
45. 0.78
46. 6
47. $2x \sec^2(x^2 + 1)$
48. $x^2 \sin x (3 \sin x + 2x \cos x)$
49. $\frac{1}{3} \tan(3x + 1) + C$
50. $\frac{1}{64}$

FINAL EXAMINATION, FORM B

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (a) | 11. (c) | 21. (a) | 31. (a) | 41. (b) |
| 2. (d) | 12. (d) | 22. (c) | 32. (d) | 42. (b) |
| 3. (a) | 13. (b) | 23. (a) | 33. (c) | 43. (a) |
| 4. (c) | 14. (a) | 24. (b) | 34. (c) | 44. (d) |
| 5. (b) | 15. (b) | 25. (a) | 35. (b) | 45. (a) |
| 6. (c) | 16. (c) | 26. (d) | 36. (b) | 46. (b) |
| 7. (c) | 17. (c) | 27. (b) | 37. (a) | 47. (a) |
| 8. (c) | 18. (b) | 28. (d) | 38. (b) | 48. (b) |
| 9. (b) | 19. (a) | 29. (c) | 39. (c) | 49. (a) |
| 10. (a) | 20. (a) | 30. (b) | 40. (c) | 50. (d) |

**SOLUTIONS
TO
ALL EXERCISES**

Chapter R

ALGEBRA REFERENCE

R.1 Polynomials

Your Turn 1

$$\begin{aligned}3(x^2 - 4x - 5) - 4(3x^2 - 5x - 7) \\&= 3x^2 - 12x - 15 - 12x^2 + 20x + 28 \\&= -9x^2 + 8x + 13\end{aligned}$$

Your Turn 2

$$\begin{aligned}(3y + 2)(4y^2 - 2y - 5) \\&= (3y)(4y^2 - 2y - 5) + (2)(4y^2 - 2y - 5) \\&= 12y^3 - 6y^2 - 15y + 8y^2 - 4y - 10 \\&= 12y^3 + 2y^2 - 19y - 10\end{aligned}$$

R.1 Exercises

- $$\begin{aligned}(2x^2 - 6x + 11) + (-3x^2 + 7x - 2) \\&= 2x^2 - 6x + 11 - 3x^2 + 7x - 2 \\&= (2 - 3)x^2 + (7 - 6)x + (11 - 2) \\&= -x^2 + x + 9\end{aligned}$$
- $$\begin{aligned}(-4y^2 - 3y + 8) - (2y^2 - 6y - 2) \\&= (-4y^2 - 3y + 8) + (-2y^2 + 6y + 2) \\&= -4y^2 - 3y + 8 - 2y^2 + 6y + 2 \\&= (-4y^2 - 2y^2) + (-3y + 6y) + (8 + 2) \\&= -6y^2 + 3y + 10\end{aligned}$$
- $$\begin{aligned}-6(2q^2 + 4q - 3) + 4(-q^2 + 7q - 3) \\&= (-12q^2 - 24q + 18) \\&\quad + (-4q^2 + 28q - 12) \\&= (-12q^2 - 4q^2) \\&\quad + (-24q + 28q) + (18 - 12) \\&= -16q^2 + 4q + 6\end{aligned}$$
- $$\begin{aligned}2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5) \\&= (6r^2 + 8r + 4) + (3r^2 - 12r + 15) \\&= (6r^2 + 3r^2) + (8r - 12r) + (4 + 15) \\&= 9r^2 - 4r + 19\end{aligned}$$

- $$\begin{aligned}(0.613x^2 - 4.215x + 0.892) - 0.47(2x^2 - 3x + 5) \\&= 0.613x^2 - 4.215x + 0.892 - 0.94x^2 \\&\quad + 1.41x - 2.35 \\&= -0.327x^2 - 2.805x - 1.458\end{aligned}$$
- $$\begin{aligned}0.5(5r^2 + 3.2r - 6) - (1.7r^2 - 2r - 1.5) \\&= (2.5r^2 + 1.6r - 3) + (-1.7r^2 + 2r + 1.5) \\&= (2.5r^2 - 1.7r^2) + (1.6r + 2r) + (-3 + 1.5) \\&= 0.8r^2 + 3.6r - 1.5\end{aligned}$$
- $$\begin{aligned}-9m(2m^2 + 3m - 1) \\&= -9m(2m^2) - 9m(3m) - 9m(-1) \\&= -18m^3 - 27m^2 + 9m\end{aligned}$$
- $$6x(-2x^3 + 5x + 6) = -12x^4 + 30x^2 + 36x$$
- $$\begin{aligned}(3t - 2y)(3t + 5y) \\&= (3t)(3t) + (3t)(5y) + (-2y)(3t) + (-2y)(5y) \\&= 9t^2 + 15ty - 6ty - 10y^2 \\&= 9t^2 + 9ty - 10y^2\end{aligned}$$
- $$\begin{aligned}(9k + q)(2k - q) \\&= (9k)(2k) + (9k)(-q) + (q)(2k) + (q)(-q) \\&= 18k^2 - 9kq + 2kq - q^2 \\&= 18k^2 - 7kq - q^2\end{aligned}$$
- $$\begin{aligned}(2 - 3x)(2 + 3x) \\&= (2)(2) + (2)(3x) + (-3x)(2) + (-3x)(3x) \\&= 4 + 6x - 6x - 9x^2 \\&= 4 - 9x^2\end{aligned}$$
- $$\begin{aligned}(6m + 5)(6m - 5) \\&= (6m)(6m) + (6m)(-5) + (5)(6m) + (5)(-5) \\&= 36m^2 - 30m + 30m - 25 \\&= 36m^2 - 25\end{aligned}$$

$$\begin{aligned}
 13. \quad & \left(\frac{2}{5}y + \frac{1}{8}z\right)\left(\frac{3}{5}y + \frac{1}{2}z\right) \\
 &= \left(\frac{2}{5}y\right)\left(\frac{3}{5}y\right) + \left(\frac{2}{5}y\right)\left(\frac{1}{2}z\right) + \left(\frac{1}{8}z\right)\left(\frac{3}{5}y\right) \\
 &\quad + \left(\frac{1}{8}z\right)\left(\frac{1}{2}z\right) \\
 &= \frac{6}{25}y^2 + \frac{1}{5}yz + \frac{3}{40}yz + \frac{1}{16}z^2 \\
 &= \frac{6}{25}y^2 + \left(\frac{8}{40} + \frac{3}{40}\right)yz + \frac{1}{16}z^2 \\
 &= \frac{6}{25}y^2 + \frac{11}{40}yz + \frac{1}{16}z^2
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right) \\
 &= \left(\frac{3}{4}r\right)\left(\frac{5}{4}r\right) + \left(\frac{3}{4}r\right)\left(\frac{1}{3}s\right) + \left(-\frac{2}{3}s\right)\left(\frac{5}{4}r\right) \\
 &\quad + \left(-\frac{2}{3}s\right)\left(\frac{1}{3}s\right) \\
 &= \frac{15}{16}r^2 + \frac{1}{4}rs - \frac{5}{6}rs - \frac{2}{9}s^2 \\
 &= \frac{15}{16}r^2 - \frac{7}{12}rs - \frac{2}{9}s^2
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & (3p - 1)(9p^2 + 3p + 1) \\
 &= (3p - 1)(9p^2) + (3p - 1)(3p) + (3p - 1)(1) \\
 &= 3p(9p^2) - 1(9p^2) + 3p(3p) \\
 &\quad - 1(3p) + 3p(1) - 1(1) \\
 &= 27p^3 - 9p^2 + 9p^2 - 3p + 3p - 1 \\
 &= 27p^3 - 1
 \end{aligned}$$

$$\begin{aligned}
 16. \quad & (3p + 2)(5p^2 + p - 4) \\
 &= (3p)(5p^2) + (3p)(p) + (3p)(-4) \\
 &\quad + (2)(5p^2) + (2)(p) + (2)(-4) \\
 &= 15p^3 + 3p^2 - 12p + 10p^2 + 2p - 8 \\
 &= 15p^3 + 13p^2 - 10p - 8
 \end{aligned}$$

$$\begin{aligned}
 17. \quad & (2m + 1)(4m^2 - 2m + 1) \\
 &= 2m(4m^2 - 2m + 1) + 1(4m^2 - 2m + 1) \\
 &= 8m^3 - 4m^2 + 2m + 4m^2 - 2m + 1 \\
 &= 8m^3 + 1
 \end{aligned}$$

$$\begin{aligned}
 18. \quad & (k + 2)(12k^3 - 3k^2 + k + 1) \\
 &= k(12k^3) + k(-3k^2) + k(k) + k(1) \\
 &\quad + 2(12k^3) + 2(-3k^2) + 2(k) + 2(1) \\
 &= 12k^4 - 3k^3 + k^2 + k + 24k^3 - 6k^2 \\
 &\quad + 2k + 2 \\
 &= 12k^4 + 21k^3 - 5k^2 + 3k + 2
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & (x + y + z)(3x - 2y - z) \\
 &= x(3x) + x(-2y) + x(-z) + y(3x) + y(-2y) \\
 &\quad + y(-z) + z(3x) + z(-2y) + z(-z) \\
 &= 3x^2 - 2xy - xz + 3xy - 2y^2 - yz + 3xz \\
 &\quad - 2yz - z^2 \\
 &= 3x^2 + xy + 2xz - 2y^2 - 3yz - z^2
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & (r + 2s - 3t)(2r - 2s + t) \\
 &= r(2r) + r(-2s) + r(t) + 2s(2r) + 2s(-2s) \\
 &\quad + 2s(t) - 3t(2r) - 3t(-2s) - 3t(t) \\
 &= 2r^2 - 2rs + rt + 4rs + 2st - 6rt + 6st \\
 &\quad - 3t^2 + 2rt - st + t^2 \\
 &= 2r^2 + 2rs - 5rt - 4s^2 + 8st - 3t^2
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & (x + 1)(x + 2)(x + 3) \\
 &= [x(x + 2) + 1(x + 2)](x + 3) \\
 &= [x^2 + 2x + x + 2](x + 3) \\
 &= [x^2 + 3x + 2](x + 3) \\
 &= x^2(x + 3) + 3x(x + 3) + 2(x + 3) \\
 &= x^3 + 3x^2 + 3x^2 + 9x + 2x + 6 \\
 &= x^3 + 6x^2 + 11x + 6
 \end{aligned}$$

$$\begin{aligned}
 22. \quad & (x - 1)(x + 2)(x - 3) \\
 &= [x(x + 2) + (-1)(x + 2)](x - 3) \\
 &= (x^2 + 2x - x - 2)(x - 3) \\
 &= (x^2 + x - 2)(x - 3) \\
 &= x^2(x - 3) + x(x - 3) + (-2)(x - 3) \\
 &= x^3 - 3x^2 + x^2 - 3x - 2x + 6 \\
 &= x^3 - 2x^2 - 5x + 6
 \end{aligned}$$

$$\begin{aligned}
 23. \quad & (x + 2)^2 = (x + 2)(x + 2) \\
 &= x(x + 2) + 2(x + 2) \\
 &= x^2 + 2x + 2x + 4 \\
 &= x^2 + 4x + 4
 \end{aligned}$$

$$\begin{aligned}
 24. \quad (2a - 4b)^2 &= (2a - 4b)(2a - 4b) \\
 &= 2a(2a - 4b) - 4b(2a - 4b) \\
 &= 4a^2 - 8ab - 8ab + 16b^2 \\
 &= 4a^2 - 16ab + 16b^2
 \end{aligned}$$

$$\begin{aligned}
 25. \quad (x - 2y)^3 &= [(x - 2y)(x - 2y)](x - 2y) \\
 &= (x^2 - 2xy - 2xy + 4y^2)(x - 2y) \\
 &= (x^2 - 4xy + 4y^2)(x - 2y) \\
 &= (x^2 - 4xy + 4y^2)x \\
 &\quad + (x^2 - 4xy + 4y^2)(-2y) \\
 &= x^3 - 4x^2y + 4xy^2 - 2x^2y + 8xy^2 - 8y^3 \\
 &= x^3 - 6x^2y + 12xy^2 - 8y^3
 \end{aligned}$$

$$\begin{aligned}
 26. \quad (3x + y)^3 &= (3x + y)(3x + y)^2 \\
 &= (3x + y)(9x^2 + 6xy + y^2) \\
 &= 27x^3 + 18x^2y + 3xy^2 + 9x^2y \\
 &\quad + 6xy^2 + y^3 \\
 &= 27x^3 + 27x^2y + 9xy^2 + y^3
 \end{aligned}$$

R.2 Factoring

Your Turn 1

Factor $4z^4 + 4z^3 + 18z^2$.

$$\begin{aligned}
 4z^4 + 4z^3 + 18z^2 &= (2z^2) \cdot 2z^2 + (2z^2) \cdot 2z + (2z^2) \cdot 9 \\
 &= (2z^2)(2z^2 + 2z + 9)
 \end{aligned}$$

Your Turn 2

Factor $6a^2 + 5ab - 4b^2$.

$$6a^2 + 5ab - 4b^2 = (2a - b)(3a + 4b)$$

R.2 Exercises

$$\begin{aligned}
 1. \quad 7a^3 + 14a^2 &= 7a^2 \cdot a + 7a^2 \cdot 2 \\
 &= 7a^2(a + 2)
 \end{aligned}$$

$$\begin{aligned}
 2. \quad 3y^3 + 24y^2 + 9y &= 3y \cdot y^2 + 3y \cdot 8y + 3y \cdot 3 \\
 &= 3y(y^2 + 8y + 3)
 \end{aligned}$$

$$\begin{aligned}
 3. \quad 13p^4q^2 - 39p^3q + 26p^2q^2 &= 13p^2q \cdot p^2q - 13p^2q \cdot 3p + 13p^2q \cdot 2q \\
 &= 13p^2q(p^2q - 3p + 2q)
 \end{aligned}$$

$$\begin{aligned}
 4. \quad 60m^4 - 120m^3n + 50m^2n^2 &= 10m^2 \cdot 6m^2 - 10m^2 \cdot 12mn + 10m^2 \cdot 5n^2 \\
 &= 10m^2(6m^2 - 12mn + 5n^2)
 \end{aligned}$$

$$\begin{aligned}
 5. \quad m^2 - 5m - 14 &= (m - 7)(m + 2) \\
 \text{since } (-7)(2) &= -14 \text{ and } -7 + 2 = -5.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad x^2 + 4x - 5 &= (x + 5)(x - 1) \\
 \text{since } 5(-1) &= -5 \text{ and } -1 + 5 = 4.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad z^2 + 9z + 20 &= (z + 4)(z + 5) \\
 \text{since } 4 \cdot 5 &= 20 \text{ and } 4 + 5 = 9.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad b^2 - 8b + 7 &= (b - 7)(b - 1) \\
 \text{since } (-7)(-1) &= 7 \text{ and } -7 + (-1) = -8.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad a^2 - 6ab + 5b^2 &= (a - b)(a - 5b) \\
 \text{since } (-b)(-5b) &= 5b^2 \text{ and } -b + (-5b) = -6b.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad s^2 + 2st - 35t^2 &= (s - 5t)(s + 7t) \\
 \text{since } (-5t)(7t) &= -35t^2 \text{ and } 7t + (-5t) = 2t.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad y^2 - 4yz - 21z^2 &= (y + 3z)(y - 7z) \\
 \text{since } (3z)(-7z) &= -21z^2 \text{ and } 3z + (-7z) = -4z.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad 3x^2 + 4x - 7 &= (3x + 7)(x - 1) \\
 \text{The possible factors of } 3x^2 &\text{ are } 3x \text{ and } x \text{ and the} \\
 \text{possible factors of } -7 &\text{ are } -7 \text{ and } 1, \text{ or } 7 \text{ and } -1. \text{ Try} \\
 \text{various combinations until one works.} & \\
 3x^2 + 4x - 7 &= (3x + 7)(x - 1)
 \end{aligned}$$

$$\begin{aligned}
 13. \quad 3a^2 + 10a + 7 &= (a + 1)(3a + 7) \\
 \text{The possible factors of } 3a^2 &\text{ are } 3a \text{ and } a \text{ and the} \\
 \text{possible factors of } 7 &\text{ are } 7 \text{ and } 1. \text{ Try various} \\
 \text{combinations until one works.} & \\
 3a^2 + 10a + 7 &= (a + 1)(3a + 7)
 \end{aligned}$$

$$14. \quad 15y^2 + y - 2 = (5y + 2)(3y - 1)$$

$$15. \quad 21m^2 + 13mn + 2n^2 = (7m + 2n)(3m + n)$$

$$\begin{aligned}
 16. \quad 6a^2 - 48a - 120 &= 6(a^2 - 8a - 20) \\
 &= 6(a - 10)(a + 2)
 \end{aligned}$$

$$\begin{aligned}
 17. \quad 3m^3 + 12m^2 + 9m &= 3m(m^2 + 4m + 3) \\
 &= 3m(m + 1)(m + 3)
 \end{aligned}$$

$$\begin{aligned}
 18. \quad 4a^2 + 10a + 6 &= 2(2a^2 + 5a + 3) \\
 &= 2(2a + 3)(a + 1)
 \end{aligned}$$

19. $24a^4 + 10a^3b - 4a^2b^2$
 $= 2a^2(12a^2 + 5ab - 2b^2)$
 $= 2a^2(4a - b)(3a + 2b)$
20. $24x^4 + 36x^3y - 60x^2y^2$
 $= 12x^2(2x^2 + 3xy - 5y^2)$
 $= 12x^2(x - y)(2x + 5y)$
21. $x^2 - 64 = x^2 - 8^2$
 $= (x + 8)(x - 8)$
22. $9m^2 - 25 = (3m)^2 - (5)^2$
 $= (3m + 5)(3m - 5)$
23. $10x^2 - 160 = 10(x^2 - 16)$
 $= 10(x^2 - 4^2)$
 $= 10(x + 4)(x - 4)$
24. $9x^2 + 64$ is the *sum* of two perfect squares. It cannot be factored. It is prime.
25. $z^2 + 14zy + 49y^2 = z^2 + 2 \cdot 7zy + 7^2y^2$
 $= (z + 7y)^2$
26. $s^2 - 10st + 25t^2 = s^2 - 2 \cdot 5st + (5t)^2$
 $= (s - 5t)^2$
27. $9p^2 - 24p + 16 = (3p)^2 - 2 \cdot 3p \cdot 4 + 4^2$
 $= (3p - 4)^2$
28. $a^3 - 216 = a^3 - 6^3$
 $= (a - 6)[(a)^2 + (a)(6) + (6)^2]$
 $= (a - 6)(a^2 + 6a + 36)$
29. $27r^3 - 64s^3 = (3r)^3 - (4s)^3$
 $= (3r - 4s)(9r^2 + 12rs + 16s^2)$
30. $3m^3 + 375 = 3(m^3 + 125)$
 $= 3(m^3 + 5^3)$
 $= 3(m + 5)(m^2 - 5m + 25)$
31. $x^4 - y^4 = (x^2)^2 - (y^2)^2$
 $= (x^2 + y^2)(x^2 - y^2)$
 $= (x^2 + y^2)(x + y)(x - y)$

32. $16a^4 - 81b^4 = (4a^2)^2 - (9b^2)^2$
 $= (4a^2 + 9b^2)(4a^2 - 9b^2)$
 $= (4a^2 + 9b^2)[(2a)^2 - (3b)^2]$
 $= (4a^2 + 9b^2)(2a + 3b)(2a - 3b)$

R.3 Rational Expressions

Your Turn 1

Write in lowest terms $\frac{z^2 + 5z + 6}{2z^2 + 7z + 3}$.

$$\frac{z^2 + 5z + 6}{2z^2 + 7z + 3} = \frac{(z + 3)(z + 2)}{(z + 3)(2z + 1)}$$

$$= \frac{z + 2}{2z + 1}$$

Your Turn 2

Perform each of the following operations.

(a) $\frac{z^2 + 5z + 6}{2z^2 - 5z - 3} \cdot \frac{2z^2 - z - 1}{z^2 + 2z - 3}$
 $= \frac{(z + 2)(z + 3)}{(2z + 1)(z - 3)} \cdot \frac{(2z + 1)(z - 1)}{(z + 3)(z - 1)}$
 $= \frac{(z + 2)\cancel{(z + 3)}\cancel{(2z + 1)}\cancel{(z - 1)}}{\cancel{(2z + 1)}(z - 3)\cancel{(z + 3)}\cancel{(z - 1)}}$
 $= \frac{z + 2}{z - 3}$

(b) $\frac{a - 3}{a^2 + 3a + 2} + \frac{5a}{a^2 - 4}$
 $= \frac{a - 3}{(a + 2)(a + 1)} + \frac{5a}{(a - 2)(a + 2)}$
 $= \frac{a - 3}{(a + 2)(a + 1)} \cdot \frac{(a - 2)}{(a - 2)}$
 $+ \frac{5a}{(a - 2)(a + 2)} \cdot \frac{(a + 1)}{(a + 1)}$
 $= \frac{(a^2 - 5a + 6) + (5a^2 + 5a)}{(a - 2)(a + 2)(a + 1)}$
 $= \frac{6a^2 + 6}{(a - 2)(a + 2)(a + 1)}$
 $= \frac{6(a^2 + 1)}{(a - 2)(a + 2)(a + 1)}$

R.3 Exercises

1. $\frac{5v^2}{35v} = \frac{5 \cdot v \cdot v}{5 \cdot 7 \cdot v} = \frac{v}{7}$

2. $\frac{25p^3}{10p^2} = \frac{5 \cdot 5 \cdot p \cdot p \cdot p}{2 \cdot 5 \cdot p \cdot p} = \frac{5p}{2}$

3. $\frac{8k+16}{9k+18} = \frac{8(k+2)}{9(k+2)} = \frac{8}{9}$
4. $\frac{2(t-15)}{(t-15)(t+2)} = \frac{2}{t+2}$
5. $\frac{4x^3-8x^2}{4x^2} = \frac{4x^2(x-2)}{4x^2} = x-2$
6. $\frac{36y^2+72y}{9y} = \frac{36y(y+2)}{9y}$
 $= \frac{9 \cdot 4 \cdot y(y+2)}{9 \cdot y}$
 $= 4(y+2)$
7. $\frac{m^2-4m+4}{m^2+m-6} = \frac{(m-2)(m-2)}{(m-2)(m+3)}$
 $= \frac{m-2}{m+3}$
8. $\frac{r^2-r-6}{r^2+r-12} = \frac{(r-3)(r+2)}{(r+4)(r-3)}$
 $= \frac{r+2}{r+4}$
9. $\frac{3x^2+3x-6}{x^2-4} = \frac{3(x+2)(x-1)}{(x+2)(x-2)} = \frac{3(x-1)}{x-2}$
10. $\frac{z^2-5z+6}{z^2-4} = \frac{(z-3)(z-2)}{(z+2)(z-2)} = \frac{z-3}{z+2}$
11. $\frac{m^4-16}{4m^2-16} = \frac{(m^2+4)(m+2)(m-2)}{4(m+2)(m-2)}$
 $= \frac{m^2+4}{4}$
12. $\frac{6y^2+11y+4}{3y^2+7y+4} = \frac{(3y+4)(2y+1)}{(3y+4)(y+1)} = \frac{2y+1}{y+1}$
13. $\frac{9k^2}{25} \cdot \frac{5}{3k} = \frac{3 \cdot 3 \cdot 5k^2}{5 \cdot 5 \cdot 3k} = \frac{3k^2}{5k} = \frac{3k}{5}$
14. $\frac{15p^3}{9p^2} \div \frac{6p}{10p^2} = \frac{15p^3}{9p^2} \cdot \frac{10p^2}{6p}$
 $= \frac{150p^5}{54p^3}$
 $= \frac{25 \cdot 6p^5}{9 \cdot 6p^3}$
 $= \frac{25p^2}{9}$
15. $\frac{3a+3b}{4c} \cdot \frac{12}{5(a+b)} = \frac{3(a+b)}{4c} \cdot \frac{3 \cdot 4}{5(a+b)}$
 $= \frac{3 \cdot 3}{c \cdot 5}$
 $= \frac{9}{5c}$
16. $\frac{a-3}{16} \div \frac{a-3}{32} = \frac{a-3}{16} \cdot \frac{32}{a-3}$
 $= \frac{a-3}{16} \cdot \frac{16 \cdot 2}{a-3}$
 $= \frac{2}{1} = 2$
17. $\frac{2k-16}{6} \div \frac{4k-32}{3} = \frac{2k-16}{6} \cdot \frac{3}{4k-32}$
 $= \frac{2(k-8)}{6} \cdot \frac{3}{4(k-8)}$
 $= \frac{1}{4}$
18. $\frac{9y-18}{6y+12} \cdot \frac{3y+6}{15y-30} = \frac{9(y-2)}{6(y+2)} \cdot \frac{3(y+2)}{15(y-2)}$
 $= \frac{27}{90} = \frac{3 \cdot 3}{10 \cdot 3} = \frac{3}{10}$
19. $\frac{4a+12}{2a-10} \div \frac{a^2-9}{a^2-a-20}$
 $= \frac{4(a+3)}{2(a-5)} \cdot \frac{(a-5)(a+4)}{(a-3)(a+3)}$
 $= \frac{2(a+4)}{a-3}$
20. $\frac{6r-18}{9r^2+6r-24} \cdot \frac{12r-16}{4r-12}$
 $= \frac{6(r-3)}{3(3r^2+2r-8)} \cdot \frac{4(3r-4)}{4(r-3)}$
 $= \frac{6(r-3)}{3(3r-4)(r+2)} \cdot \frac{4(3r-4)}{4(r-3)}$
 $= \frac{6}{3(r+2)} = \frac{2}{r+2}$
21. $\frac{k^2+4k-12}{k^2+10k+24} \cdot \frac{k^2+k-12}{k^2-9}$
 $= \frac{(k+6)(k-2)}{(k+6)(k+4)} \cdot \frac{(k+4)(k-3)}{(k+3)(k-3)}$
 $= \frac{k-2}{k+3}$

$$\begin{aligned}
 22. \quad \frac{m^2 + 3m + 2}{m^2 + 5m + 4} \div \frac{m^2 + 5m + 6}{m^2 + 10m + 24} \\
 &= \frac{m^2 + 3m + 2}{m^2 + 5m + 4} \cdot \frac{m^2 + 10m + 24}{m^2 + 5m + 6} \\
 &= \frac{(m+1)(m+2)}{(m+4)(m+1)} \cdot \frac{(m+6)(m+4)}{(m+3)(m+2)} \\
 &= \frac{m+6}{m+3}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \frac{2m^2 - 5m - 12}{m^2 - 10m + 24} \div \frac{4m^2 - 9}{m^2 - 9m + 18} \\
 &= \frac{2m^2 - 5m - 12}{m^2 - 10m + 24} \cdot \frac{m^2 - 9m + 18}{4m^2 - 9} \\
 &= \frac{(2m+3)(m-4)(m-6)(m-3)}{(m-6)(m-4)(2m-3)(2m+3)} \\
 &= \frac{m-3}{2m-3}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \frac{4n^2 + 4n - 3}{6n^2 - n - 15} \cdot \frac{8n^2 + 32n + 30}{4n^2 + 16n + 15} \\
 &= \frac{(2n+3)(2n-1)}{(2n+3)(3n-5)} \cdot \frac{2(2n+3)(2n+5)}{(2n+3)(2n+5)} \\
 &= \frac{2(2n-1)}{3n-5}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \frac{a+1}{2} - \frac{a-1}{2} &= \frac{(a+1) - (a-1)}{2} \\
 &= \frac{a+1-a+1}{2} \\
 &= \frac{2}{2} = 1
 \end{aligned}$$

$$26. \quad \frac{3}{p} + \frac{1}{2}$$

Multiply the first term by $\frac{2}{2}$ and the second by $\frac{p}{p}$.

$$\frac{2 \cdot 3}{2 \cdot p} + \frac{p \cdot 1}{p \cdot 2} = \frac{6}{2p} + \frac{p}{2p} = \frac{6+p}{2p}$$

$$27. \quad \frac{6}{5y} - \frac{3}{2} = \frac{6 \cdot 2}{5y \cdot 2} - \frac{3 \cdot 5y}{2 \cdot 5y} = \frac{12-15y}{10y}$$

$$\begin{aligned}
 28. \quad \frac{1}{6m} + \frac{2}{5m} + \frac{4}{m} &= \frac{5 \cdot 1}{5 \cdot 6m} + \frac{6 \cdot 2}{6 \cdot 5m} + \frac{30 \cdot 4}{30 \cdot m} \\
 &= \frac{5}{30m} + \frac{12}{30m} + \frac{120}{30m} \\
 &= \frac{5+12+120}{30m} \\
 &= \frac{137}{30m}
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \frac{1}{m-1} + \frac{2}{m} &= \frac{m \left(\frac{1}{m-1} \right) + \frac{m-1}{m-1} \left(\frac{2}{m} \right)}{m(m-1)} \\
 &= \frac{m+2m-2}{m(m-1)} \\
 &= \frac{3m-2}{m(m-1)}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \frac{5}{2r+3} - \frac{2}{r} &= \frac{5r}{r(2r+3)} - \frac{2(2r+3)}{r(2r+3)} \\
 &= \frac{5r-2(2r+3)}{r(2r+3)} = \frac{5r-4r-6}{r(2r+3)} \\
 &= \frac{r-6}{r(2r+3)}
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \frac{8}{3(a-1)} + \frac{2}{a-1} &= \frac{8}{3(a-1)} + \frac{3}{3} \left(\frac{2}{a-1} \right) \\
 &= \frac{8+6}{3(a-1)} \\
 &= \frac{14}{3(a-1)}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \frac{2}{5(k-2)} + \frac{3}{4(k-2)} &= \frac{4 \cdot 2}{4 \cdot 5(k-2)} + \frac{5 \cdot 3}{5 \cdot 4(k-2)} \\
 &= \frac{8}{20(k-2)} + \frac{15}{20(k-2)} \\
 &= \frac{8+15}{20(k-2)} \\
 &= \frac{23}{20(k-2)}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \frac{4}{x^2 + 4x + 3} + \frac{3}{x^2 - x - 2} \\
 &= \frac{4}{(x+3)(x+1)} + \frac{3}{(x-2)(x+1)} \\
 &= \frac{4(x-2)}{(x-2)(x+3)(x+1)} \\
 &\quad + \frac{3(x+3)}{(x-2)(x+3)(x+1)} \\
 &= \frac{4(x-2) + 3(x+3)}{(x-2)(x+3)(x+1)} \\
 &= \frac{4x-8+3x+9}{(x-2)(x+3)(x+1)} \\
 &= \frac{7x+1}{(x-2)(x+3)(x+1)}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad & \frac{y}{y^2 + 2y - 3} - \frac{1}{y^2 + 4y + 3} \\
 &= \frac{y}{(y+3)(y-1)} - \frac{1}{(y+3)(y+1)} \\
 &= \frac{y(y+1)}{(y+3)(y+1)(y-1)} \\
 &\quad - \frac{1(y-1)}{(y+3)(y+1)(y-1)} \\
 &= \frac{y(y+1) - (y-1)}{(y+3)(y+1)(y-1)} \\
 &= \frac{y^2 + y - y + 1}{(y+3)(y+1)(y-1)} \\
 &= \frac{y^2 + 1}{(y+3)(y+1)(y-1)}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad & \frac{3k}{2k^2 + 3k - 2} - \frac{2k}{2k^2 - 7k + 3} \\
 &= \frac{3k}{(2k-1)(k+2)} - \frac{2k}{(2k-1)(k-3)} \\
 &= \left(\frac{k-3}{k-3} \right) \frac{3k}{(2k-1)(k+2)} \\
 &\quad - \left(\frac{k+2}{k+2} \right) \frac{2k}{(2k-1)(k-3)} \\
 &= \frac{(3k^2 - 9k) - (2k^2 + 4k)}{(2k-1)(k+2)(k-3)} \\
 &= \frac{k^2 - 13k}{(2k-1)(k+2)(k-3)} \\
 &= \frac{k(k-13)}{(2k-1)(k+2)(k-3)}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad & \frac{4m}{3m^2 + 7m - 6} - \frac{m}{3m^2 - 14m + 8} \\
 &= \frac{4m}{(3m-2)(m+3)} - \frac{m}{(3m-2)(m-4)} \\
 &= \frac{4m(m-4)}{(3m-2)(m+3)(m-4)} \\
 &\quad - \frac{m(m+3)}{(3m-2)(m-4)(m+3)} \\
 &= \frac{4m(m-4) - m(m+3)}{(3m-2)(m-4)(m+3)} \\
 &= \frac{4m^2 - 16m - m^2 - 3m}{(3m-2)(m+3)(m-4)} \\
 &= \frac{3m^2 - 19m}{(3m-2)(m+3)(m-4)} \\
 &= \frac{m(3m-19)}{(3m-2)(m+3)(m-4)}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad & \frac{2}{a+2} + \frac{1}{a} + \frac{a-1}{a^2+2a} \\
 &= \frac{2}{a+2} + \frac{1}{a} + \frac{a-1}{a(a+2)} \\
 &= \left(\frac{a}{a} \right) \frac{2}{a+2} + \left(\frac{a+2}{a+2} \right) \frac{1}{a} + \frac{a-1}{a(a+2)} \\
 &= \frac{2a + a + 2 + a - 1}{a(a+2)} \\
 &= \frac{4a+1}{a(a+2)}
 \end{aligned}$$

$$\begin{aligned}
 38. \quad & \frac{5x+2}{x^2-1} + \frac{3}{x^2+x} - \frac{1}{x^2-x} \\
 &= \frac{5x+2}{(x+1)(x-1)} + \frac{3}{x(x+1)} - \frac{1}{x(x-1)} \\
 &= \left(\frac{x}{x} \right) \left(\frac{5x+2}{(x+1)(x-1)} \right) + \left(\frac{x-1}{x-1} \right) \left(\frac{3}{x(x+1)} \right) \\
 &\quad - \left(\frac{x+1}{x+1} \right) \left(\frac{1}{x(x-1)} \right) \\
 &= \frac{x(5x+2) + (x-1)(3) - (x+1)(1)}{x(x+1)(x-1)} \\
 &= \frac{5x^2 + 2x + 3x - 3 - x - 1}{x(x+1)(x-1)} \\
 &= \frac{5x^2 + 4x - 4}{x(x+1)(x-1)}
 \end{aligned}$$

R.4 Equations

Your Turn 1

Solve $3x - 7 = 4(5x + 2) - 7x$.

$$3x - 7 = 20x + 8 - 7x$$

$$3x - 7 = 13x + 8$$

$$-10x = 15$$

$$x = -\frac{15}{10}$$

$$x = -\frac{3}{2}$$

Your Turn 2

Solve $2m^2 + 7m = 15$.

$$2m^2 + 7m - 15 = 0$$

$$(2m-3)(m+5) = 0$$

$$2m-3 = 0 \quad \text{or} \quad m+5 = 0$$

$$m = \frac{3}{2} \quad \text{or} \quad m = -5$$

Your Turn 3Solve $z^2 + 6 = 8z$.

$$z^2 - 8z + 6 = 0$$

Use the quadratic formula with

$$a = 1, b = -8, \text{ and } c = 6.$$

$$\begin{aligned} z &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(6)}}{2(1)} \\ &= \frac{8 \pm \sqrt{64 - 24}}{2} \\ &= \frac{8 \pm \sqrt{40}}{2} \\ &= \frac{8 \pm \sqrt{4 \cdot 10}}{2} \\ &= \frac{8 \pm 2\sqrt{10}}{2} \\ &= 4 \pm \sqrt{10} \end{aligned}$$

Your Turn 4Solve $\frac{1}{x^2 - 4} + \frac{2}{x - 2} = \frac{1}{x}$.

$$\begin{aligned} \frac{1}{(x - 2)(x + 2)} + \frac{2}{x - 2} &= \frac{1}{x} \\ (x - 2)(x + 2)(x) \cdot \frac{1}{(x - 2)(x + 2)} \\ &+ (x - 2)(x + 2)(x) \cdot \frac{2}{x - 2} \\ &= (x - 2)(x + 2)(x) \cdot \frac{1}{x} \\ x + 2x^2 + 4x &= x^2 - 4 \\ x^2 + 5x + 4 &= 0 \\ (x + 1)(x + 4) &= 0 \\ x = -1 \text{ or } x = -4 \end{aligned}$$

Neither of these values makes a denominator equal to zero, so both are solutions.

R.4 Exercises

1. $2x + 8 = x - 4$

$$x + 8 = -4$$

$$x = -12$$

The solution is -12 .

2. $5x + 2 = 8 - 3x$

$$8x + 2 = 8$$

$$8x = 6$$

$$x = \frac{3}{4}$$

The solution is $\frac{3}{4}$.

3. $0.2m - 0.5 = 0.1m + 0.7$

$$10(0.2m - 0.5) = 10(0.1m + 0.7)$$

$$2m - 5 = m + 7$$

$$m - 5 = 7$$

$$m = 12$$

The solution is 12.

4. $\frac{2}{3}k - k + \frac{3}{8} = \frac{1}{2}$

Multiply both sides of the equation by 24.

$$24\left(\frac{2}{3}k\right) - 24(k) + 24\left(\frac{3}{8}\right) = 24\left(\frac{1}{2}\right)$$

$$16k - 24k + 9 = 12$$

$$-8k + 9 = 12$$

$$-8k = 3$$

$$k = -\frac{3}{8}$$

The solution is $-\frac{3}{8}$.

5. $3r + 2 - 5(r + 1) = 6r + 4$

$$3r + 2 - 5r - 5 = 6r + 4$$

$$-3 - 2r = 6r + 4$$

$$-3 = 8r + 4$$

$$-7 = 8r$$

$$-\frac{7}{8} = r$$

The solution is $-\frac{7}{8}$.

6. $5(a + 3) + 4a - 5 = -(2a - 4)$

$$5a + 15 + 4a - 5 = -2a + 4$$

$$9a + 10 = -2a + 4$$

$$11a + 10 = 4$$

$$11a = -6$$

$$a = -\frac{6}{11}$$

The solution is $-\frac{6}{11}$.

$$\begin{aligned}
7. \quad & 2[3m - 2(3 - m) - 4] = 6m - 4 \\
& 2[3m - 6 + 2m - 4] = 6m - 4 \\
& 2[5m - 10] = 6m - 4 \\
& 10m - 20 = 6m - 4 \\
& 4m - 20 = -4 \\
& 4m = 16 \\
& m = 4
\end{aligned}$$

The solution is 4.

$$\begin{aligned}
8. \quad & 4[2p - (3 - p) + 5] = -7p - 2 \\
& 4[2p - 3 + p + 5] = -7p - 2 \\
& 4[3p + 2] = -7p - 2 \\
& 12p + 8 = -7p - 2 \\
& 19p + 8 = -2 \\
& 19p = -10 \\
& p = -\frac{10}{19}
\end{aligned}$$

The solution is $-\frac{10}{19}$.

$$\begin{aligned}
9. \quad & x^2 + 5x + 6 = 0 \\
& (x + 3)(x + 2) = 0 \\
& x + 3 = 0 \quad \text{or} \quad x + 2 = 0 \\
& x = -3 \quad \text{or} \quad x = -2
\end{aligned}$$

The solutions are -3 and -2 .

$$\begin{aligned}
10. \quad & x^2 = 3 + 2x \\
& x^2 - 2x - 3 = 0 \\
& (x - 3)(x + 1) = 0 \\
& x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \\
& x = 3 \quad \text{or} \quad x = -1
\end{aligned}$$

The solutions are 3 and -1 .

$$\begin{aligned}
11. \quad & m^2 = 14m - 49 \\
& m^2 - 14m + 49 = 0 \\
& (m)^2 - 2(7m) + (7)^2 = 0 \\
& (m - 7)^2 = 0 \\
& m - 7 = 0 \\
& m = 7
\end{aligned}$$

The solution is 7.

$$\begin{aligned}
12. \quad & 2k^2 - k = 10 \\
& 2k^2 - k - 10 = 0 \\
& (2k - 5)(k + 2) = 0 \\
& 2k - 5 = 0 \quad \text{or} \quad k + 2 = 0 \\
& k = \frac{5}{2} \quad \text{or} \quad k = -2
\end{aligned}$$

The solutions are $\frac{5}{2}$ and -2 .

$$\begin{aligned}
13. \quad & 12x^2 - 5x = 2 \\
& 12x^2 - 5x - 2 = 0 \\
& (4x + 1)(3x - 2) = 0 \\
& 4x + 1 = 0 \quad \text{or} \quad 3x - 2 = 0 \\
& 4x = -1 \quad \text{or} \quad 3x = 2 \\
& x = -\frac{1}{4} \quad \text{or} \quad x = \frac{2}{3}
\end{aligned}$$

The solutions are $-\frac{1}{4}$ and $\frac{2}{3}$.

$$\begin{aligned}
14. \quad & m(m - 7) = -10 \\
& m^2 - 7m + 10 = 0 \\
& (m - 5)(m - 2) = 0 \\
& m - 5 = 0 \quad \text{or} \quad m - 2 = 0 \\
& m = 5 \quad \text{or} \quad m = 2
\end{aligned}$$

The solutions are 5 and 2.

$$\begin{aligned}
15. \quad & 4x^2 - 36 = 0 \\
& \text{Divide both sides of the equation by 4.} \\
& x^2 - 9 = 0 \\
& (x + 3)(x - 3) = 0 \\
& x + 3 = 0 \quad \text{or} \quad x - 3 = 0 \\
& x = -3 \quad \text{or} \quad x = 3
\end{aligned}$$

The solutions are -3 and 3.

$$\begin{aligned}
16. \quad & z(2z + 7) = 4 \\
& 2z^2 + 7z - 4 = 0 \\
& (2z - 1)(z + 4) = 0 \\
& 2z - 1 = 0 \quad \text{or} \quad z + 4 = 0 \\
& z = \frac{1}{2} \quad \text{or} \quad z = -4
\end{aligned}$$

The solutions are $\frac{1}{2}$ and -4 .

$$\begin{aligned}
17. \quad & 12y^2 - 48y = 0 \\
& 12y(y) - 12y(4) = 0 \\
& 12y(y - 4) = 0 \\
& 12y = 0 \quad \text{or} \quad y - 4 = 0 \\
& y = 0 \quad \text{or} \quad y = 4
\end{aligned}$$

The solutions are 0 and 4.

$$\begin{aligned}
18. \quad & 3x^2 - 5x + 1 = 0 \\
& \text{Use the quadratic formula.} \\
& x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)} \\
& = \frac{5 \pm \sqrt{25 - 12}}{6} \\
& x = \frac{5 + \sqrt{13}}{6} \quad \text{or} \quad x = \frac{5 - \sqrt{13}}{6} \\
& \approx 1.4343 \quad \approx 0.2324
\end{aligned}$$

The solutions are $\frac{5+\sqrt{13}}{6} \approx 1.4343$ and
 $\frac{5-\sqrt{13}}{6} \approx 0.2324$.

19. $2m^2 - 4m = 3$

$$2m^2 - 4m - 3 = 0$$

$$\begin{aligned} m &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{4 \pm \sqrt{40}}{4} = \frac{4 \pm \sqrt{4 \cdot 10}}{4} \\ &= \frac{4 \pm \sqrt{4}\sqrt{10}}{4} \\ &= \frac{4 \pm 2\sqrt{10}}{4} = \frac{2 \pm \sqrt{10}}{2} \end{aligned}$$

The solutions are $\frac{2+\sqrt{10}}{2} \approx 2.5811$ and
 $\frac{2-\sqrt{10}}{2} \approx -0.5811$.

20. $p^2 + p - 1 = 0$

$$\begin{aligned} p &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

The solutions are $\frac{-1+\sqrt{5}}{2} \approx 0.6180$ and
 $\frac{-1-\sqrt{5}}{2} \approx -1.6180$.

21. $k^2 - 10k = -20$

$$k^2 - 10k + 20 = 0$$

$$\begin{aligned} k &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(20)}}{2(1)} \\ k &= \frac{10 \pm \sqrt{100 - 80}}{2} \\ k &= \frac{10 \pm \sqrt{20}}{2} \\ k &= \frac{10 \pm \sqrt{4}\sqrt{5}}{2} \\ k &= \frac{10 \pm 2\sqrt{5}}{2} \\ k &= \frac{2(5 \pm \sqrt{5})}{2} \\ k &= 5 \pm \sqrt{5} \end{aligned}$$

The solutions are $5 + \sqrt{5} \approx 7.2361$ and
 $5 - \sqrt{5} \approx 2.7639$.

22. $5x^2 - 8x + 2 = 0$

$$\begin{aligned} x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(5)(2)}}{2(5)} \\ &= \frac{8 \pm \sqrt{24}}{10} \\ &= \frac{8 \pm \sqrt{4 \cdot 6}}{10} \\ &= \frac{8 \pm \sqrt{4}\sqrt{6}}{10} = \frac{8 \pm 2\sqrt{6}}{10} \\ &= \frac{4 \pm \sqrt{6}}{5} \end{aligned}$$

The solutions are $\frac{4+\sqrt{6}}{5} \approx 1.2899$ and
 $\frac{4-\sqrt{6}}{5} \approx 0.3101$.

23. $2r^2 - 7r + 5 = 0$

$$(2r - 5)(r - 1) = 0$$

$$2r - 5 = 0 \quad \text{or} \quad r - 1 = 0$$

$$2r = 5$$

$$r = \frac{5}{2} \quad \text{or} \quad r = 1$$

The solutions are $\frac{5}{2}$ and 1.

24. $2x^2 - 7x + 30 = 0$

$$\begin{aligned} x &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(30)}}{2(2)} \\ x &= \frac{7 \pm \sqrt{49 - 240}}{4} \\ x &= \frac{7 \pm \sqrt{-191}}{4} \end{aligned}$$

Since there is a negative number under the radical sign, $\sqrt{-191}$ is not a real number. Thus, there are no real number solutions.

25. $3k^2 + k = 6$

$$3k^2 + k - 6 = 0$$

$$\begin{aligned} k &= \frac{-1 \pm \sqrt{1 - 4(3)(-6)}}{2(3)} \\ &= \frac{-1 \pm \sqrt{73}}{6} \end{aligned}$$

The solutions are $\frac{-1+\sqrt{73}}{6} \approx 1.2573$ and
 $\frac{-1-\sqrt{73}}{6} \approx -1.5907$.

$$\begin{aligned}
 26. \quad & 5m^2 + 5m = 0 \\
 & 5m(m + 1) = 0 \\
 & 5m = 0 \quad \text{or} \quad m + 1 = 0 \\
 & m = 0 \quad \text{or} \quad m = -1
 \end{aligned}$$

The solutions are 0 and -1 .

$$\begin{aligned}
 27. \quad & \frac{3x - 2}{7} = \frac{x + 2}{5} \\
 & 35\left(\frac{3x - 2}{7}\right) = 35\left(\frac{x + 2}{5}\right) \\
 & 5(3x - 2) = 7(x + 2) \\
 & 15x - 10 = 7x + 14 \\
 & 8x = 24 \\
 & x = 3
 \end{aligned}$$

The solution is $x = 3$.

$$28. \quad \frac{x}{3} - 7 = 6 - \frac{3x}{4}$$

Multiply both sides by 12, the least common denominator of 3 and 4.

$$\begin{aligned}
 & 12\left(\frac{x}{3} - 7\right) = 12\left(6 - \frac{3x}{4}\right) \\
 & 12\left(\frac{x}{3}\right) - (12)(7) = (12)(6) - 12\left(\frac{3x}{4}\right) \\
 & 4x - 84 = 72 - 9x \\
 & 13x - 84 = 72 \\
 & 13x = 156 \\
 & x = 12
 \end{aligned}$$

The solution is 12.

$$\begin{aligned}
 29. \quad & \frac{4}{x - 3} - \frac{8}{2x + 5} + \frac{3}{x - 3} = 0 \\
 & \frac{4}{x - 3} + \frac{3}{x - 3} - \frac{8}{2x + 5} = 0 \\
 & \frac{7}{x - 3} - \frac{8}{2x + 5} = 0
 \end{aligned}$$

Multiply both sides by $(x - 3)(2x + 5)$. Note that $x \neq 3$ and $x \neq -\frac{5}{2}$.

$$\begin{aligned}
 & (x - 3)(2x + 5)\left(\frac{7}{x - 3} - \frac{8}{2x + 5}\right) \\
 & \quad = (x - 3)(2x + 5)(0) \\
 & 7(2x + 5) - 8(x - 3) = 0 \\
 & 14x + 35 - 8x + 24 = 0 \\
 & 6x + 59 = 0 \\
 & 6x = -59 \\
 & x = -\frac{59}{6}
 \end{aligned}$$

Note: It is especially important to check solutions of equations that involve rational expressions. Here, a check shows that $-\frac{59}{6}$ is a solution.

$$\begin{aligned}
 30. \quad & \frac{5}{p - 2} - \frac{7}{p + 2} = \frac{12}{p^2 - 4} \\
 & \frac{5}{p - 2} - \frac{7}{p + 2} = \frac{12}{(p - 2)(p + 2)}
 \end{aligned}$$

Multiply both sides by $(p - 2)(p + 2)$. Note that $p \neq 2$ and $p \neq -2$.

$$\begin{aligned}
 & (p - 2)(p + 2)\left(\frac{5}{p - 2} - \frac{7}{p + 2}\right) \\
 & \quad = (p - 2)(p + 2)\left(\frac{12}{(p - 2)(p + 2)}\right) \\
 & (p - 2)(p + 2)\left(\frac{5}{p - 2}\right) - (p - 2)(p + 2)\left(\frac{7}{p + 2}\right) \\
 & \quad = (p - 2)(p + 2)\left(\frac{12}{(p - 2)(p + 2)}\right) \\
 & (p + 2)(5) - (p - 2)(7) = 12 \\
 & 5p + 10 - 7p + 14 = 12 \\
 & -2p + 24 = 12 \\
 & -2p = -12 \\
 & p = 6
 \end{aligned}$$

The solution is 6.

$$\begin{aligned}
 31. \quad & \frac{2m}{m - 2} - \frac{6}{m} = \frac{12}{m^2 - 2m} \\
 & \frac{2m}{m - 2} - \frac{6}{m} = \frac{12}{m(m - 2)}
 \end{aligned}$$

Multiply both sides by $m(m - 2)$. Note that $m \neq 0$ and $m \neq 2$.

$$\begin{aligned}
 & m(m - 2)\left(\frac{2m}{m - 2} - \frac{6}{m}\right) \\
 & \quad = m(m - 2)\left(\frac{12}{m(m - 2)}\right) \\
 & m(2m) - 6(m - 2) = 12 \\
 & 2m^2 - 6m + 12 = 12 \\
 & 2m^2 - 6m = 0 \\
 & 2m(m - 3) = 0 \\
 & 2m = 0 \quad \text{or} \quad m - 3 = 0 \\
 & m = 0 \quad \text{or} \quad m = 3
 \end{aligned}$$

Since $m \neq 0$, 0 is not a solution. The solution is 3.

$$32. \quad \frac{2y}{y-1} = \frac{5}{y} + \frac{10-8y}{y^2-y}$$

$$\frac{2y}{y-1} = \frac{5}{y} + \frac{10-8y}{y(y-1)}$$

Multiply both sides by $y(y-1)$. Note that $y \neq 0$ and $y \neq 1$.

$$y(y-1)\left(\frac{2y}{y-1}\right) = y(y-1)\left[\frac{5}{y} + \frac{10-8y}{y(y-1)}\right]$$

$$y(y-1)\left(\frac{2y}{y-1}\right) = y(y-1)\left(\frac{5}{y}\right)$$

$$+ y(y-1)\left[\frac{10-8y}{y(y-1)}\right]$$

$$y(2y) = (y-1)(5) + (10-8y)$$

$$2y^2 = 5y - 5 + 10 - 8y$$

$$2y^2 = 5 - 3y$$

$$2y^2 + 3y - 5 = 0$$

$$(2y+5)(y-1) = 0$$

$$2y+5 = 0 \quad \text{or} \quad y-1 = 0$$

$$y = -\frac{5}{2} \quad \text{or} \quad y = 1$$

Since $y \neq 1$, 1 is not a solution. The solution is $-\frac{5}{2}$.

$$33. \quad \frac{1}{x-2} - \frac{3x}{x-1} = \frac{2x+1}{x^2-3x+2}$$

$$\frac{1}{x-2} - \frac{3x}{x-1} = \frac{2x+1}{(x-2)(x-1)}$$

Multiply both sides by $(x-2)(x-1)$.

Note that $x \neq 2$ and $x \neq 1$.

$$(x-2)(x-1)\left(\frac{1}{x-2} - \frac{3x}{x-1}\right)$$

$$= (x-2)(x-1) \cdot \left[\frac{2x+1}{(x-2)(x-1)}\right]$$

$$(x-2)(x-1)\left(\frac{1}{x-2}\right) - (x-2)(x-1) \cdot \left(\frac{3x}{x-1}\right)$$

$$= \frac{(x-2)(x-1)(2x+1)}{(x-2)(x-1)}$$

$$(x-1) - (x-2)(3x) = 2x+1$$

$$x-1-3x^2+6x = 2x+1$$

$$-3x^2+7x-1 = 2x+1$$

$$-3x^2+5x-2 = 0$$

$$3x^2-5x+2 = 0$$

$$(3x-2)(x-1) = 0$$

$$3x-2 = 0 \quad \text{or} \quad x-1 = 0$$

$$x = \frac{2}{3} \quad \text{or} \quad x = 1$$

1 is not a solution since $x \neq 1$. The solution is $\frac{2}{3}$.

$$34. \quad \frac{5}{a} + \frac{-7}{a+1} = \frac{a^2-2a+4}{a^2+a}$$

$$a(a+1)\left(\frac{5}{a} + \frac{-7}{a+1}\right)$$

$$= a(a+1)\left(\frac{a^2-2a+4}{a^2+a}\right)$$

Note that $a \neq 0$ and $a \neq -1$.

$$5(a+1) + (-7)(a) = a^2-2a+4$$

$$5a+5-7a = a^2-2a+4$$

$$5-2a = a^2-2a+4$$

$$5 = a^2+4$$

$$0 = a^2-1$$

$$0 = (a+1)(a-1)$$

$$a+1 = 0 \quad \text{or} \quad a-1 = 0$$

$$a = -1 \quad \text{or} \quad a = 1$$

Since -1 would make two denominators zero, 1 is the only solution.

$$35. \quad \frac{5}{b+5} - \frac{4}{b^2+2b} = \frac{6}{b^2+7b+10}$$

$$\frac{5}{b+5} - \frac{4}{b(b+2)} = \frac{6}{(b+5)(b+2)}$$

Multiply both sides by $b(b+5)(b+2)$. Note that $b \neq 0$, $b \neq -5$, and $b \neq -2$.

$$b(b+5)(b+2)\left(\frac{5}{b+5} - \frac{4}{b(b+2)}\right)$$

$$= b(b+5)(b+2)\left(\frac{6}{(b+5)(b+2)}\right)$$

$$5b(b + 2) - 4(b + 5) = 6b$$

$$5b^2 + 10b - 4b - 20 = 6b$$

$$5b^2 - 20 = 0$$

$$b^2 - 4 = 0$$

$$(b + 2)(b - 2) = 0$$

$$b + 2 = 0 \quad \text{or} \quad b - 2 = 0$$

$$b = -2 \quad \text{or} \quad b = 2$$

Since $b \neq -2$, -2 is not a solution. The solution is 2.

$$\begin{aligned} 36. \quad & \frac{2}{x^2 - 2x - 3} + \frac{5}{x^2 - x - 6} \\ &= \frac{1}{x^2 + 3x + 2} \\ & \frac{2}{(x - 3)(x + 1)} + \frac{5}{(x - 3)(x + 2)} \\ &= \frac{1}{(x + 2)(x + 1)} \end{aligned}$$

Multiply both sides by $(x - 3)(x + 1)(x + 2)$.

Note that $x \neq 3$, $x \neq -1$, and $x \neq -2$.

$$\begin{aligned} (x - 3)(x + 1)(x + 2) & \left(\frac{2}{(x - 3)(x + 1)} \right) \\ &+ (x - 3)(x + 1)(x + 2) \left(\frac{5}{(x - 3)(x + 2)} \right) \\ &= (x - 3)(x + 1)(x + 2) \left(\frac{1}{(x + 2)(x + 1)} \right) \\ 2(x + 2) + 5(x + 1) &= x - 3 \\ 2x + 4 + 5x + 5 &= x - 3 \\ 7x + 9 &= x - 3 \\ 6x + 9 &= -3 \\ 6x &= -12 \\ x &= -2 \end{aligned}$$

However, $x \neq -2$. Therefore there is no solution.

$$\begin{aligned} 37. \quad & \frac{4}{2x^2 + 3x - 9} + \frac{2}{2x^2 - x - 3} \\ &= \frac{3}{x^2 + 4x + 3} \\ & \frac{4}{(2x - 3)(x + 3)} + \frac{2}{(2x - 3)(x + 1)} \\ &= \frac{3}{(x + 3)(x + 1)} \end{aligned}$$

Multiply both sides by $(2x - 3)(x + 3)(x + 1)$. Note that $x \neq \frac{3}{2}$, $x \neq -3$, and $x \neq -1$.

$$(2x - 3)(x + 3)(x + 1)$$

$$\cdot \left(\frac{4}{(2x - 3)(x + 3)} + \frac{2}{(2x - 3)(x + 1)} \right)$$

$$= (2x - 3)(x + 3)(x + 1) \left(\frac{3}{(x + 3)(x + 1)} \right)$$

$$4(x + 1) + 2(x + 3) = 3(2x - 3)$$

$$4x + 4 + 2x + 6 = 6x - 9$$

$$6x + 10 = 6x - 9$$

$$10 = -9$$

This is a false statement. Therefore, there is no solution.

R.5 Inequalities

Your Turn 1

Solve $3z - 2 > 5z + 7$.

$$3z - 2 > 5z + 7$$

$$3z - 2 + 2 > 5z + 7 + 2$$

$$3z > 5z + 9$$

$$3z - 5z > 5z - 5z + 9$$

$$-2z > 9$$

$$\frac{-2z}{-2} < \frac{9}{-2}$$

$$z < -\frac{9}{2}$$

Your Turn 2

Solve $3y^2 \leq 16y + 12$.

$$3y^2 \leq 16y + 12$$

$$3y^2 - 16y - 12 \leq 0$$

First solve the equation $3y^2 - 16y - 12 = 0$.

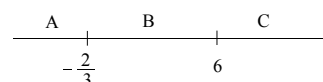
$$3y^2 - 16y - 12 = 0$$

$$(3y + 2)(y - 6) = 0$$

$$3y + 2 = 0 \quad \text{or} \quad y - 6 = 0$$

$$y = -\frac{2}{3} \quad \text{or} \quad y = 6$$

Determine three intervals on the number line and choose a test point in each interval.



Choose -1 from interval A: $3(-1)^2 - 16(-1) - 12 > 0$

Choose 0 from interval B: $3(0)^2 - 16(0) - 12 < 0$

Choose 7 from interval C: $3(7)^2 - 16(7) - 12 > 0$

The numbers in interval B satisfy the inequality, and since the sign was less than or equal to, the boundary points of interval B are also part of the solution. The solution is $[-2/3, 6]$.

Your Turn 3

Solve $\frac{k^2 - 35}{k} \geq 2$.

First solve the corresponding equation $\frac{k^2 - 35}{k} = 2$.

$$\frac{k^2 - 35}{k} = 2$$

$$k^2 - 35 = 2k$$

$$k^2 - 2k - 35 = 0$$

$$(k - 7)(k + 5) = 0$$

$$k = 7 \text{ or } k = -5$$

The denominator is 0 when $k = 0$, so there are four intervals to consider:

$(-\infty, -5)$, $(-5, 0)$, $(0, 7)$, and $(7, \infty)$.

Choose a test point in each interval.

$$k = -8; \frac{(-8)^2 - 35}{-8} < 2$$

$$k = -1; \frac{(-1)^2 - 35}{-1} > 2$$

$$k = 5; \frac{(-5)^2 - 35}{5} < 2$$

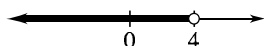
$$k = 10; \frac{(10)^2 - 35}{10} > 2$$

The second and fourth intervals are part of the solution. Since the inequality is greater than or equal to, we can include the endpoints -5 and 7 but not the endpoint 0 , which makes the denominator 0. The solution is $[-5, 0) \cup [7, \infty)$.

R.5 Exercises

1. $x < 4$

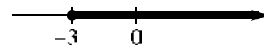
Because the inequality symbol means “less than,” the endpoint at 4 is not included. This inequality is written in interval notation as $(-\infty, 4)$. To graph this interval on a number line, place an open circle at 4 and draw a heavy arrow pointing to the left.



2. $x \geq -3$

Because the inequality sign means “greater than or equal to,” the endpoint at -3 is included. This inequality is written in interval notation as $[-3, \infty)$.

To graph this interval on a number line, place a closed circle at -3 and draw a heavy arrow pointing to the right.



3. $1 \leq x < 2$

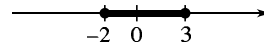
The endpoint at 1 is included, but the endpoint at 2 is not. This inequality is written in interval notation as $[1, 2)$. To graph this interval, place a closed circle at 1 and an open circle at 2; then draw a heavy line segment between them.



4. $-2 \leq x \leq 3$

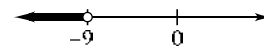
The endpoints at -2 and 3 are both included. This inequality is written in interval notation as $[-2, 3]$.

To graph this interval, place an open circle at -2 and another at 3 and draw a heavy line segment between them.



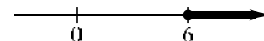
5. $-9 > x$

This inequality may be rewritten as $x < -9$, and is written in interval notation as $(-\infty, -9)$. Note that the endpoint at -9 is not included. To graph this interval, place an open circle at -9 and draw a heavy arrow pointing to the left.



6. $6 \leq x$

This inequality may be written as $x \geq 6$, and is written in interval notation as $[6, \infty)$. Note that the endpoint at 6 is included. To graph this interval, place a closed circle at 6 and draw a heavy arrow pointing to the right.



7. $[-7, -3]$

This represents all the numbers between -7 and -3 , including both endpoints. This interval can be written as the inequality $-7 \leq x \leq -3$.

8. $[4, 10)$

This represents all the numbers between 4 and 10, including 4 but not including 10. This interval can be written as the inequality $4 \leq x < 10$.

9. $(-\infty, -1]$

This represents all the numbers to the left of -1 on the number line and includes the endpoint. This interval can be written as the inequality $x \leq -1$.

10. $(3, \infty)$

This represents all the numbers to the right of 3, and does not include the endpoint. This interval can be written as the inequality $x > 3$.

11. Notice that the endpoint -2 is included, but 6 is not. The interval shown in the graph can be written as the inequality $-2 \leq x < 6$.

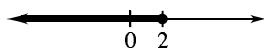
12. Notice that neither endpoint is included. The interval shown in the graph can be written as $0 < x < 8$.

13. Notice that both endpoints are included. The interval shown in the graph can be written as $x \leq -4$ or $x \geq 4$.

14. Notice that the endpoint 0 is not included, but 3 is included. The interval shown in the graph can be written as $x < 0$ or $x \geq 3$.

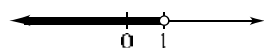
15. $6p + 7 \leq 19$
 $6p \leq 12$
 $\left(\frac{1}{6}\right)(6p) \leq \left(\frac{1}{6}\right)(12)$
 $p \leq 2$

The solution in interval notation is $(-\infty, 2]$.



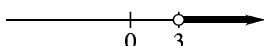
16. $6k - 4 < 3k - 1$
 $6k < 3k + 3$
 $3k < 3$
 $k < 1$

The solution in interval notation is $(-\infty, 1)$.



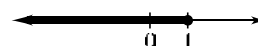
17. $m - (3m - 2) + 6 < 7m - 19$
 $m - 3m + 2 + 6 < 7m - 19$
 $-2m + 8 < 7m - 19$
 $-9m + 8 < -19$
 $-9m < -27$
 $-\frac{1}{9}(-9m) > -\frac{1}{9}(-27)$
 $m > 3$

The solution is $(3, \infty)$.



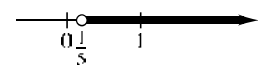
18. $-2(3y - 8) \geq 5(4y - 2)$
 $-6y + 16 \geq 20y - 10$
 $-6y + 16 + (-16) \geq 20y - 10 + (-16)$
 $-6y \geq 20y - 26$
 $-6y + (-20y) \geq 20y + (-20y) - 26$
 $-26y \geq -26$
 $-\frac{1}{26}(-26)y \leq -\frac{1}{26}(-26)$
 $y \leq 1$

The solution is $(-\infty, 1]$.



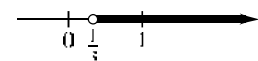
19. $3p - 1 < 6p + 2(p - 1)$
 $3p - 1 < 6p + 2p - 2$
 $3p - 1 < 8p - 2$
 $-5p - 1 < -2$
 $-5p < -1$
 $-\frac{1}{5}(-5p) > -\frac{1}{5}(-1)$
 $p > \frac{1}{5}$

The solution is $\left(\frac{1}{5}, \infty\right)$.



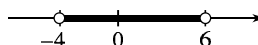
20. $x + 5(x + 1) > 4(2 - x) + x$
 $x + 5x + 5 > 8 - 4x + x$
 $6x + 5 > 8 - 3x$
 $6x > 3 - 3x$
 $9x > 3$
 $x > \frac{1}{3}$

The solution is $\left(\frac{1}{3}, \infty\right)$.



21. $-11 < y - 7 < -1$
 $-11 + 7 < y - 7 + 7 < -1 + 7$
 $-4 < y < 6$

The solution is $(-4, 6)$.



22. $8 \leq 3r + 1 \leq 13$

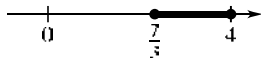
$$8 + (-1) \leq 3r + 1 + (-1) \leq 13 + (-1)$$

$$7 \leq 3r \leq 12$$

$$\frac{1}{3}(7) \leq \frac{1}{3}(3r) \leq \frac{1}{3}(12)$$

$$\frac{7}{3} \leq r \leq 4$$

The solution is $\left[\frac{7}{3}, 4\right]$.



23. $-2 < \frac{1-3k}{4} \leq 4$

$$4(-2) < 4\left(\frac{1-3k}{4}\right) \leq 4(4)$$

$$-8 < 1 - 3k \leq 16$$

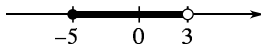
$$-9 < -3k \leq 15$$

$$-\frac{1}{3}(-9) > -\frac{1}{3}(-3k) \geq -\frac{1}{3}(15)$$

Rewrite the inequalities in the proper order.

$$-5 \leq k < 3$$

The solution is $[-5, 3)$.



24. $-1 \leq \frac{5y+2}{3} \leq 4$

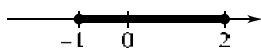
$$3(-1) \leq 3\left(\frac{5y+2}{3}\right) \leq 3(4)$$

$$-3 \leq 5y + 2 \leq 12$$

$$-5 \leq 5y \leq 10$$

$$-1 \leq y \leq 2$$

The solution is $[-1, 2]$.



25. $\frac{3}{5}(2p+3) \geq \frac{1}{10}(5p+1)$

$$10\left(\frac{3}{5}\right)(2p+3) \geq 10\left(\frac{1}{10}\right)(5p+1)$$

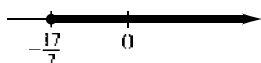
$$6(2p+3) \geq 5p+1$$

$$12p+18 \geq 5p+1$$

$$7p \geq -17$$

$$p \geq -\frac{17}{7}$$

The solution is $\left[-\frac{17}{7}, \infty\right)$.



26. $\frac{8}{3}(z-4) \leq \frac{2}{9}(3z+2)$

$$(9)\frac{8}{3}(z-4) \leq (9)\frac{2}{9}(3z+2)$$

$$24(z-4) \leq 2(3z+2)$$

$$24z - 96 \leq 6z + 4$$

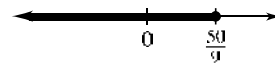
$$24z \leq 6z + 100$$

$$18z \leq 100$$

$$z \leq \frac{100}{18}$$

$$z \leq \frac{50}{9}$$

The solution is $\left(-\infty, \frac{50}{9}\right]$.



27. $(m-3)(m+5) < 0$

Solve $(m-3)(m+5) = 0$.

$$(m-3)(m+5) = 0$$

$$m = 3 \quad \text{or} \quad m = -5$$

Intervals: $(-\infty, -5)$, $(-5, 3)$, $(3, \infty)$

For $(-\infty, -5)$, choose -6 to test for m .

$$(-6-3)(-6+5) = -9(-1) = 9 \not< 0$$

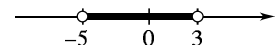
For $(-5, 3)$, choose 0 .

$$(0-3)(0+5) = -3(5) = -15 < 0$$

For $(3, \infty)$, choose 4 .

$$(4-3)(4+5) = 1(9) = 9 \not< 0$$

The solution is $(-5, 3)$.



28. $(t+6)(t-1) \geq 0$

Solve $(t+6)(t-1) = 0$.

$$(t+6)(t-1) = 0$$

$$t = -6 \quad \text{or} \quad t = 1$$

Intervals: $(-\infty, -6)$, $(-6, 1)$, $(1, \infty)$

For $(-\infty, -6)$, choose -7 to test for t .

$$(-7+6)(-7-1) = (-1)(-8) = 8 \geq 0$$

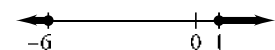
For $(-6, 1)$, choose 0 .

$$(0+6)(0-1) = (6)(-1) = -6 \not\geq 0$$

For $(1, \infty)$, choose 2 .

$$(2+6)(2-1) = (8)(1) = 8 \geq 0$$

Because the symbol $\geq$ is used, the endpoints -6 and 1 are included in the solution. The solution is $(-\infty, -6] \cup [1, \infty)$.



29. $y^2 - 3y + 2 < 0$

$$(y - 2)(y - 1) < 0$$

Solve $(y - 2)(y - 1) = 0$.

$$y = 2 \quad \text{or} \quad y = 1$$

Intervals: $(-\infty, 1)$, $(1, 2)$, $(2, \infty)$

For $(-\infty, 1)$, choose $y = 0$.

$$0^2 - 3(0) + 2 = 2 \not< 0$$

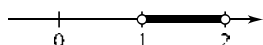
For $(1, 2)$, choose $y = \frac{3}{2}$.

$$\begin{aligned} \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 &= \frac{9}{4} - \frac{9}{2} + 2 \\ &= \frac{9 - 18 + 8}{4} \\ &= -\frac{1}{4} < 0 \end{aligned}$$

For $(2, \infty)$, choose 3.

$$3^2 - 3(3) + 2 = 2 \not< 0$$

The solution is $(1, 2)$.



30. $2k^2 + 7k - 4 > 0$

Solve $2k^2 + 7k - 4 = 0$.

$$2k^2 + 7k - 4 = 0$$

$$(2k - 1)(k + 4) = 0$$

$$k = \frac{1}{2} \quad \text{or} \quad k = -4$$

Intervals: $(-\infty, -4)$, $(-4, \frac{1}{2})$, $(\frac{1}{2}, \infty)$

For $(-\infty, -4)$, choose -5 .

$$2(-5)^2 + 7(-5) - 4 = 11 > 0$$

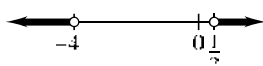
For $(-4, \frac{1}{2})$, choose 0.

$$2(0)^2 + 7(0) - 4 = -4 \not> 0$$

For $(\frac{1}{2}, \infty)$, choose 1.

$$2(1)^2 + 7(1) - 4 = 5 > 0$$

The solution is $(-\infty, -4) \cup (\frac{1}{2}, \infty)$.



31. $x^2 - 16 > 0$

Solve $x^2 - 16 = 0$.

$$x^2 - 16 = 0$$

$$(x + 4)(x - 4) = 0$$

$$x = -4 \quad \text{or} \quad x = 4$$

Intervals: $(-\infty, -4)$, $(-4, 4)$, $(4, \infty)$

For $(-\infty, -4)$, choose -5 .

$$(-5)^2 - 16 = 9 > 0$$

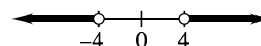
For $(-4, 4)$, choose 0.

$$0^2 - 16 = -16 \not> 0$$

For $(4, \infty)$, choose 5.

$$5^2 - 16 = 9 > 0$$

The solution is $(-\infty, -4) \cup (4, \infty)$.



32. $2k^2 - 7k - 15 \leq 0$

Solve $2k^2 - 7k - 15 = 0$.

$$2k^2 - 7k - 15 = 0$$

$$(2k + 3)(k - 5) = 0$$

$$k = -\frac{3}{2} \quad \text{or} \quad k = 5$$

Intervals: $(-\infty, -\frac{3}{2})$, $(-\frac{3}{2}, 5)$, $(5, \infty)$

For $(-\infty, -\frac{3}{2})$, choose -2 .

$$2(-2)^2 - 7(-2) - 15 = 7 \not\leq 0$$

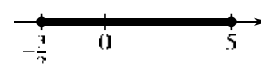
For $(-\frac{3}{2}, 5)$, choose 0.

$$2(0)^2 - 7(0) - 15 = -15 \leq 0$$

For $(5, \infty)$, choose 6.

$$2(6)^2 - 7(6) - 15 \not\leq 0$$

The solution is $[-\frac{3}{2}, 5]$.



33. $x^2 - 4x \geq 5$

Solve $x^2 - 4x = 5$.

$$x^2 - 4x = 5$$

$$x^2 - 4x - 5 = 0$$

$$(x + 1)(x - 5) = 0$$

$$x + 1 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = -1 \quad \text{or} \quad x = 5$$

Intervals: $(-\infty, -1)$, $(-1, 5)$, $(5, \infty)$

For $(-\infty, -1)$, choose -2 .

$$(-2)^2 - 4(-2) = 12 \geq 5$$

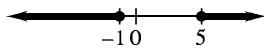
For $(-1, 5)$, choose 0.

$$0^2 - 4(0) = 0 \not\geq 5$$

For $(5, \infty)$, choose 6.

$$(6)^2 - 4(6) = 12 \geq 5$$

The solution is $(-\infty, -1] \cup [5, \infty)$.



34. $10r^2 + r \leq 2$

Solve $10r^2 + r = 2$.

$$10r^2 + r = 2$$

$$10r^2 + r - 2 = 0$$

$$(5r - 2)(2r + 1) = 0$$

$$r = \frac{2}{5} \quad \text{or} \quad r = -\frac{1}{2}$$

Intervals: $(-\infty, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{2}{5})$, $(\frac{2}{5}, \infty)$

For $(-\infty, -\frac{1}{2})$, choose -1 .

$$10(-1)^2 + (-1) = 9 \not\leq 2$$

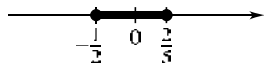
For $(-\frac{1}{2}, \frac{2}{5})$, choose 0 .

$$10(0)^2 + 0 = 0 \leq 2$$

For $(\frac{2}{5}, \infty)$, choose 1 .

$$10(1)^2 + 1 = 11 \not\leq 2$$

The solution is $[-\frac{1}{2}, \frac{2}{5}]$.



35. $3x^2 + 2x > 1$

Solve $3x^2 + 2x = 1$.

$$3x^2 + 2x = 1$$

$$3x^2 + 2x - 1 = 0$$

$$(3x - 1)(x + 1) = 0$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -1$$

Intervals: $(-\infty, -1)$, $(-1, \frac{1}{3})$, $(\frac{1}{3}, \infty)$

For $(-\infty, -1)$, choose -2 .

$$3(-2)^2 + 2(-2) = 8 > 1$$

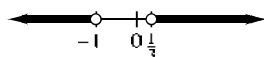
For $(-1, \frac{1}{3})$, choose 0 .

$$3(0)^2 + 2(0) = 0 \not> 1$$

For $(\frac{1}{3}, \infty)$, choose 1 .

$$3(1)^2 + 2(1) = 5 > 1$$

The solution is $(-\infty, -1) \cup (\frac{1}{3}, \infty)$.



36. $3a^2 + a > 10$

Solve $3a^2 + a = 10$.

$$3a^2 + a = 10$$

$$3a^2 + a - 10 = 0$$

$$(3a - 5)(a + 2) = 0$$

$$a = \frac{5}{3} \quad \text{or} \quad a = -2$$

Intervals: $(-\infty, -2)$, $(-2, \frac{5}{3})$, $(\frac{5}{3}, \infty)$

For $(-\infty, -2)$, choose -3 .

$$3(-3)^2 + (-3) = 24 > 10$$

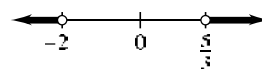
For $(-2, \frac{5}{3})$, choose 0 .

$$3(0)^2 + 0 = 0 \not> 10$$

For $(\frac{5}{3}, \infty)$, choose 2 .

$$3(2)^2 + 2 = 14 > 10$$

The solution is $(-\infty, -2) \cup (\frac{5}{3}, \infty)$.



37. $9 - x^2 \leq 0$

Solve $9 - x^2 = 0$.

$$9 - x^2 = 0$$

$$(3 + x)(3 - x) = 0$$

$$x = -3 \quad \text{or} \quad x = 3$$

Intervals: $(-\infty, -3)$, $(-3, 3)$, $(3, \infty)$

For $(-\infty, -3)$, choose -4 .

$$9 - (-4)^2 = -7 \leq 0$$

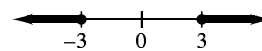
For $(-3, 3)$, choose 0 .

$$9 - (0)^2 = 9 \not\leq 0$$

For $(3, \infty)$, choose 4 .

$$9 - (4)^2 = -7 \leq 0$$

The solution is $(-\infty, -3] \cup [3, \infty)$.



38. $p^2 - 16p > 0$

Solve $p^2 - 16p = 0$.

$$p^2 - 16p = 0$$

$$p(p - 16) = 0$$

$$p = 0 \quad \text{or} \quad p = 16$$

Intervals: $(-\infty, 0)$, $(0, 16)$, $(16, \infty)$

For $(-\infty, 0)$, choose -1 .

$$(-1)^2 - 16(-1) = 17 > 0$$

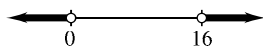
For $(0, 16)$, choose 1.

$$(1)^2 - 16(1) = -15 \not> 0$$

For $(16, \infty)$, choose 17.

$$(17)^2 - 16(17) = 17 > 0$$

The solution is $(-\infty, 0) \cup (16, \infty)$.



39. $x^3 - 4x \geq 0$

Solve $x^3 - 4x = 0$.

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$$x(x + 2)(x - 2) = 0$$

$$x = 0, \text{ or } x = -2, \text{ or } x = 2$$

Intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, $(2, \infty)$

For $(-\infty, -2)$, choose -3 .

$$(-3)^3 - 4(-3) = -15 \not\geq 0$$

For $(-2, 0)$, choose -1 .

$$(-1)^3 - 4(-1) = 3 \geq 0$$

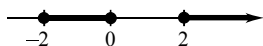
For $(0, 2)$, choose 1.

$$(1)^3 - 4(1) = -3 \not\geq 0$$

For $(2, \infty)$, choose 3.

$$(3)^3 - 4(3) = 15 \geq 0$$

The solution is $[-2, 0] \cup [2, \infty)$.



40. $x^3 + 7x^2 + 12x \leq 0$

Solve $x^3 + 7x^2 + 12x = 0$.

$$x^3 + 7x^2 + 12x = 0$$

$$x(x^2 + 7x + 12) = 0$$

$$x(x + 3)(x + 4) = 0$$

$$x = 0, \text{ or } x = -3, \text{ or } x = -4$$

Intervals: $(-\infty, -4)$, $(-4, -3)$, $(-3, 0)$, $(0, \infty)$

For $(-\infty, -4)$, choose -5 .

$$(-5)^3 + 7(-5)^2 + 12(-5) = -10 \leq 0$$

For $(-4, -3)$, choose $-\frac{7}{2}$.

$$\left(-\frac{7}{2}\right)^3 + 7\left(-\frac{7}{2}\right)^2 + 12\left(-\frac{7}{2}\right) = \frac{7}{8} \not\leq 0$$

For $(-3, 0)$, choose -1 .

$$(-1)^3 + 7(-1)^2 + 12(-1) = -6 \leq 0$$

For $(0, \infty)$, choose 1.

$$(1)^3 + 7(1)^2 + 12(1) = 20 \not\leq 0$$

The solution is $(-\infty, -4] \cup [-3, 0]$.



41. $2x^3 - 14x^2 + 12x < 0$

Solve $2x^3 - 14x^2 + 12x = 0$.

$$2x^3 - 14x^2 + 12x = 0$$

$$2x(x^2 - 7x + 6) = 0$$

$$2x(x - 1)(x - 6) = 0$$

$$x = 0, \text{ or } x = 1, \text{ or } x = 6$$

Intervals: $(-\infty, 0)$, $(0, 1)$, $(1, 6)$, $(6, \infty)$

For $(-\infty, 0)$, choose -1 .

$$2(-1)^3 - 14(-1)^2 + 12(-1) = -28 < 0$$

For $(0, 1)$, choose $\frac{1}{2}$.

$$2\left(\frac{1}{2}\right)^3 - 14\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) = \frac{11}{4} \not< 0$$

For $(1, 6)$, choose 2.

$$2(2)^3 - 14(2)^2 + 12(2) = -16 < 0$$

For $(6, \infty)$, choose 7.

$$2(7)^3 - 14(7)^2 + 12(7) = 84 \not< 0$$

The solution is $(-\infty, 0) \cup (1, 6)$.



42. $3x^3 - 9x^2 - 12x > 0$

Solve $3x^3 - 9x^2 - 12x = 0$.

$$3x^3 - 9x^2 - 12x = 0$$

$$3x(x^2 - 3x - 4) = 0$$

$$3x(x - 4)(x + 1) = 0$$

$$x = 0, \text{ or } x = 4, \text{ or } x = -1$$

Intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 4)$, $(4, \infty)$

For $(-\infty, -1)$, choose -2 .

$$3(-2)^3 - 9(-2)^2 - 12(-2) = -36 \not> 0$$

For $(-1, 0)$, choose $-\frac{1}{2}$.

$$3\left(-\frac{1}{2}\right)^3 - 9\left(-\frac{1}{2}\right)^2 - 12\left(-\frac{1}{2}\right) = \frac{27}{8} > 0$$

For $(0, 4)$, choose 1.

$$3(1)^3 - 9(1)^2 - 12(1) = -18 \not> 0$$

For $(4, \infty)$, choose 5.

$$3(5)^3 - 9(5)^2 - 12(5) = 90 > 0$$

The solution is $(-1, 0) \cup (4, \infty)$.



43. $\frac{m-3}{m+5} \leq 0$

Solve $\frac{m-3}{m+5} = 0$.

$$(m+5)\frac{m-3}{m+5} = (m+5)(0)$$

$$m-3 = 0$$

$$m = 3$$

Set the denominator equal to 0 and solve.

$$m+5 = 0$$

$$m = -5$$

Intervals: $(-\infty, -5)$, $(-5, 3)$, $(3, \infty)$

For $(-\infty, -5)$, choose -6.

$$\frac{-6-3}{-6+5} = 9 \not\leq 0$$

For $(-5, 3)$, choose 0.

$$\frac{0-3}{0+5} = -\frac{3}{5} \leq 0$$

For $(3, \infty)$, choose 4.

$$\frac{4-3}{4+5} = \frac{1}{9} \not\leq 0$$

Although the $\leq$ symbol is used, including -5 in the solution would cause the denominator to be zero.

The solution is $(-5, 3]$.

44. $\frac{r+1}{r-1} > 0$

Solve the equation $\frac{r+1}{r-1} = 0$.

$$\frac{r+1}{r-1} = 0$$

$$(r-1)\frac{r+1}{r-1} = (r-1)(0)$$

$$r+1 = 0 \quad r = -1$$

Find the value for which the denominator equals zero.

$$r-1 = 0$$

$$r = 1$$

Intervals: $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$

For $(-\infty, -1)$, choose -2.

$$\frac{-2+1}{-2-1} = \frac{-1}{-3} = \frac{1}{3} > 0$$

For $(-1, 1)$, choose 0.

$$\frac{0+1}{0-1} = \frac{1}{-1} = -1 \not> 0$$

For $(1, \infty)$, choose 2.

$$\frac{2+1}{2-1} = \frac{3}{1} = 3 > 0$$

The solution is $(-\infty, -1) \cup (1, \infty)$.

45. $\frac{k-1}{k+2} > 1$

Solve $\frac{k-1}{k+2} = 1$.

$$k-1 = k+2$$

$$-1 \neq 2$$

The equation has no solution. Solve $k+2 = 0$.

$$k = -2$$

Intervals: $(-\infty, -2)$, $(-2, \infty)$

For $(-\infty, -2)$, choose -3.

$$\frac{-3-1}{-3+2} = 4 > 1$$

For $(-2, \infty)$, choose 0.

$$\frac{0-1}{0+2} = -\frac{1}{2} \not> 1$$

The solution is $(-\infty, -2)$.

46. $\frac{a-5}{a+2} < -1$

Solve the equation $\frac{a-5}{a+2} = -1$.

$$\frac{a-5}{a+2} = -1$$

$$a-5 = -1(a+2)$$

$$a-5 = -a-2$$

$$2a = 3$$

$$a = \frac{3}{2}$$

Set the denominator equal to zero and solve for a .

$$a+2 = 0$$

$$a = -2$$

Intervals: $(-\infty, -2)$, $(-2, \frac{3}{2})$, $(\frac{3}{2}, \infty)$

For $(-\infty, -2)$, choose -3.

$$\frac{-3-5}{-3+2} = \frac{-8}{-1} = 8 \not< -1$$

For $(-2, \frac{3}{2})$, choose 0.

$$\frac{0-5}{0+2} = \frac{-5}{2} = -\frac{5}{2} < -1$$

For $(\frac{3}{2}, \infty)$, choose 2.

$$\frac{2-5}{2+2} = \frac{-3}{4} = -\frac{3}{4} \not< -1$$

The solution is $(-2, \frac{3}{2})$.

47. $\frac{2y+3}{y-5} \leq 1$

Solve $\frac{2y+3}{y-5} = 1$.

$$\begin{aligned} 2y+3 &= y-5 \\ y &= -8 \end{aligned}$$

Solve $y-5 = 0$.

$$y = 5$$

Intervals: $(-\infty, -8)$, $(-8, 5)$, $(5, \infty)$

For $(-\infty, -8)$, choose $y = -10$.

$$\frac{2(-10)+3}{-10-5} = \frac{17}{15} \not\leq 1$$

For $(-8, 5)$, choose $y = 0$.

$$\frac{2(0)+3}{0-5} = -\frac{3}{5} \leq 1$$

For $(5, \infty)$, choose $y = 6$.

$$\frac{2(6)+3}{6-5} = \frac{15}{1} \not\leq 1$$

The solution is $[-8, 5)$.

48. $\frac{a+2}{3+2a} \leq 5$

Solve the equation $\frac{a+2}{3+2a} = 5$.

$$\begin{aligned} \frac{a+2}{3+2a} &= 5 \\ a+2 &= 5(3+2a) \\ a+2 &= 15+10a \\ -9a &= 13 \\ a &= -\frac{13}{9} \end{aligned}$$

Set the denominator equal to zero and solve for a .

$$\begin{aligned} 3+2a &= 0 \\ 2a &= -3 \\ a &= -\frac{3}{2} \end{aligned}$$

Intervals: $(-\infty, -\frac{3}{2})$, $(-\frac{3}{2}, -\frac{13}{9})$, $(-\frac{13}{9}, \infty)$

For $(-\infty, -\frac{3}{2})$, choose -2 .

$$\frac{-2+2}{3+2(-2)} = \frac{0}{-1} = 0 \leq 5$$

For $(-\frac{3}{2}, -\frac{13}{9})$, choose -1.46 .

$$\frac{-1.46+2}{3+2(-1.46)} = \frac{0.54}{0.08} = 6.75 \not\leq 5$$

For $(-\frac{13}{9}, \infty)$, choose 0.

$$\frac{0+2}{3+2(0)} = \frac{2}{3} \leq 5$$

The value $-\frac{3}{2}$ cannot be included in the solution since it would make the denominator zero. The solution is $(-\infty, -\frac{3}{2}) \cup [-\frac{13}{9}, \infty)$.

49. $\frac{2k}{k-3} \leq \frac{4}{k-3}$

Solve $\frac{2k}{k-3} = \frac{4}{k-3}$.

$$\begin{aligned} \frac{2k}{k-3} &= \frac{4}{k-3} \\ \frac{2k}{k-3} - \frac{4}{k-3} &= 0 \\ \frac{2k-4}{k-3} &= 0 \\ 2k-4 &= 0 \\ k &= 2 \end{aligned}$$

Set the denominator equal to 0 and solve for k .

$$\begin{aligned} k-3 &= 0 \\ k &= 3 \end{aligned}$$

Intervals: $(-\infty, 2)$, $(2, 3)$, $(3, \infty)$

For $(-\infty, 2)$, choose 0.

$$\begin{aligned} \frac{2(0)}{0-3} &= 0 \quad \text{and} \quad \frac{4}{0-3} = -\frac{4}{3}, \text{ so} \\ \frac{2(0)}{0-3} &\not\leq \frac{4}{0-3}. \end{aligned}$$

For $(2, 3)$, choose $\frac{5}{2}$.

$$\frac{2(\frac{5}{2})}{\frac{5}{2}-3} = \frac{5}{-\frac{1}{2}} = -10$$

$$\text{and } \frac{4}{\frac{5}{2}-3} = \frac{4}{-\frac{1}{2}} = -8, \text{ so}$$

$$\frac{2(\frac{5}{2})}{\frac{5}{2}-3} \leq \frac{4}{\frac{5}{2}-3}.$$

For $(3, \infty)$, choose 4.

$$\frac{2(4)}{4-3} = 8 \quad \text{and} \quad \frac{4}{4-3} = 4, \text{ so}$$

$$\frac{2(4)}{4-3} \neq \frac{4}{4-3}.$$

The solution is $[2, 3)$.

50. $\frac{5}{p+1} > \frac{12}{p+1}$

Solve the equation $\frac{5}{p+1} = \frac{12}{p+1}$.

$$\frac{5}{p+1} = \frac{12}{p+1}$$

$$5 = 12$$

The equation has no solution.

Set the denominator equal to zero and solve for p .

$$p+1 = 0$$

$$p = -1$$

Intervals: $(-\infty, -1)$, $(-1, \infty)$

For $(-\infty, -1)$, choose -2 .

$$\frac{5}{-2+1} = -5 \quad \text{and} \quad \frac{12}{-2+1} = -12, \text{ so}$$

$$\frac{5}{-2+1} > \frac{12}{-2+1}.$$

For $(-1, \infty)$, choose 0 .

$$\frac{5}{0+1} = 5 \quad \text{and} \quad \frac{12}{0+1} = 12, \text{ so}$$

$$\frac{5}{0+1} \not> \frac{12}{0+1}.$$

The solution is $(-\infty, -1)$.

51. $\frac{2x}{x^2 - x - 6} \geq 0$

Solve $\frac{2x}{x^2 - x - 6} = 0$.

$$\frac{2x}{x^2 - x - 6} = 0$$

$$2x = 0$$

$$x = 0$$

Set the denominator equal to 0 and solve for x .

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x+2 = 0 \quad \text{or} \quad x-3 = 0$$

$$x = -2 \quad \text{or} \quad x = 3$$

Intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$, $(3, \infty)$

For $(-\infty, -2)$, choose -3 .

$$\frac{2(-3)}{(-3)^2 - (-3) - 6} = -1 \not\geq 0$$

For $(-2, 0)$, choose -1 .

$$\frac{2(-1)}{(-1)^2 - (-1) - 6} = \frac{1}{2} \geq 0$$

For $(0, 3)$, choose 2 .

$$\frac{2(2)}{2^2 - 2 - 6} = -1 \not\geq 0$$

For $(3, \infty)$, choose 4 .

$$\frac{2(4)}{4^2 - 4 - 6} = \frac{4}{3} \geq 0$$

The solution is $(-2, 0] \cup (3, \infty)$.

52. $\frac{8}{p^2 + 2p} > 1$

Solve the equation $\frac{8}{p^2 + 2p} = 1$.

$$\frac{8}{p^2 + 2p} = 1$$

$$8 = p^2 + 2p$$

$$0 = p^2 + 2p - 8$$

$$0 = (p+4)(p-2)$$

$$p+4 = 0 \quad \text{or} \quad p-2 = 0$$

$$p = -4 \quad \text{or} \quad p = 2$$

Set the denominator equal to zero and solve for p .

$$p^2 + 2p = 0$$

$$p(p+2) = 0$$

$$p = 0 \quad \text{or} \quad p+2 = 0$$

$$p = -2$$

Intervals: $(-\infty, -4)$, $(-4, -2)$, $(-2, 0)$, $(0, 2)$, $(2, \infty)$

For $(-\infty, -4)$, choose -5 .

$$\frac{8}{(-5)^2 + 2(-5)} = \frac{8}{15} \not> 1$$

For $(-4, -2)$, choose -3 .

$$\frac{8}{(-3)^2 + 2(-3)} = \frac{8}{9-6} = \frac{8}{3} > 1$$

For $(-2, 0)$, choose -1 .

$$\frac{8}{(-1)^2 + 2(-1)} = \frac{8}{-1} = -8 \not> 1$$

For $(0, 2)$, choose 1 .

$$\frac{8}{(1)^2 + 2(1)} = \frac{8}{3} > 1$$

For $(2, \infty)$, choose 3.

$$\frac{8}{(3)^2 + (2)(3)} = \frac{8}{15} \neq 1$$

The solution is $(-4, -2) \cup (0, 2)$.

53. $\frac{z^2 + z}{z^2 - 1} \geq 3$

Solve

$$\frac{z^2 + z}{z^2 - 1} = 3.$$

$$z^2 + z = 3z^2 - 3$$

$$-2z^2 + z + 3 = 0$$

$$-1(2z^2 - z - 3) = 0$$

$$-1(z + 1)(2z - 3) = 0$$

$$z = -1 \quad \text{or} \quad z = \frac{3}{2}$$

Set $z^2 - 1 = 0$.

$$z^2 = 1$$

$$z = -1 \quad \text{or} \quad z = 1$$

Intervals: $(-\infty, -1)$, $(-1, 1)$, $(1, \frac{3}{2})$, $(\frac{3}{2}, \infty)$

For $(-\infty, -1)$, choose $x = -2$.

$$\frac{(-2)^2 + 3}{(-2)^2 - 1} = \frac{7}{3} \not\geq 3$$

For $(-1, 1)$, choose $x = 0$.

$$\frac{0^2 + 3}{0^2 - 1} = -3 \not\geq 3$$

For $(1, \frac{3}{2})$, choose $x = \frac{3}{2}$.

$$\frac{(\frac{3}{2})^2 + 3}{(\frac{3}{2})^2 - 1} = \frac{21}{5} \geq 3$$

For $(\frac{3}{2}, \infty)$, choose $x = 2$.

$$\frac{2^2 + 3}{2^2 - 1} = \frac{7}{3} \not\geq 3$$

The solution is $(1, \frac{3}{2}]$.

54. $\frac{a^2 + 2a}{a^2 - 4} \leq 2$

Solve the equation $\frac{a^2 + 2a}{a^2 - 4} = 2$.

$$\frac{a^2 + 2a}{a^2 - 4} = 2$$

$$a^2 + 2a = 2(a^2 - 4)$$

$$a^2 + 2a = 2a^2 - 8$$

$$0 = a^2 - 2a - 8$$

$$0 = (a - 4)(a + 2)$$

$$a - 4 = 0 \quad \text{or} \quad a + 2 = 0$$

$$a = 4 \quad \text{or} \quad a = -2$$

But -2 is not a possible solution. Set the denominator equal to zero and solve for a .

$$a^2 - 4 = 0$$

$$(a + 2)(a - 2) = 0$$

$$a + 2 = 0 \quad \text{or} \quad a - 2 = 0$$

$$a = -2 \quad \text{or} \quad a = 2$$

Intervals: $(-\infty, -2)$, $(-2, 2)$, $(2, 4)$, $(4, \infty)$

For $(-\infty, -2)$, choose -3 .

$$\frac{(-3)^2 + 2(-3)}{(-3)^2 - 4} = \frac{9 - 6}{9 - 4} = \frac{3}{5} \leq 2$$

For $(-2, 2)$, choose 0.

$$\frac{(0)^2 + 2(0)}{0 - 4} = \frac{0}{-4} = 0 \leq 2$$

For $(2, 4)$, choose 3.

$$\frac{(3)^2 + 2(3)}{(3)^2 - 4} = \frac{9 + 6}{9 - 5} = \frac{15}{4} \not\leq 2$$

For $(4, \infty)$, choose 5.

$$\frac{(5)^2 + 2(5)}{(5)^2 - 4} = \frac{25 + 10}{25 - 4} = \frac{35}{21} \leq 2$$

The value 4 will satisfy the original inequality, but the values -2 and 2 will not since they make the denominator zero. The solution is

$$(-\infty, -2) \cup (-2, 2) \cup [4, \infty).$$

R.6 Exponents

Your Turn 1

Simplify $\left(\frac{y^2 z^{-4}}{y^{-3} z^4}\right)^{-2}$.

$$\begin{aligned} \left(\frac{y^2 z^{-4}}{y^{-3} z^4}\right)^{-2} &= \frac{(y^2 z^{-4})^{-2}}{(y^{-3} z^4)^{-2}} = \frac{y^{(2)(-2)} z^{(-4)(-2)}}{y^{(-3)(-2)} z^{(4)(-2)}} \\ &= \frac{y^{-4} z^8}{y^6 z^{-8}} = \frac{z^{8-(-8)}}{y^{6-(-4)}} = \frac{z^{16}}{y^{10}} \end{aligned}$$

Your Turn 2Factor $5z^{1/3} + 4z^{-2/3}$.

$$\begin{aligned} 5z^{1/3} + 4z^{-2/3} &= z^{-2/3} \left(5z^{(1/3)+(2/3)} + 4z^{(-2/3)+(2/3)} \right) \\ &= z^{-2/3} (5z + 4) \end{aligned}$$

R.6 Exercises

$$1. \quad 8^{-2} = \frac{1}{8^2} = \frac{1}{64}$$

$$2. \quad 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$$

$$3. \quad 5^0 = 1, \text{ by definition.}$$

$$4. \quad \left(-\frac{3}{4} \right)^0 = 1, \text{ by definition.}$$

$$5. \quad -(-3)^{-2} = -\frac{1}{(-3)^2} = -\frac{1}{9}$$

$$6. \quad -(-3^{-2}) = -\left(-\frac{1}{3^2} \right) = -\left(-\frac{1}{9} \right) = \frac{1}{9}$$

$$7. \quad \left(\frac{1}{6} \right)^{-2} = \frac{1}{\left(\frac{1}{6} \right)^2} = \frac{1}{\frac{1}{36}} = 36$$

$$8. \quad \left(\frac{4}{3} \right)^{-3} = \frac{1}{\left(\frac{4}{3} \right)^3} = \frac{1}{\frac{64}{27}} = \frac{27}{64}$$

$$9. \quad \frac{4^{-2}}{4} = 4^{-2-1} = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}$$

$$10. \quad \frac{8^9 \cdot 8^{-7}}{8^{-3}} = 8^{9+(-7)-(-3)} = 8^{9-7+3} = 8^5$$

$$\begin{aligned} 11. \quad \frac{10^8 \cdot 10^{-10}}{10^4 \cdot 10^2} &= \frac{10^{8+(-10)}}{10^{4+2}} = \frac{10^{-2}}{10^6} \\ &= 10^{-2-6} = 10^{-8} \\ &= \frac{1}{10^8} \end{aligned}$$

$$\begin{aligned} 12. \quad \left(\frac{7^{-12} \cdot 7^3}{7^{-8}} \right)^{-1} &= (7^{-12+3-(-8)})^{-1} \\ &= (7^{-12+3+8})^{-1} = (7^{-1})^{-1} \\ &= 7^{(-1)(-1)} = 7^1 = 7 \end{aligned}$$

$$13. \quad \frac{x^4 \cdot x^3}{x^5} = \frac{x^{4+3}}{x^5} = \frac{x^7}{x^5} = x^{7-5} = x^2$$

$$14. \quad \frac{y^{10} \cdot y^{-4}}{y^6} = y^{10-4-6} = y^0 = 1$$

$$\begin{aligned} 15. \quad \frac{(4k^{-1})^2}{2k^{-5}} &= \frac{4^2 k^{-2}}{2k^{-5}} = \frac{16k^{-2-(-5)}}{2} \\ &= 8k^{-2+5} = 8k^3 \\ &= 2^3 k^3 \end{aligned}$$

$$\begin{aligned} 16. \quad \frac{(3z^2)^{-1}}{z^5} &= \frac{3^{-1}(z^2)^{-1}}{z^5} = \frac{3^{-1}z^{2(-1)}}{z^5} \\ &= \frac{3^{-1}z^{-2}}{z^5} = 3^{-1}z^{-2-5} \\ &= 3^{-1}z^{-7} = \frac{1}{3} \cdot \frac{1}{z^7} = \frac{1}{3z^7} \end{aligned}$$

$$\begin{aligned} 17. \quad \frac{3^{-1} \cdot x \cdot y^2}{x^{-4} \cdot y^5} &= 3^{-1} \cdot x^{1-(-4)} \cdot y^{2-5} \\ &= 3^{-1} \cdot x^{1+4} \cdot y^{-3} \\ &= \frac{1}{3} \cdot x^5 \cdot \frac{1}{y^3} \\ &= \frac{x^5}{3y^3} \end{aligned}$$

$$\begin{aligned} 18. \quad \frac{5^{-2}m^2y^{-2}}{5^2m^{-1}y^{-2}} &= \frac{5^{-2}}{5^2} \cdot \frac{m^2}{m^{-1}} \cdot \frac{y^{-2}}{y^{-2}} \\ &= 5^{-2-2}m^{2-(-1)}y^{-2-(-2)} \\ &= 5^{-2-2}m^{2+1}y^{-2+2} \\ &= 5^{-4}m^3y^0 = \frac{1}{5^4} \cdot m^3 \cdot 1 \\ &= \frac{m^3}{5^4} \end{aligned}$$

$$\begin{aligned} 19. \quad \left(\frac{a^{-1}}{b^2} \right)^{-3} &= \frac{(a^{-1})^{-3}}{(b^2)^{-3}} = \frac{a^{(-1)(-3)}}{b^{2(-3)}} \\ &= \frac{a^3}{b^{-6}} = a^3b^6 \end{aligned}$$

$$\begin{aligned} 20. \quad \left(\frac{c^3}{7d^{-2}} \right)^{-2} &= \frac{(c^3)^{-2}}{7^{-2}(d^{-2})^{-2}} \\ &= \frac{c^{(3)(-2)}}{7^{-2}d^{(-2)(-2)}} = \frac{c^{-6}}{7^{-2}d^4} \\ &= \frac{7^2}{c^6d^4} = \frac{49}{c^6d^4} \end{aligned}$$

$$\begin{aligned}
 21. \quad a^{-1} + b^{-1} &= \frac{1}{a} + \frac{1}{b} \\
 &= \left(\frac{b}{b}\right)\left(\frac{1}{a}\right) + \left(\frac{a}{a}\right)\left(\frac{1}{b}\right) \\
 &= \frac{b}{ab} + \frac{a}{ab} \\
 &= \frac{b+a}{ab} \\
 &= \frac{a+b}{ab}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad b^{-2} - a &= \frac{1}{b^2} - a \\
 &= \frac{1}{b^2} - a\left(\frac{b^2}{b^2}\right) \\
 &= \frac{1}{b^2} - \frac{ab^2}{b^2} \\
 &= \frac{1-ab^2}{b^2}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \frac{2n^{-1} - 2m^{-1}}{m + n^2} &= \frac{\frac{2}{n} - \frac{2}{m}}{m + n^2} \\
 &= \frac{\frac{2}{n} \cdot \frac{m}{m} - \frac{2}{m} \cdot \frac{n}{n}}{(m + n^2)} \\
 &= \frac{2m - 2n}{mn(m + n^2)} \\
 &\text{or } \frac{2(m-n)}{mn(m + n^2)}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \left(\frac{m}{3}\right)^{-1} + \left(\frac{n}{2}\right)^{-2} &= \left(\frac{3}{m}\right)^1 + \left(\frac{2}{n}\right)^2 \\
 &= \frac{3}{m} + \frac{4}{n^2} \\
 &= \left(\frac{3}{m}\right)\left(\frac{n^2}{n^2}\right) + \left(\frac{4}{n^2}\right)\left(\frac{m}{m}\right) \\
 &= \frac{3n^2}{mn^2} + \frac{4m}{mn^2} \\
 &= \frac{3n^2 + 4m}{mn^2}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad (x^{-1} - y^{-1})^{-1} &= \frac{1}{\frac{1}{x} - \frac{1}{y}} \\
 &= \frac{1}{\frac{1}{x} \cdot \frac{y}{y} - \frac{1}{y} \cdot \frac{x}{x}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\frac{y}{xy} - \frac{x}{xy}} \\
 &= \frac{1}{\frac{y-x}{xy}} \\
 &= \frac{xy}{y-x}
 \end{aligned}$$

$$\begin{aligned}
 26. \quad (x \cdot y^{-1} - y^{-2})^{-2} &= \left(\frac{x}{y} - \frac{1}{y^2}\right)^{-2} \\
 &= \left[\left(\frac{x}{y}\right)\left(\frac{y}{y}\right) - \frac{1}{y^2}\right]^{-2} \\
 &= \left(\frac{xy}{y^2} - \frac{1}{y^2}\right)^{-2} \\
 &= \left(\frac{xy-1}{y^2}\right)^{-2} \\
 &= \left(\frac{y^2}{xy-1}\right)^2 \\
 &= \frac{(y^2)^2}{(xy-1)^2} \\
 &= \frac{y^4}{(xy-1)^2}
 \end{aligned}$$

$$27. \quad 121^{1/2} = (11^2)^{1/2} = 11^{2(1/2)} = 11^1 = 11$$

$$28. \quad 27^{1/3} = \sqrt[3]{27} = 3$$

$$29. \quad 32^{2/5} = (32^{1/5})^2 = 2^2 = 4$$

$$30. \quad -125^{2/3} = -(125^{1/3})^2 = -5^2 = -25$$

$$31. \quad \left(\frac{36}{144}\right)^{1/2} = \frac{36^{1/2}}{144^{1/2}} = \frac{6}{12} = \frac{1}{2}$$

This can also be solved by reducing the fraction first.

$$\left(\frac{36}{144}\right)^{1/2} = \left(\frac{1}{4}\right)^{1/2} = \frac{1^{1/2}}{4^{1/2}} = \frac{1}{2}$$

$$32. \quad \left(\frac{64}{27}\right)^{1/3} = \frac{64^{1/3}}{27^{1/3}} = \frac{4}{3}$$

$$33. \quad 8^{-4/3} = (8^{1/3})^{-4} = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$

$$34. \quad 625^{-1/4} = \frac{1}{625^{1/4}} = \frac{1}{5}$$

$$35. \left(\frac{27}{64}\right)^{-1/3} = \frac{27^{-1/3}}{64^{-1/3}} = \frac{64^{1/3}}{27^{1/3}} = \frac{4}{3}$$

$$36. \left(\frac{121}{100}\right)^{-3/2} = \frac{1}{\left(\frac{121}{100}\right)^{3/2}} = \frac{1}{\left[\left(\frac{121}{100}\right)^{1/2}\right]^3} \\ = \frac{1}{\left(\frac{11}{10}\right)^3} = \frac{1}{\frac{1331}{1000}} = \frac{1000}{1331}$$

$$37. 3^{2/3} \cdot 3^{4/3} = 3^{(2/3)+(4/3)} = 3^{6/3} = 3^2 = 9$$

$$38. 27^{2/3} \cdot 27^{-1/3} = 27^{(2/3)+(-1/3)} \\ = 27^{2/3-1/3} \\ = 27^{1/3} = 3$$

$$39. \frac{4^{9/4} \cdot 4^{-7/4}}{4^{-10/4}} = 4^{9/4-7/4-(-10/4)} \\ = 4^{12/4} = 4^3 = 64$$

$$40. \frac{3^{-5/2} \cdot 3^{3/2}}{3^{7/2} \cdot 3^{-9/2}} = 3^{(-5/2)+(3/2)-(7/2)-(-9/2)} \\ = 3^{-5/2+3/2-7/2+9/2} \\ = 3^0 = 1$$

$$41. \left(\frac{x^6 y^{-3}}{x^{-2} y^5}\right)^{1/2} = (x^{6-(-2)} y^{-3-5})^{1/2} \\ = (x^8 y^{-8})^{1/2} \\ = (x^8)^{1/2} (y^{-8})^{1/2} \\ = x^4 y^{-4} \\ = \frac{x^4}{y^4}$$

$$42. \left(\frac{a^{-7} b^{-1}}{b^{-4} a^2}\right)^{1/3} = (a^{-7-2} b^{-1-(-4)})^{1/3} \\ = (a^{-9} b^3)^{1/3} \\ = (a^{-9})^{1/3} (b^3)^{1/3} \\ = a^{-3} b^1 \\ = \frac{b}{a^3}$$

$$43. \frac{7^{-1/3} \cdot 7r^{-3}}{7^{2/3} \cdot (r^{-2})^2} = \frac{7^{-1/3+1} r^{-3}}{7^{2/3} \cdot r^{-4}} \\ = 7^{-1/3+3/3-2/3} r^{-3-(-4)} \\ = 7^0 r^{-3+4} = 1 \cdot r^1 = r$$

$$44. \frac{12^{3/4} \cdot 12^{5/4} \cdot y^{-2}}{12^{-1} \cdot (y^{-3})^{-2}} \\ = \frac{12^{3/4+5/4} \cdot y^{-2}}{12^{-1} \cdot y^{(-3)(-2)}} = \frac{12^{8/4} \cdot y^{-2}}{12^{-1} \cdot y^6} \\ = \frac{12^2 \cdot y^{-2}}{12^{-1} y^6} \\ = 12^{2-(-1)} \cdot y^{-2-6} = 12^3 y^{-8} \\ = \frac{12^3}{y^8}$$

$$45. \frac{3k^2 \cdot (4k^{-3})^{-1}}{4^{1/2} \cdot k^{7/2}} \\ = \frac{3k^2 \cdot 4^{-1} k^3}{2 \cdot k^{7/2}} \\ = 3 \cdot 2^{-1} \cdot 4^{-1} k^{2+3-(7/2)} \\ = \frac{3}{8} \cdot k^{3/2} \\ = \frac{3k^{3/2}}{8}$$

$$46. \frac{8p^{-3}(4p^2)^{-2}}{p^{-5}} = \frac{8p^{-3} \cdot 4^{-2} p^{(2)(-2)}}{p^{-5}} \\ = \frac{8p^{-3} 4^{-2} p^{-4}}{p^{-5}} \\ = 8 \cdot 4^{-2} p^{(-3)+(-4)-(-5)} \\ = 8 \cdot 4^{-2} p^{-3-4+5} \\ = 8 \cdot 4^{-2} p^{-2} \\ = 8 \cdot \frac{1}{4^2} \cdot \frac{1}{p^2} \\ = 8 \cdot \frac{1}{16} \cdot \frac{1}{p^2} \\ = \frac{8}{16p^2} = \frac{1}{2p^2}$$

$$47. \frac{a^{4/3}}{a^{2/3}} \cdot \frac{b^{1/2}}{b^{-3/2}} = a^{4/3-2/3} b^{1/2-(-3/2)} \\ = a^{2/3} b^2$$

$$48. \frac{x^{3/2} \cdot y^{4/5} \cdot z^{-3/4}}{x^{5/3} \cdot y^{-6/5} \cdot z^{1/2}} \\ = x^{3/2-(5/3)} \cdot y^{4/5-(-6/5)} \cdot z^{-3/4-(1/2)} \\ = x^{-1/6} \cdot y^2 \cdot z^{-5/4} \\ = \frac{y^2}{x^{1/6} z^{5/4}}$$

$$\begin{aligned}
 49. \quad & \frac{k^{-3/5} \cdot h^{-1/3} \cdot t^{2/5}}{k^{-1/5} \cdot h^{-2/3} \cdot t^{1/5}} \\
 &= k^{-3/5 - (-1/5)} h^{-1/3 - (-2/3)} t^{2/5 - 1/5} \\
 &= k^{-3/5 + 1/5} h^{-1/3 + 2/3} t^{2/5 - 1/5} \\
 &= k^{-2/5} h^{1/3} t^{1/5} = \frac{h^{1/3} t^{1/5}}{k^{2/5}}
 \end{aligned}$$

$$\begin{aligned}
 50. \quad & \frac{m^{7/3} \cdot n^{-2/5} \cdot p^{3/8}}{m^{-2/3} \cdot n^{3/5} \cdot p^{-5/8}} \\
 &= m^{7/3 - (-2/3)} n^{-2/5 - (3/5)} p^{3/8 - (-5/8)} \\
 &= m^{7/3 + 2/3} n^{-2/5 - 3/5} p^{3/8 + 5/8} \\
 &= m^{9/3} n^{-5/5} p^{8/8} \\
 &= m^3 n^{-1} p^1 \\
 &= \frac{m^3 p}{n}
 \end{aligned}$$

$$\begin{aligned}
 51. \quad & 3x^3(x^2 + 3x)^2 - 15x(x^2 + 3x)^2 \\
 &= 3x \cdot x^2(x^2 + 3x)^2 - 3x \cdot 5(x^2 + 3x)^2 \\
 &= 3x(x^2 + 3x)^2(x^2 - 5)
 \end{aligned}$$

$$\begin{aligned}
 52. \quad & 6x(x^3 + 7)^2 - 6x^2(3x^2 + 5)(x^3 + 7) \\
 &= 6x(x^3 + 7)(x^3 + 7) - 6x(x)(3x^2 + 5)(x^3 + 7) \\
 &= 6x(x^3 + 7)[(x^3 + 7) - x(3x^2 + 5)] \\
 &= 6x(x^3 + 7)(x^3 + 7 - 3x^3 - 5x) \\
 &= 6x(x^3 + 7)(-2x^3 - 5x + 7)
 \end{aligned}$$

$$\begin{aligned}
 53. \quad & 10x^3(x^2 - 1)^{-1/2} - 5x(x^2 - 1)^{1/2} \\
 &= 5x \cdot 2x^2(x^2 - 1)^{-1/2} - 5x(x^2 - 1)^{-1/2}(x^2 - 1)^1 \\
 &= 5x(x^2 - 1)^{-1/2}[2x^2 - (x^2 - 1)] \\
 &= 5x(x^2 - 1)^{-1/2}(x^2 + 1)
 \end{aligned}$$

$$\begin{aligned}
 54. \quad & 9(6x + 2)^{1/2} + 3(9x - 1)(6x + 2)^{-1/2} \\
 &= 3 \cdot 3(6x + 2)^{-1/2}(6x + 2)^1 \\
 &\quad + 3(9x - 1)(6x + 2)^{-1/2} \\
 &= 3(6x + 2)^{-1/2}[3(6x + 2) + (9x - 1)] \\
 &= 3(6x + 2)^{-1/2}(18x + 6 + 9x - 1) \\
 &= 3(6x + 2)^{-1/2}(27x + 5)
 \end{aligned}$$

$$\begin{aligned}
 55. \quad & x(2x + 5)^2(x^2 - 4)^{-1/2} + 2(x^2 - 4)^{1/2}(2x + 5) \\
 &= (2x + 5)^2(x^2 - 4)^{-1/2}(x) \\
 &\quad + (x^2 - 4)^1(x^2 - 4)^{-1/2}(2)(2x + 5) \\
 &= (2x + 5)(x^2 - 4)^{-1/2} \\
 &\quad \cdot [(2x + 5)(x) + (x^2 - 4)(2)] \\
 &= (2x + 5)(x^2 - 4)^{-1/2} \cdot (2x^2 + 5x + 2x^2 - 8) \\
 &= (2x + 5)(x^2 - 4)^{-1/2}(4x^2 + 5x - 8)
 \end{aligned}$$

$$\begin{aligned}
 56. \quad & (4x^2 + 1)^2(2x - 1)^{-1/2} \\
 &\quad + 16x(4x^2 + 1)(2x - 1)^{1/2} \\
 &= (4x^2 + 1)(4x^2 + 1)(2x - 1)^{-1/2} \\
 &\quad + 16x(4x^2 + 1)(2x - 1)^{-1/2}(2x - 1) \\
 &= (4x^2 + 1)(2x - 1)^{-1/2} \\
 &\quad \cdot [(4x^2 + 1) + 16x(2x - 1)] \\
 &= (4x^2 + 1)(2x - 1)^{-1/2} \\
 &\quad \cdot (4x^2 + 1 + 32x^2 - 16x) \\
 &= (4x^2 + 1)(2x - 1)^{-1/2}(36x^2 - 16x + 1)
 \end{aligned}$$

R.7 Radicals

Your Turn 1

Simplify $\sqrt{28x^9y^5}$.

$$\begin{aligned}
 \sqrt{28x^9y^5} &= \sqrt{4 \cdot x^8 \cdot y^4 \cdot 7xy} \\
 &= 2x^4y^2\sqrt{7xy}
 \end{aligned}$$

Your Turn 2

Rationalize the denominator in $\frac{5}{\sqrt{x} - \sqrt{y}}$.

$$\begin{aligned}
 \frac{5}{\sqrt{x} - \sqrt{y}} &= \frac{5}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\
 &= \frac{5(\sqrt{x} + \sqrt{y})}{x - y}
 \end{aligned}$$

R.7 Exercises

- $\sqrt[3]{125} = 5$ because $5^3 = 125$.
- $\sqrt[4]{1296} = \sqrt[4]{6^4} = 6$
- $\sqrt[5]{-3125} = -5$ because $(-5)^5 = -3125$.
- $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

5. $\sqrt{2000} = \sqrt{4 \cdot 100 \cdot 5}$
 $= 2 \cdot 10\sqrt{5}$
 $= 20\sqrt{5}$
6. $\sqrt{32y^5} = \sqrt{(16y^4)(2y)}$
 $= \sqrt{16y^4} \sqrt{2y}$
 $= 4y^2 \sqrt{2y}$
7. $\sqrt{27} \cdot \sqrt{3} = \sqrt{27 \cdot 3} = \sqrt{81} = 9$
8. $\sqrt{2} \cdot \sqrt{32} = \sqrt{2 \cdot 32} = \sqrt{64} = 8$
9. $7\sqrt{2} - 8\sqrt{18} + 4\sqrt{72}$
 $= 7\sqrt{2} - 8\sqrt{9 \cdot 2} + 4\sqrt{36 \cdot 2}$
 $= 7\sqrt{2} - 8(3)\sqrt{2} + 4(6)\sqrt{2}$
 $= 7\sqrt{2} - 24\sqrt{2} + 24\sqrt{2}$
 $= 7\sqrt{2}$
10. $4\sqrt{3} - 5\sqrt{12} + 3\sqrt{75}$
 $= 4\sqrt{3} - 5(\sqrt{4}\sqrt{3}) + 3(\sqrt{25}\sqrt{3})$
 $= 4\sqrt{3} - 5(2\sqrt{3}) + 3(5\sqrt{3})$
 $= 4\sqrt{3} - 10\sqrt{3} + 15\sqrt{3}$
 $= (4 - 10 + 15)\sqrt{3} = 9\sqrt{3}$
11. $4\sqrt{7} - \sqrt{28} + \sqrt{343}$
 $= 4\sqrt{7} - \sqrt{4}\sqrt{7} + \sqrt{49}\sqrt{7}$
 $= 4\sqrt{7} - 2\sqrt{7} + 7\sqrt{7}$
 $= (4 - 2 + 7)\sqrt{7}$
 $= 9\sqrt{7}$
12. $3\sqrt{28} - 4\sqrt{63} + \sqrt{112}$
 $= 3(\sqrt{4}\sqrt{7}) - 4(\sqrt{9}\sqrt{7}) + (\sqrt{16}\sqrt{7})$
 $= 3(2\sqrt{7}) - 4(3\sqrt{7}) + (4\sqrt{7})$
 $= 6\sqrt{7} - 12\sqrt{7} + 4\sqrt{7}$
 $= (6 - 12 + 4)\sqrt{7}$
 $= -2\sqrt{7}$
13. $\sqrt[3]{2} - \sqrt[3]{16} + 2\sqrt[3]{54}$
 $= \sqrt[3]{2} - (\sqrt[3]{8 \cdot 2}) + 2(\sqrt[3]{27 \cdot 2})$
 $= \sqrt[3]{2} - \sqrt[3]{8}\sqrt[3]{2} + 2(\sqrt[3]{27}\sqrt[3]{2})$
 $= \sqrt[3]{2} - 2\sqrt[3]{2} + 2(3\sqrt[3]{2})$
 $= \sqrt[3]{2} - 2\sqrt[3]{2} + 6\sqrt[3]{2}$
 $= 5\sqrt[3]{2}$
14. $2\sqrt[3]{5} - 4\sqrt[3]{40} + 3\sqrt[3]{135}$
 $= 2\sqrt[3]{5} - 4\sqrt[3]{8 \cdot 5} + 3\sqrt[3]{27 \cdot 5}$
 $= 2\sqrt[3]{5} - 4(2)\sqrt[3]{5} + 3(3)\sqrt[3]{5}$
 $= 2\sqrt[3]{5} - 8\sqrt[3]{5} + 9\sqrt[3]{5}$
 $= 3\sqrt[3]{5}$
15. $\sqrt{2x^3y^2z^4} = \sqrt{x^2y^2z^4} \cdot 2x = xyz^2\sqrt{2x}$
16. $\sqrt{160r^7s^9t^{12}}$
 $= \sqrt{(16 \cdot 10)(r^6 \cdot r)(s^8 \cdot s)(t^{12})}$
 $= \sqrt{(16r^6s^8t^{12})(10rs)}$
 $= \sqrt{16r^6s^8t^{12}} \sqrt{10rs}$
 $= 4r^3s^4t^6\sqrt{10rs}$
17. $\sqrt[3]{128x^3y^8z^9} = \sqrt[3]{64x^3y^6z^9} \cdot \sqrt[3]{2y^2}$
 $= \sqrt[3]{64x^3y^6z^9} \sqrt[3]{2y^2}$
 $= 4xy^2z^3\sqrt[3]{2y^2}$
18. $\sqrt[4]{x^8y^7z^{11}} = \sqrt[4]{(x^8)(y^4 \cdot y^3)(z^8z^3)}$
 $= \sqrt[4]{(x^8y^4z^8)(y^3z^3)}$
 $= \sqrt[4]{x^8y^4z^8} \sqrt[4]{y^3z^3}$
 $= x^2yz^2\sqrt[4]{y^3z^3}$
19. $\sqrt{a^3b^5} - 2\sqrt{a^7b^3} + \sqrt{a^3b^9}$
 $= \sqrt{a^2b^4ab} - 2\sqrt{a^6b^2ab} + \sqrt{a^2b^8ab}$
 $= ab^2\sqrt{ab} - 2a^3b\sqrt{ab} + ab^4\sqrt{ab}$
 $= (ab^2 - 2a^3b + ab^4)\sqrt{ab}$
 $= ab\sqrt{ab}(b - 2a^2 + b^3)$
20. $\sqrt{p^7q^3} - \sqrt{p^5q^9} + \sqrt{p^9q}$
 $= \sqrt{(p^6p)(q^2q)} - \sqrt{(p^4p)(q^8q)} + \sqrt{(p^8p)q}$
 $= \sqrt{(p^6q^2)(pq)} - \sqrt{(p^4q^8)\sqrt{(pq)}} + \sqrt{(p^8)\sqrt{pq}}$
 $= \sqrt{p^6q^2}\sqrt{pq} - \sqrt{p^4q^8}\sqrt{pq} + \sqrt{p^8}\sqrt{pq}$
 $= p^3q\sqrt{pq} - p^2q^4\sqrt{pq} + p^4\sqrt{pq}$
 $= p^2pq\sqrt{pq} - p^2q^4\sqrt{pq} + p^2p^2\sqrt{pq}$
 $= p^2\sqrt{pq}(pq - q^4 + p^2)$
21. $\sqrt{a} \cdot \sqrt[3]{a} = a^{1/2} \cdot a^{1/3}$
 $= a^{1/2+(1/3)}$
 $= a^{5/6} = \sqrt[6]{a^5}$

$$\begin{aligned}
 22. \quad \sqrt{b^3} \cdot \sqrt[4]{b^3} &= b^{3/2} \cdot b^{3/4} \\
 &= b^{3/2 + (3/4)} = b^{9/4} \\
 &= \sqrt[4]{b^9} = \sqrt[4]{b^8 \cdot b} \\
 &= \sqrt[4]{b^8} \sqrt[4]{b} = (b^2) \sqrt[4]{b}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \sqrt{16 - 8x + x^2} \\
 &= \sqrt{(4 - x)^2} \\
 &= |4 - x|
 \end{aligned}$$

$$24. \quad \sqrt{9y^2 + 30y + 25} = \sqrt{(3y + 5)^2} = |3y + 5|$$

$$25. \quad \sqrt{4 - 25z^2} = \sqrt{(2 + 5z)(2 - 5z)}$$

This factorization does not produce a perfect square, so the expression $\sqrt{4 - 25z^2}$ cannot be simplified.

$$\begin{aligned}
 26. \quad \sqrt{9k^2 + h^2} \\
 \text{The expression } 9k^2 + h^2 \text{ is the sum of two squares} \\
 \text{and cannot be factored. Therefore, } \sqrt{9k^2 + h^2} \\
 \text{cannot be simplified.}
 \end{aligned}$$

$$27. \quad \frac{5}{\sqrt{7}} = \frac{5}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{5\sqrt{7}}{7}$$

$$\begin{aligned}
 28. \quad \frac{5}{\sqrt{10}} &= \frac{5}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{5\sqrt{10}}{\sqrt{100}} \\
 &= \frac{5\sqrt{10}}{\sqrt{100}} = \frac{5\sqrt{10}}{10} \\
 &= \frac{\sqrt{10}}{2}
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \frac{-3}{\sqrt{12}} &= \frac{-3}{\sqrt{4 \cdot 3}} \\
 &= \frac{-3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-3\sqrt{3}}{6} = -\frac{\sqrt{3}}{2}
 \end{aligned}$$

$$30. \quad \frac{4}{\sqrt{8}} = \frac{4}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{\sqrt{16}} = \frac{4\sqrt{2}}{4} = \sqrt{2}$$

$$\begin{aligned}
 31. \quad \frac{3}{1 - \sqrt{2}} &= \frac{3}{1 - \sqrt{2}} \cdot \frac{1 + \sqrt{2}}{1 + \sqrt{2}} \\
 &= \frac{3(1 + \sqrt{2})}{1 - 2} \\
 &= -3(1 + \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \frac{5}{2 - \sqrt{6}} &= \frac{5}{2 - \sqrt{6}} \cdot \frac{2 + \sqrt{6}}{2 + \sqrt{6}} \\
 &= \frac{5(2 + \sqrt{6})}{4 + 2\sqrt{6} - 2\sqrt{6} - \sqrt{36}} \\
 &= \frac{5(2 + \sqrt{6})}{4 - \sqrt{36}} \\
 &= \frac{5(2 + \sqrt{6})}{4 - 6} \\
 &= \frac{5(2 + \sqrt{6})}{-2} \\
 &= -\frac{5(2 + \sqrt{6})}{2}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \frac{6}{2 + \sqrt{2}} &= \frac{6}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} \\
 &= \frac{6(2 - \sqrt{2})}{4 - 2\sqrt{2} + 2\sqrt{2} - \sqrt{4}} \\
 &= \frac{6(2 - \sqrt{2})}{4 - 2} = \frac{6(2 - \sqrt{2})}{2} \\
 &= 3(2 - \sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \frac{\sqrt{5}}{\sqrt{5} + \sqrt{2}} &= \frac{\sqrt{5}}{\sqrt{5} + \sqrt{2}} \cdot \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}} \\
 &= \frac{\sqrt{5}(\sqrt{5} - \sqrt{2})}{\sqrt{25} - \sqrt{10} + \sqrt{10} - \sqrt{4}} \\
 &= \frac{5 - \sqrt{10}}{5 - 2} \\
 &= \frac{5 - \sqrt{10}}{3}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \frac{1}{\sqrt{r} - \sqrt{3}} &= \frac{1}{\sqrt{r} - \sqrt{3}} \cdot \frac{\sqrt{r} + \sqrt{3}}{\sqrt{r} + \sqrt{3}} \\
 &= \frac{\sqrt{r} + \sqrt{3}}{r - 3}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \frac{5}{\sqrt{m} - \sqrt{5}} &= \frac{5}{\sqrt{m} - \sqrt{5}} \cdot \frac{\sqrt{m} + \sqrt{5}}{\sqrt{m} + \sqrt{5}} \\
 &= \frac{5(\sqrt{m} + \sqrt{5})}{(\sqrt{m})^2 + \sqrt{5m} - \sqrt{5m} - \sqrt{25}} \\
 &= \frac{5(\sqrt{m} + \sqrt{5})}{(\sqrt{m})^2 - \sqrt{25}} \\
 &= \frac{5(\sqrt{m} + \sqrt{5})}{m - 5}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad \frac{y-5}{\sqrt{y}-\sqrt{5}} &= \frac{y-5}{\sqrt{y}-\sqrt{5}} \cdot \frac{\sqrt{y}+\sqrt{5}}{\sqrt{y}+\sqrt{5}} \\
 &= \frac{(y-5)(\sqrt{y}+\sqrt{5})}{y-5} \\
 &= \sqrt{y}+\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \frac{\sqrt{z}-1}{\sqrt{z}-\sqrt{5}} &= \frac{\sqrt{z}-1}{\sqrt{z}-\sqrt{5}} \cdot \frac{\sqrt{z}+\sqrt{5}}{\sqrt{z}+\sqrt{5}} \\
 &= \frac{(\sqrt{z})^2 + \sqrt{5}z - \sqrt{z} - \sqrt{5}}{(\sqrt{z})^2 + \sqrt{5}z - \sqrt{5}z - \sqrt{25}} \\
 &= \frac{z + \sqrt{5}z - \sqrt{z} - \sqrt{5}}{z-5}
 \end{aligned}$$

$$\begin{aligned}
 39. \quad \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} &= \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \cdot \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}} \\
 &= \frac{x+2\sqrt{x(x+1)}+(x+1)}{x-(x+1)} \\
 &= \frac{2x+2\sqrt{x(x+1)}+1}{-1} \\
 &= -2x-2\sqrt{x(x+1)}-1
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \frac{\sqrt{p}+\sqrt{p^2-1}}{\sqrt{p}-\sqrt{p^2-1}} &= \frac{\sqrt{p}+\sqrt{p^2-1}}{\sqrt{p}-\sqrt{p^2-1}} \cdot \frac{\sqrt{p}+\sqrt{p^2-1}}{\sqrt{p}+\sqrt{p^2-1}} \\
 &= \frac{(\sqrt{p})^2 + 2\sqrt{p}\sqrt{p^2-1} + (\sqrt{p^2-1})^2}{(\sqrt{p})^2 + \sqrt{p}\sqrt{p^2-1} - \sqrt{p}\sqrt{p^2-1} - (\sqrt{p^2-1})^2} \\
 &= \frac{p+2\sqrt{p}\sqrt{p^2-1}+(p^2-1)}{p-(p^2-1)} \\
 &= \frac{p^2+p+2\sqrt{p(p^2-1)}-1}{-p^2+p+1}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \frac{1+\sqrt{2}}{2} &= \frac{(1+\sqrt{2})(1-\sqrt{2})}{2(1-\sqrt{2})} \\
 &= \frac{1-2}{2(1-\sqrt{2})} \\
 &= -\frac{1}{2(1-\sqrt{2})}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \frac{3-\sqrt{3}}{6} &= \frac{3-\sqrt{3}}{6} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}} \\
 &= \frac{9+3\sqrt{3}-3\sqrt{3}-\sqrt{9}}{6(3+\sqrt{3})} \\
 &= \frac{9-3}{6(3+\sqrt{3})} \\
 &= \frac{6}{6(3+\sqrt{3})} \\
 &= \frac{1}{3+\sqrt{3}}
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} &= \frac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \cdot \frac{\sqrt{x}-\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}} \\
 &= \frac{x-(x+1)}{x-2\sqrt{x}\cdot\sqrt{x+1}+(x+1)} \\
 &= \frac{-1}{2x-2\sqrt{x(x+1)}+1}
 \end{aligned}$$

$$\begin{aligned}
 44. \quad \frac{\sqrt{p}-\sqrt{p-2}}{\sqrt{p}} &= \frac{\sqrt{p}-\sqrt{p-2}}{\sqrt{p}} \cdot \frac{\sqrt{p}+\sqrt{p-2}}{\sqrt{p}+\sqrt{p-2}} \\
 &= \frac{(\sqrt{p})^2 + \sqrt{p}\sqrt{p-2} - \sqrt{p}\sqrt{p-2} - (\sqrt{p-2})^2}{(\sqrt{p})^2 + \sqrt{p}\sqrt{p-2}} \\
 &= \frac{p-(p-2)}{p+\sqrt{p(p-2)}} \\
 &= \frac{2}{p+\sqrt{p(p-2)}}
 \end{aligned}$$

Chapter 1

LINEAR FUNCTIONS

1.1 Slopes and Equations of Lines

Your Turn 1

Find the slope of the line through (1, 5) and (4, 6).

Let $(x_1, y_1) = (1, 5)$ and $(x_2, y_2) = (4, 6)$.

$$m = \frac{6 - 5}{4 - 1} = \frac{1}{3}$$

Your Turn 2

Find the equation of the line with x -intercept -4 and y -intercept 6.

We know that $b = 6$ and that the line crosses the axes at $(-4, 0)$ and $(0, 6)$. Use these two intercepts to find the slope m .

$$m = \frac{6 - 0}{0 - (-4)} = \frac{6}{4} = \frac{3}{2}$$

Thus the equation for the line in slope-intercept form is

$$y = \frac{3}{2}x + 6.$$

Your Turn 3

Find the slope of the line whose equation is $8x + 3y = 5$.

Solve the equation for y .

$$\begin{aligned} 8x + 3y &= 5 \\ 3y &= -8x + 5 \\ y &= -\frac{8}{3}x + \frac{5}{3} \end{aligned}$$

The slope is $-8/3$.

Your Turn 4

Find the equation (in slope-intercept form) of the line through (2, 9) and (5, 3).

First find the slope.

$$m = \frac{3 - 9}{5 - 2} = \frac{-6}{3} = -2$$

Now use the point-slope form, with $(x_1, y_1) = (5, 3)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -2(x - 5) \\ y - 3 &= -2x + 10 \\ y &= -2x + 13 \end{aligned}$$

Your Turn 5

Find (in slope-intercept form) the equation of the line that passes through the point (4, 5) and is parallel to the line $3x - 6y = 7$.

First find the slope of the line $3x - 6y = 7$ by solving this equation for y .

$$\begin{aligned} 3x - 6y &= 7 \\ 6y &= 3x - 7 \\ y &= \frac{3}{6}x - \frac{7}{6} \\ y &= \frac{1}{2}x - \frac{7}{6} \end{aligned}$$

Since the line we are to find is parallel to this line, it will also have slope $1/2$. Use the point-slope form with $(x_1, y_1) = (4, 5)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= \frac{1}{2}(x - 4) \\ y - 5 &= \frac{1}{2}x - 2 \\ y &= \frac{1}{2}x + 3 \end{aligned}$$

Your Turn 6

Find (in slope-intercept form) the equation of the line that passes through the point (3, 2) and is perpendicular to the line $2x + 3y = 4$.

First find the slope of the line $2x + 3y = 4$ by solving this equation for y .

$$\begin{aligned} 2x + 3y &= 4 \\ 3y &= -2x + 4 \\ y &= -\frac{2}{3}x + \frac{4}{3} \end{aligned}$$

Since the line we are to find is perpendicular to a line with slope $-2/3$, it will have slope $3/2$. (Note that $(-2/3)(3/2) = -1$.)

Use the point-slope form with $(x_1, y_1) = (3, 2)$.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{3}{2}(x - 3)$$

$$y - 2 = \frac{3}{2}x - \frac{9}{2}$$

$$y = \frac{3}{2}x - \frac{5}{2}$$

1.1 Exercises

1. Find the slope of the line through (4, 5) and (-1, 2).

$$\begin{aligned} m &= \frac{5 - 2}{4 - (-1)} \\ &= \frac{3}{5} \end{aligned}$$

2. Find the slope of the line through (5, -4) and (1, 3).

$$\begin{aligned} m &= \frac{3 - (-4)}{1 - 5} \\ &= \frac{3 + 4}{-4} \\ &= -\frac{7}{4} \end{aligned}$$

3. Find the slope of the line through (8, 4) and (8, -7).

$$m = \frac{4 - (-7)}{8 - 8} = \frac{11}{0}$$

The slope is undefined; the line is vertical.

4. Find the slope of the line through (1, 5) and (-2, 5).

$$m = \frac{5 - 5}{-2 - 1} = \frac{0}{-3} = 0$$

5. $y = x$

Using the slope-intercept form, $y = mx + b$, we see that the slope is 1.

6. $y = 3x - 2$

This equation is in slope-intercept form, $y = mx + b$. Thus, the coefficient of the x -term, 3, is the slope.

7. $5x - 9y = 11$

Rewrite the equation in slope-intercept form.

$$9y = 5x - 11$$

$$y = \frac{5}{9}x - \frac{11}{9}$$

The slope is $\frac{5}{9}$.

8. $4x + 7y = 1$

Rewrite the equation in slope-intercept form.

$$7y = 1 - 4x$$

$$\frac{1}{7}(7y) = \frac{1}{7}(1) - \frac{1}{7}(4x)$$

$$y = \frac{1}{7} - \frac{4}{7}x$$

$$y = -\frac{4}{7}x + \frac{1}{7}$$

The slope is $-\frac{4}{7}$.

9. $x = 5$

This is a vertical line. The slope is undefined.

10. The x -axis is the horizontal line $y = 0$.

Horizontal lines have a slope of 0.

11. $y = 8$

This is a horizontal line, which has a slope of 0.

12. $y = -6$

By rewriting this equation in the slope-intercept form, $y = mx + b$, we get $y = 0x - 6$, with the slope, m , being 0.

13. Find the slope of a line parallel to $6x - 3y = 12$.

Rewrite the equation in slope-intercept form.

$$-3y = -6x + 12$$

$$y = 2x - 4$$

The slope is 2, so a parallel line will also have slope 2.

14. Find the slope of a line perpendicular to $8x = 2y - 5$.

First, rewrite the given equation in slope-intercept form.

$$8x = 2y - 5$$

$$8x + 5 = 2y$$

$$4x + \frac{5}{2} = y \quad \text{or} \quad y = 4x + \frac{5}{2}$$

Let m be the slope of any line perpendicular to the given line. Then,

$$4 \cdot m = -1$$

$$m = -\frac{1}{4}.$$

15. The line goes through (1, 3), with slope $m = -2$. Use point-slope form.

$$y - 3 = -2(x - 1)$$

$$y = -2x + 2 + 3$$

$$y = -2x + 5$$

16. The line goes through (2, 4), with slope $m = -1$.
Use point-slope form.

$$y - 4 = -1(x - 2)$$

$$y - 4 = -x + 2$$

$$y = -x + 6$$

17. The line goes through $(-5, -7)$ with slope $m = 0$. Use point-slope form.

$$y - (-7) = 0[x - (-5)]$$

$$y + 7 = 0$$

$$y = -7$$

18. The line goes through $(-8, 1)$, with undefined slope. Since the slope is undefined, the line is vertical. The equation of the vertical line passing through $(-8, 1)$ is $x = -8$.

19. The line goes through (4, 2) and (1, 3). Find the slope, then use point-slope form with either of the two given points.

$$m = \frac{3 - 2}{1 - 4} = -\frac{1}{3}$$

$$y - 3 = -\frac{1}{3}(x - 1)$$

$$y = -\frac{1}{3}x + \frac{1}{3} + 3$$

$$y = -\frac{1}{3}x + \frac{10}{3}$$

20. The line goes through (8, -1) and (4, 3). Find the slope, then use point-slope form with either of the two given points.

$$m = \frac{3 - (-1)}{4 - 8}$$

$$= \frac{3 + 1}{-4}$$

$$= \frac{4}{-4} = -1$$

$$y - (-1) = -1(x - 8)$$

$$y + 1 = -x + 8$$

$$y = -x + 7$$

21. The line goes through $(\frac{2}{3}, \frac{1}{2})$ and $(\frac{1}{4}, -2)$.

$$m = \frac{-2 - \frac{1}{2}}{\frac{1}{4} - \frac{2}{3}} = \frac{-\frac{4}{2} - \frac{1}{2}}{\frac{3}{12} - \frac{8}{12}}$$

$$m = \frac{-\frac{5}{2}}{-\frac{5}{12}} = \frac{60}{10} = 6$$

$$y - (-2) = 6\left(x - \frac{1}{4}\right)$$

$$y + 2 = 6x - \frac{3}{2}$$

$$y = 6x - \frac{3}{2} - 2$$

$$y = 6x - \frac{3}{2} - \frac{4}{2}$$

$$y = 6x - \frac{7}{2}$$

22. The line goes through $(-2, \frac{3}{4})$ and $(\frac{2}{3}, \frac{5}{2})$.

$$m = \frac{\frac{5}{2} - \frac{3}{4}}{\frac{2}{3} - (-2)} = \frac{\frac{10}{4} - \frac{3}{4}}{\frac{2}{3} + \frac{6}{3}}$$

$$= \frac{\frac{7}{4}}{\frac{8}{3}} = \frac{21}{32}$$

$$y - \frac{3}{4} = \frac{21}{32}[x - (-2)]$$

$$y - \frac{3}{4} = \frac{21}{32}x + \frac{42}{32}$$

$$y = \frac{21}{32}x + \frac{42}{32} + \frac{3}{4}$$

$$y = \frac{21}{32}x + \frac{21}{16} + \frac{12}{16}$$

$$y = \frac{21}{32}x + \frac{33}{16}$$

23. The line goes through $(-8, 4)$ and $(-8, 6)$.

$$m = \frac{4 - 6}{-8 - (-8)} = \frac{-2}{0};$$

which is undefined.

This is a vertical line; the value of x is always -8 .

The equation of this line is $x = -8$.

24. The line goes through $(-1, 3)$ and $(0, 3)$.

$$m = \frac{3 - 3}{-1 - 0} = \frac{0}{-1} = 0$$

This is a horizontal line; the value of y is always 3.

The equation of this line is $y = 3$.

25. The line has x -intercept -6 and y -intercept -3 .

Two points on the line are $(-6, 0)$ and $(0, -3)$.

Find the slope; then use slope-intercept form.

$$m = \frac{-3 - 0}{0 - (-6)} = \frac{-3}{6} = -\frac{1}{2}$$

$$b = -3$$

$$y = -\frac{1}{2}x - 3$$

$$2y = -x - 6$$

$$x + 2y = -6$$

26. The line has x -intercept -2 and y -intercept 4 . Two points on the line are $(-2, 0)$ and $(0, 4)$. Find the slope; then use slope-intercept form.

$$m = \frac{4 - 0}{0 - (-2)} = \frac{4}{2} = 2$$

$$y = mx + b$$

$$y = 2x + 4$$

$$2x - y = -4$$

27. The vertical line through $(-6, 5)$ goes through the point $(-6, 0)$, so the equation is $x = -6$.

28. The line is horizontal, through $(8, 7)$. The line has an equation of the form $y = k$ where k is the y -coordinate of the point. In this case, $k = 7$, so the equation is $y = 7$.

29. Write an equation of the line through $(-4, 6)$, parallel to $3x + 2y = 13$.

Rewrite the equation of the given line in slope-intercept form.

$$3x + 2y = 13$$

$$2y = -3x + 13$$

$$y = -\frac{3}{2}x + \frac{13}{2}$$

The slope is $-\frac{3}{2}$.

Use $m = -\frac{3}{2}$ and the point $(-4, 6)$ in the point-slope form.

$$y - 6 = -\frac{3}{2}[x - (-4)]$$

$$y = -\frac{3}{2}(x + 4) + 6$$

$$y = -\frac{3}{2}x - 6 + 6$$

$$y = -\frac{3}{2}x$$

$$2y = -3x$$

$$3x + 2y = 0$$

30. Write the equation of the line through $(2, -5)$, parallel to $y - 4 = 2x$. Rewrite the equation in slope-intercept form.

$$y - 4 = 2x$$

$$y = 2x + 4$$

The slope of this line is 2 .

Use $m = 2$ and the point $(2, -5)$ in the point-slope form.

$$y - (-5) = 2(x - 2)$$

$$y + 5 = 2x - 4$$

$$y = 2x - 9$$

$$2x - y = 9$$

31. Write an equation of the line through $(3, -4)$, perpendicular to $x + y = 4$.

Rewrite the equation of the given line as

$$y = -x + 4.$$

The slope of this line is -1 . To find the slope of a perpendicular line, solve

$$-1m = -1.$$

$$m = 1$$

Use $m = 1$ and $(3, -4)$ in the point-slope form.

$$y - (-4) = 1(x - 3)$$

$$y = x - 3 - 4$$

$$y = x - 7$$

$$x - y = 7$$

32. Write the equation of the line through $(-2, 6)$, perpendicular to $2x - 3y = 5$.

Rewrite the equation in slope-intercept form.

$$2x - 3y = 5$$

$$-3y = -2x + 5$$

$$y = \frac{2}{3}x - \frac{5}{3}$$

The slope of this line is $\frac{2}{3}$. To find the slope of a perpendicular line, solve

$$\frac{2}{3}m = -1.$$

$$m = -\frac{3}{2}$$

Use $m = -\frac{3}{2}$ and $(-2, 6)$ in the point-slope form.

$$y - 6 = -\frac{3}{2}[x - (-2)]$$

$$y - 6 = -\frac{3}{2}(x + 2)$$

$$y - 6 = -\frac{3}{2}x - 3$$

$$y = -\frac{3}{2}x + 3$$

$$2y = -3x + 6$$

$$3x + 2y = 6$$

33. Write an equation of the line with y -intercept 4, perpendicular to $x + 5y = 7$.

Find the slope of the given line.

$$x + 5y = 7$$

$$5y = -x + 7$$

$$y = -\frac{1}{5}x + \frac{7}{5}$$

The slope is $-\frac{1}{5}$, so the slope of the perpendicular line will be 5. If the y -intercept is 4, then using the slope-intercept form we have

$$y = mx + b$$

$$y = 5x + 4, \text{ or } 5x - y = -4$$

34. Write the equation of the line with x -intercept $-\frac{2}{3}$, perpendicular to $2x - y = 4$.

Find the slope of the given line.

$$2x - y = 4$$

$$2x - 4 = y$$

The slope of this line is 2. Since the lines are perpendicular, the slope of the needed line is $-\frac{1}{2}$.

The line also has an x -intercept of $-\frac{2}{3}$. Thus, it passes through the point $(-\frac{2}{3}, 0)$.

Using the point-slope form, we have

$$y - 0 = -\frac{1}{2}\left[x - \left(-\frac{2}{3}\right)\right]$$

$$y = -\frac{1}{2}\left(x + \frac{2}{3}\right)$$

$$y = -\frac{1}{2}x - \frac{1}{3}$$

$$6y = -3x - 2$$

$$3x + 6y = -2$$

35. Do the points $(4, 3)$, $(2, 0)$, and $(-18, -12)$ lie on the same line?

Find the slope between $(4, 3)$ and $(2, 0)$.

$$m = \frac{0 - 3}{2 - 4} = \frac{-3}{-2} = \frac{3}{2}$$

Find the slope between $(4, 3)$ and $(-18, -12)$.

$$m = \frac{-12 - 3}{-18 - 4} = \frac{-15}{-22} = \frac{15}{22}$$

Since these slopes are not the same, the points do not lie on the same line.

36. (a) Write the given line in slope-intercept form.

$$2x + 3y = 6$$

$$3y = -2x + 6$$

$$y = -\frac{2}{3}x + 2$$

This line has a slope of $-\frac{2}{3}$. The desired line has a slope of $-\frac{2}{3}$ since it is parallel to the given line. Use the definition of slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$-\frac{2}{3} = \frac{2 - (-1)}{k - 4}$$

$$-\frac{2}{3} = \frac{3}{k - 4}$$

$$-2(k - 4) = (3)(3)$$

$$-2k + 8 = 9$$

$$-2k = 1$$

$$k = -\frac{1}{2}$$

- (b) Write the given line in slope-intercept form.

$$5x - 2y = -1$$

$$2y = 5x + 1$$

$$y = \frac{5}{2}x + \frac{1}{2}$$

This line has a slope of $\frac{5}{2}$. The desired line has a slope of $-\frac{2}{5}$ since it is perpendicular to the given line. Use the definition of slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2 - (-1)}{k - 4}$$

$$-\frac{2}{5} = \frac{2 + 1}{k - 4}$$

$$-\frac{2}{5} = \frac{3}{k - 4}$$

$$-2(k - 4) = (3)(5)$$

$$-2k + 8 = 15$$

$$-2k = 7$$

$$k = -\frac{7}{2}$$

37. A parallelogram has 4 sides, with opposite sides parallel. The slope of the line through $(1, 3)$ and $(2, 1)$ is

$$m = \frac{3 - 1}{1 - 2}$$

$$= \frac{2}{-1}$$

$$= -2.$$

The slope of the line through $\left(-\frac{5}{2}, 2\right)$ and $\left(-\frac{7}{2}, 4\right)$ is

$$m = \frac{2 - 4}{-\frac{5}{2} - \left(-\frac{7}{2}\right)} = \frac{-2}{1} = -2.$$

Since these slopes are equal, these two sides are parallel.

The slope of the line through $\left(-\frac{7}{2}, 4\right)$ and $(1, 3)$ is

$$m = \frac{4 - 3}{-\frac{7}{2} - 1} = \frac{1}{-\frac{9}{2}} = -\frac{2}{9}.$$

Slope of the line through $\left(-\frac{5}{2}, 2\right)$ and $(2, 1)$ is

$$m = \frac{2 - 1}{-\frac{5}{2} - 2} = \frac{1}{-\frac{9}{2}} = -\frac{2}{9}.$$

Since these slopes are equal, these two sides are parallel.

Since both pairs of opposite sides are parallel, the quadrilateral is a parallelogram.

38. Two lines are perpendicular if the product of their slopes is -1 .

The slope of the diagonal containing $(4, 5)$ and $(-2, -1)$ is

$$m = \frac{5 - (-1)}{4 - (-2)} = \frac{6}{6} = 1.$$

The slope of the diagonal containing $(-2, 5)$ and $(4, -1)$ is

$$m = \frac{5 - (-1)}{-2 - 4} = \frac{6}{-6} = -1.$$

The product of the slopes is $(1)(-1) = -1$, so the diagonals are perpendicular.

39. The line goes through $(0, 2)$ and $(-2, 0)$

$$m = \frac{2 - 0}{0 - (-2)} = \frac{2}{2} = 1$$

The correct choice is (a).

40. The line goes through $(1, 3)$ and $(2, 0)$.

$$m = \frac{3 - 0}{1 - 2} = \frac{3}{-1} = -3$$

The correct choice is (f).

41. The line appears to go through $(0, 0)$ and $(-1, 4)$.

$$m = \frac{4 - 0}{-1 - 0} = \frac{4}{-1} = -4$$

42. The line goes through $(-2, 0)$ and $(0, 1)$.

$$m = \frac{1 - 0}{0 - (-2)} = \frac{1}{2}$$

43. (a) See the figure in the textbook.

Segment MN is drawn perpendicular to segment PQ . Recall that MQ is the length of segment MQ .

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{MQ}{PQ}$$

From the diagram, we know that $PQ = 1$.

Thus, $m_1 = \frac{MQ}{1}$, so MQ has length m_1 .

$$(b) \quad m_2 = \frac{\Delta y}{\Delta x} = \frac{-QN}{PQ} = \frac{-QN}{1}$$

$$QN = -m_2$$

- (c) Triangles MPQ , PNQ , and MNP are right triangles by construction. In triangles MPQ and MNP ,

$$\text{angle } M = \text{angle } M,$$

and in the right triangles PNQ and MNP ,

$$\text{angle } N = \text{angle } N.$$

Since all right angles are equal, and since triangles with two equal angles are similar, triangle MPQ is similar to triangle MNP and triangle PNQ is similar to triangle MNP .

Therefore, triangles MNQ and PNQ are similar to each other.

- (d) Since corresponding sides in similar triangles are proportional,

$$MQ = k \cdot PQ \quad \text{and} \quad PQ = k \cdot QN.$$

$$\frac{MQ}{PQ} = \frac{k \cdot PQ}{k \cdot QN}$$

$$\frac{MQ}{PQ} = \frac{PQ}{QN}$$

From the diagram, we know that $PQ = 1$.

$$MQ = \frac{1}{QN}$$

From (a) and (b), $m_1 = MQ$ and

$$-m_2 = QN.$$

Substituting, we get

$$m_1 = \frac{1}{-m_2}.$$

Multiplying both sides by m_2 , we have

$$m_1 m_2 = -1.$$

44. (a) Multiplying both sides of the equation

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{by } ab, \text{ we have}$$

$$ab\left(\frac{x}{a}\right) + ab\left(\frac{y}{b}\right) = ab(1)$$

$$bx + ay = ab.$$

Solve this equation for y .

$$bx + ay = ab$$

$$ay = ab - bx$$

$$y = \frac{ab - bx}{a}$$

$$y = -\frac{b}{a}x + b$$

If we let $m = -\frac{b}{a}$, then the equation becomes

$$y = mx + b.$$

(b) Let $y = 0$.

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{a} + 0 = 1$$

$$\frac{x}{a} = 1$$

$$x = a$$

The x -intercept is a .

Let $x = 0$.

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$0 + \frac{y}{b} = 1$$

$$\frac{y}{b} = 1$$

$$y = b$$

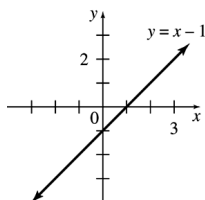
The y -intercept is b .

(c) If the equation of a line is written as

$\frac{x}{a} + \frac{y}{b} = 1$, we immediately know the intercepts of the line, which are a and b .

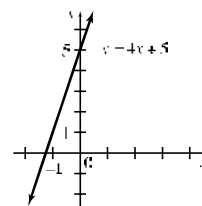
45. $y = x - 1$

Three ordered pairs that satisfy this equation are $(0, -1)$, $(1, 0)$, and $(4, 3)$. Plot these points and draw a line through them.



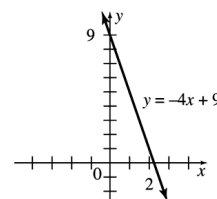
46. $y = 4x + 5$

Three ordered pairs that satisfy this equation are $(-2, -3)$, $(-1, 1)$, and $(0, 5)$. Plot these points and draw a line through them.



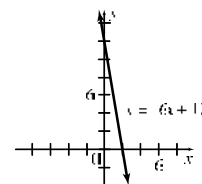
47. $y = -4x + 9$

Three ordered pairs that satisfy this equation are $(0, 9)$, $(1, 5)$, and $(2, 1)$. Plot these points and draw a line through them.



48. $y = -6x + 12$

Three ordered pairs that satisfy this equation are $(0, 12)$, $(1, 6)$, and $(2, 0)$. Plot these points and draw a line through them.



49. $2x - 3y = 12$

Find the intercepts.

If $y = 0$, then

$$2x - 3(0) = 12$$

$$2x = 12$$

$$x = 6$$

so the x -intercept is 6.

If $x = 0$, then

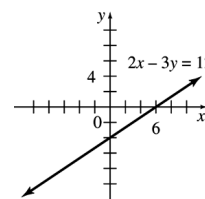
$$2(0) - 3y = 12$$

$$-3y = 12$$

$$y = -4$$

so the y -intercept is -4 .

Plot the ordered pairs $(6, 0)$ and $(0, -4)$ and draw a line through these points. (A third point may be used as a check.)



50. $3x - y = -9$

Find the intercepts.

If $y = 0$, then

$$3x - 0 = -9$$

$$3x = -9$$

$$x = -3$$

If $x = 0$, then

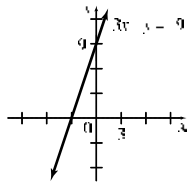
$$3(0) - y = -9$$

$$-y = -9$$

$$y = 9$$

so the y -intercept is 9.

Plot the ordered pairs $(-3, 0)$ and $(0, 9)$ and draw a line through these points. (A third point may be used as a check.)



51. $3y - 7x = -21$

Find the intercepts.

If $y = 0$, then

$$3(0) + 7x = -21$$

$$-7x = -21$$

$$x = 3$$

so the x -intercept is 3.

If $x = 0$, then

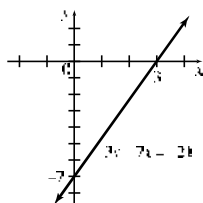
$$3y - 7(0) = -21$$

$$3y = -21$$

$$y = -7$$

So the y -intercept is -7 .

Plot the ordered pairs $(3, 0)$ and $(0, -7)$ and draw a line through these points. (A third point may be used as a check.)



52. $5y + 6x = 11$

Find the intercepts.

If $y = 0$, then

$$5(0) + 6x = 11$$

$$6x = 11$$

$$x = \frac{11}{6}$$

so the x -intercept is $\frac{11}{6}$.

If $x = 0$, then

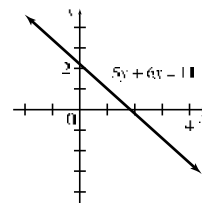
$$5y + 6(0) = 11$$

$$5y = 11$$

$$y = \frac{11}{5}$$

so the y -intercept is $\frac{11}{5}$.

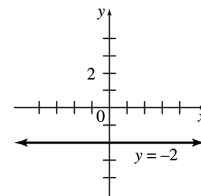
Plot the ordered pairs $(\frac{11}{6}, 0)$ and $(0, \frac{11}{5})$ and draw a line through these points. (A third point may be used as a check.)



53. $y = -2$

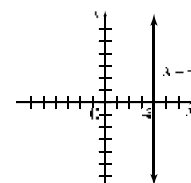
The equation $y = -2$, or, equivalently,

$y = 0x - 2$, always gives the same y -value, -2 , for any value of x . The graph of this equation is the horizontal line with y -intercept -2 .



54. $x = 4$

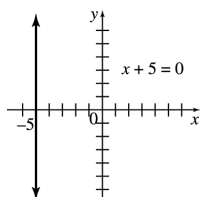
For any value of y , the x -value is 4. Because all ordered pairs that satisfy this equation have the same first number, this equation does not represent a function. The graph is the vertical line with x -intercept 4.



55. $x + 5 = 0$

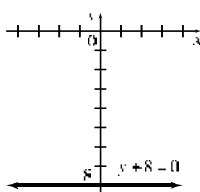
This equation may be rewritten as $x = -5$. For any value of y , the x -value is -5 . Because all ordered pairs that satisfy this equation have the

same first number, this equation does not represent a function. The graph is the vertical line with x -intercept -5 .



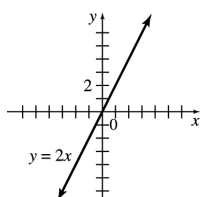
56. $y + 8 = 0$

This equation may be rewritten as $y = -8$, or, equivalently, $y = 0x + -8$. The y -value is -8 for any value of x . The graph is the horizontal line with y -intercept -8 .



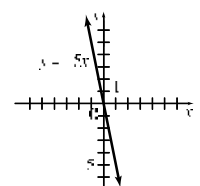
57. $y = 2x$

Three ordered pairs that satisfy this equation are $(0, 0)$, $(-2, -4)$, and $(2, 4)$. Use these points to draw the graph.



58. $y = -5x$

Three ordered pairs that satisfy this equation are $(0, 0)$, $(-1, 5)$, and $(1, -5)$. Use these points to draw the graph.

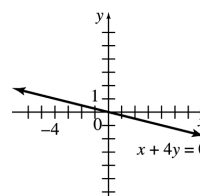


59. $x + 4y = 0$

If $y = 0$, then $x = 0$, so the x -intercept is 0. If $x = 0$, then $y = 0$, so the y -intercept is 0. Both intercepts give the same ordered pair, $(0, 0)$. To get a second point, choose some other value of x (or y). For example if $x = 4$, then

$$\begin{aligned} x + 4y &= 0 \\ 4 + 4y &= 0 \\ 4y &= -4 \\ y &= -1, \end{aligned}$$

giving the ordered pair $(4, -1)$. Graph the line through $(0, 0)$ and $(4, -1)$.



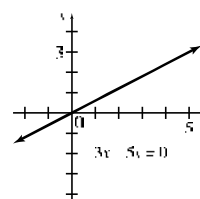
60. $3x - 5y = 0$

If $y = 0$, then $x = 0$, so the x -intercept is 0. If $x = 0$, then $y = 0$, so the y -intercept is 0. Both intercepts give the same ordered pair $(0, 0)$.

To get a second point, choose some other value of x (or y). For example, if $x = 5$, then

$$\begin{aligned} 3x - 5y &= 0 \\ 3(5) - 5y &= 0 \\ 15 - 5y &= 0 \\ -5y &= -15 \\ y &= 3 \end{aligned}$$

giving the ordered pair $(5, 3)$. Graph the line through $(0, 0)$ and $(5, 3)$.



61. (a) The line goes through $(2, 27,000)$ and $(5, 63,000)$.

$$\begin{aligned} m &= \frac{63,000 - 27,000}{5 - 2} \\ &= 12,000 \end{aligned}$$

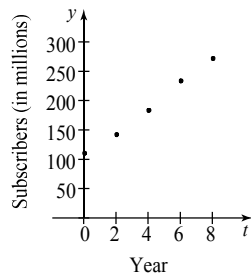
$$\begin{aligned} y - 27,000 &= 12,000(x - 2) \\ y - 27,000 &= 12,000x - 24,000 \\ y &= 12,000x + 3,000 \end{aligned}$$

(b) Let $y = 100,000$; find x .

$$\begin{aligned} 100,000 &= 12,000x + 3,000 \\ 97,000 &= 12,000x \\ 8.08 &= x \end{aligned}$$

Sales would surpass \$100,000 after 8 years, 1 month.

62. (a)



- (b) The line goes through (0, 109.48) and (8, 270.33).

$$m = \frac{270.33 - 109.48}{8 - 0} \approx 20.106$$

$$b = 109.48$$

$$y = 20.106t + 109.48$$

- (c) The line goes through (2, 140.77) and (8, 270.33).

$$m = \frac{270.33 - 140.77}{8 - 2} \approx 21.593$$

Use (2, 140.77) and the point-slope form.

$$y - 140.77 = 21.593(t - 2)$$

$$y = 21.593t - 43.186 + 140.77$$

$$y = 21.593t + 97.58$$

- (d) The data is *approximately* linear because all the data points do not fall on a straight line. So the lines between different pairs of points have different slopes that are close in value.

- (e) $y = 20.106t + 109.48$
 $y = 20.106(7) + 109.48$
 $y = 250.22$ million subscribers
 $y = 21.593t + 97.58$
 $y = 21.593(7) + 97.58$
 $y = 248.73$ million subscribers

Both estimated values are slightly less than the actual number of subscribers of 255.40 million.

63. (a) The line goes through (3, 100) and (28, 215.3).

$$m = \frac{215.3 - 100}{28 - 3} \approx 4.612$$

Use the point (3, 100) and the point-slope form.

$$y - 100 = 4.612(t - 3)$$

$$y = 4.612t - 13.836 + 100$$

$$y = 4.612t + 86.164$$

- (b) The year 2000 corresponds to $t = 2000 - 1980 = 20$.

$$y = 4.612(20) + 86.164$$

$$y \approx 178.4$$

The predicted value is slightly more than the actual CPI of 172.2.

- (c) The annual CPI is increasing at a rate of approximately 4.6 units per year.

64. (a) The line goes through (4, 0.17) and (7, 0.33).

$$m = \frac{0.33 - 0.17}{7 - 4} = \frac{0.16}{3} \approx 0.053$$

$$y - 0.33 = \frac{0.16}{3}(t - 7)$$

$$y - 0.33 = 0.053t - 0.373$$

$$y \approx 0.053t - 0.043$$

- (b) Let $y = 0.5$; solve for t .

$$0.5 = 0.053t - 0.043$$

$$0.543 = 0.053t$$

$$10.2 = t$$

In about 10.2 years, half of these patients will have AIDS.

65. (a) Let $x =$ age.

$$u = 0.85(220 - x) = 187 - 0.85x$$

$$l = 0.7(200 - x) = 154 - 0.7x$$

- (b) $u = 187 - 0.85(20) = 170$

$$l = 154 - 0.7(20) = 140$$

The target heart rate zone is 140 to 170 beats per minute.

- (c) $u = 187 - 0.85(40) = 153$

$$l = 154 - 0.7(40) = 126$$

The target heart rate zone is 126 to 153 beats per minute.

- (d) $154 - 0.7x = 187 - 0.85(x + 36)$

$$154 - 0.7x = 187 - 0.85x - 30.6$$

$$154 - 0.7x = 156.4 - 0.85x$$

$$0.15x = 2.4$$

$$x = 16$$

The younger woman is 16; the older woman is $16 + 36 = 52$. $l = 0.7(220 - 16) \approx 143$ beats per minute.

66. Let x represent the force and y represent the speed. The linear function contains the points (0.75, 2) and (0.93, 3).

$$m = \frac{3 - 2}{0.93 - 0.75} = \frac{1}{0.18} = \frac{1}{\frac{18}{100}} = \frac{100}{18} = \frac{50}{9}$$

Use point-slope form to write the equation.

$$\begin{aligned} y - 2 &= \frac{50}{9}(x - 0.75) \\ y - 2 &= \frac{50}{9}x - \frac{50}{9}(0.75) \\ y &= \frac{50}{9}x - \frac{75}{18} + 2 \\ y &= \frac{50}{9}x - \frac{13}{6} \end{aligned}$$

Now determine y , the speed, when x , the force, is 1.16.

$$\begin{aligned} y &= \frac{50}{9}(1.16) - \frac{13}{6} \\ &= \frac{58}{9} - \frac{13}{6} = \frac{77}{18} \approx 4.3 \end{aligned}$$

The pony switches from a trot to a gallop at approximately 4.3 meters per second.

67. Let $x = 0$ correspond to 1900. Then the “life expectancy from birth” line contains the points (0, 46) and (104, 77.8).

$$m = \frac{77.8 - 46}{104 - 0} = \frac{31.8}{104} \approx 0.306$$

Since (0, 46) is one of the points, the line is given by the equation.

$$y = 0.306x + 46.$$

The “life expectancy from age 65” line contains the points (0, 76) and (104, 83.7).

$$m = \frac{83.7 - 76}{104 - 0} = \frac{7.7}{104} \approx 0.074$$

Since (0, 76) is one of the points, the line is given by the equation

$$y = 0.074x + 76.$$

Set the two expressions for y equal to determine where the lines intersect. At this point, life expectancy should increase no further.

$$\begin{aligned} 0.306x + 46 &= 0.074x + 76 \\ 0.232x &= 30 \\ x &\approx 129 \end{aligned}$$

Determine the y -value when $x = 129$. Use the first equation.

$$\begin{aligned} y &= 0.306(129) + 46 \\ &= 39.474 + 46 \\ &= 85.474 \end{aligned}$$

Thus, the maximum life expectancy for humans is about 86 years.

68. (a) The line goes through (90, 90) and (108, 65).

$$m = \frac{65 - 90}{108 - 90} \approx -1.389$$

Use the point (90, 90) and the point-slope form.

$$\begin{aligned} y - 90 &= -1.389(t - 90) \\ y &= -1.389t + 125.01 + 90 \\ y &= -1.389t + 215 \end{aligned}$$

- (b) Let $y = 50$.

$$\begin{aligned} 50 &= -1.389t + 215 \\ -165 &= -1.389t \\ 119 &\approx t \end{aligned}$$

The mortality rate will drop to 50 or below in the year $1900 + 119 = 2019$.

69. (a) The line goes through (9, 17.2) and (18, 20.3).

$$m = \frac{20.3 - 17.2}{18 - 9} \approx 0.344$$

Use the point (9, 17.2) and the point-slope form.

$$\begin{aligned} y - 17.2 &= 0.344(t - 9) \\ y &= 0.344t - 3.096 + 17.2 \\ y &= 0.344t + 14.1 \end{aligned}$$

- (b) Let $y = 25$.

$$\begin{aligned} 25 &= 0.344t + 14.1 \\ 10.9 &= 0.344t \\ 32 &\approx t \end{aligned}$$

The percentage of adults without health insurance would be at least 25% in the year $1990 + 32 = 2022$.

70. (a) The line (for the data for men) goes through (0, 24.7) and (25, 27.1).

$$m = \frac{27.1 - 24.7}{25 - 0} \approx 0.096$$

Use the point (0, 24.7) and the point-slope form.

$$\begin{aligned} y - 24.7 &= 0.096(t - 0) \\ y &= 0.096t + 24.7 \end{aligned}$$

- (b) The line (for the data for women) goes through (0, 22.0) and (25, 25.3).

$$m = \frac{25.3 - 22.0}{25 - 0} \approx 0.132$$

Use the point (0, 22.0) and the point-slope form.

$$\begin{aligned} y - 22.0 &= 0.132(t - 0) \\ y &= 0.132t + 22.0 \end{aligned}$$

- (c) Since $0.132 > 0.096$, women seem to have the faster increase in median age at first marriage.

- (d) Let
- $y = 30$
- .

$$30 = 0.096t + 24.7$$

$$5.3 = 0.096t$$

$$55.208 \approx t$$

The median age at first marriage for men will reach 30 in the year $1980 + 55 = 2035$ or $1980 + 56 = 2036$, depending on how the computations were rounded.

- (e) Let
- $t = 55$
- .

$$y = 0.132(55.208) + 22.0$$

$$y \approx 29.3$$

The median age at first marriage for women will reach be 29.3 when the median age for men is 30. (The answer will be 29.4 if the year 2036 is used as the answer for part (d).)

71. (a) The line goes through $(50, 249,187)$ and $(108, 1,107,126)$.

$$m = \frac{1,107,126 - 249,187}{108 - 50}$$

$$\approx 14,792.05$$

Use the point $(50, 249,187)$ and the point-slope form.

$$y - 249,187 = 14,792.05(t - 50)$$

$$y = 14,792.05t - 739,602.5 + 249,187$$

$$y = 14,792.05t - 490,416$$

- (b) The year 2015 corresponds to
- $t = 115$
- .

$$y = 14,792.05(115) - 490,416$$

$$y \approx 1,210,670$$

The number of immigrants admitted to the United States in 2015 will be about 1,210,670.

- (c) The equation $y = 14,792.05t - 490,416$ has $-490,416$ for the y -intercept, indicating that the number of immigrants admitted in the year 1900 was $-490,416$. Realistically, the number of immigrants cannot be a negative value, so the equation cannot be used for valid predicted values.

72. (a) If the temperature rises 0.3°C per decade, it rises 0.03°C per year.

$$m = 0.03$$

$$b = 15, \text{ since a point is } (0, 15).$$

$$T = 0.03t + 15$$

- (b) Let
- $T = 19$
- ; find
- t
- .

$$19 = 0.03t + 15$$

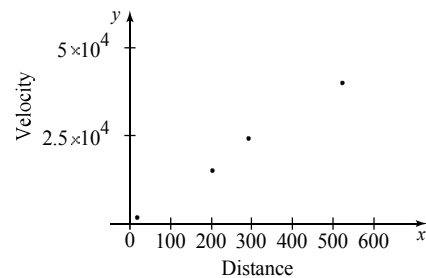
$$4 = 0.03t$$

$$t = 133.3 \approx 133$$

So, $1970 + 133 = 2103$.

The temperature will rise to 19°C in about the year 2103.

73. (a) Plot the points $(15, 1600)$, $(200, 15,000)$, $(290, 24,000)$, and $(520, 40,000)$.



The points lie approximately on a line, so there appears to be a linear relationship between distance and time.

- (b) The graph of any equation of the form $y = mx$ goes through the origin, so the line goes through $(520, 40,000)$ and $(0, 0)$.

$$m = \frac{40,000 - 0}{520 - 0} \approx 76.9$$

$$b = 0$$

$$y = 76.9x + 0$$

$$y = 76.9x$$

- (c) Let
- $y = 60,000$
- ; solve for
- x
- .

$$60,000 = 76.9x$$

$$780.23 \approx x$$

Hydra is about 780 megaparsecs from earth.

- (d) $A = \frac{9.5 \times 10^{11}}{m}, m = 76.9$

$$A = \frac{9.5 \times 10^{11}}{76.9}$$

$$= 12.4 \text{ billion years}$$

74. (a) The line goes through $(1, 1139)$ and $(7, 1370)$.

$$m = \frac{1370 - 1139}{7 - 1} \approx 38.5$$

Use the point $(1, 1139)$ and the point-slope form.

$$y - 1139 = 38.5(t - 1)$$

$$y = 38.5t - 38.5 + 1139$$

$$y = 38.5t + 1100.5$$

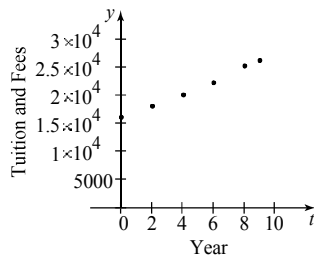
- (b) The year 2008 corresponds to $t = 8$.

$$y = 38.5(8) + 1100.5$$

$$y \approx 1408.5$$

The number of stations carrying news/talk radio in 2008 will be about 1408 or 1409. Since the actual number of stations in 2008 is 2046, which is almost twice the predicted value of 1409, the linear trend from 2001 to 2007 did not continue in 2008.

75. (a)



Yes, the data appear to lie roughly along a straight line.

- (b) The line goes through $(0, 16,072)$ and $(9, 26,273)$.

$$m = \frac{26,273 - 16,072}{9 - 0} \approx 1133.4$$

$$b = 16,072$$

$$y = 1133.4t + 16,072$$

The slope 1133.4 indicates that tuition and fees have increased approximately \$1133 per year.

- (c) The year 2025 is too far in the future to rely on this equation to predict costs; too many other factors may influence these costs by then.

1.2 Linear Functions and Applications

Your Turn 1

For $g(x) = -4x + 5$, calculate $g(-5)$.

$$g(x) = -4x + 5$$

$$g(-5) = -4(-5) + 5$$

$$= 20 + 5$$

$$= 25$$

Your Turn 2

For the demand and supply functions given in Example 2, find the quantity of watermelon demanded and supplied at a price of \$3.30 per watermelon.

$$p = D(q) = 9 - 0.75q$$

$$3.30 = 9 - 0.75q$$

$$0.75q = 5.7$$

$$q = \frac{5.7}{0.75} = 7.6$$

Since the quantity is in thousands, 7600 watermelon are demanded at a price of \$3.30.

$$p = S(q) = 0.75q$$

$$3.30 = 0.75q$$

$$q = \frac{3.3}{0.75} = 4.4$$

Since the quantity is in thousands, 4400 watermelon are supplied at a price of \$3.30.

Your Turn 3

Set the two price expressions equal and solve for the equilibrium quantity q .

$$10 - 0.85q = 0.4q$$

$$10 = 1.25q$$

$$q = \frac{10}{1.25} = 8$$

The equilibrium quantity is 8000 watermelon. Use either price expression to find the equilibrium price p .

$$p = 0.4q$$

$$p = 0.4(8) = 3.2$$

The equilibrium price is \$3.20 per watermelon.

Your Turn 4

The marginal cost is the slope of the cost function $C(x)$, so this function has the form $C(x) = 15x + b$. To find b , use the fact that producing 80 batches costs \$1930.

$$C(x) = 15x + b$$

$$C(80) = 15(80) + b$$

$$1930 = 1200 + b$$

$$b = 730$$

Thus the cost function is $C(x) = 15x + 730$.

Your Turn 5

The cost function is $C(x) = 35x + 250$ and the revenue function is $R(x) = 58x$. Thus the profit function is

$$P(x) = R(x) - C(x)$$

$$= 58x - (35x + 250)$$

$$= 23x - 250$$

The profit is to be \$8030.

$$P(x) = 23x - 250$$

$$8030 = 23x - 250$$

$$23x = 8280$$

$$x = \frac{8280}{23} = 360$$

Sale of 360 units will produce \$8030 profit.

1.2 Exercises

1. $f(2) = 7 - 5(2) = 7 - 10 = -3$
2. $f(4) = 7 - 5(4) = 7 - 20 = -13$
3. $f(-3) = 7 - 5(-3) = 7 + 15 = 22$
4. $f(-1) = 7 - 5(-1) = 7 + 5 = 12$
5. $g(1.5) = 2(1.5) - 3 = 3 - 3 = 0$
6. $g(2.5) = (2.5) - 3 = 5 - 3 = 2$
7. $g\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right) - 3 = -1 - 3 = -4$
8. $g\left(-\frac{3}{4}\right) = 2\left(-\frac{3}{4}\right) - 3 = -\frac{3}{2} - 3 = -\frac{9}{2}$
9. $f(t) = 7 - 5(t) = 7 - 5t$
10. $g(k^2) = 2(k^2) - 3 = 2k^2 - 3$
11. This statement is true.
When we solve $y = f(x) = 0$, we are finding the value of x when $y = 0$, which is the x -intercept.
When we evaluate $f(0)$, we are finding the value of y when $x = 0$, which is the y -intercept.
12. This statement is false.
The graph of $f(x) = -5$ is a horizontal line.
13. This statement is true.
Only a vertical line has an undefined slope, but a vertical line is not the graph of a function.
Therefore, the slope of a linear function cannot be undefined.
14. This statement is true.
For any value of a ,
$$f(0) = a \cdot 0 = 0,$$
so the point $(0, 0)$, which is the origin, lies on the line.
15. The fixed cost is constant for a particular product and does not change as more items are made. The marginal cost is the rate of change of cost at a specific level of production and is equal to the slope of the cost function at that specific value; it approximates the cost of producing one additional item.
19. \$10 is the fixed cost and \$2.25 is the cost per hour.
Let x = number of hours;
 $R(x)$ = cost of renting a snowboard for x hours.
Thus,

$$\begin{aligned} R(x) &= \text{fixed cost} + (\text{cost per hour}) \cdot (\text{number of hours}) \\ R(x) &= 10 + (2.25)(x) \\ &= 2.25x + 10 \end{aligned}$$

20. \$10 is the fixed cost and \$0.99 is the cost per downloaded song—the marginal cost.
Let x = the number of downloaded songs and
 $C(x)$ = cost of downloading x songs. Then,
 $C(x)$ = (marginal cost) $\cdot$ (number of downloaded songs)
+ fixed cost
$$C(x) = 0.99x + 10.$$
21. \$2 is the fixed cost and \$0.75 is the cost per half-hour.
Let x = the number of half-hours;
 $C(x)$ = the cost of parking a car for x half-hours.
Thus,
$$\begin{aligned} C(x) &= 2 + 0.75x \\ &= 0.75x + 2 \end{aligned}$$
22. \$44 is the fixed cost and \$0.28 is the cost per mile.
Let x = the number of miles;
 $R(x)$ = the cost of renting for x miles.
Thus,
 $R(x)$ = fixed cost + (cost per mile) $\cdot$ (number of miles)
$$R(x) = 44 + 0.28x.$$
23. Fixed cost, \$100; 50 items cost \$1600 to produce.
Let $C(x)$ = cost of producing x items.
 $C(x) = mx + b$, where b is the fixed cost.
$$C(x) = mx + 100$$
Now,
 $C(x) = 1600$ when $x = 50$, so
$$\begin{aligned} 1600 &= m(50) + 100 \\ 1500 &= 50m \\ 30 &= m. \end{aligned}$$
Thus, $C(x) = 30x + 100$.
24. Fixed cost: \$35; 8 items cost \$395.
Let $C(x)$ = cost of x items
 $C(x) = mx + b$, where b is the fixed cost
$$C(x) = mx + 35$$
Now, $C(x) = 395$ when $x = 8$, so
$$\begin{aligned} 395 &= m(8) + 35 \\ 360 &= 8m \\ 45 &= m. \end{aligned}$$
Thus, $C(x) = 45x + 35$.

25. Marginal cost: \$75; 50 items cost \$4300.

$$C(x) = 75x + b$$

Now, $C(x) = 4300$ when $x = 50$.

$$4300 = 75(50) + b$$

$$4300 = 3750 + b$$

$$550 = b$$

Thus, $C(x) = 75x + 550$.

26. Marginal cost, \$120; 700 items cost \$96,500 to produce.

$$C(x) = 120x + b$$

Now, $C(x) = 96,500$ when $x = 700$.

$$96,500 = 120(700) + b$$

$$96,500 = 84,000 + b$$

$$12,500 = b$$

Thus, $C(x) = 120x + 12,500$.

27. $D(q) = 16 - 1.25q$

(a) $D(0) = 16 - 1.25(0) = 16 - 0 = 16$

When 0 watches are demanded, the price is \$16.

(b) $D(4) = 16 - 1.25(4) = 16 - 5 = 11$

When 400 watches are demanded, the price is \$11.

(c) $D(8) = 16 - 1.25(8) = 16 - 10 = 6$

When 800 watches are demanded, the price is \$6.

(d) Let $D(q) = 8$. Find q .

$$8 = 16 - 1.25q$$

$$\frac{5}{4}q = 8$$

$$q = 6.4$$

When the price is \$8, the number of watches demanded is 640.

(e) Let $D(q) = 10$. Find q .

$$10 = 16 - 1.25q$$

$$\frac{5}{4}q = 6$$

$$q = 4.8$$

When the price is \$10, the number of watches demanded is 480.

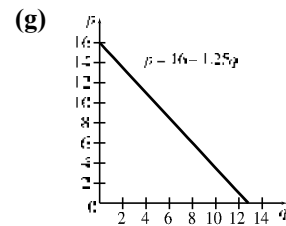
(f) Let $D(q) = 12$. Find q .

$$12 = 16 - 1.25q$$

$$\frac{5}{4}q = 4$$

$$q = 3.2$$

When the price is \$12, the number of watches demanded is 320.



(h) $S(q) = 0.75q$

Let $S(q) = 0$. Find q .

$$0 = 0.75q$$

$$0 = q$$

When the price is \$0, the number of watches supplied is 0.

(i) Let $S(q) = 10$. Find q .

$$10 = 0.75q$$

$$\frac{40}{3} = q$$

$$q = 13.\bar{3}$$

When the price is \$10, The number of watches supplied is about 1333.

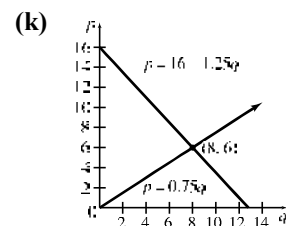
(j) Let $S(q) = 20$. Find q .

$$20 = 0.75q$$

$$\frac{80}{3} = q$$

$$q = 26.\bar{6}$$

When the price is \$20, the number of watches demanded is about 2667.



(l) $D(q) = S(q)$

$$16 - 1.25q = 0.75q$$

$$16 = 2q$$

$$8 = q$$

$$S(8) = 0.75(8) = 6$$

The equilibrium quantity is 800 watches, and the equilibrium price is \$6.

28. $D(q) = 5 - 0.25q$

(a) $D(0) = 5 - 0.25(0) = 5 - 0 = 5$

When 0 quarts are demanded, the price is \$5.

(b) $D(4) = 5 - 0.25(4) = 5 - 1 = 4$

When 400 quarts are demanded, the price is \$4.

(c) $D(8.4) = 5 - 0.25(8.4) = 5 - 2.1 = 2.9$

When 840 quarts are demanded, the price is \$2.90.

(d) Let $D(q) = 4.5$. Find q .

$$4.5 = 5 - 0.25q$$

$$0.25q = 0.5$$

$$q = 2$$

When the price is \$4.50, 200 quarts are demanded.

(e) Let $D(q) = 3.25$. Find q .

$$3.25 = 5 - 0.25q$$

$$0.25q = 1.75$$

$$q = 7$$

When the price is \$3.25, 700 quarts are demanded.

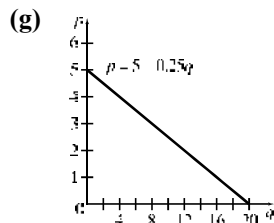
(f) Let $D(q) = 2.4$. Find q .

$$2.4 = 5 - 0.25q$$

$$0.25q = 2.6$$

$$q = 10.4$$

When the price is \$2.40, 1040 quarts are demanded.



(h) $S(q) = 0.25q$

Let $S(q) = 0$. Find q .

$$0 = 0.25q$$

$$q = 0$$

When the price is \$0, 0 quarts are supplied.

(i) Let $S(q) = 2$. Find q .

$$2 = 0.25q$$

$$q = 8$$

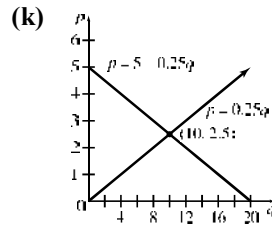
When the price is \$2, 800 quarts are supplied.

(j) Let $S(q) = 4.5$. Find q .

$$4.5 = 0.25q$$

$$q = 18$$

When the price is \$4.50, 1800 quarts are supplied.



(l) $D(q) = S(q)$

$$5 - 0.25q = 0.25q$$

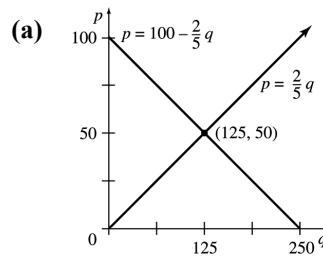
$$5 = 0.5q$$

$$10 = q$$

$$S(10) = 0.25(10) = 2.5$$

The equilibrium quantity is 1000 quarts and the equilibrium price is \$2.50.

29. $p = S(q) = \frac{2}{5}q$; $p = D(q) = 100 - \frac{2}{5}q$



(b) $S(q) = D(q)$

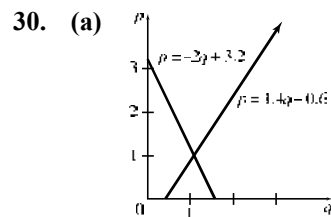
$$\frac{2}{5}q = 100 - \frac{2}{5}q$$

$$\frac{4}{5}q = 100$$

$$q = 125$$

$$S(125) = \frac{2}{5}(125) = 50$$

The equilibrium quantity is 125, the equilibrium price is \$50.



(b) $S(q) = p = 1.4q - 0.6$

$$D(q) = p = -2q + 3.2$$

Set supply equal to demand and solve for q .

$$1.4q - 0.6 = -2q + 3.2$$

$$1.4q + 2q = 0.6 + 3.2$$

$$3.4q = 3.8$$

$$q = \frac{3.8}{3.4} \approx 1.12$$

$$S(1.12) = 1.4(1.12) - 0.6 = 0.968$$

The equilibrium quantity is about 1120 pounds; the equilibrium price is about \$0.96

- 31.** Use the supply function to find the equilibrium quantity that corresponds to the given equilibrium price of \$4.50.

$$S(q) = p = 0.3q + 2.7$$

$$4.50 = 0.3q + 2.7$$

$$1.8 = 0.3q$$

$$6 = q$$

The line that represents the demand function goes through the given point (2, 6.10) and the equilibrium point (6, 4.50).

$$m = \frac{4.50 - 6.10}{6 - 2} = -0.4$$

Use point-slope form and the point (2, 6.10).

$$D(q) - 6.10 = -0.4(q - 2)$$

$$D(q) = -0.4q + 0.8 + 6.10$$

$$D(q) = -0.4q + 6.9$$

- 32.** Use the supply function to find the equilibrium quantity that corresponds to the given equilibrium price of \$5.85.

$$p = S(q)$$

$$5.85 = 0.25q + 3.6$$

$$2.25 = 0.25q$$

$$9 = q$$

The line that represents the demand function goes through the given point (4, 7.60) and the equilibrium point (9, 5.85).

$$m = \frac{5.85 - 7.60}{9 - 4} = -0.35$$

Use point-slope form and the point (4, 7.60).

$$D(q) - 7.60 = -0.35(q - 4)$$

$$D(q) = -0.35q + 1.4 + 7.60$$

$$D(q) = -0.35q + 9$$

- 33. (a)** $C(x) = mx + b$; $m = 3.50$; $C(60) = 300$

$$C(x) = 3.50x + b$$

Find b .

$$300 = 3.50(60) + b$$

$$300 = 210 + b$$

$$90 = b$$

$$C(x) = 3.50x + 90$$

(b) $R(x) = 9x$

$$C(x) = R(x)$$

$$3.50x + 90 = 9x$$

$$90 = 5.5x$$

$$16.36 = x$$

Joanne must produce and sell 17 shirts.

(c) $P(x) = R(x) - C(x)$; $P(x) = 500$

$$500 = 9x - (3.50x + 90)$$

$$500 = 5.5x - 90$$

$$590 = 5.5x$$

$$107.27 = x$$

To make a profit of \$500, Joanne must produce and sell 108 shirts.

34. (a) $C(x) = mx + b$

$$C(1000) = 2675$$
; $b = 525$

Find m .

$$2675 = m(1000) + 525$$

$$2150 = 1000m$$

$$2.15 = m$$

$$C(x) = 2.15x + 525$$

(b) $R(x) = 4.95x$

$$C(x) = R(x)$$

$$2.15x + 525 = 4.95x$$

$$525 = 2.80x$$

$$187.5 = x$$

In order to break even, he must produce and sell 188 books.

(c) $P(x) = R(x) - C(x)$; $P(x) = 1000$

$$1000 = 4.95x - (2.15x + 525)$$

$$1000 = 4.95x - 2.15x - 525$$

$$1000 = 2.80x - 525$$

$$1525 = 2.80x$$

$$544.6 = x$$

In order to make a profit of \$1000, he must produce and sell 545 books.

- 35. (a)** Using the points (100, 11.02) and (400, 40.12),

$$m = \frac{40.12 - 11.02}{400 - 100}$$

$$= \frac{29.1}{300} = 0.097.$$

$$y - 11.02 = 0.097(x - 100)$$

$$y - 11.02 = 0.097x - 9.7$$

$$y = 0.097x + 1.32$$

$$C(x) = 0.097x + 1.32$$

- (b)** The fixed cost is given by the constant in $C(x)$. It is \$1.32.

$$\begin{aligned} \text{(c)} \quad C(1000) &= 0.097(1000) + 1.32 \\ &= 97 + 1.32 \\ &= 98.32 \end{aligned}$$

The total cost of producing 1000 cups is \$98.32.

$$\begin{aligned} \text{(d)} \quad C(1001) &= 0.097(1001) + 1.32 \\ &= 97.097 + 1.32 \\ &= 98.417 \end{aligned}$$

The total cost of producing 10001 cups is \$98.42.

$$\begin{aligned} \text{(e)} \quad \text{Marginal cost} &= 98.417 - 98.32 \\ &= \$0.097 \quad \text{or} \quad 9.7\text{¢} \end{aligned}$$

(f) The marginal cost for *any* cup is the slope, \$0.097 or 9.7¢. This means the cost of producing one additional cup of coffee would be 9.7¢.

$$36. \quad C(10,000) = 547,500; C(50,000) = 737,500$$

$$\text{(a)} \quad C(x) = mx + b$$

$$\begin{aligned} m &= \frac{737,500 - 547,500}{50,000 - 10,000} \\ &= \frac{190,000}{40,000} \\ &= 4.75 \end{aligned}$$

$$y - 547,500 = 4.75(x - 10,000)$$

$$y - 547,500 = 4.75x - 47,500$$

$$y = 4.75x + 500,000$$

$$C(x) = 4.75x + 500,000$$

(b) The fixed cost is \$500,000.

$$\begin{aligned} \text{(c)} \quad C(100,000) &= 4.75(100,000) + 500,000 \\ &= 475,000 + 500,000 \\ &= 975,000 \end{aligned}$$

The total cost to produce 100,000 items is \$975,000.

(d) Since the slope of the cost function is 4.75, the marginal cost is \$4.75. This means that the cost of producing one additional item at this production level is \$4.75.

$$37. \quad C(x) = 5x + 20; R(x) = 15x$$

$$\text{(a)} \quad C(x) = R(x)$$

$$5x + 20 = 15x$$

$$20 = 10x$$

$$2 = x$$

The break-even quantity is 2 units.

$$\text{(b)} \quad P(x) = R(x) - C(x)$$

$$P(x) = 15x - (5x + 20)$$

$$P(100) = 15(100) - (5 \cdot 100 + 20)$$

$$= 1500 - 520$$

$$= 980$$

The profit from 100 units is \$980.

$$\text{(c)} \quad P(x) = 500$$

$$15x - (5x + 20) = 500$$

$$10x - 20 = 500$$

$$10x = 520$$

$$x = 52$$

For a profit of \$500, 52 units must be produced.

$$38. \quad C(x) = 12x + 39; R(x) = 25x$$

$$\text{(a)} \quad C(x) = R(x)$$

$$12x + 39 = 25x$$

$$39 = 13x$$

$$3 = x$$

The break-even quantity is 3 units.

$$\text{(b)} \quad P(x) = R(x)$$

$$P(x) = 25x - (12x + 39)$$

$$P(x) = 13x - 39$$

$$P(250) = 13(250) - 39$$

$$= 3250 - 39$$

$$= 3211$$

The profit from 250 units is \$3211.

$$\text{(c)} \quad P(x) = \$130; \text{ find } x.$$

$$130 = 13x - 39$$

$$169 = 13x$$

$$13 = x$$

For a profit of \$130, 13 units must be produced.

$$39. \quad C(x) = 85x + 900$$

$$R(x) = 105x$$

Set $C(x) = R(x)$ to find the break-even quantity.

$$85x + 900 = 105x$$

$$900 = 20x$$

$$45 = x$$

The break-even quantity is 45 units. You should decide not to produce since no more than 38 units can be sold.

$$P(x) = R(x) - C(x)$$

$$= 105x - (85x + 900)$$

$$= 20x - 900$$

The profit function is $P(x) = 20x - 900$.

40. $C(x) = 105x + 6000$

$$R(x) = 250x$$

Set $C(x) = R(x)$ to find the break-even quantity.

$$105x + 6000 = 250x$$

$$6000 = 145x$$

$$41.38 \approx x$$

The break-even quantity is about 41 units, so you should decide to produce.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 250x - (105x + 6000) \\ &= 145x - 6000 \end{aligned}$$

The profit function is $P(x) = 145x - 6000$.

41. $C(x) = 70x + 500$

$$R(x) = 60x$$

$$70x + 500 = 60x$$

$$10x = -500$$

$$x = -50$$

This represents a break-even quantity of -50 units. It is impossible to make a profit when the break-even quantity is negative. Cost will always be greater than revenue.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 60x - (70x + 500) \\ &= -10x - 500 \end{aligned}$$

The profit function is $P(x) = -10x - 500$.

42. $C(x) = 1000x + 5000$

$$R(x) = 900x$$

$$900x = 1000x + 5000$$

$$-5000 = 100x$$

$$-50 = x$$

It is impossible to make a profit when the break-even quantity is negative. Cost will always be greater than revenue.

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 900x - (1000x + 5000) \\ &= -100x - 5000 \end{aligned}$$

The profit function is $P(x) = -100x - 5000$ (always a loss).

43. Since the fixed cost is \$400, the cost function is $C(x) = mx + 100$, where m is the cost per unit. The revenue function is $R(x) = px$, where p is the price per unit. The profit $P(x) = R(x) - C(x)$ is 0 at the given break-even quantity of 80.

$$P(x) = px - (mx + 400)$$

$$P(x) = px - mx - 400$$

$$P(x) = Mx - 400 \quad (\text{Let } M = p - m.)$$

$$P(80) = M \cdot 80 - 400$$

$$0 = 80M - 400$$

$$400 = 80M$$

$$5 = M$$

So, the linear profit function is $P(x) = 5x - 400$, and the marginal profit is 5.

44. Since the fixed cost is \$650, the cost function is $C(x) = mx + 650$, where m is the cost per unit. The revenue function is $R(x) = px$, where p is the price per unit. The profit $P(x) = R(x) - C(x)$ is 0 at the given break-even quantity of 25.

$$P(x) = px - (mx + 650)$$

$$P(x) = px - mx - 650$$

$$P(x) = Mx - 650 \quad (\text{Let } M = p - m.)$$

$$P(25) = M \cdot 25 - 650$$

$$0 = 25M - 650$$

$$650 = 25M$$

$$26 = M$$

So, the linear profit function is $P(x) = 26x - 650$, and the marginal profit is 26.

45. Use the formula derived in Example 7 in this section of the textbook.

$$F = \frac{9}{5}C + 32 \quad \text{or} \quad C = \frac{5}{9}(F - 32)$$

- (a) $F = 58$; find C .

$$C = \frac{5}{9}(58 - 32)$$

$$C = \frac{5}{9}(26) = 14.4$$

The temperature is 14.4°C .

- (b) $F = -20$; find C .

$$C = \frac{5}{9}(F - 32)$$

$$C = \frac{5}{9}(-20 - 32)$$

$$C = \frac{5}{9}(-52) = -28.9$$

The temperature is -28.9°C .

- (c)
- $C = 50$
- ; find
- F
- .

$$F = \frac{9}{5}C + 32$$

$$F = \frac{9}{5}(50) + 32$$

$$F = 90 + 32 = 122$$

The temperature is 122°F .

46. Use the formula derived in Example 7 in this section of the textbook.

$$F = \frac{9}{5}C + 32 \quad \text{or} \quad C = \frac{5}{9}(F - 32)$$

- (a)
- $C = 37$
- ; find
- F
- .

$$F = \frac{9}{5}(37) + 32$$

$$F = \frac{333}{5} + 32 = 98.6$$

The Fahrenheit equivalent of 37°C is 98.6°F .

- (b)
- $C = 36.5$
- ; find
- F
- .

$$F = \frac{9}{5}(36.5) + 32$$

$$F = 65.7 + 32 = 97.7$$

$C = 37.5$; find F .

$$F = \frac{9}{5}(37.5) + 32$$

$$= 67.5 + 32 = 99.5$$

The range is between 97.7°F and 99.5°F .

47. If the temperatures are numerically equal, then $F = C$.

$$F = \frac{9}{5}C + 32$$

$$C = \frac{9}{5}C + 32$$

$$-\frac{4}{5}C = 32$$

$$C = -40$$

The Celsius and Fahrenheit temperatures are numerically equal at -40° .

48. (a) $m = 1140$

$$b = 486,000$$

$$C(x) = mx + b$$

$$C(x) = 1140x + 486,000$$

- (b)
- $C(500) = 1140 \cdot 500 + 486,000$

$$= 1,056,000$$

The total cost for 500 students will be \$1,056,000.

- (c) Let
- $C(x) = 1,000,000$
- .

$$1,000,000 = 1140x + 486,000$$

$$514,000 = 1140x$$

$$450.88 = x$$

The maximum number of students that each center can support for \$1 million in costs is 450 students.

1.3 The Least Squares Line

Your Turn 1

Rather than recompute all the numbers in the solution table for Example 1, we record the changes to the totals. Note that we have $(90)(40.2) = 3618$, which replaces the next to last value in the xy column. Also note that we have $40.2^2 = 1616.04$, which replaces the next to last value in the y^2 column. The new totals are as follows:

$$\Sigma x = 550 - 100 = 450$$

$$\Sigma y = 595.5 - 34.0 - 36.9 + 40.2 = 564.8$$

$$\Sigma xy = 28,135 - 3400 - 3321 + 3618 = 25,032$$

$$\Sigma x^2 = 38,500 - 10,000 = 28,500$$

$$\begin{aligned} \Sigma y^2 &= 38,249.41 - 1156.00 - 1361.61 + 1616.04 \\ &= 37,347.84 \end{aligned}$$

The number of data points n is now 9 rather than 10. Put the new column totals into the formulas for the slope and intercept.

$$m = \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{n(\Sigma x^2) - (\Sigma x)^2}$$

$$m = \frac{9(25,032) - (450)(564.8)}{9(28,500) - (450)^2} \approx -0.534667$$

$$m \approx -0.535$$

$$b = \frac{\Sigma y - m(\Sigma x)}{n}$$

$$= \frac{564.8 - (-0.534667)(450)}{9} \approx 89.5$$

The least squares line is $Y = -0.535x + 89.5$.

Your Turn 2

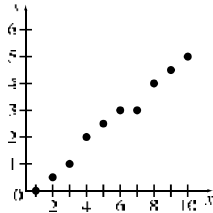
Use the new column totals computed in Your Turn 1.

$$\begin{aligned} r &= \frac{n(\Sigma xy) - (\Sigma x)(\Sigma y)}{\sqrt{n(\Sigma x^2) - (\Sigma x)^2} \cdot \sqrt{n(\Sigma y^2) - (\Sigma y)^2}} \\ &= \frac{9(25,032) - (450)(564.8)}{\sqrt{9(28,500) - (450)^2} \cdot \sqrt{9(37,347.84) - (564.8)^2}} \\ &\approx -0.949 \end{aligned}$$

1.3 Exercises

2. For the set of points (1, 4), (2, 5), and (3, 6),
 $Y = x + 3$. For the set (4, 1), (5, 2), and (6, 3),
 $Y = x - 3$.

3. (a)



(b)

x	y	xy	x^2	y^2
1	0	0	1	0
2	0.5	1	4	0.25
3	1	3	9	1
4	2	8	16	4
5	2.5	12.5	25	6.25
6	3	18	36	9
7	3	21	49	9
8	4	32	64	16
9	4.5	40.5	81	20.25
10	5	50	100	25
55	25.5	186	385	90.75

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$= \frac{10(186) - (55)(25.5)}{\sqrt{10(385) - (55)^2} \cdot \sqrt{10(90.75) - (25.5)^2}}$$

$$\approx 0.993$$

- (c) The least squares line is of the form
 $Y = mx + b$. First solve for m .

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{10(186) - (55)(25.5)}{10(385) - (55)^2}$$

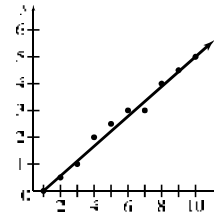
$$= 0.5545454545 \approx 0.555$$

Now find b .

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{25.5 - 0.5545454545(55)}{10}$$

$$= -0.5$$

Thus, $Y = 0.555x - 0.5$.

- (d) Let $x = 11$. Find Y .

$$Y = 0.55(11) - 0.5 = 5.6$$

4.

x	y	xy	x^2	y^2
6.8	0.8	5.44	46.24	0.64
7.0	1.2	8.4	49.0	1.44
7.1	0.9	6.39	50.41	0.81
7.2	0.9	6.48	51.84	0.81
7.4	1.5	11.1	54.76	2.25
35.5	5.3	37.81	252.25	5.95

$$r = \frac{5(37.81) - (35.5)(5.3)}{\sqrt{5(252.25) - (35.5)^2} \cdot \sqrt{5(5.95) - (5.3)^2}}$$

$$\approx 0.6985$$

$$r^2 = (0.6985)^2 \approx 0.5$$

The answer is choice (c).

5.

x	y	xy	x^2	y^2
1	1	1	1	1
1	2	2	1	4
2	1	2	4	1
2	2	4	4	4
9	9	81	81	81
15	15	90	91	91

- (a) $n = 5$

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{5(90) - (15)(15)}{5(91) - (15)^2}$$

$$= 0.9782608 \approx 0.9783$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{15 - (0.9782608)(15)}{5} \approx 0.0652$$

Thus, $Y = 0.9783x + 0.0652$.

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$= \frac{5(90) - (15)(15)}{\sqrt{5(91) - (15)^2} \cdot \sqrt{5(91) - (15)^2}} \approx 0.9783$$

(b)

x	y	xy	x^2	y^2
1	1	1	1	1
1	2	2	1	4
2	1	2	4	1
2	2	4	4	4
6	6	9	10	10

$$n = 4$$

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{4(9) - (6)(6)}{4(10) - (6)^2} = 0$$

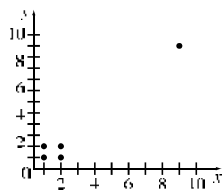
$$b = \frac{\sum y - m(\sum x)}{n} = \frac{6 - (0)(6)}{4} = 1.5$$

Thus, $Y = 0x + 1.5$, or $Y = 1.5$.

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$= \frac{4(9) - (6)(6)}{\sqrt{4(10) - (6)^2} \cdot \sqrt{4(10) - (6)^2}} = 0$$

(c)



The point (9, 9) is an outlier that has a strong effect on the least squares line and the correlation coefficient.

6.

x	y	xy	x^2	y^2
1	1	1	1	1
2	2	4	4	4
3	3	9	9	9
4	4	16	16	16
9	-20	-180	81	400
19	-10	-150	111	430

(a) $n = 5$

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{5(-150) - (19)(-10)}{5(111) - (19)^2}$$

$$= -2.886597 \approx -2.887$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{-10 - (-2.886597)(19)}{5}$$

$$= 8.969069 \approx 8.969$$

Thus, $Y = -2.887x + 8.969$.

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$= \frac{5(-150) - (19)(-10)}{\sqrt{5(111) - (19)^2} \cdot \sqrt{5(430) - (-10)^2}}$$

$$= -0.887994$$

$$\approx -0.8880$$

(b)

x	y	xy	x^2	y^2
1	1	1	1	1
2	2	4	4	4
3	3	9	9	9
4	4	16	16	16
10	10	30	30	30

$$n = 4$$

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

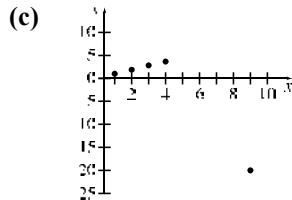
$$= \frac{4(30) - (10)(10)}{4(30) - (10)^2} = 1$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{10 - (1)(10)}{4} = 0$$

Thus, $Y = 1x + 0$, or $Y = x$.

$$\begin{aligned}
 r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\
 &= \frac{4(30) - (10)(10)}{\sqrt{4(30) - (10)^2} \cdot \sqrt{4(30) - 10^2}} \\
 &= 1
 \end{aligned}$$



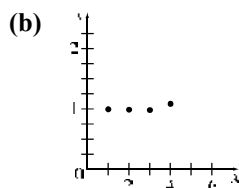
The point $(9, -20)$ is an outlier that has a strong effect on the least squares line and the correlation coefficient.

7.

x	y	xy	x^2	y^2
1	1	1	1	1
2	1	2	4	1
3	1	3	9	1
4	1.1	4.4	16	1.21
10	4.1	10.4	30	4.21

(a) $n = 4$

$$\begin{aligned}
 r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\
 &= \frac{4(10.4) - (10)(4.1)}{\sqrt{4(30) - (10)^2} \cdot \sqrt{4(4.21) - (4.1)^2}} \\
 &= 0.7745966 \approx 0.7746
 \end{aligned}$$



(c) Yes; because the data points are either on or very close to the horizontal line $y = 1$, it seems that the data should have a strong linear relationship. The correlation coefficient does not describe well a linear relationship if the data points fit a horizontal line.

8.

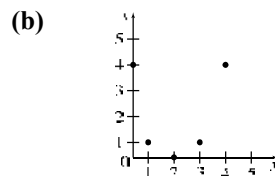
x	y	xy	x^2	y^2
0	4	0	0	16
1	1	1	1	1
2	0	0	4	0
3	1	3	9	1
4	4	16	16	16
10	10	20	30	34

(a) $n = 5$

$$\begin{aligned}
 m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\
 &= \frac{5(20) - (10)(10)}{5(30) - (10)^2} = 0 \\
 b &= \frac{\sum y - m(\sum x)}{n} \\
 &= \frac{10 - (0)(10)}{5} = 2
 \end{aligned}$$

Thus, $Y = 0x + 2$, or $Y = 2$.

$$\begin{aligned}
 r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\
 &= \frac{5(20) - (10)(10)}{\sqrt{5(30) - (10)^2} \cdot \sqrt{5(34) - (10)^2}} = 0
 \end{aligned}$$



(c) No; a correlation coefficient of 0 means that there isn't a linear relationship between the x and y values. A parabola (a quadratic relationship) seems to fit the given data points.

9. $nb + (\sum x)m = \sum y$

$$\begin{aligned}
 (\sum x)b + (\sum x^2)m &= \sum xy \\
 nb + (\sum x)m &= \sum y \\
 nb &= (\sum y) - (\sum x)m \\
 b &= \frac{\sum y - m(\sum x)}{n}
 \end{aligned}$$

$$(\sum x) \left(\frac{\sum y - m(\sum x)}{n} \right) + (\sum x^2)m = \sum xy$$

$$(\sum x)[(\sum y) - m(\sum x)] + nm(\sum x^2) = n(\sum xy)$$

$$(\sum x)(\sum y) - m(\sum x)^2 + nm(\sum x^2) = n(\sum xy)$$

$$nm(\sum x^2) - m(\sum x)^2 = n(\sum xy) - (\sum x)(\sum y)$$

$$m[n(\sum x^2) - (\sum x)^2] = n(\sum xy) - (\sum x)(\sum y)$$

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$\begin{aligned} 10. \quad (a) \quad m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\ &= \frac{7(147.1399) - (35)(28.4269)}{7(203) - (35)^2} \\ &= 0.178764 \approx 0.1788 \end{aligned}$$

$$\begin{aligned} b &= \frac{\sum y - m(\sum x)}{n} \\ &= \frac{28.4269 - (0.1788)(35)}{7} \\ &= 3.1669857 \approx 3.167 \end{aligned}$$

Thus, $Y = 0.1788x + 3.167$.

- (b) The year 2015 corresponds to $x = 15$.
 $Y = 0.1788(15) + 3.167 = 5.849$

The total value of consumer durable goods in 2015 will be about \$5.849 trillion.

- (c) Let $Y = 6$ and find x .
 $6 = 0.1788x + 3.167$
 $2.833 = 0.1788x$
 $15.84 = x$
 $16 \approx x$

The total value of consumer durable goods will reach at least \$6 trillion in the year
 $2000 + 16 = 2016$.

$$\begin{aligned} (d) \quad r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\ &= \frac{7(147.1399) - (35)(28.4269)}{\sqrt{7(203) - (35)^2} \cdot \sqrt{7(116.3396) - (28.4269)^2}} \\ &\approx 0.9980 \end{aligned}$$

Since r is very close to 1, the data has a strong linear relationship, and the least squares line fits the data very well.

$$\begin{aligned} 11. \quad (a) \quad m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\ &= \frac{10(1810.095) - (235)(77.564)}{10(5605) - (235)^2} \\ &\approx -0.1534 \\ b &= \frac{\sum y - m(\sum x)}{n} \\ &= \frac{77.564 - (-0.1534)(235)}{10} \\ &\approx 11.36 \end{aligned}$$

Thus, $Y = -0.1534x + 11.36$.

- (b) The year 2020 corresponds to $x = 2020 - 1990 = 30$.

$$Y = -0.1534(30) + 11.36 = 6.758$$

If the trend continues linearly, there will be about 6760 banks in 2020.

- (c) Let $Y = 6$ (since Y is the number of banks in thousands) and find x .

$$6 = -0.1534x + 11.36$$

$$-5.36 = -0.1534x$$

$$34.94 = x$$

$$35 \approx x$$

The number of U.S. banks will drop below 6000 in the year $1990 + 35 = 2025$.

$$\begin{aligned} (d) \quad r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\ &= \frac{10(1810.095) - (235)(77.564)}{\sqrt{10(5605) - (235)^2} \cdot \sqrt{10(603.60324) - (77.564)^2}} \\ &\approx -0.9890 \end{aligned}$$

This means that the least squares line fits the data points very well. The negative sign indicates that the number of banks is decreasing as the years increase.

12.	x	y	xy	x^2	y^2
	0	8.5	0	0	72.25
	2	19.3	38.6	4	372.49
	4	25.4	101.6	16	645.16
	6	32.6	195.6	36	1062.76
	8	40.4	323.2	64	1632.16
	20	126.2	659	120	3784.82

$$\begin{aligned}
 \text{(a)} \quad m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\
 &= \frac{5(659) - (20)(126.2)}{5(120) - (20)^2} \\
 &\approx 3.855 \\
 b &= \frac{\sum y - m(\sum x)}{n} \\
 &= \frac{126.2 - (3.855)(20)}{5} \\
 &\approx 9.82
 \end{aligned}$$

Thus, $Y = 3.855x + 9.82$.

- (b) The number of subscribers to digital cable television is growing at a rate of about 3.855 million per year.

- (c) The year 2012 corresponds to $x = 12$.

$$Y = 3.855(12) + 9.82 = 56.08$$

If the trend continues linearly, the number of subscribers will be about 56.08 million in 2012.

- (d) Let $Y = 70$ and find x .

$$70 = 3.855x + 9.82$$

$$60.18 = 3.855x$$

$$15.6 = x$$

$$16 \approx x$$

The number of subscribers will exceed 70 million in the year $2000 + 16 = 2016$.

$$\begin{aligned}
 \text{(e)} \quad r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\
 &= \frac{5(659) - (20)(126.2)}{\sqrt{5(120) - (20)^2} \cdot \sqrt{5(3784.82) - (126.2)^2}} \\
 &\approx 0.9957
 \end{aligned}$$

This means that the least squares line fits the data points extremely well.

13.

x	y	xy	x^2	y^2
4	2219.5	8878.0	16	4,926,180.25
5	2319.8	11,599.0	25	5,381,472.04
6	2415.0	14,490.0	36	5,832,225.00
7	2551.9	17,863.3	49	6,512,193.61
8	2592.1	20,786.8	64	6,718,982.41
30	12,098.3	73,567.1	190	29,371,053.31

$$\begin{aligned}
 \text{(a)} \quad m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\
 &= \frac{5(73,567.1) - (30)(12,098.3)}{5(190) - (30)^2} \approx 97.73 \\
 b &= \frac{\sum y - m(\sum x)}{n} \\
 &= \frac{12,098.3 - (97.73)(30)}{5} \approx 1833.3
 \end{aligned}$$

Thus, $Y = 97.73x + 1833.3$.

- (b) The total amount of consumer credit is increasing at a rate of about \$97.73 billion per year.

- (c) The year 2015 corresponds to $x = 15$.

$$Y = 97.73(15) + 1833.3 = 3299.25$$

If the trend continues linearly, the total amount of consumer credit will be about \$3299 billion in 2015.

- (d) Let $Y = 4000$ and find x .

$$4000 = 97.73x + 1833.3$$

$$2166.7 = 97.73x$$

$$22.17 \approx x$$

The total debt will exceed \$4000 billion in the year $2000 + 23 = 2023$.

$$\begin{aligned}
 \text{(e)} \quad r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\
 &= \frac{5(73,567.1) - (30)(12,098.3)}{\sqrt{5(190) - (30)^2} \cdot \sqrt{5(29,371,053.31) - (12,098.3)^2}} \\
 &\approx 0.9909
 \end{aligned}$$

This means that the least squares line fits the data points extremely well.

14.

x	y	xy	x^2	y^2
0	17.3	0.0	0	299.29
1	17.1	17.1	1	292.41
2	16.8	33.6	4	282.24
3	16.6	49.8	9	275.56
4	16.9	67.6	16	285.61
5	16.9	84.5	25	285.61
6	16.5	99.0	36	272.25
7	16.1	112.7	49	259.21
28	134.2	464.3	140	2252.18

$$\begin{aligned}
 \text{(a)} \quad m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\
 &= \frac{8(464.3) - (28)(134.2)}{8(140) - (28)^2} \approx -0.1286 \\
 b &= \frac{\sum y - m(\sum x)}{n} \\
 &= \frac{134.2 - (-0.1286)(28)}{8} \approx 17.225
 \end{aligned}$$

Thus, $Y = -0.1286x + 17.225$.

$$\begin{aligned}
 r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\
 &= \frac{8(464.3) - (28)(134.2)}{\sqrt{8(140) - (28)^2} \cdot \sqrt{8(2252.18) - (134.2)^2}} \\
 &\approx -0.8439
 \end{aligned}$$

(b)

x	y	xy	x^2	y^2
0	17.3	0.0	0	299.29
2	16.8	33.6	4	282.24
4	16.9	67.6	16	285.61
6	16.5	99.0	36	272.25
12	67.5	200.2	56	1139.39

$$\begin{aligned}
 m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\
 &= \frac{4(200.2) - (12)(67.5)}{4(56) - (12)^2} \approx -0.115
 \end{aligned}$$

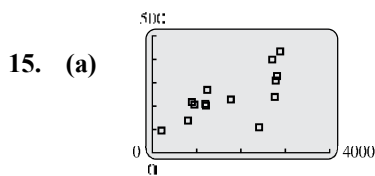
$$\begin{aligned}
 b &= \frac{\sum y - m(\sum x)}{n} \\
 &= \frac{67.5 - (-0.115)(12)}{4} \approx 17.22
 \end{aligned}$$

Thus, $Y = -0.115x + 17.22$.

$$\begin{aligned}
 r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\
 &= \frac{4(200.2) - (12)(67.5)}{\sqrt{4(56) - (12)^2} \cdot \sqrt{4(1139.39) - (67.5)^2}} \\
 &\approx -0.8987
 \end{aligned}$$

- (c)** The two least squares lines have approximately the same y -intercept and slope. The correlation coefficient of the four data points is slightly closer to 1 in absolute value than the correlation coefficient of the eight data points. Since the correlation coefficient for the eight-point data set is fairly close to -1 , the data points lie

approximately along a straight line, and we would expect a subset of this data set (in this case containing four points) to yield approximately the same least squares line.



Yes, the data points lie in a linear pattern.

(b)

x	y	xy	x^2	y^2
206	95	19,570	42,436	9025
802	138	110,676	643,204	19,044
1771	228	403,788	3,136,441	51,984
1198	209	250,382	1,435,204	43,681
1238	269	333,022	1,532,644	72,361
2786	309	860,874	1,761,796	95,481
1207	202	243,814	1,456,849	40,804
892	217	193,564	795,664	47,089
2411	109	262,799	5,812,921	11,881
2885	434	1,252,090	8,323,225	188,356
2705	399	1,079,295	7,317,025	159,201
948	206	195,288	898,704	42,436
2762	239	660,118	7,628,644	57,121
2815	329	926,135	7,924,255	108,241
24,626	3383	6,791,415	54,708,982	946,705

$$\begin{aligned}
 r &= \frac{14(6,791,415) - (24,626)(3383)}{\sqrt{14(54,708,982) - 24,626^2} \cdot \sqrt{14(946,705) - 3383^2}} \\
 &\approx 0.693
 \end{aligned}$$

There is a positive correlation between the price and the distance.

(c)

$$\begin{aligned}
 m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\
 m &= \frac{14(6,791,415) - (24,626)(3383)}{14(54,708,982) - 24,626^2} \\
 m &= 0.0737999664 \approx 0.0738
 \end{aligned}$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$b = \frac{3383 - (0.0737999664)(24,626)}{14}$$

$$\approx 111.83$$

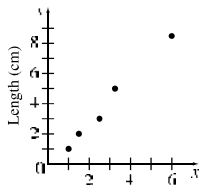
$$Y = 0.0738x + 111.83$$

The marginal cost is about 7.38 cents per mile.

- (d) In 2000, the marginal cost was 2.43 cents per mile. It increased to 7.38 cents per mile by 2006.

- (e) Phoenix is the outlier.

16. (a)



Yes, the data appear to be linear.

(b)

x	y	xy	x^2	y^2
5.8	8.6	49.88	33.64	73.96
1.5	1.9	2.85	2.25	3.61
2.3	3.1	7.13	5.29	9.61
1.0	1.0	1.0	1.0	1.0
3.3	5.0	16.5	10.89	25.0
13.9	19.6	77.36	53.07	113.18

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

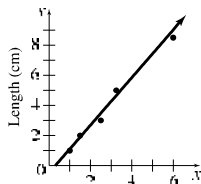
$$= \frac{5(77.36) - (13.9)(19.6)}{5(53.07) - 13.9^2}$$

$$= 1.585250901 \approx 1.585$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{19.6 - 1.585250901(13.9)}{5} \approx -0.487$$

$$Y = 1.585x - 0.487$$

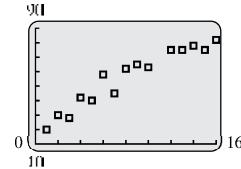


- (c) No, it gives negative values for small widths.

(d)
$$r = \frac{5(77.36) - (13.9)(19.6)}{\sqrt{5(53.07) - 13.9^2} \cdot \sqrt{5(113.18) - 19.6^2}}$$

$$\approx 0.999$$

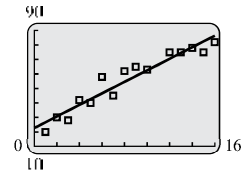
17. (a)



Yes, the points lie in a linear pattern.

- (b) Using a calculator's STAT feature, the correlation coefficient is found to be $r \approx 0.959$. This indicates that the percentage of successful hunts does trend to increase with the size of the hunting party.

(c) $Y = 3.98x + 22.7$



18. (a)

x	y	xy	x^2	y^2
88.6	20.0	1772	7849.96	400.0
71.6	16.0	1145.6	5126.56	256.0
93.3	19.8	1847.34	8704.89	392.04
84.3	18.4	1551.12	7106.49	338.56
80.6	17.1	1378.26	6496.36	292.41
75.2	15.5	1165.6	5655.04	240.25
69.7	14.7	1024.59	4858.09	216.09
82.0	17.1	1402.2	6724	292.41
69.4	15.4	1068.76	4816.36	237.16
83.3	16.2	1349.46	3938.89	262.44
79.6	15.0	1194	6336.16	225
82.6	17.2	1420.72	6822.76	295.84
80.6	16.0	1289.6	6496.36	256.0
83.5	17.0	1419.5	6972.25	289.0
76.3	14.4	1098.72	5821.69	207.36
1200.6	249.8	20,127.47	96,725.86	4200.56

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{15(20,127.47) - (1200.6)(249.8)}{15(96,725.86) - 1200.6^2}$$

$$= 0.211925009 \approx 0.212$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{249.8 - 0.211925(1200.6)}{15} \approx -0.309$$

$$Y = 0.212x - 0.309$$

- (b) Let
- $x = 73$
- ; find
- Y
- .

$$Y = 0.212(73) - 0.309 \approx 15.2$$

If the temperature were 73°F , you would expect to hear 15.2 chirps per second.

- (c) Let
- $Y = 18$
- ; find
- x
- .

$$18 = 0.212x - 0.309$$

$$18.309 = 0.212x$$

$$86.4 \approx x$$

When the crickets are chirping 18 times per second, the temperature is 86.4°F .

- (d)

$$r = \frac{15(20,127) - (1200.6)(249.8)}{\sqrt{15(96,725.86) - (1200.6)^2} \cdot \sqrt{15(4200.56) - (249.8)^2}} = 0.835$$

19.

x	y	xy	x^2	y^2
0	17.4	0	0	302.76
4	17.7	70.8	16	313.29
8	16.9	135.2	64	285.61
12	16.2	194.4	144	262.44
16	15.8	252.8	256	249.64
40	84.0	653.2	480	1413.74

$$(a) \quad m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$= \frac{5(653.2) - (40)(84)}{5(480) - (40)^2}$$

$$\approx -0.1175$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$= \frac{84 - (-0.1175)(40)}{5}$$

$$\approx 17.74$$

Thus, $Y = -0.1175x + 17.74$.

- (b) The year 2020 corresponds to
- $x = 2020 - 1990 = 30$
- .

$$Y = -0.1175(30) + 17.74 = 14.215 \approx 14.2$$

If the trend continues linearly, the pupil-teacher ratio will be about 14.2 in 2020.

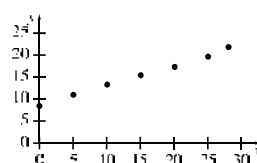
$$(c) \quad r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} = \frac{5(653.2) - (40)(84)}{\sqrt{5(480) - (40)^2} \cdot \sqrt{5(1413.74) - (84)^2}} \approx -0.9326$$

The value indicates a strong linear correlation.

20.

x	y	xy	x^2	y^2
0	8.414	0.000	0	70.7954
5	10.989	54.945	25	120.7581
10	13.359	133.590	100	178.4629
15	15.569	233.535	225	242.3938
20	17.604	352.080	400	309.9008
25	19.961	499.025	625	398.4415
28	22.207	621.796	784	493.1508
103	108.103	1894.971	2159	1813.9033

- (a)



Yes, the data appear to lie along a straight line.

- (b)

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} = \frac{7(1894.971) - (103)(108.103)}{\sqrt{7(2159) - (103)^2} \cdot \sqrt{7(1813.9033) - (108.103)^2}} = 0.9982$$

The value indicates a strong linear correlation.

$$(c) \quad m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} = \frac{7(1894.971) - (103)(108.103)}{7(2159) - (103)^2} \approx 0.4730$$

$$b = \frac{\sum y - m(\sum x)}{n} = \frac{108.103 - (0.4730)(103)}{7} \approx 8.484$$

Thus, $Y = 0.4730x + 8.484$.

- (d) The year 2018 corresponds to
- $x = 2018 - 1980 = 38$
- .

$$Y = 0.4730(38) + 8.484 = 26.456$$

The predicted poverty level in the year 2018 is \$26,460.

21.

x	y	xy	x^2	y^2
59	66	3894	3481	4356
62	71	4402	3844	5041
66	72	4752	4356	5184
68	73	4964	4624	5329
71	75	5325	5041	5625
67	63	4221	4489	3969
70	63	4410	4900	3969
71	67	4757	5041	4489
73	66	4818	5329	4356
75	66	4950	5625	4356
682	682	46,493	46,730	46,674

$$\begin{aligned} \text{(a)} \quad m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\ &= \frac{10(46,493) - (682)(682)}{10(46,730) - (682)^2} \approx -0.08915 \end{aligned}$$

$$\begin{aligned} b &= \frac{\sum y - m(\sum x)}{n} \\ &= \frac{682 - (-0.08915)(682)}{10} \approx 74.28 \end{aligned}$$

Thus, $Y = -0.08915x + 74.28$.

$$\begin{aligned} r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\ &= \frac{10(46,493) - (682)(682)}{\sqrt{10(46,730) - (682)^2} \cdot \sqrt{10(46,674) - (682)^2}} \\ &\approx -0.1035 \end{aligned}$$

The taller the student, the shorter the ideal partner's height is.

(b) Data for female students:

x	y	xy	x^2	y^2
59	66	3894	3481	4356
62	71	4402	3844	5041
66	72	4752	4356	5184
68	73	4964	4624	5329
71	75	5325	5041	5625
326	357	23,337	21,346	25,535

$$\begin{aligned} m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\ &= \frac{5(23,337) - (326)(357)}{5(21,346) - (326)^2} \approx 0.6674 \end{aligned}$$

$$\begin{aligned} b &= \frac{\sum y - m(\sum x)}{n} \\ &= \frac{357 - (0.6674)(326)}{5} \approx 27.89 \end{aligned}$$

Thus, $Y = 0.6674x + 27.89$.

$$\begin{aligned} r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\ &= \frac{5(23,337) - (326)(357)}{\sqrt{5(21,346) - (326)^2} \cdot \sqrt{5(25,535) - (357)^2}} \\ &\approx 0.9459 \end{aligned}$$

Data for male students:

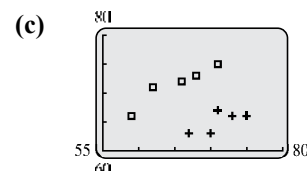
x	y	xy	x^2	y^2
67	63	4221	4489	3969
70	63	4419	4900	3969
71	67	4757	5041	4489
73	66	4818	5329	4356
75	66	4950	5625	4356
356	325	23,156	25,384	21,139

$$\begin{aligned} m &= \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \\ &= \frac{5(23,156) - (356)(325)}{5(25,384) - (356)^2} \approx 0.4348 \end{aligned}$$

$$\begin{aligned} b &= \frac{\sum y - m(\sum x)}{n} \\ &= \frac{325 - (0.4348)(356)}{5} \approx 34.04 \end{aligned}$$

Thus, $Y = 0.4348x + 34.04$.

$$\begin{aligned} r &= \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \cdot \sqrt{n(\sum y^2) - (\sum y)^2}} \\ &= \frac{5(23,156) - (356)(325)}{\sqrt{5(25,384) - (356)^2} \cdot \sqrt{5(21,139) - (325)^2}} \\ &\approx 0.7049 \end{aligned}$$



There is no linear relationship among all 10 data pairs. However, there is a linear relationship among the first five data pairs (female students) and a separate linear relationship among the second five data pairs (male students).

22. (a)

x	y	xy	x^2	y^2
540	20	10,800	291,600	400
510	16	8160	260,100	256
490	10	4900	240,100	100
560	8	4480	313,600	64
470	12	5640	220,900	144
600	11	6600	360,000	121
540	10	5400	291,600	100
580	8	4640	336,400	64
680	15	10,200	462,400	225
560	8	4480	313,600	64
560	13	7280	313,600	169
500	14	7000	250,000	196
470	10	4700	220,900	100
440	10	4400	193,600	100
520	11	5720	270,400	121
620	11	6820	384,400	121
680	8	5440	462,400	64
550	8	4400	302,500	64
620	7	4340	384,400	49
10,490	210	115,400	5,872,500	2522

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$m = \frac{19(115,400) - (10,490)(210)}{19(5,872,500) - 10,490^2}$$

$$m = -0.0066996227 \approx -0.0067$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$b = \frac{210 - (-0.0066996227)(10,490)}{19} \approx 14.75$$

$$Y = -0.0067x + 14.75$$

(b) Let $x = 420$; find Y .

$$Y = -0.0067(420) + 14.75$$

$$= 11.936 \approx 12$$

(c) Let $x = 620$; find Y .

$$Y = -0.0067(620) + 14.75$$

$$= 10.596 \approx 11$$

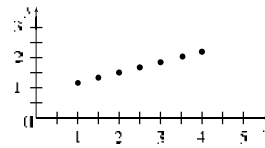
(d)

$$r = \frac{19(115,400) - (10,490)(210)}{\sqrt{19(5,872,500) - (10,490)^2} \cdot \sqrt{19(2522) - 210^2}}$$

$$\approx -0.13$$

(e) There is no linear relationship between a student's math SAT and mathematics placement test scores.

23. (a)



(b)

L	T	LT	L^2	T^2
1.0	1.11	1.11	1	1.2321
1.5	1.36	2.04	2.25	1.8496
2.0	1.57	3.14	4	2.4649
2.5	1.76	4.4	6.25	3.0976
3.0	1.92	5.76	9	3.6864
3.5	2.08	7.28	12.25	4.3264
4.0	2.22	8.88	16	4.9844
17.5	12.02	32.61	50.75	21.5854

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$m = \frac{7(32.61) - (17.5)(12.02)}{7(50.75) - 17.5^2}$$

$$m = 0.3657142857$$

$$\approx 0.366$$

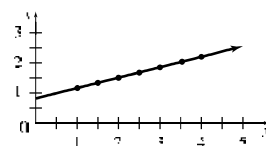
$$b = \frac{\sum T - m(\sum L)}{n}$$

$$b = \frac{12.02 - 0.3657142857(17.5)}{7}$$

$$\approx 0.803$$

$$Y = 0.366x + 0.803$$

The line seems to fit the data.



(c)

$$r = \frac{7(32.61) - (17.5)(12.02)}{\sqrt{7(50.75) - 17.5^2} \cdot \sqrt{7(21.5854) - 12.02^2}}$$

$$= 0.995,$$

which is a good fit and confirms the conclusion in part (b).

24. (a)

x	y	xy	x^2	y^2
150	5000	750,000	22,500	25,000,000
175	5500	962,500	30,625	30,250,000
215	6000	1,290,000	46,225	36,000,000
250	6500	1,625,000	62,500	42,250,000
280	7000	1,960,000	78,400	49,000,000
310	7500	2,325,000	96,100	56,250,000
350	8000	2,800,000	122,500	64,000,000
370	8500	3,145,000	136,900	72,250,000
420	9000	3,780,000	176,400	81,000,000
450	9500	4,275,000	202,500	90,250,000
2970	72,500	22,912,500	974,650	546,250,000

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$m = \frac{10(22,912,500) - (2970)(72,500)}{10(974,650) - 2970^2}$$

$$m = 14.90924806$$

$$\approx 14.9$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$b = \frac{72,500 - 14.9(2970)}{10}$$

$$b \approx 2820$$

$$Y = 14.9x + 2820$$

(b) Let $x = 150$; find Y .

$$Y = 14.9(150) + 2820$$

$$Y \approx 5060, \text{ compared to actual } 5000$$

Let $x = 280$; Find Y .

$$Y = 14.9(280) + 2820$$

$$\approx 6990, \text{ compared to actual } 7000$$

Let $x = 420$; find Y .

$$Y = 14.9(420) + 2820$$

$$\approx 9080, \text{ compared to actual } 9000$$

(c) Let $x = 230$; find Y .

$$Y = 14.9(230) + 2820$$

$$\approx 6250$$

Adam would need to buy a 6500 BTU air conditioner.

25. (a)

$$r = \frac{10(399.16) - (500)(20.668)}{\sqrt{10(33,250) - 500^2} \cdot \sqrt{10(91.927042) - (20.668)^2}}$$

$$= -0.995$$

Yes, there does appear to be a linear correlation.

$$\text{(b)} \quad m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$m = \frac{10(399.16) - (500)(20.668)}{10(33,250) - 500^2}$$

$$m = -0.0768775758 \approx -0.0769$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$b = \frac{20.668 - (-0.0768775758)(500)}{10}$$

$$\approx 5.91$$

$$Y = -0.0769x + 5.91$$

(c) Let $x = 50$

$$Y = -0.0769(50) + 5.91 \approx 2.07$$

The predicted number of points expected when a team is at the 50 yard line is 2.07 points.

26. (a) Use a calculator's statistical features to obtain the least squares line.

$$Y = -0.1358x + 113.94$$

$$\text{(b)} \quad Y = -0.3913x + 148.98$$

(c) Set the two expressions for Y equal and solve for x .

$$-0.1358x + 113.94 = -0.3913x + 148.98$$

$$0.2555x = 35.04$$

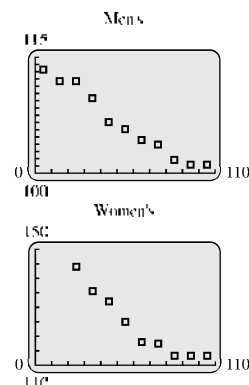
$$x \approx 137$$

The women's record will catch up with the men's record in 1900 + 137, or in the year 2037.

$$\text{(d)} \quad r_{\text{men}} \approx -0.9823$$

$$r_{\text{women}} \approx -0.9487$$

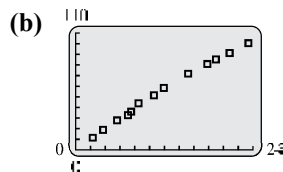
Both sets of data points closely fit a line with negative slope.

(e)

27.

x	y
0.00	0.0
2.3167	11.5
3.7167	18.9
5.6000	27.8
7.0833	32.8
7.5000	36.0
8.5000	43.9
10.6000	51.5
11.9333	58.4
15.2333	71.8
17.8167	80.9
18.9667	85.2
20.8333	91.3
23.3833	100.5
153.4833	710.5

- (a) Skaggs' average speed was $100.5/23.3833 \approx 4.298$ miles per hour.



The data appear to lie approximately on a straight line.

- (c) Using a graphing calculator,

$$Y = 4.317x + 3.419.$$

- (d) Using a graphing calculator,

$$r \approx 0.9971$$

Yes, the least squares line is a very good fit to the data.

- (e) A good value for Skaggs' average speed would be the slope of the least squares line, or

$$m = 4.317 \text{ miles per hour.}$$

This value is faster than the average speed found in part (a). The value 4.317 miles per hour is most likely the better value because it takes into account all 14 data pairs.

Chapter 1 Review Exercises

- False; a line can have only one slant, so its slope is unique.
- False; the equation $y = 3x + 4$ has slope 3.
- True; the point $(3, -1)$ is on the line because $-1 = -2(3) + 5$ is a true statement.
- False; the points $(2, 3)$ and $(2, 5)$ do not have the same y -coordinate.
- True; the points $(4, 6)$ and $(5, 6)$ do have the same y -coordinate.
- False; the x -intercept of the line $y = 8x + 9$ is $-\frac{9}{8}$.
- True; $f(x) = \pi x + 4$ is a linear function because it is in the form $y = mx + b$, where m and b are real numbers.
- False; $f(x) = 2x^2 + 3$ is not linear function because it isn't in the form $y = mx + b$, and it is a second-degree equation.
- False; the line $y = 3x + 17$ has slope 3, and the line $y = -3x + 8$ has slope -3 . Since $3 \cdot -3 \neq -1$, the lines cannot be perpendicular.
- False; the line $4x + 3y = 8$ has slope $-\frac{4}{3}$, and the line $4x + y = 5$ has slope -4 . Since the slopes are not equal, the lines cannot be parallel.
- False; a correlation coefficient of zero indicates that there is no linear relationship among the data.
- True; a correlation coefficient always will be a value between -1 and 1 .
- Marginal cost is the rate of change of the cost function; the fixed cost is the initial expenses before production begins.
- To compute the coefficient of correlation, you need the following quantities: $\sum x$, $\sum y$, $\sum xy$, $\sum x^2$, $\sum y^2$, and n .
- Through $(-3, 7)$ and $(2, 12)$

$$m = \frac{12 - 7}{2 - (-3)} = \frac{5}{5} = 1$$

16. Through $(4, -1)$ and $(3, -3)$.

$$\begin{aligned} m &= \frac{-3 - (-1)}{3 - 4} \\ &= \frac{-3 + 1}{-1} \\ &= \frac{-2}{-1} = 2 \end{aligned}$$

17. Through the origin and $(11, -2)$

$$m = \frac{-2 - 0}{11 - 0} = -\frac{2}{11}$$

18. Through the origin and $(0, 7)$

$$m = \frac{7 - 0}{0 - 0} = \frac{7}{0}$$

The slope of the line is undefined.

19. $4x + 3y = 6$

$$3y = -4x + 6$$

$$y = -\frac{4}{3}x + 2$$

Therefore, the slope is $m = -\frac{4}{3}$.

20. $4x - y = 7$

$$-y = -4x + 7$$

$$y = 4x - 7$$

$$m = 4$$

21. $y + 4 = 9$

$$y = 5$$

$$y = 0x + 5$$

$$m = 0$$

22. $3y - 1 = 14$

$$3y = 14 + 1$$

$$3y = 15$$

$$y = 5$$

This is a horizontal line. The slope of a horizontal line is 0.

23. $y = 5x + 4$

$$m = 5$$

24. $x = 5y$

$$\frac{1}{5}x = y$$

$$m = \frac{1}{5}$$

25. Through $(5, -1)$; slope $\frac{2}{3}$

Use point-slope form.

$$y - (-1) = \frac{2}{3}(x - 5)$$

$$y + 1 = \frac{2}{3}(x - 5)$$

$$3(y + 1) = 2(x - 5)$$

$$3y + 3 = 2x - 10$$

$$3y = 2x - 13$$

$$y = \frac{2}{3}x - \frac{13}{3}$$

26. Through $(8, 0)$, with slope $-\frac{1}{4}$

Use point-slope form.

$$y - 0 = -\frac{1}{4}(x - 8)$$

$$y = -\frac{1}{4}x + 2$$

27. Through $(-6, 3)$ and $(2, -5)$

$$m = \frac{-5 - 3}{2 - (-6)} = \frac{-8}{8} = -1$$

Use point-slope form.

$$y - 3 = -1[x - (-6)]$$

$$y - 3 = -x - 6$$

$$y = -x - 3$$

28. Through $(2, -3)$ and $(-3, 4)$

$$m = \frac{4 - (-3)}{-3 - 2} = -\frac{7}{5}$$

Use point-slope form.

$$y - (-3) = -\frac{7}{5}(x - 2)$$

$$y + 3 = -\frac{7}{5}x + \frac{14}{5}$$

$$y = -\frac{7}{5}x + \frac{14}{5} - 3$$

$$y = -\frac{7}{5}x + \frac{14}{5} - \frac{15}{5}$$

$$y = -\frac{7}{5}x - \frac{1}{5}$$

- 29.** Through $(2, -10)$, perpendicular to a line with undefined slope

A line with undefined slope is a vertical line. A line perpendicular to a vertical line is a horizontal line with equation of the form $y = k$. The desired line passed through $(2, -10)$, so $k = -10$. Thus, an equation of the desired line is $y = -10$.

- 30.** Through $(-2, 5)$, with slope 0

Horizontal lines have 0 slope and an equation of the form $y = k$.

The line passes through $(-2, 5)$ so $k = 5$. An equation of the line is $y = 5$.

- 31.** Through $(3, -4)$ parallel to $4x - 2y = 9$

Solve $4x - 2y = 9$ for y .

$$-2y = -4x + 9$$

$$y = 2x - \frac{9}{2}$$

$$m = 2$$

The desired line has the same slope. Use the point-slope form.

$$y - (-4) = 2(x - 3)$$

$$y + 4 = 2x - 6$$

$$y = 2x - 10$$

Rearrange.

$$2x - y = 10$$

- 32.** Through $(0, 5)$, perpendicular to $8x + 5y = 3$

Find the slope of the given line first.

$$8x + 5y = 3$$

$$5y = -8x + 3$$

$$y = \frac{-8}{5}x + \frac{3}{5}$$

$$m = -\frac{8}{5}$$

The perpendicular line has $m = \frac{5}{8}$.

Use point-slope form.

$$y - 5 = \frac{5}{8}(x - 0)$$

$$y = \frac{5}{8}x + 5$$

Rearrange.

$$8y = 5x + 40$$

$$5x - 8y = -40$$

- 33.** Through $(-1, 4)$; undefined slope

Undefined slope means the line is vertical.

The equation of the vertical line through $(-1, 4)$ is $x = -1$.

- 34.** Through $(7, -6)$, parallel to a line with undefined slope.

A line with undefined slope has the form $x = a$ (a vertical line). The vertical line that goes through $(7, -6)$ is the line $x = 7$.

- 35.** Through $(3, -5)$, parallel to $y = 4$

Find the slope of the given line.

$y = 0x + 4$, so $m = 0$, and the required line will also have slope 0.

Use the point-slope form.

$$y - (-5) = 0(x - 3)$$

$$y + 5 = 0$$

$$y = -5$$

- 36.** Through $(-3, 5)$, perpendicular to $y = -2$ The given line, $y = -2$, is a horizontal line. A line perpendicular to a horizontal line is a vertical line with equation of the form $x = h$.

The desired line passes through $(-3, 5)$, so

$h = -3$. Thus, an equation of the desired line is $x = -3$.

- 37.** $y = 4x + 3$

Let $x = 0$: $y = 4(0) + 3$

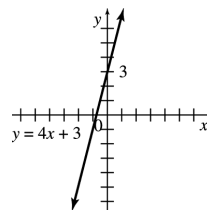
$$y = 3$$

Let $y = 0$: $0 = 4x + 3$

$$-3 = 4x$$

$$-\frac{3}{4} = x$$

Draw the line through $(0, 3)$ and $(-\frac{3}{4}, 0)$.



- 38.** $y = 6 - 2x$

Find the intercepts.

Let $x = 0$.

$$y = 6 - 2(0) = 6$$

The y -intercept is 6.

Let $y = 0$.

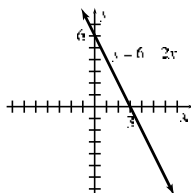
$$0 = 6 - 2x$$

$$2x = 6$$

$$x = 3$$

The x -intercept is 3.

Draw the line through $(0, 6)$ and $(3, 0)$.



39. $3x - 5y = 15$

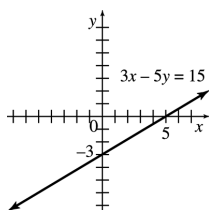
$$-5y = -3x + 15$$

$$y = \frac{3}{5}x - 3$$

When $x = 0$, $y = -3$.

When $y = 0$, $x = 5$.

Draw the line through $(0, -3)$ and $(5, 0)$.



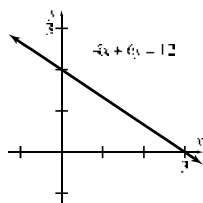
40. $4x + 6y = 12$

Find the intercepts.

When $x = 0$, $y = 2$, so the y -intercept is 2.

When $y = 0$, $x = 3$, so the x -intercept is 3.

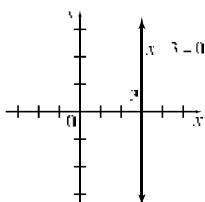
Draw the line through $(0, 2)$ and $(3, 0)$.



41. $x - 3 = 0$

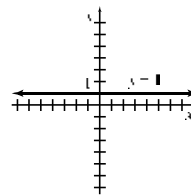
$$x = 3$$

This is the vertical line through $(3, 0)$.



42. $y = 1$

This is the horizontal line passing through $(0, 1)$.

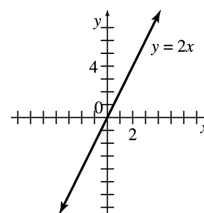


43. $y = 2x$

When $x = 0$, $y = 0$.

When $x = 1$, $y = 2$.

Draw the line through $(0, 0)$ and $(1, 2)$.

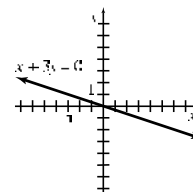


44. $x + 3y = 0$

When $x = 0$, $y = 0$.

When $x = 3$, $y = -1$.

Draw the line through $(0, 0)$ and $(3, -1)$.



45. (a) $E = 352 + 42x$ (where x is in thousands)

(b) $R = 130x$ (where x is in thousands)

(c) $R > E$

$$130x > 352 + 42x$$

$$88x > 352$$

$$x > 4$$

For a profit to be made, more than 4000 chips must be sold.

46. $S(q) = 6q + 3$; $D(q) = 19 - 2q$

(a) $S(q) = D(q) = 10$

$$10 = 6q + 3 \qquad 10 = 19 - 2q$$

$$7 = 6q \qquad -9 = -2q$$

$$\frac{7}{6} = q \text{ (supply)} \qquad \frac{9}{2} = q \text{ (demand)}$$

When the price is \$10 per pound, the supply is $\frac{7}{6}$ pounds per day, and the demand is $\frac{9}{2}$ pounds per day.

(b) $S(q) = D(q) = 15$

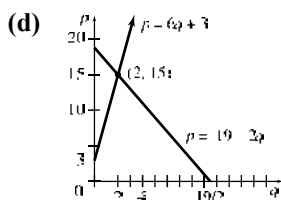
$$\begin{aligned} 15 &= 6q + 3 & 15 &= 19 - 2q \\ 12 &= 6q & -4 &= -2q \\ 2 &= q \text{ (supply)} & 2 &= q \text{ (demand)} \end{aligned}$$

When the price is \$15 per pound, the supply is 2 pounds per day, and the demand is 2 pounds per day.

(c) $S(q) = D(q) = \$18$

$$\begin{aligned} 18 &= 6q + 3 & 18 &= 19 - 2q \\ 15 &= 6q & -1 &= -2q \\ \frac{5}{2} &= q \text{ (supply)} & \frac{1}{2} &= q \text{ (demand)} \end{aligned}$$

When the price is \$18 per pound, the supply is $\frac{5}{2}$ pounds per day, the demand is $\frac{1}{2}$ pound per day.



(e) The graph shows that the lines representing the supply and demand functions intersect at the point (2, 15). The y -coordinate of this point gives the equilibrium price. Thus, the equilibrium price is \$15.

(f) The x -coordinate of the intersection point gives the equilibrium quantity. Thus, the equilibrium quantity is 2, representing 2 pounds of crabmeat per day.

47. Using the points (60, 40) and (100, 60),

$$\begin{aligned} m &= \frac{60 - 40}{100 - 60} = \frac{20}{40} = 0.5. \\ p - 40 &= 0.5(q - 60) \\ p - 40 &= 0.5q - 30 \\ p &= 0.5q + 10 \\ S(q) &= 0.5q + 10 \end{aligned}$$

48. Using the points (50, 47.50) and (80, 32.50),

$$\begin{aligned} m &= \frac{47.50 - 32.50}{50 - 80} \\ &= \frac{15}{-30} = \frac{-1}{2} = -0.5. \\ p - 47.50 &= -0.5(q - 50) \\ p - 47.50 &= -0.5q + 25 \\ p &= -0.5q + 72.50 \\ D(q) &= -0.5q + 72.50 \end{aligned}$$

49. $S(q) = D(q)$

$$\begin{aligned} 0.5q + 10 &= -0.5q + 72.50 \\ q &= 62.5 \end{aligned}$$

$$\begin{aligned} S(62.5) &= 0.5(62.5) + 10 \\ &= 31.25 + 10 \\ &= 41.25 \end{aligned}$$

The equilibrium price is \$41.25, and the equilibrium quantity is 62.5 diet pills.

50. Eight units cost \$300; fixed cost is \$60.

The fixed cost is the cost if zero units are made.

(8, 300) and (0, 60) are points on the line.

$$m = \frac{60 - 300}{0 - 8} = 30$$

Use slope-intercept form.

$$y = 30x + 60$$

$$C(x) = 30x + 60$$

51. Fixed cost is \$2000; 36 units cost \$8480.

Two points on the line are (0, 2000) and (36, 8480), so

$$m = \frac{8480 - 2000}{36 - 0} = \frac{6480}{36} = 180.$$

Use point-slope form.

$$y = 180x + 2000$$

$$C(x) = 180x + 2000$$

52. Twelve units cost \$445; 50 units cost \$1585.

Points on the line are (12, 445) and (50, 1585).

$$m = \frac{1585 - 445}{50 - 12} = 30$$

Use point-slope form.

$$y - 445 = 30(x - 12)$$

$$y - 445 = 30x - 360$$

$$y = 30x + 85$$

$$C(x) = 30x + 85$$

53. Thirty units cost \$1500; 120 units cost \$5640. Two points on the line are (30, 1500), (120, 5640), so

$$m = \frac{5640 - 1500}{120 - 30} = \frac{4140}{90} = 46.$$

Use point-slope form.

$$y - 1500 = 46(x - 30)$$

$$y = 46x - 1380 + 1500$$

$$y = 46x + 120$$

$$C(x) = 46x + 120$$

54. $C(x) = 200x + 1000$

$R(x) = 400x$

(a) $C(x) = R(x)$

$$200x + 1000 = 400x$$

$$1000 = 200x$$

$$5 = x$$

The break-even quantity is 5 cartons.

(b) $R(5) = 400(5) = 2000$

The revenue from 5 cartons of CD's is \$2000.

55. (a) $C(x) = 3x + 160$; $R(x) = 7x$

$$C(x) = R(x)$$

$$3x + 160 = 7x$$

$$160 = 4x$$

$$40x = x$$

The break-even quantity is 40 pounds.

(b) $R(40) = 7 \cdot 40 = \$280$

The revenue for 40 pounds is \$280.

56. Let y represent imports from China in billions of dollars. Using the points (1, 102) and (8, 338),

$$m = \frac{338 - 102}{8 - 1} = \frac{236}{7} = 33.7$$

$$y - 102 = 33.7(t - 1)$$

$$y - 102 = 33.7t - 33.7$$

$$y = 33.7t + 68.3$$

57. Let y represent imports to China in billions of dollars. Using the points (1, 19.1) and (8, 69.7)

$$m = \frac{69.7 - 19.1}{8 - 1} = \frac{50.6}{7} \approx 7.23$$

$$y - 19.1 = 7.23(t - 1)$$

$$y - 19.1 = 7.23t - 7.23$$

$$y = 7.23t + 11.9$$

58. Using the points (88, 47,614) and (108, 50,303)

$$m = \frac{50,303 - 47,614}{108 - 88}$$

$$= \frac{2689}{20} \approx 134.45$$

$$I - 47,614 = 134.45(t - 88)$$

$$I - 47,614 = 134.45t - 11,831.6$$

$$I(t) = 134.45t + 35,782.4$$

59.

x	y
0	7500
5	12,000
10	16,000
15	20,450
20	24,900
25	28,400

- (a) Use the points (0, 7500) and (25, 28,400) to find the slope.

$$m = \frac{28,400 - 7500}{25 - 0} = 836$$

$$b = 7500$$

The linear equation for the average new car cost since 1980 is $y = 836x + 7500$.

- (b) Use the points (15, 20,450) and (25, 28,400) to find the slope.

$$m = \frac{28,400 - 20,450}{25 - 15} = 795$$

$$y - 20,450 = 795(x - 15)$$

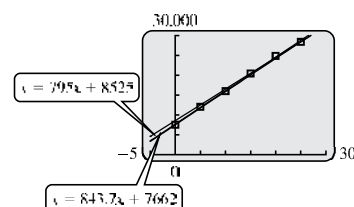
$$y - 20,450 = 795x - 11,925$$

$$y = 8525$$

The linear equation for the average new car cost since 1980 is $y = 795x + 8525$.

- (c) Using a graphing calculator, the least square line is $Y = 843.7x + 7662$.

(d)

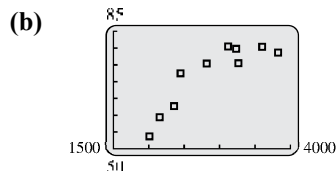


- (e) The least squares line best describes the data. Since the data seems to fit a straight line, a linear model describes the data very well.
- (f) Using a graphing calculator, $r \approx 0.9995$.

60.

x	y
2818	75.4
2155	59.4
3602	80.4
2358	62.6
3235	75.5
3265	79.8
2450	72.5
3120	80.5
2010	53.7
3826	78.7

- (a) Using a graphing calculator, $r \approx 0.8807$. Yes, the data seem to fit a straight line.



- (c) Using a graphing calculator, $Y = 0.01376x + 32.17$. The data somewhat fit a straight, but a curve would fit the data better.

- (d) Let $x = 3426$. Find Y .

$$Y = 0.01376(3426) + 32.17 \approx 79.3$$

The predicted life expectancy in the United Kingdom, with a daily calorie supply of 3426, is about 79.3 years. This agrees with the actual value of 79.0 years.

- (e) The higher daily calorie supply most likely contains more healthy nutrients, which might result in a longer life expectancy.

61. (a)

x	y	xy	x^2	y^2
130	170	22,100	16,900	28,900
138	160	22,080	19,044	25,600
142	173	24,566	20,164	29,929
159	181	28,779	25,281	32,761
165	201	33,165	27,225	40,401
200	192	38,400	40,000	36,864
210	240	50,400	44,100	57,600
250	290	72,500	62,500	84,100
1394	1607	291,990	255,214	336,155

$$m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$m = \frac{8(291,990) - (1394)(1607)}{8(225,214) - 1394^2}$$

$$m = 0.9724399854 \approx 0.9724$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$b = \frac{1607 - 0.9724(1394)}{8} \approx 31.43$$

$$Y = 0.9724x + 31.43$$

- (b) Let $x = 190$; find Y .

$$Y = 0.9724(190) + 31.43$$

$$Y = 216.19 \approx 216$$

The cholesterol level for a person whose blood sugar level is 190 would be about 216.

(c)
$$r = \frac{8(291,990) - (1394)(1607)}{\sqrt{8(225,214) - 1394^2} \cdot \sqrt{8(336,155) - 1607^2}}$$

$$= 0.933814 \approx 0.93$$

62. Using the points (24, 115.7) and (57, 92.9),

$$m = \frac{92.9 - 115.7}{57 - 24} = \frac{-22.8}{33} \approx -0.691$$

$$y - 115.7 = -0.691(t - 24)$$

$$y - 115.7 = -0.691t + 16.6$$

$$y = -0.691t + 132.3$$

63. Using the points (5, 55) and (19, 72.1),

$$m = \frac{72.1 - 55}{19 - 5} = \frac{17.1}{14} \approx 1.22$$

$$y - 55 = 1.22(t - 5)$$

$$y - 55 = 1.22t - 6.1$$

$$y = 1.22t + 48.9$$

64. (a) Use the points (0, 6400) and (8, 8147) to find the slope.

$$m = \frac{8147 - 6400}{8 - 0} = 218.375$$

$$b = 6400$$

The linear equation for the average new car cost since 1980 is $y = 218.375x + 6400$.

- (b) Use the points (4, 7623) and (8, 8147) to find the slope.

$$m = \frac{8147 - 7623}{8 - 4} = 131$$

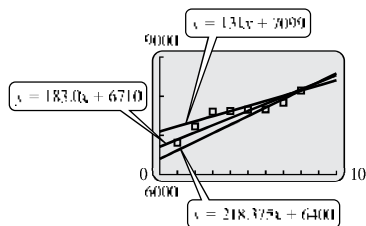
$$y - 7623 = 131(x - 4)$$

$$y - 7623 = 131x - 524$$

$$y = 131x + 7099$$

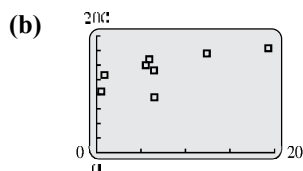
The linear equation for the average new car cost since 1980 is $y = 131x + 7099$.

- (c) Using a graphing calculator, the least squares line is $Y = 183.03x + 6710$.



- (d) The least squares line best describes the data. Since the data seems to fit a straight line, a linear model describes the data well.
- (e) Using a graphing calculator, $r \approx 0.9277$.

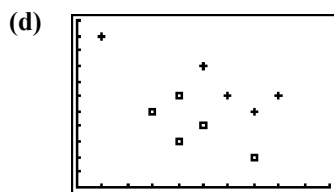
65. (a) Using a graphing calculator, $r = 0.6998$. The data seem to fit a line but the fit is not very good.



- (b) Using a graphing calculator, $Y = 3.396x + 117.2$.
- (d) The slope is 3.396 thousand (or 3396). On average, the governor's salary increases \$3396 for each additional million in population.

66. Use a graphing calculator to find these correlations.

- (a) Correlation between years since 2000 and length: $r = 0.4529$
- (b) Correlation between length and rating: $r = 0.3955$
- (c) Correlation between years since 2000 and rating: $r = -0.4768$



This calculator graph plots year (on the horizontal axis) versus rating (on the vertical axis). Squares represent movies with lengths no more than 110 minutes, and plus signs represent movies with lengths 115 minutes or more.

Extended Application: Using Extrapolation to Predict Life Expectancy

1.	x	y	xy	x^2	y^2
	1970	74.7	147,159.0	3,880,900	5580.09
	1975	76.6	151,285.0	3,900,625	5867.56
	1980	77.4	153,252.0	3,920,400	5990.76
	1985	78.2	155,227.0	3,940,225	6115.24
	1990	78.8	156,812.0	3,960,100	6209.44
	1995	78.9	157,405.5	3,980,025	6225.21
	2000	79.3	158,600.0	4,000,000	6288.49
	2005	79.9	160,199.5	4,020,025	6384.01
	15,900	623.8	1,239,940	31,602,300	48,660.8

$$m = \frac{m(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$$

$$m = \frac{8(1,239,940) - (15,900)(623.8)}{8(31,602,300) - 15,900^2} \approx 0.1310$$

$$b = \frac{\sum y - m(\sum x)}{n}$$

$$b = \frac{623.8 - 0.1310(15,900)}{8} \approx -182.3875$$

$$Y = 0.1310x - 182.3875$$

2. Let $x = 1900$. Find Y .

$$Y = 0.1310(1900) - 182.3875 \approx 66.5$$

From the equation, the guess for the life expectancy of females born in 1900 is 66.5 years.

3. The poor prediction isn't surprising, since we were extrapolating far beyond the range of the original data.

4.	x	y	Predicted value	Residual
	1970	74.7	75.84	-1.14
	1975	76.6	76.495	0.105
	1980	77.4	77.15	0.25
	1985	78.2	77.805	0.395
	1990	78.8	78.46	0.34
	1995	78.9	79.115	-0.215
	2000	79.3	79.77	-0.47
	2005	79.9	80.425	-0.525

5. It's not clear that any simple smooth function will fit this data. This will make it difficult to predict the life expectancy for females born in 2015.
6. You'll get 0 slope and 0 intercept, because you've already subtracted out the linear component of the data.
7. They used a regression equation of some type to predict this value.

Chapter 2

NONLINEAR FUNCTIONS

2.1 Properties of Functions

Your Turn 1

$$y = \frac{1}{\sqrt{x^2 - 4}}$$

Since the denominator cannot be zero and the radicand cannot be negative, the domain includes only those values of x satisfying $x^2 - 4 > 0$. Using the methods for solving a quadratic inequality, we get the domain $(-\infty, -2) \cup (2, \infty)$. Since the denominator can never be negative, y cannot be negative or zero. So, the range is $(0, \infty)$.

Your Turn 2

$$f(x) = 2x^2 - 3x - 4$$

$$\begin{aligned} \text{(a)} \quad f(x+h) &= 2(x+h)^2 - 3(x+h) - 4 \\ &= 2(x^2 + 2xh + h^2) - 3(x+h) - 4 \\ &= 2x^2 + 4xh + 2h^2 - 3x - 3h - 4 \end{aligned}$$

$$\text{(b)} \quad f(x) = -5$$

$$2x^2 - 3x - 4 = -5$$

$$2x^2 - 3x + 1 = 0$$

$$(2x-1)(x+1) = 0$$

$$x = \frac{1}{2} \text{ or } x = -1$$

2.1 Exercises

1. The x -value of 82 corresponds to two y -values, 93 and 14. In a function, each value of x must correspond to exactly one value of y .
The rule is not a function.
2. Each x -value corresponds to exactly one y -value.
The rule is a function.
3. Each x -value corresponds to exactly one y -value.
The rule is a function.
4. 9 corresponds to 3 and -3 , 4 corresponds to 2 and -2 , and 1 corresponds to -1 and 1.
The rule is not a function.

$$5. \quad y = x^3 + 2$$

Each x -value corresponds to exactly one y -value.
The rule is a function.

$$6. \quad y = \sqrt{x}$$

Each x -value corresponds to exactly one y -value.
The rule is a function.

$$7. \quad x = |y|$$

Each value of x (except 0) corresponds to two y -values.
The rule is not a function.

$$8. \quad x = y^2 + 4$$

Solve the rule for y .

$$y^2 = x - 4 \text{ or } y = \pm\sqrt{x-4}$$

Each value of x (greater than 4) corresponds to two y -values

$$y = \sqrt{x-4} \text{ and } y = -\sqrt{x-4}.$$

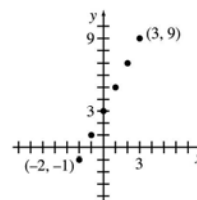
The rule is not a function.

$$9. \quad y = 2x + 3$$

x	-2	-1	0	1	2	3
y	-1	1	3	5	7	9

Pairs: $(-2, -1)$, $(-1, 1)$, $(0, 3)$,
 $(1, 5)$, $(2, 7)$, $(3, 9)$

Range: $\{-1, 1, 3, 5, 7, 9\}$

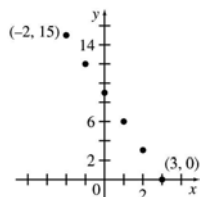


$$10. \quad y = -3x + 9$$

x	-2	-1	0	1	2	3
y	15	12	9	6	3	0

Pairs: $(-2, 15), (-1, 12), (0, 9),$
 $(1, 6), (2, 3), (3, 0)$

Range: $\{0, 3, 6, 9, 12, 15\}$



11. $2y - x = 5$

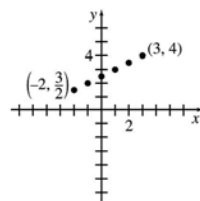
$$2y = 5 + x$$

$$y = \frac{1}{2}x + \frac{5}{2}$$

x	-2	-1	0	1	2	3
y	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4

Pairs: $(-2, \frac{3}{2}), (-1, 2), (0, \frac{5}{2}), (1, 3),$
 $(2, \frac{7}{2}), (3, 4).$

Range: $\{\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$



12. $6x - y = -1$

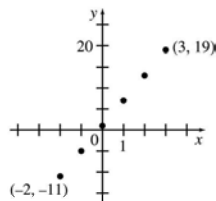
$$-y = -6x - 1$$

$$y = 6x + 1$$

x	-2	-1	0	1	2	3
y	-11	-5	1	7	13	19

Pairs: $(-2, -11), (-1, -5), (0, 1),$
 $(1, 7), (2, 13), (3, 19)$

Range: $\{-11, -5, 1, 7, 13, 19\}$

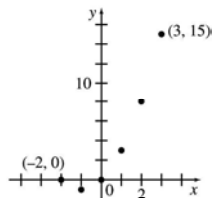


13. $y = x(x + 2)$

x	-2	-1	0	1	2	3
y	0	-1	0	3	8	15

Pairs: $(-2, 0), (-1, -1), (0, 0), (1, 3), (2, 8), (3, 15)$

Range: $\{-1, 0, 3, 8, 15\}$

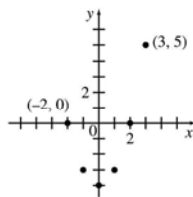


14. $y = (x - 2)(x + 2)$

x	-2	-1	0	1	2	3
y	0	-3	-4	-3	0	5

Pairs: $(-2, 0), (-1, -3), (0, -4),$
 $(1, -3), (2, 0), (3, 5)$

Range: $\{-4, -3, 0, 5\}$

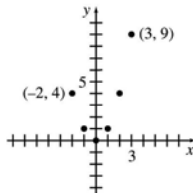


15. $y = x^2$

x	-2	-1	0	1	2	3
y	4	1	0	1	4	9

Pairs: $(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)$

Range: $\{0, 1, 4, 9\}$

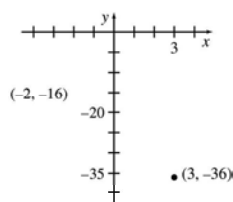


16. $y = -4x^2$

x	-2	-1	0	1	2	3
y	-16	-4	0	-4	-16	-36

Pairs: $(-2, -16), (-1, -4), (0, 0),$
 $(1, -4), (2, -16), (3, -36)$

Range: $\{-36, -16, -4, 0\}$



17. $f(x) = 2x$

x can take on any value, so the domain is the set of real numbers, $(-\infty, \infty)$.

18. $f(x) = 2x + 3$

x can take on any value, so the domain is the set of real numbers, which is written $(-\infty, \infty)$.

19. $f(x) = x^4$

x can take on any value, so the domain is the set of real numbers, $(-\infty, \infty)$.

20. $f(x) = (x + 3)^2$

x can take on any value, so the domain is the set of real numbers, $(-\infty, \infty)$.

21. $f(x) = \sqrt{4 - x^2}$

For $f(x)$ to be a real number, $4 - x^2 \geq 0$.
Solve $4 - x^2 = 0$.

$$(2 - x)(2 + x) = 0$$

$$x = 2 \text{ or } x = -2$$

The numbers form the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

Values in the interval $(-2, 2)$ satisfy the inequality; $x = 2$ and $x = -2$ also satisfy the inequality. The domain is $[-2, 2]$.

22. $f(x) = |3x - 6|$

x can take on any value, so the domain is the set of real numbers, $(-\infty, \infty)$.

23. $f(x) = (x - 3)^{1/2} = \sqrt{x - 3}$

For $f(x)$ to be a real number,

$$x - 3 \geq 0$$

$$x \geq 3.$$

The domain is $[3, \infty)$.

24. $f(x) = (3x + 5)^{1/2} = \sqrt{3x + 5}$

For $f(x)$ to be a real number,

$$3x + 5 \geq 0$$

$$3x \geq -5$$

$$\frac{1}{3}(3x) \geq \frac{1}{3}(-5)$$

$$x \geq -\frac{5}{3}.$$

In interval notation, the domain is $\left[-\frac{5}{3}, \infty\right)$.

25. $f(x) = \frac{2}{1 - x^2} = \frac{2}{(1 - x)(1 + x)}$

Since division by zero is not defined,
 $(1 - x) \cdot (1 + x) \neq 0$.

When $(1 - x)(1 + x) = 0$,

$$1 - x = 0 \quad \text{or} \quad 1 + x = 0$$

$$x = 1 \quad \text{or} \quad x = -1$$

Thus, x can be any real number except ± 1 .
The domain is

$$(-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

26. $f(x) = \frac{-8}{x^2 - 36}$

In order for $f(x)$ to be a real number, $x^2 - 36$ cannot be equal to 0.

When $x^2 - 36 = 0$,

$$x^2 = 36$$

$$x = 6 \text{ or } x = -6.$$

Thus, the domain is any real number except 6 or -6 . In interval notation, the domain is

$$(-\infty, -6) \cup (-6, 6) \cup (6, \infty).$$

27. $f(x) = -\sqrt{\frac{2}{x^2 - 16}} = -\sqrt{\frac{2}{(x - 4)(x + 4)}}$

$(x - 4) \cdot (x + 4) > 0$, since $(x - 4) \cdot (x + 4) < 0$ would produce a negative radicand and $(x - 4) \cdot (x + 4) = 0$ would lead to division by zero.

Solve $(x - 4) \cdot (x + 4) = 0$.

$$x - 4 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = 4 \quad \text{or} \quad x = -4$$

Use the values -4 and 4 to divide the number line into 3 intervals, $(-\infty, -4)$, $(-4, 4)$ and $(4, \infty)$.

Only the values in the intervals $(-\infty, -4)$ and $(4, \infty)$ satisfy the inequality.

The domain is $(-\infty, -4) \cup (4, \infty)$.

$$28. f(x) = -\sqrt{\frac{5}{x^2 + 36}}$$

x can take on any value. No choice for x will produce a zero in the denominator. Also, no choice for x will produce a negative number under the radical. The domain is $(-\infty, \infty)$.

$$29. f(x) = \sqrt{x^2 - 4x - 5} = \sqrt{(x - 5)(x + 1)}$$

See the method used in Exercise 21.

$$(x - 5)(x + 1) \geq 0$$

when $x \geq 5$ and when $x \leq -1$. The domain is $(-\infty, -1] \cup [5, \infty)$.

$$30. f(x) = \sqrt{15x^2 + x - 2}$$

The expression under the radical must be nonnegative.

$$\begin{aligned} 15x^2 + x - 2 &\geq 0 \\ (5x + 2)(3x - 1) &\geq 0 \end{aligned}$$

Solve $(5x + 2)(3x - 1) = 0$.

$$\begin{aligned} 5x + 2 &= 0 & \text{or} & & 3x - 1 &= 0 \\ 5x &= -2 & & & 3x &= 1 \\ x &= -\frac{2}{5} & \text{or} & & x &= \frac{1}{3} \end{aligned}$$

Use these numbers to divide the number line into 3 intervals, $(-\infty, -\frac{2}{5})$, $(-\frac{2}{5}, \frac{1}{3})$, and $(\frac{1}{3}, \infty)$.

Only values in the intervals $(-\infty, -\frac{2}{5})$ and $(\frac{1}{3}, \infty)$ satisfy the inequality. The domain is $(-\infty, -\frac{2}{5}] \cup [\frac{1}{3}, \infty)$.

$$31. f(x) = \frac{1}{\sqrt{3x^2 + 2x - 1}} = \frac{1}{\sqrt{(3x - 1)(x + 1)}}$$

$(3 - 2)(x + 1) > 0$, since the radicand cannot be negative and the denominator of the function cannot be zero.

Solve $(3 - 1)(x + 1) = 0$.

$$\begin{aligned} 3 - 1 &= 0 & \text{or} & & x + 1 &= 0 \\ x &= \frac{1}{3} & \text{or} & & x &= -1 \end{aligned}$$

Use the values -1 and $\frac{1}{3}$ to divide the number line into 3 intervals, $(-\infty, -1)$, $(-1, \frac{1}{3})$ and $(\frac{1}{3}, \infty)$.

Only the values in the intervals $(-\infty, -1)$ and $(\frac{1}{3}, \infty)$ satisfy the inequality.

The domain is $(-\infty, -1) \cup (\frac{1}{3}, \infty)$.

$$32. f(x) = \sqrt{\frac{x^2}{3 - x}}$$

For $f(x)$ to be a real number,

$$\begin{aligned} x^2 &\geq 0 & \text{and} & & 3 - x &> 0 \\ -\infty < x < \infty & & \text{and} & & x &< 3 \\ & & & & x &< 3. \end{aligned}$$

The domain is $(-\infty, 3)$.

33. By reading the graph, the domain is all numbers greater than or equal to -5 and less than 4 . The range is all numbers greater than or equal to -2 and less than or equal to 6 .

Domain: $[-5, 4)$; range: $[-2, 6]$

34. By reading the graph, the domain is all numbers greater than or equal to -5 . The range is all numbers greater than or equal to 0 .

Domain: $[-5, \infty)$ range: $[0, \infty)$

35. By reading the graph, x can take on any value, but y is less than or equal to 12 .

Domain: $(-\infty, \infty)$; range: $(-\infty, 12]$

36. By reading the graph, both x and y can take on any values.

Domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

37. The domain is all real numbers between the end points of the curve, or $[-2, 4]$.

The range is all real numbers between the minimum and maximum values of the function or $[0, 4]$.

$$(a) f(-2) = 0$$

$$(b) f(0) = 4$$

$$(c) f\left(\frac{1}{2}\right) = 3$$

(d) From the graph, $f(x) = 1$ when $x = -1.5, 1.5$, or 2.5 .

38. The domain is all real numbers between the end points of the curve, or $[-2, 4]$.

The range is all real numbers between the minimum and maximum values of the function or $[0, 5]$.

(a) $f(-2) = 5$

(b) $f(0) = 0$

(c) $f\left(\frac{1}{2}\right) = 1$

(d) From the graph, if $f(x) = 1$, $x = -0.2, 0.5, 1.2$, or 2.8 .

39. The domain is all real numbers between the endpoints of the curve, or $[-2, 4]$.

The range is all real numbers between the minimum and maximum values of the function or $[-3, 2]$.

(a) $f(-2) = -3$

(b) $f(0) = -2$

(c) $f\left(\frac{1}{2}\right) = -1$

(d) From the graph, $f(x) = 1$ when $x = 2.5$.

40. The domain is all real numbers between the endpoints of the curve, or $[-2, 4]$.

The range is all real numbers between the minimum and maximum values of the function, or in this case, $\{3\}$.

(a) $f(-2) = 3$

(b) $f(0) = 3$

(c) $f\left(\frac{1}{2}\right) = 3$

(d) From the graph, $f(x)$ is 1 nowhere.

41. $f(x) = 3x^2 - 4x + 1$

(a) $f(4) = 3(4)^2 - 4(4) + 1$
 $= 48 - 16 + 1$
 $= 33$

(b) $f\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 1$
 $= \frac{3}{4} + 2 + 1$
 $= \frac{15}{4}$

(c) $f(a) = 3(a)^2 - 4(a) + 1$
 $= 3a^2 - 4a + 1$

(d) $f\left(\frac{2}{m}\right) = 3\left(\frac{2}{m}\right)^2 - 4\left(\frac{2}{m}\right) + 1$
 $= \frac{12}{m^2} - \frac{8}{m} + 1$
or $\frac{12 - 8m + m^2}{m^2}$

(e) $f(x) = 1$

$$3x^2 - 4x + 1 = 1$$

$$3x^2 - 4x = 0$$

$$x(3x - 4) = 0$$

$$x = 0 \text{ or } x = \frac{4}{3}$$

42. $f(x) = (x + 3)(x - 4)$

(a) $f(4) = (4 + 3)(4 - 4) = (7)(0) = 0$

(b) $f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2} + 3\right)\left(-\frac{1}{2} - 4\right)$
 $= \left(\frac{5}{2}\right)\left(-\frac{9}{2}\right) = -\frac{45}{4}$

(c) $f(a) = [(a) + 3][(a) - 4] = (a + 3)(a - 4)$

(d) $f\left(\frac{2}{m}\right) = \left(\frac{2}{m} + 3\right)\left(\frac{2}{m} - 4\right)$
 $= \left(\frac{2 + 3m}{m}\right)\left(\frac{2 - 4m}{m}\right)$
 $= \frac{(2 + 3m)(2 - 4m)}{m^2}$
or $\frac{2(2 + 3m)(1 - 2m)}{m^2}$

(e) $f(x) = 1$

$$(x + 3)(x - 4) = 1$$

$$x^2 - x - 12 = 1$$

$$x^2 - x - 13 = 0$$

$$x = \frac{1 \pm \sqrt{1 + 52}}{2}$$

$$x = \frac{1 \pm \sqrt{53}}{2}$$

$$x \approx -3.140 \text{ or } x \approx 4.140$$

$$43. f(x) = \begin{cases} \frac{2x+1}{x-4} & \text{if } x \neq 4 \\ 7 & \text{if } x = 4 \end{cases}$$

$$(a) f(4) = 7$$

$$(b) f\left(-\frac{1}{2}\right) = \frac{2\left(-\frac{1}{2}\right) + 1}{\left(-\frac{1}{2}\right) - 4} = \frac{0}{-\frac{9}{2}} = 0$$

$$(c) f(a) = \frac{2a+1}{a-4} \text{ if } a \neq 4 \\ f(a) = 7 \text{ if } a = 4$$

$$(d) f\left(\frac{2}{m}\right) = \frac{2\left(\frac{2}{m}\right) + 1}{\frac{2}{m} - 4} = \frac{\frac{4}{m} + 1}{\frac{2}{m} - 4} \\ = \frac{\frac{4+m}{m}}{\frac{2-4m}{m}} = \frac{4+m}{2-4m} \text{ if } m \neq \frac{1}{2} \\ f\left(\frac{2}{m}\right) = 7 \text{ if } m = \frac{1}{2}$$

$$(e) \frac{2x+1}{x-4} = 1 \\ 2x+1 = x-4 \\ x = -5$$

$$44. f(x) = \begin{cases} \frac{x-4}{2x+1} & \text{if } x \neq -\frac{1}{2} \\ 10 & \text{if } x = -\frac{1}{2} \end{cases}$$

$$(a) f(4) = \frac{4-4}{2(4)+1} = \frac{0}{9} = 0$$

$$(b) f\left(-\frac{1}{2}\right) = 10$$

$$(c) f(a) = \frac{a-4}{2a+1} \text{ if } a \neq -\frac{1}{2} \\ f(a) = 10 \text{ if } a = -\frac{1}{2}$$

$$(d) f\left(\frac{2}{m}\right) = \frac{\frac{2}{m} - 4}{2\left(\frac{2}{m}\right) + 1} = \frac{\frac{2-4m}{m}}{\frac{4+m}{m}} \\ = \frac{2-4m}{4+m} \text{ if } m \neq -4 \\ f\left(\frac{2}{m}\right) = 10 \text{ if } m = -4$$

$$(e) \frac{x-4}{2x+1} = 1 \\ x-4 = 2x+1 \\ x = -5$$

$$45. f(x) = 6x^2 - 2$$

$$f(t+1) = 6(t+1)^2 - 2 \\ = 6(t^2 + 2t + 1) - 2 \\ = 6t^2 + 12t + 6 - 2 \\ = 6t^2 + 12t + 4$$

$$46. f(x) = 6x^2 - 2$$

$$f(2-r) = 6(2-r)^2 - 2 \\ = 6(4 - 4r + r^2) - 2 \\ = 24 - 24r + 6r^2 - 2 \\ = 6r^2 - 24r + 22$$

$$47. g(r+h)$$

$$= (r+h)^2 - 2(r+h) + 5 \\ = r^2 + 2hr + h^2 - 2r - 2h + 5$$

$$48. g(z-p)$$

$$= (z-p)^2 - 2(z-p) + 5 \\ = z^2 - 2zp + p^2 - 2z + 2p + 5$$

$$49. g\left(\frac{3}{q}\right) = \left(\frac{3}{q}\right)^2 - 2\left(\frac{3}{q}\right) + 5 \\ = \frac{9}{q^2} - \frac{6}{q} + 5 \\ \text{or } \frac{9 - 6q + 5q^2}{q^2}$$

$$50. g\left(-\frac{5}{z}\right) = \left(-\frac{5}{z}\right)^2 - 2\left(-\frac{5}{z}\right) + 5 \\ = \frac{25}{z^2} + \frac{10}{z} + 5 \\ = \frac{25}{z^2} + \frac{10z}{z^2} + \frac{5z^2}{z^2} \\ = \frac{25 + 10z + 5z^2}{z^2}$$

$$51. f(x) = 2x + 1$$

$$(a) f(x+h) = 2(x+h) + 1 \\ = 2x + 2h + 1$$

$$\begin{aligned}
 \text{(b)} \quad f(x+h) - f(x) &= 2x + 2h + 1 - 2x - 1 \\
 &= 2h
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{f(x+h) - f(x)}{h} &= \frac{2h}{h} \\
 &= 2
 \end{aligned}$$

$$52. \quad f(x) = x^2 - 3$$

$$\begin{aligned}
 \text{(a)} \quad f(x+h) &= (x+h)^2 - 3 \\
 &= (x^2 + 2xh + h^2) - 3 \\
 &= x^2 + 2xh + h^2 - 3
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad f(x+h) - f(x) &= (x^2 + 2xh + h^2 - 3) - (x^2 - 3) \\
 &= x^2 + 2xh + h^2 - 3 - x^2 + 3 \\
 &= 2xh + h^2
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{f(x+h) - f(x)}{h} &= \frac{2xh + h^2}{h} \\
 &= \frac{h(2x + h)}{h} \\
 &= 2x + h
 \end{aligned}$$

$$53. \quad f(x) = 2x^2 - 4x - 5$$

$$\begin{aligned}
 \text{(a)} \quad f(x+h) &= 2(x+h)^2 - 4(x+h) - 5 \\
 &= 2(x^2 + 2xh + h^2) - 4x - 4h - 5 \\
 &= 2x^2 + 4hx + 2h^2 - 4x - 4h - 5
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad f(x+h) - f(x) &= 2x^2 + 4hx + 2h^2 - 4x - 4h - 5 \\
 &\quad - (2x^2 - 4x - 5) \\
 &= 2x^2 + 4hx + 2h^2 - 4x - 4h - 5 \\
 &\quad - 2x^2 + 4x + 5 \\
 &= 4hx + 2h^2 - 4h
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{f(x+h) - f(x)}{h} &= \frac{4hx + 2h^2 - 4h}{h} \\
 &= \frac{h(4x + 2h - 4)}{h} \\
 &= 4x + 2h - 4
 \end{aligned}$$

$$54. \quad f(x) = -4x^2 + 3x + 2$$

$$\begin{aligned}
 \text{(a)} \quad f(x+h) &= -4(x+h)^2 + 3(x+h) + 2 \\
 &= -4(x^2 + 2xh + h^2) + 3x + 3h + 2 \\
 &= -4x^2 - 8hx - 4h^2 + 3x + 3h + 2
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad f(x+h) - f(x) &= -4x^2 - 8hx - 4h^2 + 3x + 3h + 2 \\
 &\quad - (-4x^2 + 3x + 2) \\
 &= -4x^2 - 8hx - 4h^2 + 3x + 3h + 2 \\
 &\quad + 4x^2 - 3x - 2 \\
 &= -8hx - 4h^2 + 3h
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{f(x+h) - f(x)}{h} &= \frac{-8hx - 4h^2 + 3h}{h} \\
 &= \frac{h(-8x - 4h + 3)}{h} \\
 &= -8x - 4h + 3
 \end{aligned}$$

$$55. \quad f(x) = \frac{1}{x}$$

$$\text{(a)} \quad f(x+h) = \frac{1}{x+h}$$

$$\begin{aligned}
 \text{(b)} \quad f(x+h) - f(x) &= \frac{1}{x+h} - \frac{1}{x} \\
 &= \left(\frac{x}{x}\right) \frac{1}{x+h} - \frac{1}{x} \left(\frac{x+h}{x+h}\right) \\
 &= \frac{x - (x+h)}{x(x+h)} \\
 &= \frac{-h}{x(x+h)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{f(x+h) - f(x)}{h} &= \frac{1}{h} \left[\frac{-h}{x(x+h)} \right] \\
 &= \frac{-1}{x(x+h)}
 \end{aligned}$$

56. $f(x) = -\frac{1}{x^2}$

(a) $f(x+h) = -\frac{1}{(x+h)^2}$

$$= -\frac{1}{x^2 + 2xh + h^2}$$

(b) $f(x+h) - f(x)$

$$= -\frac{1}{x^2 + 2xh + h^2} - \left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2 + 2xh + h^2} + \frac{1}{x^2}$$

$$= -\frac{x^2}{x^2(x^2 + 2xh + h^2)} + \frac{(x^2 + 2xh + h^2)}{x^2(x^2 + 2xh + h^2)}$$

$$= \frac{-x^2 + x^2 + 2xh + h^2}{x^2(x^2 + 2xh + h^2)}$$

$$= \frac{2xh + h^2}{x^2(x^2 + 2xh + h^2)}$$

(c) $\frac{f(x+h) - f(x)}{h} = \frac{\frac{h(2x+h)}{x^2(x^2 + 2xh + h^2)}}{h}$

$$= \frac{2x + h}{x^2(x^2 + 2xh + h^2)}$$

57. A vertical line drawn anywhere through the graph will intersect the graph in only one place. The graph represents a function.

58. A vertical line drawn anywhere through the graph will intersect the graph in only one place. The graph represents a function.

59. A vertical line drawn through the graph may intersect the graph in two places. The graph does not represent a function.

60. A vertical line drawn through the graph may intersect the graph in two or more places. The graph does not represent a function.

61. A vertical line drawn anywhere through the graph will intersect the graph in only one place. The graph represents a function.

62. A vertical line is not the graph of a function since the one x -value in the domain corresponds to more than one, in fact, infinitely many y -values. The graph does not represent a function.

63. $f(x) = 3x$
 $f(-x) = 3(-x)$
 $= -(3x)$
 $= -f(x)$

The function is odd.

64. $f(x) = 5x$
 $f(-x) = 5(-x)$
 $= -(5x)$
 $= -f(x)$

The function is odd.

65. $f(x) = 2x^2$
 $f(-x) = 2(-x)^2$
 $= 2x^2$
 $= f(x)$

The function is even.

66. $f(x) = x^2 - 3$
 $f(-x) = (-x)^2 - 3$
 $= x^2 - 3$
 $= f(x)$

The function is even.

67. $f(x) = \frac{1}{x^2 + 4}$
 $f(-x) = \frac{1}{(-x)^2 + 4}$
 $= \frac{1}{x^2 + 4}$
 $= f(x)$

The function is even.

68. $f(x) = x^3 + x$
 $f(-x) = (-x)^3 + (-x)$
 $= -x^3 - x$
 $= -(x^3 + x)$
 $= -f(x)$

The function is odd.

$$\begin{aligned}
 69. \quad f(x) &= \frac{x}{x^2 - 9} \\
 f(-x) &= \frac{-x}{(-x)^2 - 9} \\
 &= -\frac{x}{x^2 - 9} \\
 &= -f(x)
 \end{aligned}$$

The function is odd.

$$\begin{aligned}
 70. \quad f(x) &= |x - 2| \\
 f(-x) &= |-x - 2| \\
 &= |-(x + 2)| \\
 &= |x + 2|
 \end{aligned}$$

$f(-x)$ does not equal either $f(x)$ or $f(-x)$.
The function is neither even nor odd.

71. If x is a whole number of days, the cost of renting a saw in dollars is $S(x) = 28x + 8$. For x in whole days and a fraction of a day, substitute the next whole number for x in $28x + 8$, because a fraction of a day is charged as a whole day.

$$(a) \quad S\left(\frac{1}{2}\right) = S(1) = 28(1) + 8 = 36$$

The cost is \$36.

$$(b) \quad S(1) = 28(1) + 8 = 36$$

The cost is \$36.

$$\begin{aligned}
 (c) \quad S\left(1\frac{1}{4}\right) &= S(2) = 28(2) + 8 \\
 &= 56 + 8 = 64
 \end{aligned}$$

The cost is \$64.

$$\begin{aligned}
 (d) \quad S\left(3\frac{1}{2}\right) &= S(4) = 28(4) + 8 \\
 &= 112 + 8 = 120
 \end{aligned}$$

The cost is \$120.

$$(e) \quad S(4) = 28(4) + 8 = 112 + 8 = 120$$

The cost is \$120.

$$\begin{aligned}
 (f) \quad S\left(4\frac{1}{10}\right) &= S(5) = 28(5) + 8 \\
 &= 140 + 8 = 148
 \end{aligned}$$

The cost is \$148.

$$\begin{aligned}
 (g) \quad S\left(4\frac{9}{10}\right) &= S(5) = 28(5) + 8 \\
 &= 140 + 8 = 148
 \end{aligned}$$

The cost is \$148.

- (h) To continue the graph, continue the horizontal bars up and to the right.
(i) The independent variable is x , the number of full and partial days.
(j) The dependent variable is S , the cost of renting a saw.
(k) S is not a linear function. Its graph is not a continuous straight line. S is a step function.

72. If x is a whole number of days, the cost of renting a car is given by

$$C(x) = 54x + 44.$$

For x in whole days plus a fraction of a day, substitute the next whole number for x in $54x + 44$ because a fraction of a day is charged as a whole day.

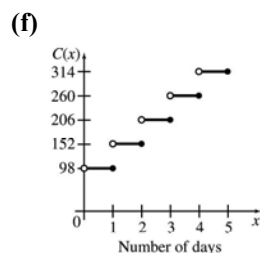
$$\begin{aligned}
 (a) \quad C\left(\frac{3}{4}\right) &= C(1) \\
 &= 54(1) + 44 \\
 &= \$98
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad C\left(\frac{9}{10}\right) &= C(1) \\
 &= \$98
 \end{aligned}$$

$$(c) \quad C(1) = \$98$$

$$\begin{aligned}
 (d) \quad C\left(1\frac{5}{8}\right) &= C(2) \\
 &= 54(2) + 44 \\
 &= \$152
 \end{aligned}$$

$$\begin{aligned}
 (e) \quad C(2.4) &= C(3) \\
 &= 54(3) + 44 \\
 &= \$206
 \end{aligned}$$



- (g) Yes, C is a function.
(h) No, C is not a linear function.
(i) The independent variable is x , the number of full or partial days.
(j) The dependent variable is C , the cost of renting the car.

73. (a)

$$f(250,000) = 0.40(150,000) + 0.333(100,000) \\ = 93,300$$

The maximum amount that an attorney can receive for a \$250,000 jury award is \$93,300.

$$(b) \quad f(350,000) = 0.40(150,000) \\ + 0.333(150,000) \\ + 0.30(50,000) \\ = 124,950$$

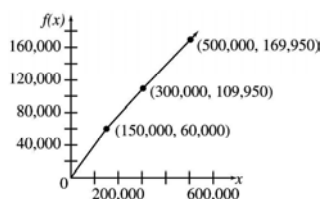
The maximum amount that an attorney can receive for a \$350,000 jury award is \$124,950.

(c)

$$f(550,000) = 0.40(150,000) + 0.333(150,000) \\ + 0.30(200,000) + 0.24(50,000) \\ = 181,950$$

The maximum amount that an attorney can receive for a \$550,000 jury award is \$181,950.

(d)



74. (a)

$$f(10,000) = 0.04(8000) + 0.045(2000) = 410$$

The tax due on an income of \$10,000 was \$410.

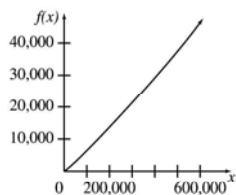
$$(b) \quad f(12,000) = 0.04(8000) + 0.045(3000) \\ + 0.525(1000) \\ = 507.50$$

The tax due on an income of \$12,000 was \$507.50.

$$(c) \quad f(12,000) = 0.04(8000) + 0.045(3000) \\ + 0.525(2000) + 0.059(5000) \\ = 855$$

tax due on an income of \$18,000 was \$855.

(d)



75. (a) The curve in the graph crosses the point with x -coordinate 17:37 and y -coordinate of approximately 140. So, at time 17 hours, 37 minutes the whale reaches a depth of about 140 m.

(b) The curve in the graph crosses the point with x -coordinate 17:39 and y -coordinate of approximately 240. So, at time 17 hours, 39 minutes the whale reaches a depth of about 250 m.

$$76. (a) (i) \quad y = f(5) = 19.7(5)^{0.753} \\ \approx 66 \text{ kcal/day}$$

$$(ii) \quad y = f(25) = 19.7(25)^{0.753} \\ \approx 222 \text{ kcal/day}$$

(b) Since 1 pound equals 0.454 kg, then $x = g(z) = 0.454z$ is the number of kilograms equal in weight to z pounds.

$$(c) \quad f(g(z)) = f(0.454z) = 19.7(0.454z)^{0.753} \\ = 19.7(0.454^{0.753})z^{0.753} \\ \approx 10.9z^{0.753}$$

77. (a) (i) By the given function f , a muskrat weighing 800 g expends

$$f(800) = 0.01(800)^{0.88} \\ \approx 3.6, \text{ or approximately}$$

3.6 kcal/km when swimming at the surface of the water.

(ii) A sea otter weighing 20,000 g expends

$$f(20,000) = 0.01(20,000)^{0.88} \\ \approx 61, \text{ or approximately}$$

61 kcal/km when swimming at the surface of the water.

(b) If z is the number of kilograms of an animal's weight, then $x = g(z) = 1000z$ is the number of grams since 1 kilogram equals 1000 grams.

$$(c) \quad f(g(z)) = f(1000z) \\ = 0.01(1000z)^{0.88} \\ = 0.01(1000^{0.88})z^{0.88} \\ \approx 4.4z^{0.88}$$

78. (a) In the graph, the curves representing wood and coal intersect approximately at the point (1880, 50). So, in 1880 use of wood and coal were both about 50% of the global energy consumption.

- (b) In the graph, the curves representing oil and coal intersect approximately at the point (1967, 35). So, in 1967 use of oil and coal were both about 35% of the global energy consumption.

79. (a) $P = 2l + 2w$

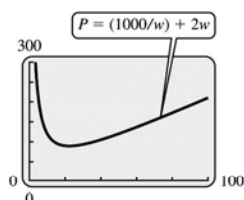
However, $lw = 500$, so $l = \frac{500}{w}$.

$$P(w) = 2\left(\frac{500}{w}\right) + 2w$$

$$P(w) = \frac{1000}{w} + 2w$$

- (b) Since $l = \frac{500}{w}$, $w \neq 0$ but w could be any positive value. Therefore, the domain of P is $0 < w < \infty$, or $(0, \infty)$.

(c)



80. (a) Let w = the width of the field;
 l = the length.

The perimeter of the field is 6000 ft, so

$$2l + 2w = 6000$$

$$l + w = 3000$$

$$l = 3000 - w.$$

Thus, the area of the field is given by

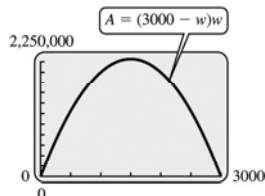
$$A = lw$$

$$A = (3000 - w)w.$$

- (b) Since $l = 3000 - w$ and w cannot be negative, $0 \leq w \leq 3000$.

The domain of A is $0 \leq w \leq 3000$.

(c)



2.2 Quadratic Functions; Translation and Reflection

Your Turn 1

$$f(x) = 2x^2 - 6x - 1$$

$$\begin{aligned} \text{(a)} \quad y &= 2x^2 - 6x - 1 \\ &= 2(x^2 - 3x) - 1 \\ &= 2\left(x^2 - 3x + \frac{9}{4}\right) - 1 - 2\left(\frac{9}{4}\right) \\ &= 2\left(x - \frac{3}{2}\right)^2 - \frac{11}{2} \end{aligned}$$

$$\text{(b)} \quad y = 2(0)^2 - 6(0) - 1 = -1$$

The y -intercept is -1 .

(c)

$$0 = 2\left(x - \frac{3}{2}\right)^2 - \frac{11}{2}$$

$$\frac{11}{2} = 2\left(x - \frac{3}{2}\right)^2$$

$$\frac{11}{4} = \left(x - \frac{3}{2}\right)^2$$

$$\pm \frac{\sqrt{11}}{2} = x - \frac{3}{2}$$

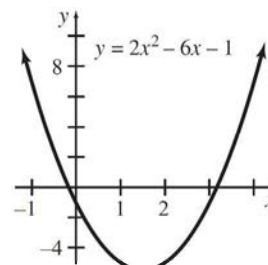
$$\frac{3 \pm \sqrt{11}}{2} = x$$

- (d) Since $y = 2\left(x - \frac{3}{2}\right)^2 - \frac{11}{2}$ is in the form

$$y = a(x - h)^2 + k, \text{ we get } h = \frac{3}{2} \text{ and}$$

$$k = -\frac{11}{2}. \text{ So the vertex is } \left(\frac{3}{2}, -\frac{11}{2}\right).$$

(e)



Your Turn 2

Let x be the number of \$40 decreases in the price.

$$\text{Price per person} = 1650 - 40x$$

$$\text{Number of people} = 900 + 80x$$

$$\begin{aligned}
 R(x) &= (1650 - 40x)(900 + 80x) \\
 &= 1,485,000 + 96,000x - 3200x^2 \\
 x &= \frac{-b}{2a} = \frac{-96,000}{2(-3200)} = \frac{-96,000}{-6400} = 15
 \end{aligned}$$

The y -coordinate is

$$\begin{aligned}
 y &= 1,485,000 + 96,000(15) - 3200(15)^2 \\
 &= 2,205,000
 \end{aligned}$$

The maximum revenue is \$2,205,000 when $1650 - 40(15) = \$1050$ per person is charged.

Your Turn 3

$$R(x) = -x^2 + 40x \text{ and } C(x) = 8x + 102$$

(a) $R(x) = C(x)$

$$-x^2 + 40x = 8x + 192$$

$$0 = x^2 - 32x + 192$$

$$0 = (x - 24)(x - 8)$$

The two graphs cross when $x = 8$ or $x = 20$.
The minimum break-even quantity is $x = 8$, so the deli owner must sell 8 lb of cream cheese to break even.

(b) The maximum revenue for $R(x) = -x^2 + 40x$ is at

$$x = \frac{-b}{2a} = \frac{-40}{2(-1)} = 20.$$

The maximum revenue is

$$R(20) = -20^2 + 40(20) = 400, \text{ or } \$400.$$

(c) $P(x) = R(x) - C(x)$

$$= -x^2 + 40x - 8x - 192$$

$$= -x^2 + 32x - 192$$

The maximum point of $P(x)$ is at

$$x = \frac{-b}{2a} = \frac{-32}{2(-1)} = 16.$$

$$P(16) = -(16)^2 + 32(16) - 192 = 64$$

So, the maximum profit is \$64.

2.2 Exercises

- The graph of $y = x^2 - 3$ is the graph of $y = x^2$ translated 3 units downward.
This is graph **D**.
- The graph of $y = (x - 3)^2$ is the graph of $y = x^2$ translated 3 units to the right.
This is graph **F**.
- The graph of $y = (x - 3)^2 + 2$ is the graph of $y = x^2$ translated 3 units to the right and 2 units upward.
This is graph **A**.
- The graph of $y = (x + 3)^2 + 2$ is the graph of $y = x^2$ translated 3 units to the left and 2 units upward.
This is graph **B**.
- The graph of $y = -(3 - x)^2 + 2$ is the same as the graph of $y = -(x - 3)^2 + 2$. This is the graph of $y = x^2$ reflected in the x -axis, translated 3 units to the right, and translated 2 units upward.
This is graph **C**.
- The graph of $y = -(x + 3)^2 + 2$ is the graph of $y = x^2$ reflected in the x -axis, translated three units to the left, and two units upward.
This is graph **E**.
- $$\begin{aligned}
 y &= 3x^2 + 9x + 5 \\
 &= 3(x^2 + 3x) + 5 \\
 &= 3\left(x^2 + 3x + \frac{9}{4}\right) + 5 - 3\left(\frac{9}{4}\right) \\
 &= 3\left(x + \frac{3}{2}\right)^2 - \frac{7}{4}
 \end{aligned}$$

The vertex is $\left(-\frac{3}{2}, -\frac{7}{4}\right)$.
- $$\begin{aligned}
 y &= 4x^2 - 20x - 7 \\
 &= 4(x^2 - 5x) - 7 \\
 &= 4\left(x^2 - 5x + \frac{25}{4}\right) - 7 - 4\left(\frac{25}{4}\right) \\
 &= 4\left(x - \frac{5}{2}\right)^2 - 32
 \end{aligned}$$

The vertex is $\left(\frac{5}{2}, -32\right)$.

$$\begin{aligned}
 11. \quad y &= -2x^2 + 8x - 9 \\
 &= -2(x^2 - 4x) - 9 \\
 &= -2(x^2 - 4x + 4) - 9 - (-2)(4) \\
 &= -2(x - 2)^2 - 1
 \end{aligned}$$

The vertex is $(2, -1)$.

$$\begin{aligned}
 12. \quad y &= -5x^2 - 8x + 3 \\
 &= -5\left(x^2 + \frac{8}{5}x\right) + 3 \\
 &= -5\left(x^2 + \frac{8}{5}x + \frac{16}{25}\right) + 3 - (-5)\left(\frac{16}{25}\right) \\
 &= -5\left(x + \frac{4}{5}\right)^2 + \frac{31}{5}
 \end{aligned}$$

The vertex is $\left(-\frac{4}{5}, \frac{31}{5}\right)$.

$$\begin{aligned}
 13. \quad y &= x^2 + 5x + 6 \\
 y &= (x + 3)(x + 2)
 \end{aligned}$$

Set $y = 0$ to find the x -intercepts.

$$\begin{aligned}
 0 &= (x + 3)(x + 2) \\
 x &= -3, x = -2
 \end{aligned}$$

The x -intercepts are -3 and -2 . Set $x = 0$ to find the y -intercept.

$$\begin{aligned}
 y &= 0^2 + 5(0) + 6 \\
 y &= 6
 \end{aligned}$$

The y -intercept is 6 .

The x -coordinate of the vertex is

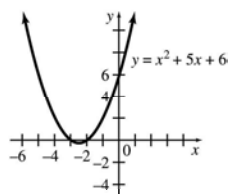
$$x = \frac{-b}{2a} = \frac{-5}{2} = -\frac{5}{2}.$$

Substitute to find the y -coordinate.

$$y = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6 = \frac{25}{4} - \frac{25}{2} + 6 = -\frac{1}{4}$$

The vertex is $\left(-\frac{5}{2}, -\frac{1}{4}\right)$

The axis is $x = -\frac{5}{2}$, the vertical line through the vertex.



$$\begin{aligned}
 14. \quad y &= x^2 + 4x - 5 \\
 &= (x + 5)(x - 1)
 \end{aligned}$$

Set $y = 0$ to find the x -intercepts.

$$\begin{aligned}
 0 &= (x + 5)(x - 1) \\
 x &= -5 \text{ or } x = 1
 \end{aligned}$$

The x -intercepts are -5 and 1 .

Set $x = 0$ to find the y -intercept.

$$\begin{aligned}
 y &= 0^2 + 4(0) - 5 \\
 y &= -5
 \end{aligned}$$

The y -intercept is -5 .

The x -coordinate of the vertex is

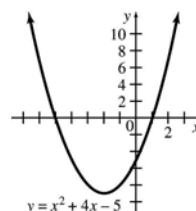
$$x = \frac{-b}{2a} = \frac{-4}{2} = -2.$$

Substitute to find the y -coordinate.

$$\begin{aligned}
 y &= (-2)^2 + 4(-2) - 5 \\
 &= 4 - 8 - 5 \\
 &= -9
 \end{aligned}$$

The vertex is $(-2, -9)$.

The axis is $x = -2$, the vertical line through the vertex.



$$\begin{aligned}
 15. \quad y &= -2x^2 - 12x - 16 \\
 &= -2(x^2 + 6x + 8) \\
 &= -2(x + 4)(x + 2)
 \end{aligned}$$

Let $y = 0$.

$$\begin{aligned}
 0 &= -2(x + 4)(x + 2) \\
 x &= -4, x = -2
 \end{aligned}$$

-4 and -2 are the x -intercepts. Let $x = 0$.

$$y = -2(0)^2 + 12(0) - 16$$

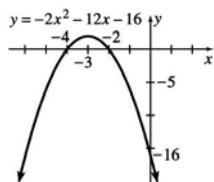
-16 is the y -intercept.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{12}{-4} = -3$$

$$\begin{aligned} y &= -2(-3)^2 - 12(-3) - 16 \\ &= -18 + 36 - 16 = 2 \end{aligned}$$

The vertex is $(-3, 2)$.

The axis is $x = -3$, the vertical line through the vertex.



16. $y = -3x^2 - 6x + 4$

Let $y = 0$.

The equation

$$0 = -3x^2 - 6x + 4$$

cannot be solved by factoring.

Use the quadratic formula.

$$\begin{aligned} 0 &= -3x^2 - 6x + 4 \\ x &= \frac{6 \pm \sqrt{6^2 - 4(-3)(4)}}{2(-3)} \\ &= \frac{6 \pm \sqrt{36 + 48}}{-6} \\ &= \frac{6 \pm \sqrt{84}}{-6} = \frac{6 \pm 2\sqrt{21}}{-6} \\ &= -1 \pm \frac{\sqrt{21}}{3} \end{aligned}$$

The x -intercepts are $-1 + \frac{\sqrt{21}}{3} \approx 0.53$ and

$$-1 - \frac{\sqrt{21}}{3} \approx -2.53$$

Let $x = 0$.

$$\begin{aligned} y &= -3(0)^2 - 6(0) + 4 \\ y &= 4 \end{aligned}$$

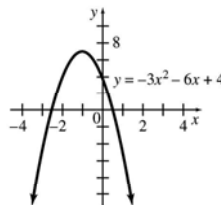
4 is the y -intercept.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{6}{2(-3)} = \frac{6}{-6} = -1$$

$$\begin{aligned} y &= -3(-1)^2 - 6(-1) + 4 \\ &= -3 + 6 + 4 \\ &= 7 \end{aligned}$$

The vertex is $(-1, 7)$.

The axis is $x = -1$, the vertical line through the vertex.



17. $y = 2x^2 + 8x - 8$

Let $y = 0$.

$$2x^2 + 8x - 8 = 0$$

$$x^2 + 4x - 4 = 0$$

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-4)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{32}}{2} = \frac{-4 \pm 4\sqrt{2}}{2} \\ &= -2 \pm 2\sqrt{2} \end{aligned}$$

The x -intercepts are $-2 \pm 2\sqrt{2} \approx 0.83$ or -4.83 .

Let $x = 0$.

$$y = 2(0)^2 + 8(0) - 8 = -8$$

The y -intercept is -8 .

The x -coordinate of the vertex is

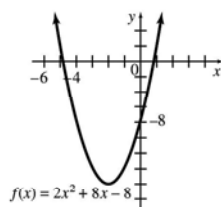
$$x = \frac{-b}{2a} = -\frac{8}{4} = -2.$$

If $x = -2$,

$$\begin{aligned} y &= 2(-2)^2 + 8(-2) - 8 \\ &= 8 - 16 - 8 = -16. \end{aligned}$$

The vertex is $(-2, -16)$.

The axis is $x = -2$.



18. $f(x) = -x^2 + 6x - 6$

Let $f(x) = 0$.

$$0 = -x^2 + 6x - 6$$

$$\begin{aligned}
 x &= \frac{-6 \pm \sqrt{6^2 - 4(-1)(-6)}}{2(-1)} \\
 &= \frac{-6 \pm \sqrt{36 - 24}}{-2} \\
 &= \frac{-6 \pm \sqrt{12}}{-2} \\
 &= \frac{-6 \pm 2\sqrt{3}}{-2} \\
 &= 3 \pm \sqrt{3}
 \end{aligned}$$

The x -intercepts are $3 + \sqrt{3} \approx 4.73$
and $3 - \sqrt{3} \approx 1.27$.

Let $x = 0$.

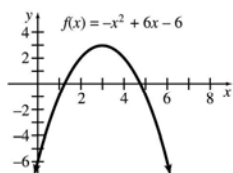
$$y = -0^2 + 6(0) - 6$$

-6 is the y -intercept.

$$\begin{aligned}
 \text{Vertex: } x &= \frac{-b}{2a} = \frac{-6}{2(-1)} = \frac{-6}{-2} = 3 \\
 y &= -3^2 + 6(3) - 6 \\
 &= -9 + 18 - 6 \\
 &= 3
 \end{aligned}$$

The vertex is $(3, 3)$.

The axis is $x = 3$.



19. $f(x) = 2x^2 - 4x + 5$

Let $f(x) = 0$.

$$\begin{aligned}
 0 &= 2x^2 - 4x + 5 \\
 x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(5)}}{2(2)} \\
 &= \frac{4 \pm \sqrt{16 - 40}}{4} \\
 &= \frac{4 \pm \sqrt{-24}}{4}
 \end{aligned}$$

Since the radicand is negative, there are no x -intercepts.

Let $x = 0$.

$$\begin{aligned}
 y &= 2(0)^2 - 4(0) + 5 \\
 y &= 5
 \end{aligned}$$

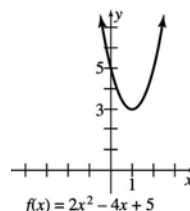
5 is the y -intercept.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-(-4)}{2(2)} = \frac{4}{4} = 1$$

$$y = 2(1)^2 - 4(1) + 5 = 2 - 4 + 5 = 3$$

The vertex is $(1, 3)$.

The axis is $x = 1$.



20. $f(x) = \frac{1}{2}x^2 + 6x + 24$

Let $f(x) = 0$.

$$0 = \frac{1}{2}x^2 + 6x + 24$$

Multiply by 2 to clear fractions.

$$0 = x^2 + 12x + 48$$

Use the quadratic formula.

$$\begin{aligned}
 x &= \frac{-12 \pm \sqrt{12^2 - 4(1)(48)}}{2(1)} \\
 &= \frac{-12 \pm \sqrt{144 - 192}}{2} \\
 &= \frac{-12 \pm \sqrt{-48}}{2}
 \end{aligned}$$

Since the radicand is negative, there are no x -intercepts.

Let $x = 0$.

$$\begin{aligned}
 y &= \frac{1}{2}(0)^2 + 6(0) + 24 \\
 y &= 24
 \end{aligned}$$

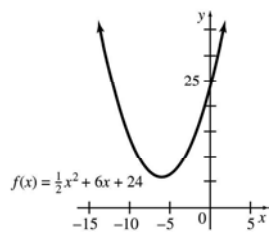
24 is the y -intercept.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-6}{2(\frac{1}{2})} = \frac{-6}{1} = -6$$

$$\begin{aligned}
 y &= \frac{1}{2}(-6)^2 + 6(-6) + 24 \\
 &= 18 - 36 + 24 = 6
 \end{aligned}$$

The vertex is $(-6, 6)$.

The axis $x = -6$.



21. $f(x) = -2x^2 + 16x - 21$

Let $f(x) = 0$

Use the quadratic formula.

$$\begin{aligned} x &= \frac{-16 \pm \sqrt{16^2 - 4(-2)(-21)}}{2(-2)} \\ &= \frac{-16 \pm \sqrt{88}}{-4} \\ &= \frac{-16 \pm 2\sqrt{22}}{-4} \\ &= 4 \pm \frac{\sqrt{22}}{2} \end{aligned}$$

The x-intercepts are $4 + \frac{\sqrt{22}}{2} \approx 6.35$

and $4 - \frac{\sqrt{22}}{2} \approx 1.65$.

Let $x = 0$.

$$\begin{aligned} y &= -2(0)^2 + 16(0) - 21 \\ y &= -21 \end{aligned}$$

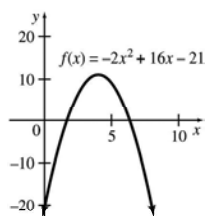
-21 is the y-intercept.

Vertex: $x = \frac{-b}{2a} = \frac{-16}{2(-2)} = \frac{-16}{-4} = 4$

$$\begin{aligned} y &= -2(4)^2 + 16(4) - 21 \\ &= -32 + 64 - 21 = 11 \end{aligned}$$

The vertex is (4, 11).

The axis is $x = 4$.



22. $f(x) = \frac{3}{2}x^2 - x - 4$

Let $f(x) = 0$.

$$0 = \frac{3}{2}x^2 - x - 4$$

Multiply by 2 to clear fractions, then solve.

$$\begin{aligned} 0 &= 3x^2 - 2x - 8 \\ &= (3x + 4)(x - 2) \\ x &= -\frac{4}{3}, 2 \end{aligned}$$

The x-intercepts are $-\frac{4}{3}$ and 2.

Let $x = 0$.

$$\begin{aligned} y &= \frac{3}{2}(0)^2 - 0 - 4 \\ y &= -4 \end{aligned}$$

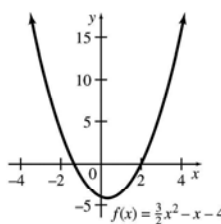
The y-intercept is -4.

Vertex: $x = \frac{-b}{2a} = \frac{-(-1)}{2(\frac{3}{2})} = \frac{1}{3}$

$$y = \frac{3}{2}\left(\frac{1}{3}\right)^2 - \frac{1}{3} - 4 = \frac{1}{6} - \frac{2}{6} - \frac{24}{6} = -\frac{25}{6}$$

The vertex is $\left(\frac{1}{3}, -\frac{25}{6}\right)$.

The axis $x = \frac{1}{3}$.



23. $y = \frac{1}{3}x^2 - \frac{8}{3}x + \frac{1}{3}$

Let $y = 0$.

$$0 = \frac{1}{3}x^2 - \frac{8}{3}x + \frac{1}{3}$$

Multiply by 3.

$$\begin{aligned} 0 &= x^2 - 8x + 1 \\ x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{8 \pm \sqrt{64 - 4}}{2} = \frac{8 \pm \sqrt{60}}{2} \\ &= \frac{8 \pm 2\sqrt{15}}{2} = 4 \pm \sqrt{15} \end{aligned}$$

The x-intercepts are $4 + \sqrt{15} \approx 7.87$
and $4 - \sqrt{15} \approx 0.13$.

Let $x = 0$.

$$y = \frac{1}{3}(0)^2 - \frac{8}{3}(0) + \frac{1}{3}$$

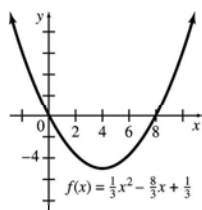
$\frac{1}{3}$ is the y -intercept.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-\left(-\frac{8}{3}\right)}{2\left(\frac{1}{3}\right)} = \frac{\frac{8}{3}}{\frac{2}{3}} = 4$$

$$\begin{aligned} y &= \frac{1}{3}(4)^2 - \frac{8}{3}(4) + \frac{1}{3} \\ &= \frac{16}{3} - \frac{32}{3} + \frac{1}{3} = -\frac{15}{3} = -5 \end{aligned}$$

The vertex is $(4, -5)$.

The axis is $x = 4$.



24. $y = -\frac{1}{2}x^2 - x - \frac{7}{2}$

Let $y = 0$.

$$0 = -\frac{1}{2}x^2 - x - \frac{7}{2}$$

Multiply by -2 to clear fractions.

$$0 = x^2 + 2x + 7$$

Use the quadratic formula.

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(1)(7)}}{2} \\ &= \frac{-2 \pm \sqrt{-24}}{2} \end{aligned}$$

Since the radicand is negative, there are no x -intercepts.

Let $x = 0$.

$$y = -\frac{1}{2}(0)^2 - 0 - \frac{7}{2} = -\frac{7}{2}$$

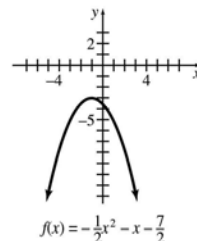
$-\frac{7}{2}$ is the y -intercept.

$$\text{Vertex: } \frac{-b}{2a} = \frac{-(-1)}{2\left(-\frac{1}{2}\right)} = \frac{1}{-1} = -1$$

$$\begin{aligned} y &= -\frac{1}{2}(-1)^2 - (-1) - \frac{7}{2} \\ &= -\frac{1}{2} + 1 - \frac{7}{2} \\ &= -3 \end{aligned}$$

The vertex is $(-1, -3)$.

The axis is $x = -1$.

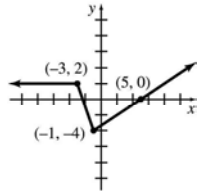


25. The graph of $y = \sqrt{x+2} - 4$ is the graph of $y = \sqrt{x}$ translated 2 units to the left and 4 units downward. This is graph **D**.
26. The graph of $y = \sqrt{x-2} - 4$ is the graph of $y = \sqrt{x}$ translated 2 units to the right and 4 units downward. This is graph **A**.
27. The graph of $y = \sqrt{-x+2} - 4$ is the graph of $y = \sqrt{-(x-2)} - 4$, which is the graph of $y = \sqrt{x}$ reflected in the y -axis, translated 2 units to the right, and translated 4 units downward. This is graph **C**.
28. The graph of $y = \sqrt{-x-2} - 4$ is the same as the graph of $y = \sqrt{-(x+2)} - 4$. This is the graph of $y = \sqrt{x}$ reflected in the y -axis, translated 2 units to the left, and translated 4 units downward. This is graph **B**.
29. The graph of $y = -\sqrt{x+2} - 4$ is the graph of $y = \sqrt{x}$ reflected in the x -axis, translated 2 units to the left, and translated 4 units downward. This is graph **E**.

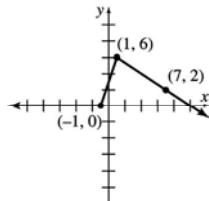
30. The graph of $y = -\sqrt{x-2} - 4$ is the graph of $y = \sqrt{x}$ reflected in the x -axis, translated 2 units to the right, and translated 4 units downward.

This is graph F.

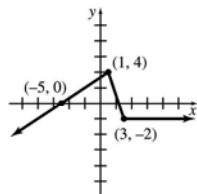
31. The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected in the x -axis.



32. The graph of $y = f(x-2) + 2$ is the graph of $y = f(x)$ translated 2 units to the right and 2 units upward.

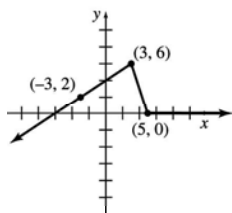


33. The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected in the y -axis.



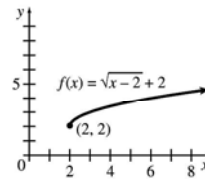
34. $y = f(2-x) + 2$
 $y = f[-(x-2)] + 2$

This is the graph of $y = f(x)$ reflected in the y -axis, translated 2 units to the right, and translated 2 units upward.



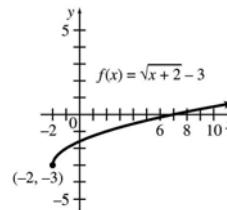
35. $f(x) = \sqrt{x-2} + 2$

Translate the graph of $f(x) = \sqrt{x}$ 2 units right and 2 units up.



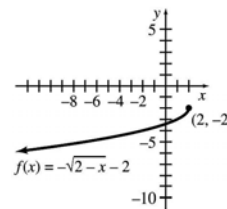
36. $f(x) = \sqrt{x+2} - 3$

Translate the graph of $f(x) = \sqrt{x}$ 2 units left and 3 units down.



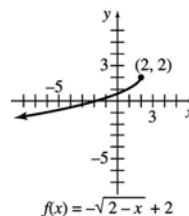
37. $f(x) = -\sqrt{2-x} - 2$
 $= -\sqrt{-(x-2)} - 2$

Reflect the graph of $f(x)$ vertically and horizontally. Translate the graph 2 units right and 2 units down.

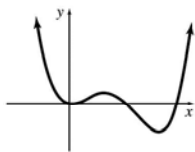


38. $f(x) = -\sqrt{2-x} + 2 = -\sqrt{-(x-2)} + 2$

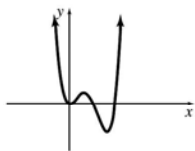
Reflect the graph of $f(x)$ vertically and horizontally. Translate the graph 2 units right and 2 units up.



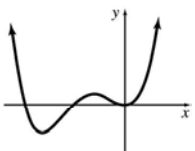
39. If $0 < a < 1$, the graph of $f(ax)$ will be the graph of $f(x)$ stretched horizontally.



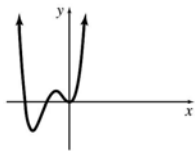
40. If $1 < a$, the graph of $f(ax)$ will be the graph of $f(x)$ compressed horizontally.



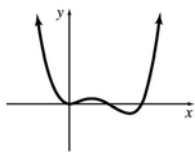
41. If $-1 < a < 0$, the graph of $f(ax)$ will be reflected horizontally, since a is negative. It will be stretched horizontally.



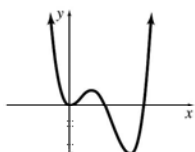
42. If $a < -1$, the graph of $f(ax)$ will be reflected horizontally, since a is negative. It will also be compressed horizontally.



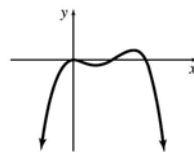
43. If $0 < a < 1$, the graph of $af(x)$ will be flatter than the graph of $f(x)$. Each y -value is only a fraction of the height of the original y -values.



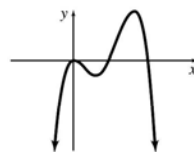
44. If $1 < a$, the graph of $af(x)$ will be taller than the graph of $f(x)$. The absolute value of the y -value will be larger than the original y -values, while the x -values will remain the same.



45. If $-1 < a < 0$, the graph will be reflected vertically, since a will be negative. Also, because a is a fraction, the graph will be flatter because each y -value will only be a fraction of its original height.

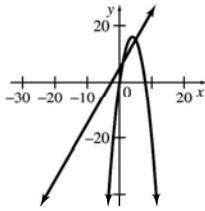


46. If $a < -1$, the graph of $af(x)$ will be reflected vertically. It will also be taller than the graph of $f(x)$ since the absolute value of each y -value will be larger than the original y -values, while the x -values stay the same.



47. (a) Since the graph of $y = f(x)$ is reflected vertically to obtain the graph of $y = -f(x)$, the x -intercept is unchanged. The x -intercept of the graph of $y = f(x)$ is r .
- (b) Since the graph of $y = f(x)$ is reflected horizontally to obtain the graph of $y = f(-x)$, the x -intercept of the graph of $y = f(-x)$ is $-r$.
- (c) Since the graph of $y = f(x)$ is reflected both horizontally and vertically to obtain the graph of $y = -f(-x)$, the x -intercept of the graph of $y = -f(-x)$ is $-r$.
48. (a) Since the graph of $y = f(x)$ is reflected vertically to obtain the graph of $y = -f(x)$, the y -intercept of the graph of $y = -f(x)$ is $-b$.
- (b) Since the graph of $y = f(x)$ is reflected horizontally to obtain the graph of $y = f(-x)$, the y -intercept is unchanged. The y -intercept of the graph of $y = f(-x)$ is b .
- (c) Since the graph of $y = f(x)$ is reflected both horizontally and vertically to obtain the graph of $y = -f(-x)$, the y -intercept of the graph of $y = -f(-x)$ is $-b$.

49. (a)



- (b) Break-even quantities are values of x = batches of widgets for which revenue and cost are equal.

Set $R(x) = C(x)$ and solve for x .

$$-x^2 + 8x = 2x + 5$$

$$x^2 - 6x + 5 = 0$$

$$(x - 5)(x - 1) = 0$$

$$x - 5 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 5 \quad \text{or} \quad x = 1$$

So, the break-even quantities are 1 and 5.

The minimum break-even quantity is 1 batch of widgets.

- (c) The maximum revenue occurs at the vertex of R . Since $R(x) = -x^2 + 8x$, then the x -coordinate of the vertex is

$$x = -\frac{b}{2a} = -\frac{8}{2(-1)} = 4.$$

So, the maximum revenue is

$$R(4) = -4^2 + 8(4) = 16, \text{ or } \$16,000.$$

- (d) The maximum profit is the maximum difference $R(x) - C(x)$. Since

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -x^2 + 8x - (2x + 5) \\ &= -x^2 + 6x - 5 \end{aligned}$$

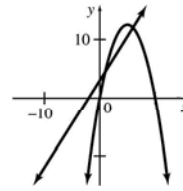
is a quadratic function, we can find the maximum profit by finding the vertex of P . This occurs at

$$x = -\frac{b}{2a} = \frac{-6}{2(-1)} = 3.$$

Therefore, the maximum profit is

$$P(3) = -(3)^2 + 6(3) - 5 = 4, \text{ or } \$4000.$$

50. (a)



- (b) Break-even quantities are values of x = batches of widgets for which revenue and cost are equal. Set $R(x) = C(x)$ and solve for x .

$$-\frac{x^2}{2} + 5x = \frac{3}{2}x + 3$$

$$-x^2 + 10x = 3x + 6$$

$$x^2 - 7x + 6 = 0$$

$$(x - 1)(x - 6) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x - 6 = 0$$

$$x = 1 \quad \text{or} \quad x = 6$$

So, the break-even quantities are 1 and 6.

The minimum break-even quantity is 1 batch of widgets.

- (c) The maximum revenue occurs at the vertex of R . Since $R(x) = -\frac{x^2}{2} + 5x$, then the x -coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-5}{2\left(-\frac{1}{2}\right)} = 5.$$

So, the maximum revenue is

$$R(5) = -\frac{5^2}{2} + 5(5) = 12.5, \text{ or } \$12,500.$$

- (d) The maximum profit is the maximum difference $R(x) - C(x)$. Since

$$\begin{aligned} P(x) &= R(x) - C(x) = -\frac{x^2}{2} + 5x - \left(\frac{3}{2}x + 3\right) \\ &= -\frac{x^2}{2} + \frac{7}{2}x - 3 \end{aligned}$$

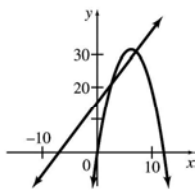
is a quadratic function, we can find the maximum profit by finding the vertex of P . This occurs at

$$x = -\frac{b}{2a} = \frac{\frac{7}{2}}{2\left(-\frac{1}{2}\right)} = \frac{7}{2}.$$

Therefore, the maximum profit is

$$P\left(\frac{7}{2}\right) = \frac{-\left(\frac{7}{2}\right)^2}{2} + \frac{7}{2}\left(\frac{7}{2}\right) - 3 = 3.125, \text{ or } \$3125.$$

51. (a)



- (b) Break-even quantities are values of x = batches of widgets for which revenue equals cost.

Set $R(x) = C(x)$ and solve for x .

$$-\frac{4}{5}x^2 + 10x = 2x + 15$$

$$\frac{4}{5}x^2 - 8x + 15 = 0$$

$$4x^2 - 40x + 75 = 0$$

$$4x^2 - 10x - 30x + 75 = 0$$

$$2x(2x - 5) - 15(2x - 5) = 0$$

$$(2x - 5)(2x - 15) = 0$$

$$2x - 5 = 0 \quad \text{or} \quad 2x - 15 = 0$$

$$x = 2.5 \quad \text{or} \quad x = 7.5$$

So, the break-even quantities are 2.5 and 7.5 with the minimum break-even quantity being 2.5 batches of widgets.

- (c) The maximum revenue occurs at the vertex of R . Since $R(x) = -\frac{4}{5}x^2 + 10x$, then the x -coordinate of the vertex is

$$x = -\frac{b}{2a} = -\frac{10}{2\left(-\frac{4}{5}\right)} = 6.25.$$

So, the maximum revenue is

$$R(6.25) = 31.25, \text{ or } \$31,250.$$

- (d) The maximum profit is the maximum difference $R(x) - C(x)$. Since

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -\frac{4}{5}x^2 + 10x - (2x + 15) \\ &= -\frac{4}{5}x^2 + 8x - 15 \end{aligned}$$

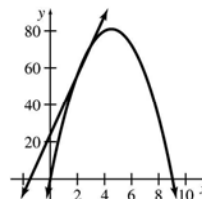
is a quadratic function, we can find the maximum profit by finding the vertex of P . This occurs at

$$x = -\frac{b}{2a} = -\frac{8}{2\left(-\frac{4}{5}\right)} = 5.$$

Therefore, the maximum profit is

$$P(5) = -\frac{4}{5}5^2 + 8(5) - 15 = 5, \text{ or } \$5000.$$

52. (a)



- (b) Break-even quantities are values of x for which revenue equals cost.

Set $R(x) = C(x)$ and solve for x .

$$-4x^2 + 36x = 16x + 24$$

$$4x^2 - 20x + 24 = 0$$

$$4(x - 2)(x - 3) = 0$$

$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 2 \quad \text{or} \quad x = 3$$

So, the break-even quantities are 2 and 3. The minimum break-even quantity is 2 batches of widgets.

- (c) The maximum revenue occurs at the vertex of R . Since $R(x) = -4x^2 + 36x$, then x -coordinate of the vertex is

$$x = -\frac{b}{2a} = \frac{-36}{2(-4)} = \frac{9}{2}.$$

So, the maximum revenue is

$$R(5) = -4\left(\frac{9}{2}\right)^2 + 36\left(\frac{9}{2}\right) = 81, \text{ or } \$81,000.$$

- (d) The maximum profit is the maximum difference $R(x) - C(x)$.

$$\begin{aligned} \text{Since } P(x) &= R(x) - C(x) \\ &= -4x^2 + 36x - (16x + 24) \\ &= -4x^2 + 20x - 24 \end{aligned}$$

is a quadratic function, we can find the maximum profit by finding the vertex of P . This occurs at

$$x = -\frac{b}{2a} = -\frac{20}{2(-4)} = \frac{5}{2} = 2.5.$$

Therefore, the maximum profit is

$$P\left(\frac{5}{2}\right) = -4\left(\frac{5}{2}\right)^2 + 20\left(\frac{5}{2}\right) - 24 = 1, \text{ or } \$1000.$$

$$\begin{aligned} 53. \quad R(x) &= 8000 + 70x - x^2 \\ &= -x^2 + 70x + 8000 \end{aligned}$$

The maximum revenue occurs at the vertex.

$$x = \frac{-b}{2a} = \frac{-70}{2(-1)} = 35$$

$$\begin{aligned} y &= 8000 + 70(35) - (35)^2 \\ &= 8000 + 2450 - 1225 \\ &= 9225 \end{aligned}$$

The vertex is (35, 9225).

The maximum revenue of \$9225 is realized when 35 seats are left unsold.

54. (a) The revenue is

$$R(x) = (\text{Price per ticket}) \cdot (\text{Number of people flying}).$$

$$\text{Number of people flying} = 100 - x$$

$$\text{Price per ticket} = 200 + 4x$$

$$\begin{aligned} R(x) &= (200 + 4x)(100 - x) \\ &= 20,000 + 200x - 4x^2 \end{aligned}$$

(b) $R(x) = -4x^2 + 200x + 20,000$

x -intercepts:

$$0 = (200 + 4x)(100 - x)$$

$$x = -50 \quad \text{or} \quad x = 100$$

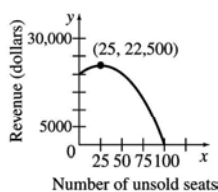
y -intercept:

$$\begin{aligned} y &= -4(0)^2 + 200(0) + 20,000 \\ &= 20,000 \end{aligned}$$

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-200}{-8} = 25$$

$$y = -4(25)^2 + 200(25) + 20,000 = 22,500$$

This is a parabola which opens downward.
The vertex is at (25, 22,500).



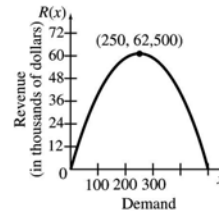
- (c) The maximum revenue occurs at the vertex, (25, 22,500).
This will happen when $x = 25$, or there are 25 unsold seats.
- (d) The maximum revenue is \$22,500, as seen from the graph.

55. $p = 500 - x$

- (a) The revenue is

$$\begin{aligned} R(x) &= px \\ &= (500 - x)(x) \\ &= 500x - x^2. \end{aligned}$$

- (b)



- (c) From the graph, the vertex is halfway between $x = 0$ and $x = 500$, so $x = 250$ units corresponds to maximum revenue. Then the price is

$$\begin{aligned} p &= 500 - x \\ &= 500 - 250 = \$250. \end{aligned}$$

Note that price, p , cannot be read directly from the graph of

$$R(x) = 500x - x^2.$$

(d) $R(x) = 500x - x^2$
 $= -x^2 + 500x$

Find the vertex.

$$x = \frac{-b}{2a} = \frac{-500}{2(-1)} = 250$$

$$\begin{aligned} y &= -(250)^2 + 500(250) \\ &= 62,500 \end{aligned}$$

The vertex is (250, 62,500).

The maximum revenue is \$62,500.

56. Let x = the number of weeks to wait.

- (a) Income per pound (in cents):

$$80 - 4x$$

- (b) Yield in pounds per tree:

$$100 + 5x$$

- (c) Revenue per tree (in cents):

$$R(x) = (100 + 5x)(80 - 4x)$$

$$R(x) = 8000 - 20x^2$$

- (d) Find the vertex.

$$x = \frac{-b}{2a} = \frac{0}{-20} = 0$$

$$y = 8000 - 20(0)^2 = 8000$$

The vertex is (0, 8000).

To produce maximum revenue, wait 0 weeks.

Pick the peaches now.

(e) $R(0) = 8000 - 20(0)^2 = 8000$

or the maximum revenue is 8000 cents per tree or \$80.00 per tree.

57. Let x = the number of \$25 increases.

(a) Rent per apartment: $800 + 25x$

(b) Number of apartments rented: $80 - x$

(c) Revenue:

$$\begin{aligned} R(x) &= (\text{number of apartments rented}) \\ &\quad \times (\text{rent per apartment}) \\ &= (80 - x)(800 + 25x) \\ &= -25x^2 + 1200x + 64,000 \end{aligned}$$

(d) Find the vertex:

$$\begin{aligned} x &= \frac{-b}{2a} = \frac{-1200}{2(-25)} = 24 \\ y &= -25(24)^2 + 1200(24) + 64,000 \\ &= 78,400 \end{aligned}$$

The vertex is (24, 78,400). The maximum revenue occurs when $x = 24$.

(e) The maximum revenue is the y -coordinate of the vertex, or \$78,400.

58. $S(x) = -\frac{1}{4}(x - 10)^2 + 40$ for $0 \leq x \leq 10$

(a) $S(0) = -\frac{1}{4}(0 - 10)^2 + 40$
 $= -25 + 40 = 15$

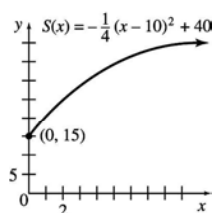
The increase in sales is \$15,000.

(b) For \$10,000, $x = 10$.

$$\begin{aligned} S(10) &= -\frac{1}{4}(10 - 10)^2 + 40 \\ S(10) &= 40 \end{aligned}$$

The increase in sales is \$40,000.

(c) The graph of $S(x) = -\frac{1}{4}(x - 10)^2 + 40$ is the graph of $y = -\frac{1}{4}x^2$ translated 10 units to the right and 40 units upward.



59. $S(x) = 1 - 0.058x - 0.076x^2$

(a) $0.50 = 1 - 0.058x - 0.076x^2$

$$0.076x^2 + 0.058x - 0.50 = 0$$

$$76x^2 + 58x - 500 = 0$$

$$38x^2 + 29x - 250 = 0$$

$$x = \frac{-29 \pm \sqrt{(29)^2 - 4(38)(-250)}}{2(38)}$$

$$= \frac{-29 \pm \sqrt{38,841}}{76}$$

$$\frac{-29 - \sqrt{38,841}}{76} \approx -2.97$$

$$\text{and } \frac{-29 + \sqrt{38,841}}{76} \approx 2.21$$

We ignore the negative value.

The value $x = 2.2$ represents 2.2 decades or 22 years, and 22 years after 65 is 87.

The median length of life is 87 years.

(b) If nobody lives, $S(x) = 0$.

$$1 - 0.058x - 0.076x^2 = 0$$

$$76x^2 + 58x - 1000 = 0$$

$$38x^2 + 29x - 500 = 0$$

$$x = \frac{-29 \pm \sqrt{(29)^2 - 4(38)(-500)}}{2(38)}$$

$$= \frac{-29 \pm \sqrt{76,841}}{76}$$

$$\frac{-29 - \sqrt{76,841}}{76} \approx -4.03$$

$$\text{and } \frac{-29 + \sqrt{76,841}}{76} \approx 3.27$$

We ignore the negative value.

The value $x = 3.3$ represents 3.3 decades or 33 years, and 33 years after 65 is 98.

Virtually nobody lives beyond 98 years.

60. (a) When $t = 14$, $L = 2.024$, so the length of the crown is 2.024 mm at 14 weeks. When $t = 24$, $L = 6.104$, so the length is 6.104 mm at 24 weeks.

(b) The vertex occurs at

$$t = \frac{-b}{2a} = \frac{-0.788}{2(-0.01)} \approx 39.4.$$

When $t = 39.4$, $L \approx 8.48$, so the maximum length is 8.48 mm and occurs 39.4 weeks after conception.

61. (a) The vertex of the quadratic function

$y = 0.057x - 0.001x^2$ is at

$$x = -\frac{b}{2a} = -\frac{0.057}{2(-0.001)} = 28.5.$$

Since the coefficient of the leading term, -0.001 , is negative, then the graph of the function opens downward, so a maximum is reached at 28.5 weeks of gestation.

- (b) The maximum splenic artery resistance reached at the vertex is

$$y = 0.057(28.5) - 0.001(28.5)^2 \approx 0.81.$$

- (c) The splenic artery resistance equals 0, when $y = 0$.

$$0.057x - 0.001x^2 = 0 \quad \text{Substitute in the expression in } x \text{ for } y$$

$$x(0.057 - 0.001x) = 0 \quad \text{Factor}$$

$$x = 0 \text{ or } 0.057 - 0.001x = 0 \quad \text{Set each factor equal to 0.}$$

$$x = \frac{0.057}{0.001} = 57$$

So, the splenic artery resistance equals 0 at 0 weeks or 57 weeks of gestation.

No, this is not reasonable because at $x = 0$ or 57 weeks, the fetus does not exist.

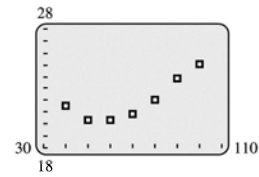
62. The vertex of the quadratic function $y = f(t) = -0.040194t^2 + 2.1493t + 23.921$ is at

$$t = \frac{-b}{2a} = \frac{-2.1493}{2(-0.040194)} \approx 26.7.$$

Since the coefficient of the leading term is negative, the graph of the function opens downward, so the maximum is reached when $t \approx 27$, or about $1975 + 27 = 2002$.

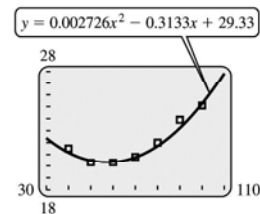
Since the graph opens downward, the function values increase to the left of the maximum point and decrease to the right of the maximum point. So, the incidence rate was increasing during the years 1975–2002 and was decreasing during 2002–2007.

63. (a)



- (b) Quadratic

(c) $y = 0.002726x^2 - 0.3113x + 29.33$



- (d) Given that $(h, k) = (60, 20.3)$, the equation has the form

$$y = a(x - 60)^2 + 20.3.$$

Since $(100, 25.1)$ is also on the curve.

$$25.1 = a(100 - 60)^2 + 20.3$$

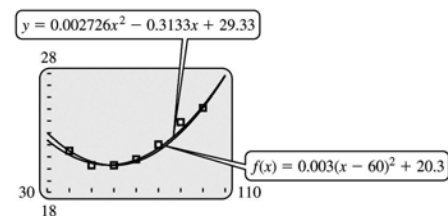
$$4.8 = 1600a$$

$$a = 0.003$$

A quadratic function that models the data is

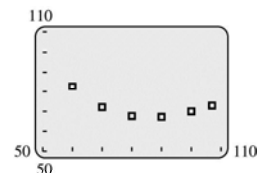
$$f(x) = 0.003(x - 60)^2 + 20.3.$$

- (e)



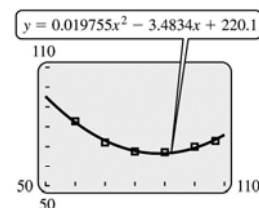
The two graphs are very close.

64. (a)



- (b) Quadratic

(c) $f(x) = 0.019755x^2 - 3.4834x + 220.1$



- (d) Given that $(h, k) = (90, 67.2)$, the equation has the form $y = a(x - 90)^2 + 67.2$.
The point $(60, 82.8)$ is also on the curve.

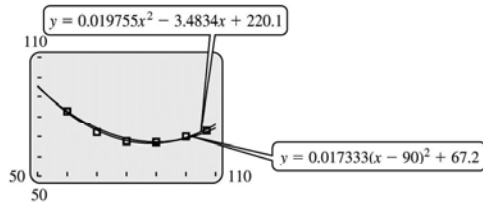
$$82.8 = a(60 - 90)^2 + 67.2$$

$$15.6 = 900a$$

$$a = 0.017333$$

A quadratic function that models the data is $f(x) = 0.017333(x - 90)^2 + 67.2$.

(e)



The two graphs are very close.

(f)

$$f(104) = 0.019755(104)^2 - 3.4834(104) + 220.1$$

$$71.96 \approx 71.5$$

$$f(104) = 0.017333(104 - 90)^2 + 67.2$$

$$= 70.597 \approx 70.6$$

The quadratic regression function that best fits the data is closer to the actual value of 71.7.

65. $f(x) = 60.0 - 2.28x + 0.0232x^2$

$$\frac{-b}{2a} = -\frac{-2.28}{2(0.0232)} \approx 49.1$$

The minimum value occurs when $x \approx 49.1$.
The age at which the accident rate is a minimum is 49 years. The minimum rate is

$$f(49.1) = 60.0 - 2.28(49.1) + 0.0232(49.1)^2$$

$$= 60.0 - 111.948 + 55.930792$$

$$\approx 3.98.$$

66. $h = 32t - 16t^2$

$$= -16t^2 + 32t$$

(a) Find the vertex.

$$x = \frac{-b}{2a} = \frac{-32}{-32} = 1$$

$$y = -16(1)^2 + 32(1)$$

$$= 16$$

The vertex is $(1, 16)$, so the maximum height is 16 ft.

(b) When the object hits the ground, $h = 0$, so

$$32t - 16t^2 = 0$$

$$16t(2 - t) = 0$$

$$t = 0 \text{ or } t = 2.$$

When $t = 0$, the object is thrown upward.

When $t = 2$, the object hits the ground; that is, after 2 sec.

67. $y = 0.056057x^2 + 1.06657x$

(a) If $x = 25$ mph,

$$y = 0.056057(25)^2 + 1.06657(25)$$

$$y \approx 61.7.$$

At 25 mph, the stopping distance is approximately 61.7 ft.

(b) $0.056057x^2 + 1.06657x = 150$

$$0.056057x^2 + 1.06657x - 150 = 0$$

$$x = \frac{-1.06657 \pm \sqrt{(1.06657)^2 - 4(0.056057)(-150)}}{2(0.056057)}$$

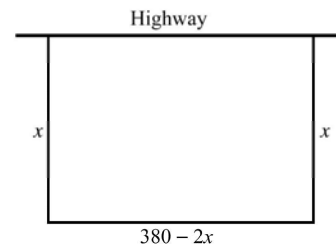
$$x \approx 43.08 \text{ or } x \approx -62.11$$

We ignore the negative value.

To stop within 150 ft, the fastest speed you can drive is 43.08 mph or about 43 mph.

68. Let x = the width.

Then $380 - 2x$ = the length.



$$\text{Area} = x(380 - 2x) = -2x^2 + 380x$$

Find the vertex:

$$x = \frac{-b}{2a} = \frac{-380}{-4} = 95$$

$$y = -2(95)^2 + 380(95) = 18,050$$

The graph of the area function is a parabola with vertex $(95, 18,050)$.

The maximum area of 18,050 sq ft occurs when the width is 95 ft and the length is

$$380 - 2x = 380 - 2(95) = 190 \text{ ft.}$$

69. Let x = the length of the lot and y = the width of the lot.

The perimeter is given by

$$\begin{aligned} P &= 2x + 2y \\ 380 &= 2x + 2y \\ 190 &= x + y \\ 190 - x &= y. \end{aligned}$$

Area = xy (quantity to be maximized)

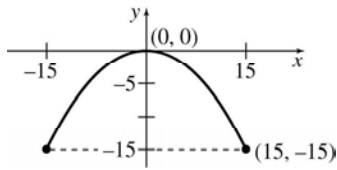
$$\begin{aligned} A &= x(190 - x) \\ &= 190x - x^2 \\ &= -x^2 + 190x \end{aligned}$$

Find the vertex: $\frac{-b}{2a} = \frac{-190}{-2} = 95$

$$\begin{aligned} y &= -(95)^2 + 190(95) \\ &= 9025 \end{aligned}$$

This is a parabola with vertex $(95, 9025)$ that opens downward. The maximum area is the value of A at the vertex, or 9025 sq ft.

70. Draw a sketch of the arch with the vertex at the origin.



Since the arch is a parabola that opens downward, the equation of the parabola is the form $y = a(x - h)^2 + k$, where the vertex $(h, k) = (0, 0)$ and $a < 0$. That is, the equation is of the form $y = ax^2$.

Since the arch is 30 meters wide at the base and 15 meters high, the points $(15, -15)$ and $(-15, -15)$ are on the parabola. Use $(15, -15)$ as one point on the parabola.

$$\begin{aligned} -15 &= a(15)^2 \\ a &= \frac{-15}{15^2} = -\frac{1}{15} \end{aligned}$$

So, the equation is

$$y = -\frac{1}{15}x^2.$$

Ten feet from the ground (the base) is at $y = -5$.

Substitute -5 for y and solve for x .

$$-5 = -\frac{1}{15}x^2$$

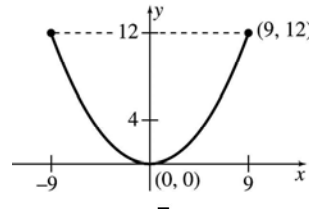
$$x^2 = -5(-15) = 75$$

$$x = \pm\sqrt{75} = \pm 5\sqrt{3}$$

The width of the arch ten feet from the ground is then

$$\begin{aligned} 5\sqrt{3} - (-5\sqrt{3}) &= 10\sqrt{3} \text{ meters} \\ &\approx 17.32 \text{ meters.} \end{aligned}$$

71. Sketch the culvert on the xy -axes as a parabola that opens upward with vertex at $(0, 0)$.



The equation is of the form $y = ax^2$. Since the culvert is 18 ft wide at 12 ft from its vertex, the points $(9, 12)$ and $(-9, 12)$ are on the parabola. Use $(9, 12)$ as one point on the parabola.

$$\begin{aligned} 12 &= a(9)^2 \\ 12 &= 81a \\ \frac{12}{81} &= a \\ \frac{4}{27} &= a \end{aligned}$$

Thus, $y = \frac{4}{27}x^2$.

To find the width 8 feet from the top, find the points with

$$y\text{-value} = 12 - 8 = 4.$$

Thus,

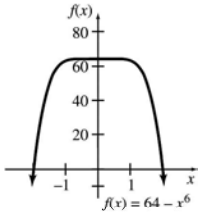
$$\begin{aligned} 4 &= \frac{4}{27}x^2 \\ 108 &= 4x^2 \\ 27 &= x^2 \\ x^2 &= 27 \\ x &= \pm\sqrt{27} \\ x &= \pm 3\sqrt{3}. \end{aligned}$$

The width of the culvert is $2(3\sqrt{3}) = 6\sqrt{3}$ ft ≈ 10.39 ft.

2.3 Polynomial and Rational Functions

Your Turn 1

Graph $f(x) = 64 - x^6$.



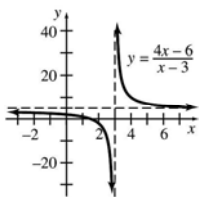
Your Turn 2

Graph $y = \frac{4x-6}{x-3}$.

The value $x = 3$ makes the denominator 0, so 3 is not in the domain and the line $x = 3$ is a vertical asymptote. Let x get larger and larger.

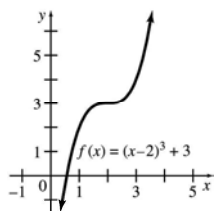
Then $\frac{4x-6}{x-3} \approx \frac{4x}{x} = 4$ as x gets larger and larger.

So, the line $y = 4$ is a horizontal asymptote.

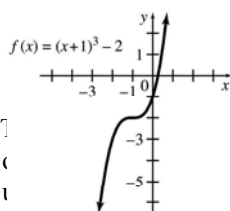


2.3 Exercises

3. The graph of $f(x) = (x-2)^3 + 3$ is the graph of $y = x^3$ translated 2 units to the right and

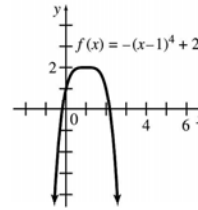


4. The graph of $f(x) = (x+1)^3 - 2$ is the graph of $y = x^3$ translated 1 unit to the left and 2 units downward.



5. The graph of $f(x) = -(x+3)^4 + 1$ is the graph of $y = x^4$ horizontally translated 3 units to the left and 1 unit upward.

6. The graph of $f(x) = -(x-1)^4 + 2$ is the graph of $y = x^4$ reflected in the x -axis, and translated 1 unit to the right and 2 units upward.



7. The graph of $y = x^3 - 7x - 9$ has the right end up, the left end down, at most two turning points, and a y -intercept of -9 .

This is graph D.

8. The graph of $y = -x^3 + 4x^2 + 3x - 8$ has the right end down, the left end up, at most two turning points, and a y -intercept of -8 .

This is graph C.

9. The graph of $y = -x^3 - 4x^2 + x + 6$ has the right end down, the left end up, at most two turning points, and a y -intercept of 6.

This is graph E.

10. The graph of $y = 2x^3 + 4x + 5$ has the right end up, the left end down, at most two turning points, and a y -intercept of 5.

This is graph B.

11. The graph of $y = x^4 - 5x^2 + 7$ has both ends up, at most three turning points, and a y -intercept of 7.

This is graph I.

12. The graph of $y = x^4 + 4x^3 - 20$ has both ends up, at most three turning points, and a y -intercept of -20 .

This is graph F.

13. The graph of $y = -x^4 + 2x^3 + 10x + 15$ has both ends down, at most three turning points, and a y -intercept of 15.

This is graph G.

14. The graph of $y = 0.7x^5 - 2.5x^4 - x^3 + 8x^2 + x + 2$ has the right end up, the left end down, at most four turning points, and a y -intercept of 2.

This is graph H.

15. The graph of $y = -x^5 + 4x^4 + x^3 - 16x^2 + 12x + 5$ has the right end down, the left end up, at most four turning points, and a y -intercept of 5.

This is graph A.

16. The graph of $y = \frac{2x^2 + 3}{x^2 - 1}$ has the lines with equations $x = 1$ and $x = -1$ as vertical asymptotes, the line with equation $y = 2$ as a horizontal asymptote, and a y -intercept of -3 .

This is graph B.

17. The graph of $y = \frac{2x^2 + 3}{x^2 + 1}$ has no vertical asymptote, the line with equation $y = 2$ as a horizontal asymptote, and a y -intercept of 3.

This is graph D.

18. The graph of $y = \frac{-2x^2 - 3}{x^2 - 1}$ has the lines with equations $x = 1$ and $x = -1$ as vertical asymptotes, the line with equation $y = -2$ as a horizontal asymptote, and a y -intercept of 3.

This is graph A.

19. The graph $y = \frac{-2x^2 - 3}{x^2 + 1}$ has no vertical asymptote, the line with equation $y = -2$ as a horizontal asymptote, and a y -intercept of -3 .

This is graph E.

20. The graph of $y = \frac{2x^2 + 3}{x^3 - 1}$ has the line with equation $x = 1$ as a vertical asymptote, the x -axis as a horizontal asymptote, and a y -intercept of -3 .

This is graph C.

21. The right end is up and the left end is up. There are three turning points.

The degree is an even integer equal to 4 or more.

The x^n term has a $+$ sign.

22. The graph has the right end up, the left end down, and four turning points. The degree is an odd integer equal to 5 or more. The x^n term has a $+$ sign.

23. The right end is up and the left end is down. There are four turning points. The degree is an odd integer equal to 5 or more. The x^n term has a $+$ sign.

24. Both ends are down, and there are five turning points. The degree is an even integer equal to 6 or more. The x^n term has a $-$ sign.

25. The right end is down and the left end is up. There are six turning points. The degree is an odd integer equal to 7 or more. The x^n term has a $-$ sign.

26. The right end is up, the left end is down, and there are six turning points. The degree is an odd integer equal to 7 or more. The x^n term has a $+$ sign.

27. $y = \frac{-4}{x + 2}$

The function is undefined for $x = -2$, so the line $x = -2$ is a vertical asymptote.

x	-102	-12	-7	-5	-3	-1	8	98
$x + 2$	-100	-10	-5	-3	-1	1	10	100
y	0.04	0.4	0.8	1.3	4	-4	-0.4	-0.04

The graph approaches $y = 0$, so the line $y = 0$ (the x -axis) is a horizontal asymptote.

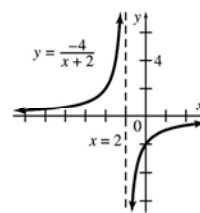
Asymptotes: $y = 0$, $x = -2$

x -intercept:

none

y -intercept:

-2 , the value when $x = 0$



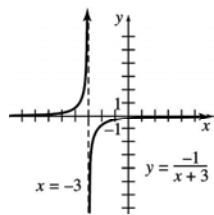
28. $y = \frac{-1}{x + 3}$

A vertical asymptote occurs when $x + 3 = 0$ or when $x = -3$, since this value makes the denominator 0.

x	-6	-5	-4	-2	-1	0
$x + 3$	-3	-2	-1	1	2	3
y	$\frac{1}{3}$	$\frac{1}{2}$	1	-1	$-\frac{1}{2}$	$-\frac{1}{3}$

As $|x|$ gets larger, $\frac{-1}{x+3}$ approaches 0, so $y = 0$ is a horizontal asymptote.

Asymptotes: $y = 0$, $x = -3$



x -intercept:

none

$-\frac{1}{3}$, the value when $x = 0$

29. $y = \frac{2}{3+2x}$

$3+2x = 0$ when $2x = -3$ or $x = -\frac{3}{2}$, so the line $x = -\frac{3}{2}$ is a vertical asymptote.

x	-51.5	-6.5	-2	-1	3.5	48.5
$3+2x$	-100	-10	-1	1	10	100
y	-0.02	-0.2	-2	2	0.2	0.02

The graph approaches $y = 0$, so the line $y = 0$ (the x -axis) is a horizontal asymptote.

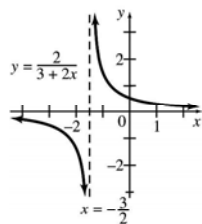
Asymptote: $y = 0$, $x = -\frac{3}{2}$

x -intercept:

none

y -intercept:

$\frac{2}{3}$, the value when $x = 0$



30. $y = \frac{8}{5-3x}$

Undefined for

$$5-3x = 0$$

$$3x = 5$$

$$x = \frac{5}{3}$$

Since $x = \frac{5}{3}$ causes the denominator to equal 0, $x = \frac{5}{3}$ is a vertical asymptote.

x	-1	0	1	2	3	4
$5-3x$	8	5	2	-1	-4	-7
y	0.5	0.8	2	-4	-1	-0.571

The graph approaches $y = 0$, so the line $y = 0$ is a horizontal asymptote.

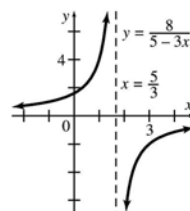
Asymptotes: $y = 0$, $x = \frac{5}{3}$

x -intercept:

none

y -intercept:

$\frac{8}{5}$, the value when $x = 0$



31. $y = \frac{2x}{x-3}$

$x-3 = 0$ when $x = 3$, so the line $x = 3$ is a vertical asymptote.

x	-97	-7	-1	1	2	2.5
$2x$	-194	-14	-2	2	4	5
$x-3$	-100	-10	-4	-2	-1	-0.5
y	1.94	1.4	0.5	-1	-4	-10

x	3.5	4	5	7	11	103
$2x$	7	8	10	14	22	206
$x-3$	0.5	1	2	4	8	100
y	14	8	5	3.5	2.75	2.06

As x gets larger,

$$\frac{2x}{x-3} \approx \frac{2x}{x} = 2.$$

Thus, $y = 2$ is a horizontal asymptote.

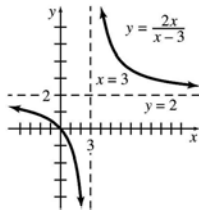
Asymptotes: $y = 2$, $x = 3$

x -intercept:

0, the value when $y = 0$

y -intercept:

0, the value when $x = 0$



32. $y = \frac{4x}{3 - 2x}$

Since $x = \frac{3}{2}$ causes the denominator to equal 0, $x = \frac{3}{2}$ is a vertical asymptote.

x	-3	-2	-1	0	1
$4x$	-12	-8	-4	0	4
$3 - 2x$	9	7	5	3	1
y	-1.33	-1.14	-0.8	0	4

x	2	3	4
$4x$	8	12	16
$3 - 2x$	-1	-3	-5
y	-8	-4	-3.2

As x gets larger,

$$\frac{4x}{3 - 2x} \approx \frac{4x}{-2x} = -2.$$

Thus, the line $y = -2$ is a horizontal asymptote.

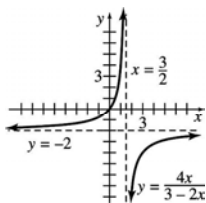
Asymptotes: $y = -2$, $x = \frac{3}{2}$

x -intercept:

0, the value when $y = 0$

y -intercept:

0, the value when $x = 0$



33. $y = \frac{x + 1}{x - 4}$

$x - 4 = 0$ when $x = 4$, so $x = 4$ is a vertical asymptote.

x	-96	-6	-1	0	3
$x + 1$	-95	-5	0	1	4
$x - 4$	-100	-10	-5	-4	-1
y	0.95	0.5	0	-0.25	-4

x	3.5	4.5	5	14	104
$x + 1$	4.5	5.5	6	15	105
$x - 4$	-0.5	0.5	1	10	100
y	-9	11	6	1.5	1.05

As x gets larger,

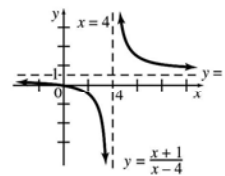
$$\frac{x + 1}{x - 4} \approx \frac{x}{x} = 1.$$

Thus, $y = 1$ is a horizontal asymptote.

Asymptotes: $y = 1$, $x = 4$

x -intercept: -1, the value when $y = 0$

y -intercept: $-\frac{1}{4}$, the value when $x = 0$



34. $y = \frac{x - 4}{x + 1}$

Since $x = -1$ causes the denominator to equal 0, $x = -1$ is a vertical asymptote.

x	-4	-3	-2	0	1
$x - 4$	-8	-7	-6	-4	-3
$x + 1$	-3	-2	-1	1	2
y	2.67	3.5	6	-4	-1.5

x	2	3	4
$x - 4$	-2	-1	0
$x + 1$	3	4	5
y	-0.67	-0.25	0

As x gets larger,

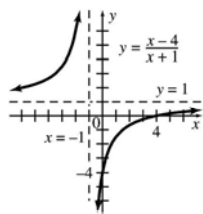
$$\frac{x - 4}{x + 1} \approx \frac{x}{x} = 1.$$

Thus, the line $y = 1$ is a horizontal asymptote.

Asymptotes: $y = 1$, $x = -1$

x -intercept: 4, the value when $y = 0$

y -intercept: -4, the value when $x = 0$



35. $y = \frac{3-2x}{4x+20}$

$4x + 20 = 0$ when $4x = -20$ or $x = -5$, so the line $x = -5$ is a vertical asymptote.

x	-8	-7	-6	-4	-3	-2
$3 - 2x$	-26	-23	-20	-14	-11	-8
$4x + 20$	-12	-8	-4	4	8	12
y	2.17	2.88	5	-3.5	-1.38	-0.67

As x gets larger,

$$\frac{3-2x}{4x+20} \approx \frac{-2x}{4x} = -\frac{1}{2}.$$

Thus, the line $y = -\frac{1}{2}$ is a horizontal asymptote.

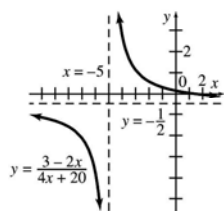
Asymptotes: $x = -5$, $y = -\frac{1}{2}$

x -intercept:

$\frac{3}{2}$, the value when $y = 0$

y -intercept:

$\frac{3}{20}$, the value when $x = 0$



36. $y = \frac{6-3x}{4x+12}$

$4x + 12 = 0$ when $4x = -12$ or $x = -3$, so $x = -3$ is a vertical asymptote.

x	-6	-5	-4	-2	-1	0
$6 - 3x$	24	21	18	12	9	6
$4x + 12$	-12	-8	-4	4	8	12
y	-2	-2.625	-4.5	3	1.125	0.5

As x gets larger,

$$\frac{6-3x}{4x+12} \approx \frac{-3x}{4x} = -\frac{3}{4}.$$

The line $y = -\frac{3}{4}$ is a horizontal asymptote.

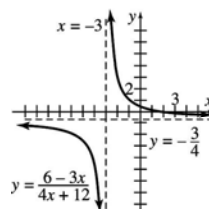
Asymptotes: $y = -\frac{3}{4}$, $x = -3$

x -intercept:

2, the value when $y = 0$

y -intercept:

$\frac{1}{2}$, the value when $x = 0$



37. $y = \frac{-x-4}{3x+6}$

$3x + 6 = 0$ when $3x = -6$ or $x = -2$, so the line $x = -2$ is a vertical asymptote.

x	-5	-4	-3	-1	0	1
$-x - 4$	1	0	-1	-3	-4	-5
$3x + 6$	-9	-6	-3	3	6	9
y	-0.11	0	0.33	-1	-0.67	-0.56

As x gets larger,

$$\frac{-x-4}{3x+6} \approx \frac{-x}{3x} = -\frac{1}{3}.$$

The line $y = -\frac{1}{3}$ is a horizontal asymptote.

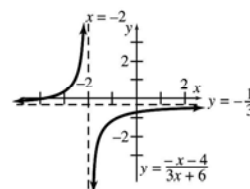
Asymptotes: $y = -\frac{1}{3}$, $x = -2$

x -intercept:

-4, the value when $y = 0$

y -intercept:

$-\frac{2}{3}$, the value when $x = 0$



38. $y = \frac{-2x+5}{x+3}$

$x + 3 = 0$ when $x = -3$, so $x = -3$ is a vertical asymptote.

x	-6	-5	-4	-2	-1	0
$-2x + 5$	17	15	13	9	7	5
$x + 3$	-3	-2	-1	1	2	3
y	-5.67	-7.5	-13	9	3.5	1.67

As x gets larger,

$$\frac{-2x + 5}{x + 3} \approx \frac{-2x}{x} = -2.$$

The line $y = -2$ is a horizontal asymptote.

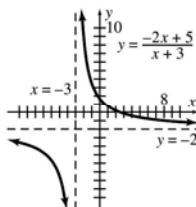
Asymptotes: $y = -2$, $x = -3$

x -intercept:

$\frac{5}{2}$, the value when $y = 0$

y -intercept:

$\frac{5}{2}$, the value when $x = 0$

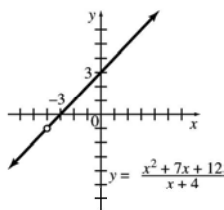


$$\begin{aligned} 39. \quad y &= \frac{x^2 + 7x + 12}{x + 4} \\ &= \frac{(x + 3)(x + 4)}{x + 4} \\ &= x + 3, x \neq -4 \end{aligned}$$

There are no asymptotes, but there is a hole at $x = -4$.

x -intercept: -3 , the value when $y = 0$.

y -intercept: 3 , the value when $x = 0$.

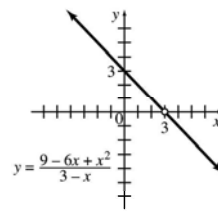


$$\begin{aligned} 40. \quad y &= \frac{9 - 6x + x^2}{3 - x} \\ &= \frac{(3 - x)(3 - x)}{3 - x} \\ &= 3 - x, x \neq 3 \end{aligned}$$

There are no asymptotes, but there is a hole at $x = 3$.

There is no x -intercept, since $3 - x = 0$ implies $x = 3$, but there is a hole at $x = 3$.

y -intercept: 3 , the value when $x = 0$.



41. For a vertical asymptote at $x = 1$, put $x - 1$ in the denominator. For a horizontal asymptote at $y = 2$, the degree of the numerator must equal the degree of the denominator and the quotient of their leading terms must equal 2 . So, $2x$ in the numerator would cause y to approach 2 as x gets larger.

So, one possible answer is $y = \frac{2x}{x - 1}$.

42. For a vertical asymptote at $x = -2$, put $x + 2$ in the denominator. For a horizontal asymptote at $y = 0$, the only condition is that the degree of the numerator is less than the degree of the denominator. If the degree of the denominator is 1 , then put a constant in the numerator to make y approach 0 as x gets larger. So, one possible solution is $y = \frac{3}{x + 2}$.

$$43. \quad f(x) = (x - 1)(x - 2)(x + 3),$$

$$g(x) = x^3 + 2x^2 - x - 2,$$

$$h(x) = 3x^3 + 6x^2 - 3x - 6$$

$$(a) \quad f(1) = (0)(-1)(4) = 0$$

$$(b) \quad f(x) \text{ is zero when } x = 2 \text{ and when } x = -3.$$

$$(c) \quad \begin{aligned} g(-1) &= (-1)^3 + 2(-1)^2 - (-1) - 2 \\ &= -1 + 2 + 1 - 2 = 0 \end{aligned}$$

$$\begin{aligned} g(1) &= (1)^3 + 2(1)^2 - (1) - 2 \\ &= 1 + 2 - 1 - 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(-2) &= (-2)^3 + 2(-2)^2 - (-2) - 2 \\ &= -8 + 8 + 2 - 2 \\ &= 0 \end{aligned}$$

$$(d) \quad \begin{aligned} g(x) &= [x - (-1)](x - 1)[x - (-2)] \\ g(x) &= (x + 1)(x - 1)(x + 2) \end{aligned}$$

$$(e) \quad \begin{aligned} h(x) &= 3g(x) \\ &= 3(x + 1)(x - 1)(x + 2) \end{aligned}$$

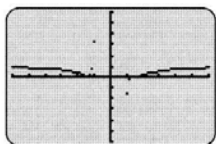
- (f) If f is a polynomial and $f(a) = 0$ for some number a , then one factor of the polynomial is $x - a$.

44. Graph

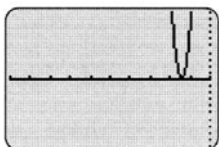
$$f(x) = \frac{x^7 - 4x^5 - 3x^4 + 4x^3 + 12x^2 - 12}{x^7}$$

using a graphing calculator with the indicated viewing windows.

- (a) There appear to be two x -intercepts, one at $x = -1.4$ and one at $x = 1.4$.

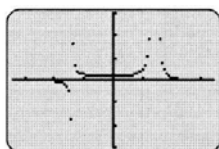


- (b) There appear to be three x -intercepts, one at $x = -1.414$, one at $x = 1.414$, and one at $x = 1.442$.

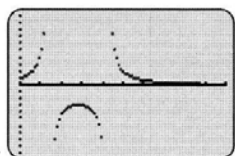
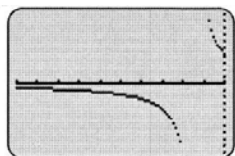


45. $f(x) = \frac{1}{x^5 - 2x^3 - 3x^2 + 6}$

- (a) Two vertical asymptotes appear, one at $x = -1.4$ and one at $x = 1.4$.



- (b) Three vertical asymptotes appear, one at $x = -1.414$, one at $x = 1.414$, and one at $x = 1.442$.



46. $\bar{C}(x) = \frac{600}{x + 20}$

(a) $\bar{C}(10) = \frac{600}{10 + 20}$
 $= \frac{600}{30} = \$20$

$\bar{C}(20) = \frac{600}{20 + 20}$
 $= \frac{600}{40} = \$15$

$\bar{C}(50) = \frac{600}{50 + 20}$
 $= \frac{600}{70} \approx \8.57

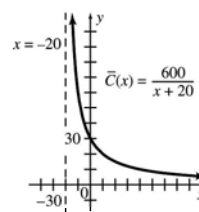
$\bar{C}(75) = \frac{600}{75 + 20}$
 $= \frac{600}{95} \approx \6.32

$\bar{C}(100) = \frac{600}{100 + 20}$
 $= \frac{600}{120} = \$5$

- (b) $(0, \infty)$ would be a more reasonable domain for average cost than $[0, \infty)$. If zero were included in the domain, there would be no units produced. It is not reasonable to discuss the average cost per unit of zero units.

- (c) The graph has a vertical asymptote at $x = -20$, a horizontal asymptote at $y = 0$ (the x -axis), and y -intercept $\frac{600}{20} \approx 30$.

- (d)



$$47. \quad \bar{C}(x) = \frac{220,000}{x + 475}$$

(a) If $x = 25$,

$$\bar{C}(25) = \frac{220,000}{25 + 475} = \frac{220,000}{500} = \$440.$$

If $x = 50$,

$$\bar{C}(50) = \frac{220,000}{50 + 475} = \frac{220,000}{525} \approx \$419.$$

If $x = 100$,

$$\bar{C}(100) = \frac{220,000}{100 + 475} = \frac{220,000}{575} \approx \$383.$$

If $x = 200$,

$$\bar{C}(200) = \frac{220,000}{200 + 475} = \frac{220,000}{675} \approx \$326.$$

If $x = 300$,

$$\bar{C}(300) = \frac{220,000}{300 + 475} = \frac{220,000}{775} \approx \$284.$$

If $x = 400$,

$$\bar{C}(400) = \frac{220,000}{400 + 475} = \frac{220,000}{875} \approx \$251.$$

(b) A vertical asymptote occurs when the denominator is 0.

$$x + 475 = 0$$

$$x = -475$$

A horizontal asymptote occurs when $\bar{C}(x)$ approaches a value as x gets larger. In this case, $\bar{C}(x)$ approaches 0.

The asymptotes are $x = -475$ and $y = 0$.

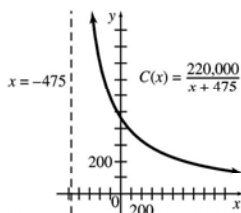
(c) x -intercepts:

$$0 = \frac{220,000}{x + 475}; \text{ no such } x, \text{ so no } x\text{-intercepts}$$

y -intercepts:

$$\bar{C}(0) = \frac{220,000}{0 + 475} \approx 463.2$$

(d) Use the following ordered pairs: (25,440), (50,419), (100,383), (200,326), (300,284), (400,251).



$$48. \quad y = x(100 - x)(x^2 + 500),$$

x = tax rate;

y = tax revenue in hundreds of thousands of dollars.

(a) $x = 10$

$$\begin{aligned} y &= 10(100 - 10)(10^2 + 500) \\ &= 10(90)(600) \\ &= \$54 \text{ billion} \end{aligned}$$

(b) $x = 40$

$$\begin{aligned} y &= 40(100 - 40)(40^2 + 500) \\ &= 40(60)(2100) \\ &= \$504 \text{ billion} \end{aligned}$$

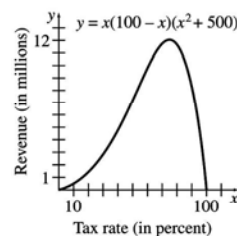
(c) $x = 50$

$$\begin{aligned} y &= 50(100 - 50)(50^2 + 500) \\ &= 50(50)(3000) \\ &= \$750 \text{ billion} \end{aligned}$$

(d) $x = 80$

$$\begin{aligned} y &= 80(100 - 80)(80^2 + 500) \\ &= 80(20)(6900) \\ &= \$1104 \text{ billion} \end{aligned}$$

(e)



49. Quadratic functions with roots at $x = 0$ and $x = 100$ are of the form $f(x) = ax(100 - x)$.

$f_1(x)$ has a maximum of 100, which occurs at the vertex. The x -coordinate of the vertex lies between the two roots.

The vertex is (50, 100).

$$100 = a(50)(100 - 50)$$

$$100 = a(50)(50)$$

$$\frac{100}{2500} = a$$

$$\frac{1}{25} = a$$

$$f_1(x) = \frac{1}{25}x(100 - x) \text{ or } \frac{x(100 - x)}{25}$$

$f_2(x)$ has a maximum of 250, occurring at (50, 250).

$$250 = a(50)(100 - 50)$$

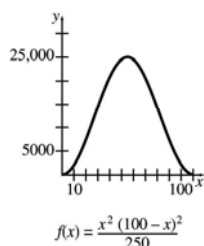
$$250 = a(50)(50)$$

$$\frac{250}{2500} = a$$

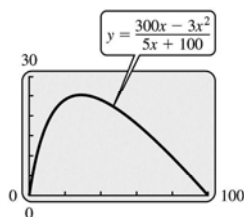
$$\frac{1}{10} = a$$

$$f_2(x) = \frac{1}{10}x(100 - x) \text{ or } \frac{x(100 - x)}{10}$$

$$\begin{aligned} f_1(x) \cdot f_2(x) &= \left[\frac{x(100 - x)}{25} \right] \cdot \left[\frac{x(100 - x)}{10} \right] \\ &= \frac{x^2(100 - x)^2}{250} \end{aligned}$$



50. (a)



(b) The tax rate for maximum revenue is 29.0%.
The maximum revenue is \$25.2 million.

51. $y = \frac{6.7x}{100 - x},$

Let x = percent of pollutant;
 y = cost in thousands.

(a) $x = 50$

$$y = \frac{6.7(50)}{100 - 50} = 6.7$$

The cost is \$6700.

$$x = 70$$

$$y = \frac{6.7(70)}{100 - 70} \approx 15.6$$

The cost is \$15,600.

$$x = 80$$

$$y = \frac{6.7(80)}{100 - 80} = 26.8$$

The cost is \$26,800.

$$x = 90$$

$$y = \frac{6.7(90)}{100 - 90} = 60.3$$

The cost is \$60,300.

$$x = 95$$

$$y = \frac{6.7(95)}{100 - 95}$$

The cost is \$127,300.

$$x = 98$$

$$y = \frac{6.7(98)}{100 - 98} = 328.3$$

The cost is \$328,300.

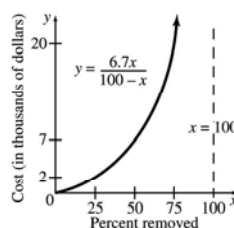
$$x = 99$$

$$y = \frac{6.7(99)}{100 - 99} = 668.3$$

The cost is \$668,300.

(b) No, because $x = 100$ makes the denominator zero, so $x = 100$ is a vertical asymptote.

(c)



52. $y = \frac{6.5x}{102 - x}$

y = percent of pollutant;

x = cost in thousands of dollars.

(a) $x = 0$

$$y = \frac{6.5(0)}{102 - 0} = \frac{0}{102} = \$0$$

$$x = 50$$

$$y = \frac{6.5(50)}{102 - 50} = \frac{325}{52} = 6.25$$

$$6.25(1000) = \$6250$$

$$x = 80$$

$$y = \frac{6.5(80)}{102 - 80} = \frac{520}{22} = 23.636$$

$$\begin{aligned} (23.636)(1000) &= \$23,636 \\ &\approx \$23,600 \end{aligned}$$

$$x = 90$$

$$y = \frac{6.5(90)}{102 - 90} = \frac{585}{12} = 48.75$$

$$(48.75)(1000) = \$48,750$$

$$\approx \$48,800$$

$$x = 95$$

$$y = \frac{6.5(95)}{102 - 95} = \frac{617.5}{7} = 88.214$$

$$(88.214)(1000) = 88,214$$

$$\approx \$88,200$$

$$x = 99$$

$$y = \frac{6.5(99)}{102 - 99} = \frac{643.5}{3} = 214.5$$

$$(214.500)(1000) = 214,500$$

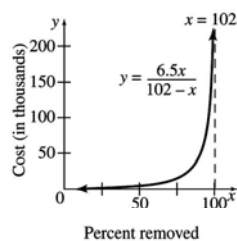
$$\approx \$214,500$$

$$x = 100$$

$$y = \frac{6.5(100)}{102 - 100} = \frac{650}{2} = 325$$

$$(325)(1000) = \$325,000$$

(b)



53. (a) $a = \frac{k}{d}$

$$k = ad$$

d	a	$k = ad$
36.000	9.37	337.32
36.125	9.34	337.4075
36.250	9.31	337.4875
36.375	9.27	337.19625
36.500	9.24	337.26
36.625	9.21	337.31625
36.750	9.18	337.365
36.875	9.15	337.40625
37.000	9.12	337.44

We find the average of the nine values of k by adding them and dividing by 9. This gives 337.35, or, rounding to the nearest integer, $k = 337$. Therefore,

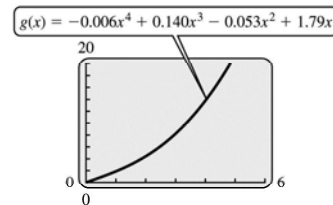
$$a = \frac{337}{d}.$$

(b) When $d = 40.50$,

$$a = \frac{337}{40.50} \approx 8.32.$$

The strength for 40.50 diopter lenses is 8.32 mm of arc.

54. (a)

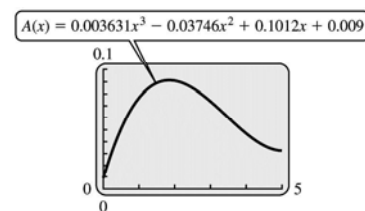


(b) Because the leading coefficient is negative and the degree of the polynomial is even, the graph will have right end down, so it cannot keep increasing forever.

55. $A(x) = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$

(a)

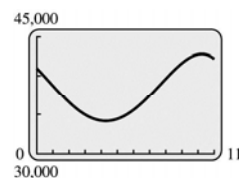
x	0	1	2	3	4	5
$A(x)$	0.009	0.076	0.091	0.073	0.047	0.032



(b) The peak of the curve comes at about $x = 2$ hours.

(c) The curve rises to a y -value of 0.08 at about $x = 1.1$ hours and stays at or above that level until about $x = 2.7$ hours.

56. (a)



(b) The minimum number of applicants is at about $t = 4.18$, and the maximum number of applicants is at about $t = 10.24$. So, the number of medical school applicants increased during $1998 + 5$ to $1998 + 10$, or 2003 to 2008.

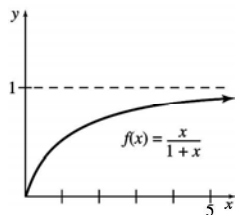
57. $f(x) = \frac{\lambda x}{1 + (ax)^b}$

- (a) A reasonable domain for the function is $[0, \infty)$.

Populations are not measured using negative numbers and they may get extremely large.

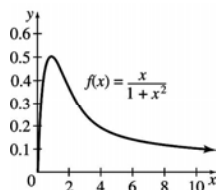
- (b) If $\lambda = a = b = 1$, the function becomes

$$f(x) = \frac{x}{1 + x^2}.$$



- (c) If $\lambda = a = 1$ and $b = 2$, the function becomes

$$f(x) = \frac{x}{1 + x^2}.$$



- (d) As seen from the graphs, when b increases, the population of the next generation, $f(x)$, gets smaller when the current generation, x , is larger.

58. (a) A reasonable domain for the function is $[0, \infty)$. Populations are not measured using negative numbers, and they may get extremely large.

(b) $f(x) = \frac{Kx}{A + x}$

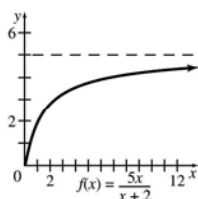
When $K = 5$ and $A = 2$,

$$f(x) = \frac{5x}{2 + x}.$$

The graph has a horizontal asymptote at $y = 5$ since

$$\frac{5x}{2 + x} \approx \frac{5x}{x} = 5$$

as x gets larger.



(c) $f(x) = \frac{Kx}{A + x}$

As x gets larger,

$$\frac{Kx}{A + x} \approx \frac{Kx}{x} = K.$$

Thus, $y = K$ will always be a horizontal asymptote for this function.

- (d) K represents the maximum growth rate. The function approaches this value asymptotically, showing that although the growth rate can get very close to K , it can never reach the maximum, K .

(e) $f(x) = \frac{Kx}{A + x}$

Let $A = x$, the quantity of food present.

$$f(x) = \frac{KA}{A + A} = \frac{KA}{2A} = \frac{K}{2}$$

K is the maximum growth rate, so $\frac{K}{2}$ is half the maximum. Thus, A represents the quantity of food for which the growth rate is half of its maximum.

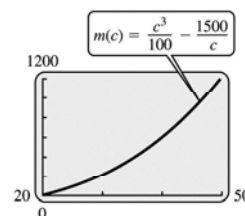
59. (a) When $c = 30$, $w = \frac{30^3}{100} - \frac{1500}{30} = 220$,

so the brain weighs 220 g when its circumference measures 30 cm. When

$c = 40$, $w = \frac{40^3}{100} - \frac{1500}{40} = 602.5$, so the brain weighs 602.5 g when its circumference is 40 cm. When $c = 50$, $w = \frac{50^3}{100} - \frac{1500}{50} = 1220$, so the brain weighs 1220 g when its circumference is 50 cm.

- (b) Set the window of a graphing calculator so you can trace to the positive x -intercept of the function. Using a “root” or “zero” program, this x -intercept is found to be approximately 19.68. Notice in the graph that positive c values less than 19.68 correspond to negative w values. Therefore, the answer is $c < 19.68$.

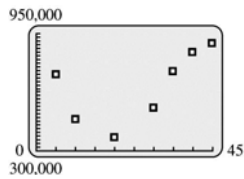
(c)



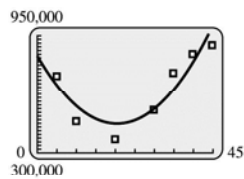
- (d) One method is to graph the line $y = 700$ on the graph found in part (c) and use an “intercept” program to find the point of intersection of the two graphs. This point has the approximate

coordinates (41.9, 700). Therefore, an infant has a brain weighing 700 g when the circumference measures 41.9 cm.

60. (a)

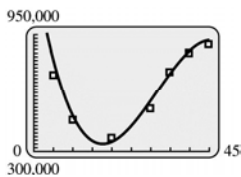


(b) $y = 890.37x^2 - 36,370x + 830,144$



(c)

$$y = -52.954x^3 + 5017.88x^2 - 127,714x + 1,322,606$$



61. (a) Using a graphing calculator with the given data, for the 4.0-foot pendulum and for $n = 1$,

$$4.0 = k(2.22)^1$$

$$k = 1.80;$$

$n = 2$,

$$4.0 = k(2.22)^2$$

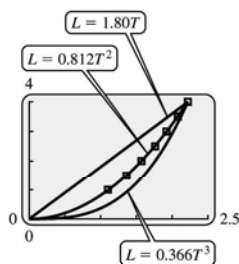
$$k = 0.812;$$

$n = 3$

$$4.0 = k(2.22)^3$$

$$k = 0.366.$$

(b)



From the graphs, $L = 0.812T^2$ is the best fit.

(c) $5.0 = 0.812T^2$

$$T^2 = \frac{5.0}{0.812}$$

$$T = \sqrt{\frac{5.0}{0.812}}$$

$$T \approx 2.48$$

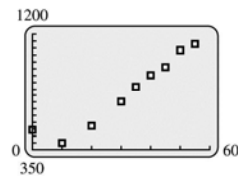
The period will be 2.48 seconds.

(d) $L = 0.812T^2$

If L doubles, T^2 doubles, and T increases by factor of $\sqrt{2}$.

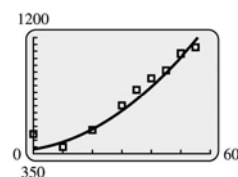
(e) $L \approx 0.822T^2$ which is very close to $L = 0.812T^2$.

62. (a)



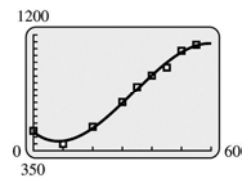
(b) $f(x) = 0.19327x^2 + 3.0039x + 431.30$

(c)



(d) $f(x) = -0.010883x^3 + 1.1079x^2 - 16.432x + 485.45$

(e)



(f) The function in part (d) appears to be a better fit to the data.

2.4 Exponential Functions

Your Turn 1

$$25^{x/2} = 125^{x+3}$$

$$(5^2)^{x/2} = (5^3)^{x+3}$$

$$5^x = 5^{3x+9}$$

$$x = 3x + 9$$

$$2x = -9$$

$$x = -\frac{9}{2}$$

Your Turn 2

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

$$= 4400\left(1 + \frac{0.0325}{4}\right)^{5(4)}$$

$$= 5172.97$$

The interest amounts to $5172.97 - 4400 = \$772.97$.

Your Turn 3

$$A = Pe^{rt}$$

$$= 800e^{4(0.0315)}$$

$$= 907.43$$

The amount after 4 years will be \$907.43.

2.4 Exercises

1.

number of folds	1	2	3	4	5 ...	10 ...	50
layers of paper	2	4	8	16	32 ...	1024 ...	2^{50}

$$2^{50} = 1.125899907 \times 10^{15}$$

2. 500 sheets are 2 inches high

$$\frac{500}{2 \text{ in.}} = \frac{2^{50}}{x \text{ in.}}$$

$$x = \frac{2 \cdot 2^{50}}{500}$$

$$= 4.503599627 \times 10^{12} \text{ in.}$$

$$= 71,079,539.57 \text{ mi}$$

3. The graph of $y = 3^x$ is the graph of an exponential function $y = a^x$ with $a > 1$.

This is graph E.

4. The graph of $y = 3^{-x}$ is the graph of $y = 3^x$ reflected in the y -axis.

This is graph D.

5. The graph of $y = \left(\frac{1}{3}\right)^{1-x}$ is the graph of $y = (3^{-1})^{1-x}$ or $y = 3^{x-1}$. This is the graph of $y = 3^x$ translated 1 unit to the right.

This is graph C.

6. The graph of $y = 3^{x+1}$ is the graph of $y = 3^x$ translated 1 unit to the left.

This is graph F.

7. The graph of $y = 3(3)^x$ is the same as the graph of $y = 3^{x+1}$. This is the graph of $y = 3^x$ translated 1 unit to the left.

This is graph F.

8. The graph of $y = \left(\frac{1}{3}\right)^x$ is the graph of $y = (3^{-1})^x = 3^{-x}$. This is the graph of $y = 3^x$ reflected in the y -axis.

This is graph D.

9. The graph of $y = 2 - 3^{-x}$ is the same as the graph of $y = -3^{-x} + 2$. This is the graph of $y = 3^x$ reflected in the x -axis, reflected in the y -axis, and translated up 2 units.

This is graph A.

10. The graph of $y = -2 + 3^{-x}$ is the same as the graph of $y = 3^{-x} - 2$. This is the graph of $y = 3^x$ reflected in the y -axis and translated 2 units downward.

This is graph B.

11. The graph of $y = 3^{x-1}$ is the graph of $y = 3^x$ translated 1 unit to the right.

This is graph C.

$$\begin{aligned} 13. \quad 2^x &= 32 \\ 2^x &= 2^5 \\ x &= 5 \end{aligned}$$

$$\begin{aligned} 14. \quad 4^x &= 64 \\ 4^x &= 4^3 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} 15. \quad 3^x &= \frac{1}{81} \\ 3^x &= \frac{1}{3^4} \\ 3^x &= 3^{-4} \\ x &= -4 \end{aligned}$$

$$\begin{aligned} 16. \quad e^x &= \frac{1}{e^5} \\ e^x &= e^{-5} \\ x &= -5 \end{aligned}$$

$$\begin{aligned} 17. \quad 4^x &= 8^{x+1} \\ (2^2)^x &= (2^3)^{x+1} \\ 2^{2x} &= 2^{3x+3} \\ 2x &= 3x + 3 \\ -x &= 3 \\ x &= -3 \end{aligned}$$

$$\begin{aligned} 18. \quad 25^x &= 125^{x+2} \\ (5^2)^x &= (5^3)^{x+2} \\ 5^{2x} &= 5^{3x+6} \\ 2x &= 3x + 6 \\ -6 &= x \end{aligned}$$

$$\begin{aligned} 19. \quad 16^{x+3} &= 64^{2x-5} \\ (2^4)^{x+3} &= (2^6)^{2x-5} \\ 2^{4x+12} &= 2^{12x-30} \\ 4x + 12 &= 12x - 30 \\ 42 &= 8x \\ \frac{21}{4} &= x \end{aligned}$$

$$\begin{aligned} 20. \quad (e^3)^{-2x} &= e^{-x+5} \\ e^{-6x} &= e^{-x+5} \\ -6x &= -x + 5 \\ -5x &= 5 \\ x &= -1 \end{aligned}$$

$$\begin{aligned} 21. \quad e^{-x} &= (e^4)^{x+3} \\ e^{-x} &= e^{4x+12} \\ -x &= 4x + 12 \\ -5x &= 12 \\ x &= -\frac{12}{5} \end{aligned}$$

$$\begin{aligned} 22. \quad 2^{|x|} &= 8 \\ 2^{|x|} &= 2^3 \\ |x| &= 3 \\ x &= 3 \text{ or } x = -3 \end{aligned}$$

$$\begin{aligned} 23. \quad 5^{-|x|} &= \frac{1}{25} \\ 5^{-|x|} &= 5^{-2} \\ |x| &= 2 \\ x &= 2 \text{ or } x = -2 \end{aligned}$$

$$\begin{aligned} 24. \quad 2x^{2-4x} &= \left(\frac{1}{16}\right)^{x-4} \\ 2x^{2-4x} &= (2^{-4})^{x-4} \\ 2x^{2-4x} &= 2^{-4x+16} \\ x^2 - 4x &= -4x + 16 \\ x^2 - 16 &= 0 \\ (x + 4)(x - 4) &= 0 \\ x &= -4 \text{ or } x = 4 \end{aligned}$$

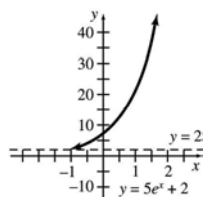
$$\begin{aligned} 25. \quad 5^{x^2+x} &= 1 \\ 5^{x^2+x} &= 5^0 \\ x^2 + x &= 0 \\ x(x + 1) &= 0 \\ x = 0 \text{ or } x + 1 &= 0 \\ x = 0 \text{ or } x &= -1 \end{aligned}$$

$$\begin{aligned}
 26. \quad & 8x^2 = 2^{x+4} \\
 & (2^3)^{x^2} = 2^{x+4} \\
 & 2^{3x^2} = 2^{x+4} \\
 & 3x^2 = x + 4 \\
 & 3x^2 - x - 4 = 0 \\
 & (3x - 4)(x + 1) = 0 \\
 & x = \frac{4}{3} \quad \text{or} \quad x = -1
 \end{aligned}$$

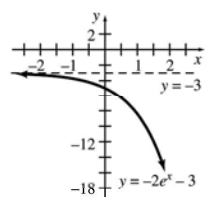
$$\begin{aligned}
 27. \quad & 27^x = 9^{x^2+x} \\
 & (3^3)^x = (3^2)^{x^2+x} \\
 & 3^{3x} = 3^{2x^2+2x} \\
 & 3x = 2x^2 + 2x \\
 & 0 = 2x^2 - x \\
 & 0 = x(2x - 1) \\
 & x = 0 \quad \text{or} \quad 2x - 1 = 0 \\
 & x = 0 \quad \text{or} \quad x = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & e^{x^2+5x+6} = 1 \\
 & e^{x^2+5x+6} = e^0 \\
 & x^2 + 5x + 6 = 0 \\
 & (x + 3)(x + 2) = 0 \\
 & x + 3 = 0 \quad \text{or} \quad x + 2 = 0 \\
 & x = -3 \quad \text{or} \quad x = -2
 \end{aligned}$$

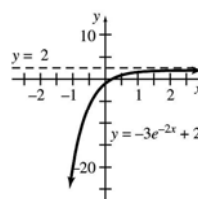
29. Graph of $y = 5e^x + 2$



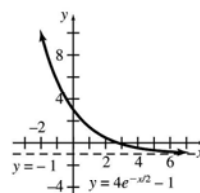
30. Graph of $y = -2e^x - 3$



31. Graph of $y = -3e^{-2x} + 2$

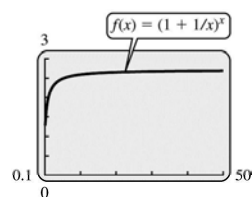


32. Graph of $y = 4e^{-x/2} - 1$



34. 4 and 6 cannot be easily written as powers of the same base, so the equation $4^x = 6$ cannot be solved using this approach.

36.



$f(x)$ approaches $e \approx 2.71828$.

37. $A = P\left(1 + \frac{r}{m}\right)^{tm}$, $P = 10,000$, $r = 0.04$, $t = 5$

(a) annually, $m = 1$

$$\begin{aligned}
 A &= 10,000\left(1 + \frac{0.04}{1}\right)^{5(1)} \\
 &= 10,000(1.04)^5 \\
 &= \$12,166.53 \\
 \text{Interest} &= \$12,166.53 - \$10,000 \\
 &= \$2,166.53
 \end{aligned}$$

(b) semiannually, $m = 2$

$$\begin{aligned}
 A &= 10,000\left(1 + \frac{0.04}{2}\right)^{5(2)} \\
 &= 10,000(1.02)^{10} \\
 &= \$12,189.94 \\
 \text{Interest} &= \$12,189.94 - \$10,000 \\
 &= \$2,189.94
 \end{aligned}$$

(c) quarterly, $m = 4$

$$\begin{aligned}
 A &= 10,000 \left(1 + \frac{0.04}{4} \right)^{5(4)} \\
 &= 10,000(1.01)^{20} \\
 &= \$12,201.90 \\
 \text{Interest} &= \$12,201.90 - \$10,000 \\
 &= \$2201.90
 \end{aligned}$$

(d) monthly, $m = 12$

$$\begin{aligned}
 A &= 10,000 \left(1 + \frac{0.04}{12} \right)^{5(12)} \\
 &= 10,000(1.00\bar{3})^{60} \\
 &= \$12,209.97 \\
 \text{Interest} &= \$12,209.97 - \$10,000 \\
 &= \$2209.97
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad A &= 10,000e^{(0.04)(5)} = \$12,214.03 \\
 \text{Interest} &= \$12,214.03 - \$10,000 \\
 &= \$2214.03
 \end{aligned}$$

$$\begin{aligned}
 38. \quad A &= P \left(1 + \frac{r}{m} \right)^{tm}, \quad P = 26,000, \quad r = 0.06, \\
 t &= 4
 \end{aligned}$$

(a) annually, $m = 1$

$$\begin{aligned}
 A &= 26,000 \left(1 + \frac{0.06}{1} \right)^{4(1)} \\
 &= 26,000(1.06)^4 \\
 &= \$32,824.40 \\
 \text{Interest} &= \$32,824.40 - \$26,000 \\
 &= \$6824.40
 \end{aligned}$$

(b) semiannually, $m = 2$

$$\begin{aligned}
 A &= 26,000 \left(1 + \frac{0.06}{2} \right)^{4(2)} \\
 &= 26,000(1.03)^8 \\
 &= \$32,936.02 \\
 \text{Interest} &= \$32,936.02 - \$26,000 \\
 &= \$6936.02
 \end{aligned}$$

(c) quarterly, $m = 4$

$$\begin{aligned}
 A &= 26,000 \left(1 + \frac{0.06}{4} \right)^{4(4)} \\
 &= 26,000(1.015)^{16} \\
 &= \$32,993.62 \\
 \text{Interest} &= \$32,993.62 - \$26,000 \\
 &= \$6993.62
 \end{aligned}$$

(d) monthly, $m = 12$

$$\begin{aligned}
 A &= 26,000 \left(1 + \frac{0.06}{12} \right)^{4(12)} \\
 &= 26,000(1.005)^{48} \\
 &= \$33,032.72 \\
 \text{Interest} &= \$33,032.72 - \$26,000 \\
 &= \$7032.72
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad A &= 26,000e^{(0.06)(4)} \\
 &= \$33,052.48 \\
 \text{Interest} &= \$33,052.48 - \$26,000 \\
 &= \$7052.48
 \end{aligned}$$

39. For 6% compounded annually for 2 years,

$$\begin{aligned}
 A &= 18,000(1 + 0.06)^2 \\
 &= 18,000(1.06)^2 \\
 &= 20,224.80
 \end{aligned}$$

For 5.9% compounded monthly for 2 years,

$$\begin{aligned}
 A &= 18,000 \left(1 + \frac{0.059}{12} \right)^{12(2)} \\
 &= 18,000 \left(\frac{12.059}{12} \right)^{24} \\
 &= 20,248.54
 \end{aligned}$$

The 5.9% investment is better. The additional interest is

$$\$20,248.54 - \$20,224.80 = \$23.74.$$

$$\begin{aligned}
 40. \quad A &= P \left(1 + \frac{r}{m} \right)^{tm}, \quad P = 5000, \quad A = \$7500, \\
 t &= 5
 \end{aligned}$$

(a) $m = 1$

$$\begin{aligned}
 7500 &= 5000 \left(1 + \frac{r}{1} \right)^{5(1)} \\
 \frac{3}{2} &= (1 + r)^5 \\
 \left(\frac{3}{2} \right)^{1/5} - 1 &= r \\
 0.084 &\approx r
 \end{aligned}$$

The interest rate is about 8.4%.

(b) $m = 4$

$$7500 = 5000 \left(1 + \frac{r}{4} \right)^{5(4)}$$

$$\frac{3}{2} = \left(1 + \frac{r}{4} \right)^{20}$$

$$\left(\frac{3}{2} \right)^{1/20} - 1 = \frac{r}{4}$$

$$4 \left[\left(\frac{3}{2} \right)^{1/20} - 1 \right] \approx r$$

$$0.082 = r$$

The interest rate is about 8.2%

41. $A = Pe^{rt}$

(a) $r = 3\%$

$$A = 10e^{0.03(3)} = \$10.94$$

(b) $r = 4\%$

$$A = 10e^{0.04(3)} = \$11.27$$

(c) $r = 5\%$

$$A = 10e^{0.05(3)} = \$11.62$$

42. $P = \$25,000$, $r = 5.5\%$

Use the formula for continuous compounding,

$$A = Pe^{rt}.$$

(a) $t = 1$

$$\begin{aligned} A &= 25,000e^{0.055(1)} \\ &= \$26,413.52 \end{aligned}$$

(b) $t = 5$

$$\begin{aligned} A &= 25,000e^{0.055(5)} \\ &= \$32,913.27 \end{aligned}$$

(c) $t = 10$

$$\begin{aligned} A &= 25,000e^{0.055(10)} \\ &= \$43,331.33 \end{aligned}$$

43. $1200 = 500 \left(1 + \frac{r}{4} \right)^{(14)(4)}$

$$\frac{1200}{500} = \left(1 + \frac{r}{4} \right)^{56}$$

$$2.4 = \left(1 + \frac{r}{4} \right)^{56}$$

$$1 + \frac{r}{4} = (2.4)^{1/56}$$

$$4 + r = 4(2.4)^{1/56}$$

$$r = 4(2.4)^{1/56} - 4$$

$$r \approx 0.0630$$

The required interest rate is 6.30%.

44. (a) $30,000 = 10,500 \left(1 + \frac{r}{4} \right)^{(4)(12)}$

$$\frac{300}{105} = \left(1 + \frac{r}{4} \right)^{48}$$

$$1 + \frac{r}{4} = \left(\frac{300}{105} \right)^{1/48}$$

$$4 + r = 4 \left(\frac{300}{105} \right)^{1/48}$$

$$r = 4 \left(\frac{300}{105} \right)^{1/48} - 4$$

$$r \approx 0.0884$$

The required interest rate is 8.84%.

(b) $30,000 = 10,500e^{12r}$

$$\frac{300}{105} = e^{12r}$$

$$12r = \ln \left(\frac{300}{105} \right)$$

$$r = \frac{\ln \left(\frac{300}{105} \right)}{12} \approx 0.0875$$

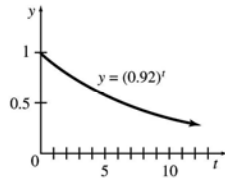
The required interest rate is 8.75%.

45. $y = (0.92)^t$

(a)

t	y
0	$(0.92)^0 = 1$
1	$(0.92)^1 = 0.92$
2	$(0.92)^2 \approx 0.85$
3	$(0.92)^3 \approx 0.78$
4	$(0.92)^4 \approx 0.72$
5	$(0.92)^5 \approx 0.66$
6	$(0.92)^6 \approx 0.61$
7	$(0.92)^7 \approx 0.56$
8	$(0.92)^8 \approx 0.51$
9	$(0.92)^9 \approx 0.47$
10	$(0.92)^{10} \approx 0.43$

(b)

(c) Let x = the cost of the house in 10 years.

$$\text{Then, } 0.43x = 165,000$$

$$x \approx 383,721.$$

In 10 years, the house will cost about \$384,000.

(d) Let x = the cost of the book in 8 years.

$$\text{Then, } 0.51x = 50$$

$$x \approx 98$$

In 8 years, the textbook will cost about \$98.

$$46. \quad A = P \left(1 + \frac{r}{m} \right)^{tm}$$

$$A = 1000 \left(1 + \frac{j}{2} \right)^{5(2)} = 1000 \left(1 + \frac{j}{2} \right)^{10}$$

This represents the amount in Bank X on January 1, 2005.

$$\begin{aligned} A &= P \left(1 + \frac{r}{m} \right)^{tm} \\ &= \left[1000 \left(1 + \frac{j}{2} \right)^{10} \right] \left(1 + \frac{k}{4} \right)^{3(4)} \\ &= 1000 \left(1 + \frac{j}{2} \right)^{10} \left(1 + \frac{k}{4} \right)^{12} \end{aligned}$$

This represents the amount in Bank Y on January 1, 2008, \$1990.76.

$$\begin{aligned} A &= P \left(1 + \frac{r}{m} \right)^{tm} = 1000 \left(1 + \frac{k}{4} \right)^{8.4} \\ &= 1000 \left(1 + \frac{k}{4} \right)^{32} \end{aligned}$$

This represents the amount he could have had from January 1, 2000, to January 1, 2008, at a rate of k per annum compounded quarterly, \$2203.76.

So,

$$1000 \left(1 + \frac{j}{2} \right)^{10} \left(1 + \frac{k}{4} \right)^{12} = 1990.76$$

$$\text{and } 1000 \left(1 + \frac{k}{4} \right)^{32} = 2203.76.$$

$$\left(1 + \frac{k}{4} \right)^{32} = 2.20376$$

$$1 + \frac{k}{4} = (2.20376)^{1/32}$$

$$1 + \frac{k}{4} = 1.025$$

$$\frac{k}{4} = 0.025$$

$$k = 0.1 \quad \text{or} \quad 10\%$$

Substituting, we have

$$1000 \left(1 + \frac{j}{2} \right)^{10} \left(1 + \frac{1.0}{4} \right)^{12} = 1990.76$$

$$1000 \left(1 + \frac{j}{2} \right)^{10} (1.025)^{12} = 1990.76$$

$$\left(1 + \frac{j}{2} \right)^{10} = 1.480$$

$$1 + \frac{j}{2} = (1.480)^{1/10}$$

$$1 + \frac{j}{2} = 1.04$$

$$\frac{j}{2} = 0.04$$

$$j = 0.08 \quad \text{or} \quad 8\%$$

The ratio $\frac{k}{j} = \frac{0.1}{0.08} = 1.25$, is choice (a).

$$47. \quad A(t) = 3100e^{0.0166t}$$

(a) 1970: $t = 10$

$$A(20) = 3100e^{(0.0166)(10)}$$

$$= 3100e^{0.166}$$

$$\approx 3659.78$$

The function gives a population of about 3660 million in 1970.

This is very close to the actual population of about 3686 million.

- (b) 2000:
- $t = 40$

$$\begin{aligned} A(50) &= 3100e^{0.0166(40)} \\ &= 3100e^{0.664} \\ &\approx 6021.90 \end{aligned}$$

The function gives a population of 6022 million in 2000.

- (c) 2015:
- $t = 55$

$$\begin{aligned} &= 3100e^{0.0166(55)} \\ &= 3100e^{0.913} \\ &= 7724.54 \end{aligned}$$

From the function, we estimate that the world population in 2015 will be 7725 million.

48. $f(x) = 500 \cdot 2^{3x}$

- (a) After 1 hour:

$$f(1) = 500 \cdot 2^{3(1)} = 500 \cdot 8 = 4000 \text{ bacteria}$$

- (b) initially:

$$f(0) = 500 \cdot 2^{3(0)} = 500 \cdot 1 = 500 \text{ bacteria}$$

- (c) The bacteria double every
- $3x = 1$
- hour, or every
- $\frac{1}{3}$
- hour, or 20 minutes.

- (d) When does
- $f(x) = 32,000$
- ?

$$\begin{aligned} 32,000 &= 500 \cdot 2^{3x} \\ 64 &= 2^{3x} \\ 2^6 &= 2^{3x} \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

The number of bacteria will increase to 32,000 in 2 hours.

49. (a) Hispanic population:

$$\begin{aligned} h(t) &= 37.79(1.021)^t \\ h(5) &= 37.79(1.021)^5 \\ &\approx 41.93 \end{aligned}$$

The projected Hispanic population in 2005 is 41.93 million, which is slightly less than the actual value of 42.69 million.

- (b) Asian population:

$$\begin{aligned} h(t) &= 11.14(1.023)^t \\ h(5) &= 11.14(1.023)^5 \\ &\approx 12.48 \end{aligned}$$

The projected Asian population in 2005 is 12.48 million, which is very close to the actual value of 12.69 million.

- (c) Annual Hispanic percent increase:

$$1.021 - 1 = 0.021 = 2.1\%$$

Annual Asian percent increase:

$$1.023 - 1 = 0.023 = 2.3\%$$

The Asian population is growing at a slightly faster rate.

- (d) Black population:

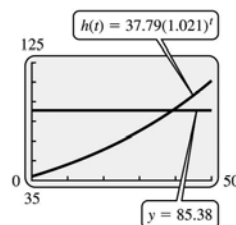
$$\begin{aligned} b(t) &= 0.5116t + 35.43 \\ b(5) &= 0.5116(5) + 35.43 \\ &\approx 37.99 \end{aligned}$$

The projected Black population in 2005 is 37.99 million, which is extremely close to the actual value of 37.91 million.

- (e) Hispanic population:

Double the actual 2005 value is

$$2(42.69) = 85.38 \text{ million.}$$

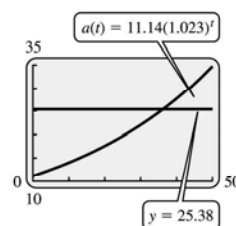


The doubling point is reached when $t \approx 39$, or in the year 2039.

Asian population:

Double the actual 2005 value is

$$2(12.69) = 25.38 \text{ million.}$$

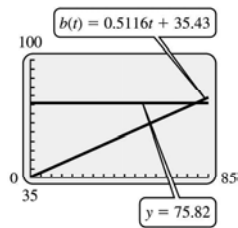


The doubling point is reached when $t \approx 36$, or in the year 2036.

Black population:

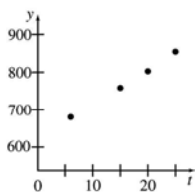
Double the actual 2005 value is

$$2(37.91) = 75.82 \text{ million.}$$



The doubling point is reached when $t \approx 79$, or in the year 2079.

50. (a)



The data appears to fit an exponential curve.

(b) $f(x) = Ce^{kx}$

$$f(6) = Ce^{k(6)}$$

$$680.5 = Ce^{6k}$$

$$C = \frac{680.5}{e^{6k}}$$

$$f(15) = Ce^{15k}$$

$$758.6 = Ce^{15k}$$

$$C = \frac{758.6}{e^{15k}}$$

$$\frac{680.5}{e^{6k}} = \frac{758.6}{e^{15k}}$$

$$\frac{e^{15k}}{e^{6k}} = \frac{758.6}{680.5}$$

$$e^{9k} = \frac{758.6}{680.5}$$

$$9k = \ln\left(\frac{758.6}{680.5}\right)$$

$$k = \frac{1}{9} \ln\left(\frac{758.6}{680.5}\right) \approx 0.01207186$$

$$C = \frac{680.5}{e^{6(0.01207186)}} \approx 632.95$$

$$f(x) = 632.95e^{0.01207x}$$

(c) $t = 20$ corresponds to 2020 and $t = 25$ corresponds to 2025.

$$f(t) = 632.95e^{0.01207t}$$

$$f(20) = 632.95e^{0.01207(20)} \approx 805.8$$

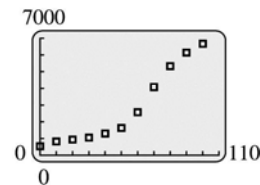
$$f(25) = 632.95e^{0.01207(25)} \approx 855.9$$

The demand for physicians in 2020 will be about 805,800 and in 2020 will be about 855,900. These are very close to the values in the data.

(d) The exponential regression function is

$$f(t) = 631.81e^{0.01224t}. \text{ This is close to the function found in part (b).}$$

51. (a)



The emissions appear to grow exponentially.

(b) $f(x) = f_0 a^x$

$$f_0 = 534$$

Use the point (100, 6672) to find a .

$$6672 = 534a^{100}$$

$$a^{100} = \frac{6672}{534}$$

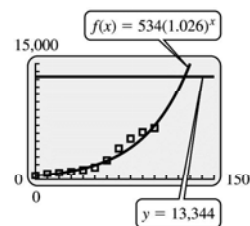
$$a = 100\sqrt[100]{\frac{6672}{534}}$$

$$\approx 1.026$$

$$f(x) = 534(1.026)^x$$

(c) $1.026 - 1 = 0.026 = 2.6\%$

(d) Double the 2000 value is $2(6672) = 13,344$.



The doubling point is reached when $x \approx 125.4$. The first year in which emissions equal or exceed that threshold is 2026.

52. $Q(t) = 1000(5^{-0.3t})$

(a) $Q(6) = 1000[5^{-0.3(6)}]$

$$= 1000(5^{-1.8})$$

$$= 55$$

The amount present in 6 months will be 55 grams.

$$(b) \quad 8 = 1000(5^{-0.3t})$$

$$\frac{1}{125} = 5^{-0.3t}$$

$$5^{-3} = 5^{-0.3t}$$

$$-3 = -0.3t$$

$$10 = t$$

It will take 10 months to reduce the substance to 8 grams.

$$53. (a) \text{ When } x = 0, P = 1013.$$

$$\text{When } x = 10,000, P = 265.$$

First we fit $P = ae^{kx}$.

$$1013 = ae^0$$

$$a = 1013$$

$$P = 1013e^{kx}$$

$$265 = 1013e^{k(10,000)}$$

$$\frac{265}{1013} = e^{10,000k}$$

$$10,000k = \ln\left(\frac{265}{1013}\right)$$

$$k = \frac{\ln\left(\frac{265}{1013}\right)}{10,000} \approx -1.34 \times 10^{-4}$$

$$\text{Therefore } P = 1013e^{(-1.34 \times 10^{-4})x}.$$

Next we fit $P = mx + b$.

We use the points (0, 1013) and (10,000, 265).

$$m = \frac{265 - 1013}{10,000 - 0} = -0.0748$$

$$b = 1013$$

$$\text{Therefore } P = -0.0748x + 1013.$$

Finally, we fit $P = \frac{1}{ax+b}$.

$$1013 = \frac{1}{a(0) + b}$$

$$b = \frac{1}{1013} \approx 9.87 \times 10^{-4}$$

$$P = \frac{1}{ax + \frac{1}{1013}}$$

$$265 = \frac{1}{10,000a + \frac{1}{1013}}$$

$$\frac{1}{265} = 10,000a + \frac{1}{1013}$$

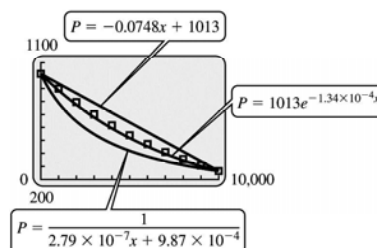
$$10,000a = \frac{1}{265} - \frac{1}{1013}$$

$$a = \frac{\frac{1}{265} - \frac{1}{1013}}{10,000} \approx 2.79 \times 10^{-7}$$

Therefore,

$$P = \frac{1}{(2.79 \times 10^{-7})x + (9.87 \times 10^{-4})}.$$

(b)



$$P = 1013e^{(-1.34 \times 10^{-4})x} \text{ is the best fit.}$$

$$(c) \quad P(1500) = 1013e^{-1.34 \times 10^{-4}(1500)} \approx 829$$

$$P(11,000) = 1013e^{-1.34 \times 10^{-4}(11,000)} \approx 232$$

We predict that the pressure at 1500 meters will be 829 millibars, and at 11,000 meters will be 232 millibars.

(d) Using exponential regression, we obtain

$P = 1038(0.99998661)^x$ which differs slightly from the function found in part (b) which can be rewritten as

$$P = 1013(0.99998660)^x.$$

$$54. (a) \quad y = mt + b$$

$$b = 0.275$$

Use the points (0, 0.275) and (24, 1900) to find m .

$$m = \frac{1900 - 0.275}{24 - 0} = \frac{1899.725}{24} \approx 79.155$$

$$y = 79.155t + 0.275$$

$$y = at^2 + b$$

$$b = 0.275$$

Use the point (24, 1900) to find a .

$$1900 = a(24)^2 + 0.275$$

$$1899.725 = 576a$$

$$a = \frac{1899.725}{576} \approx 3.298$$

$$y = 3.298t^2 + 0.275$$

$$y = ab^t$$

$$a = 0.275$$

Use the point (24, 1900) to find b .

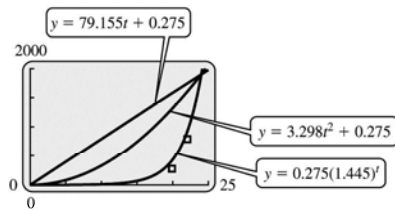
$$1900 = 0.275b^{24}$$

$$b^{24} = \frac{1900}{0.275}$$

$$b = \sqrt[24]{\frac{1900}{0.275}} \approx 1.445$$

$$y = 0.275(1.445)^t$$

(b)



The function $y = 0.275(1.445)^t$ is the best fit.

(c) $x = 30$ corresponds to 1985.

$$y = 0.275(1.445)^{30}$$

$$= 17193.75098$$

$$\approx 17,200$$

(d) The regression function is $y = 0.1787(1.441)^t$. This is close to the function in part (b).

55. (a) $C = mt + b$

Use the points (1, 24,322) and (9, 159,213) to find m .

$$m = \frac{159,213 - 24,322}{9 - 1} = \frac{134,891}{8} \approx 16,861$$

Use the point (1, 24,322) to find b .

$$y = 16,861t + b$$

$$24,322 = 16,861(1) + b$$

$$b = 7461$$

$$C = 16,861t + 7461$$

$$C = at^2 + b$$

Use the points (1, 24,322) and (9, 159,213) to find a and b .

$$24,322 = a(1)^2 + b \quad 159,213 = a(9)^2 + b$$

$$24,322 = a + b \quad 159,213 = 81a + b$$

$$81a + b = 159,213$$

$$a + b = 24,322$$

$$80a = 134,891$$

$$a = \frac{134,891}{80} \approx 1686$$

$$a + b = 24,322$$

$$b = 24,322 - 1686$$

$$b = 22,636$$

$$C = 1686t^2 + 22,636$$

$$C = ab^t$$

Use the points (1, 24,322) and (9, 159,213) to find a and b .

$$24,322 = ab^1 = ab \quad 159,213 = ab^9$$

$$\frac{159,213}{24,322} = \frac{ab^9}{ab} = b^8$$

$$b = \sqrt[8]{\frac{159,213}{24,322}} \approx 1.2647$$

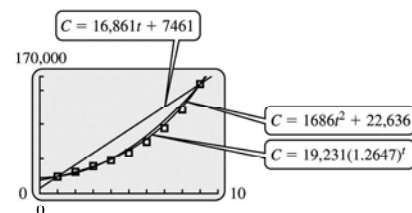
$$ab = 24,322$$

$$(1.2647)a = 24,322$$

$$a = \frac{24,322}{1.2647} \approx 19,231$$

$$C = 19,231(1.2647)^t$$

(b)



The function $C = 19,231(1.2647)^t$ is the best fit.

(c) The regression function is $C = 19,250(1.2579)^t$. This is very close to the function in part (b).

(d) $x = 10$ corresponds to 2010.

$$\begin{aligned} C &= 16,861t + 7461 \\ &= 16,861(10) + 7461 \\ &= 176,071 \\ &\approx 176,100 \end{aligned}$$

$$\begin{aligned} C &= 1686t^2 + 22,636 \\ &= 1686(10)^2 + 22,636 \\ &= 191,236 \\ &\approx 191,200 \end{aligned}$$

$$\begin{aligned} C &= 19,231(1.2647)^t \\ &= 19,231(1.2647)^{10} \\ &= 201,315.43 \\ &\approx 201,300 \end{aligned}$$

$$\begin{aligned} C &= 19,250(1.2579)^t \\ &= 19,250(1.2579)^{10} \\ &= 190,937.80 \\ &\approx 190,900 \end{aligned}$$

2.5 Logarithmic Functions

Your Turn 1

$$5^{-2} = \frac{1}{25} \text{ means } \log_5\left(\frac{1}{25}\right) = -2$$

Your Turn 2

$$\log_3\left(\frac{1}{81}\right)$$

We seek a number x such that

$$\begin{aligned} 3^x &= \frac{1}{81} \\ 3^x &= (3)^{-4} \\ x &= -4 \end{aligned}$$

Your Turn 3

$$\begin{aligned} \log_a\left(\frac{x^2}{y^3}\right) &= \log_a x^2 - \log_a y^3 \\ &= 2\log_a x - 3\log_a y \end{aligned}$$

Your Turn 4

$$\log_3 50 = \frac{\ln 50}{\ln 3} \approx 3.561$$

Your Turn 5

$$\begin{aligned} \log_2 x + \log_2 (x + 2) &= 3 \\ \log_2 x(x + 2) &= 3 \\ x(x + 2) &= 2^3 \\ x^2 + 2x &= 8 \\ x^2 + 2x - 8 &= 0 \\ x = -4 \text{ or } x &= 2 \end{aligned}$$

Since the logarithm of a negative number is undefined, the only solution is $x = 2$.

Your Turn 6

$$\begin{aligned} 2^{x+1} &= 3^x \\ \ln 2^{x+1} &= \ln 3^x \\ (x + 1)\ln 2 &= x\ln 3 \\ x\ln 2 + \ln 2 &= x\ln 3 \\ x\ln 3 - x\ln 2 &= \ln 2 \\ x(\ln 3 - \ln 2) &= \ln 2 \\ x\ln\left(\frac{3}{2}\right) &= \ln 2 \\ x &= \frac{\ln 2}{\ln 1.5} \approx 1.7095 \end{aligned}$$

Your Turn 7

$$e^{0.025x} = (e^{0.025})^x \approx 1.0253^x$$

2.5 Exercises

1. $5^3 = 125$

Since $a^y = x$ means $y = \log_a x$, the equation in logarithmic form is

$$\log_5 125 = 3.$$

2. $7^2 = 49$

Since $a^y = x$ means $y = \log_a x$, the equation in logarithmic form is

$$\log_7 49 = 2.$$

3. $3^4 = 81$

The equation in logarithmic form is

$$\log_3 81 = 4.$$

4. $2^7 = 128$

Since $a^y = x$ means $y = \log_a x$, the equation in logarithmic form is

$$\log_2 128 = 7.$$

5. $3^{-2} = \frac{1}{9}$

The equation in logarithmic form is

$$\log_3 \frac{1}{9} = -2.$$

6. $\left(\frac{5}{4}\right)^{-2} = \frac{16}{25}$

Since $a^y = x$ means $y = \log_a x$, the equation in logarithmic form is

$$\log_{5/4} \frac{16}{25} = -2.$$

7. $\log_2 32 = 5$

Since $y = \log_a x$ means $a^y = x$, the equation in exponential form is

$$2^5 = 32.$$

8. $\log_3 81 = 4$

Since $y = \log_a x$ means $a^y = x$, the equation in exponential form is

$$3^4 = 81.$$

9. $\ln \frac{1}{e} = -1$

The equation in exponential form is

$$e^{-1} = \frac{1}{e}.$$

10. $\log_2 \frac{1}{8} = -3$

The equation in exponential form is

$$2^{-3} = \frac{1}{8}.$$

11. $\log 100,000 = 5$

$$\log_{10} 100,000 = 5$$

$$10^5 = 100,000$$

When no base is written, $\log_{10}$ is understood.

12. $\log 0.001 = -3$

$$\log_{10} 0.001 = -3$$

$$10^{-3} = 0.001$$

When no base is written, $\log_{10}$ is understood.

13. Let $\log_8 64 = x$.

$$\text{Then, } 8^x = 64$$

$$8^x = 8^2$$

$$x = 2.$$

$$\text{Thus, } \log_8 64 = 2.$$

14. Let $\log_9 81 = x$.

$$\text{Then, } 9^x = 81$$

$$9^x = 9^2$$

$$x = 2.$$

$$\text{Thus } \log_9 81 = 2.$$

15. $\log_4 64 = x$

$$4^x = 64$$

$$4^x = 4^3$$

$$x = 3$$

16. $\log_3 27 = x$

$$3^x = 27$$

$$3^x = 3^3$$

$$x = 3$$

17. $\log_2 \frac{1}{16} = x$

$$2^x = \frac{1}{16}$$

$$2^x = 2^{-4}$$

$$x = -4$$

18. $\log_3 \frac{1}{81} = x$

$$3^x = \frac{1}{81}$$

$$3^x = 3^{-4}$$

$$x = -4$$

19. $\log_2 \sqrt[3]{\frac{1}{4}} = x$

$$2^x = \left(\frac{1}{4}\right)^{1/3}$$

$$2^x = \left(\frac{1}{2^2}\right)^{1/3}$$

$$2^x = 2^{-2/3}$$

$$x = -\frac{2}{3}$$

20. $\log_8 \sqrt[4]{\frac{1}{2}} = x$

$$8^x = \sqrt[4]{\frac{1}{2}} = \left(\frac{1}{2}\right)^{1/4}$$

$$(2^3)^x = 2^{-1/4}$$

$$3x = -\frac{1}{4}$$

$$x = -\frac{1}{12}$$

21. $\ln e = x$

Recall that $\ln$ means $\log_e$.

$$e^x = e$$

$$x = 1$$

22. $\ln e^3 = x$

Recall that $\ln$ means $\log_e$.

$$e^x = e^3$$

$$x = 3$$

23. $\ln e^{5/3} = x$

$$e^x = e^{5/3}$$

$$x = \frac{5}{3}$$

24. $\ln 1 = x$

$$e^x = 1$$

$$e^x = e^0$$

$$x = 0$$

25. The logarithm to the base 3 of 4 is written $\log_3 4$.

The subscript denotes the base.

27. $\log_5 (3k) = \log_5 3 + \log_5 k$

28. $\log_9 (4m) = \log_9 4 + \log_9 m$

29. $\log_3 \frac{3p}{5k}$

$$= \log_3 3p - \log_3 5k$$

$$= (\log_3 3 + \log_3 p) - (\log_3 5 + \log_3 k)$$

$$= 1 + \log_3 p - \log_3 5 - \log_3 k$$

30. $\log_7 \frac{15p}{7y}$

$$= \log_7 15p - \log_7 7y$$

$$= (\log_7 15 + \log_7 p) - (\log_7 7 + \log_7 y)$$

$$= \log_7 15 + \log_7 p - \log_7 7 - \log_7 y$$

$$= \log_7 15 + \log_7 p - 1 - \log_7 y$$

31. $\ln \frac{3\sqrt{5}}{\sqrt[3]{6}}$

$$= \ln 3\sqrt{5} - \ln \sqrt[3]{6}$$

$$= \ln 3 \cdot 5^{1/2} - \ln 6^{1/3}$$

$$= \ln 3 + \ln 5^{1/2} - \ln 6^{1/3}$$

$$= \ln 3 + \frac{1}{2} \ln 5 - \frac{1}{3} \ln 6$$

32. $\ln \frac{9\sqrt[3]{5}}{4\sqrt{3}}$

$$= \ln 9\sqrt[3]{5} - \ln 4\sqrt{3}$$

$$= \ln 9 \cdot 5^{1/3} - \ln 3^{1/4}$$

$$= \ln 9 + \ln 5^{1/3} - \ln 3^{1/4}$$

$$= \ln 9 + \frac{1}{3} \ln 5 - \frac{1}{4} \ln 3$$

33. $\log_b 32 = \log_b 2^5$
 $= 5 \log_b 2$
 $= 5a$

34. $\log_b 18$
 $= \log_b (2 \cdot 9)$
 $= \log_b (2 \cdot 3^2)$
 $= \log_b 2 + \log_b 3^2$
 $= \log_b 2 + 2 \log_b 3$
 $= a + 2c$

$$\begin{aligned}
 35. \quad \log_b 72b &= \log_b 72 + \log_b b \\
 &= \log_b 72 + 1 \\
 &= \log_b 2^3 \cdot 3^3 + 1 \\
 &= \log_b 2^3 + \log_b 3^2 + 1 \\
 &= 3 \log_b 2 + 2 \log_b 3 + 1 \\
 &= 3a + 2c + 1
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \log_b (9b^2) &= \log_b 9 + \log_b b^2 \\
 &= \log_b 3^2 + \log_b b^2 \\
 &= 2 \log_b 3 + 2 \log_b b \\
 &= 2c + 2(1) \\
 &= 2c + 2
 \end{aligned}$$

$$\begin{aligned}
 37. \quad \log_5 30 &= \frac{\ln 30}{\ln 5} \\
 &\approx \frac{3.4012}{1.6094} \\
 &\approx 2.113
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \log_{12} 210 &= \frac{\ln 210}{\ln 12} \\
 &\approx 2.152
 \end{aligned}$$

$$\begin{aligned}
 39. \quad \log_{1.2} 0.95 &= \frac{\ln 0.95}{\ln 1.2} \\
 &\approx -0.281
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \log_{2.8} 0.12 &= \frac{\ln 0.12}{\ln 2.8} \\
 &\approx -2.059
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \log_x 36 &= -2 \\
 x^{-2} &= 36 \\
 (x^{-2})^{-1/2} &= 36^{-1/2} \\
 x &= \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \log_9 27 &= m \\
 9^m &= 27 \\
 (3^2)^m &= 3^3 \\
 3^{2m} &= 3^3 \\
 2m &= 3 \\
 m &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \log_8 16 &= z \\
 8^z &= 16 \\
 (2^3)^z &= 2^4 \\
 2^{3z} &= 2^4 \\
 3z &= 4 \\
 z &= \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 44. \quad \log_y 8 &= \frac{3}{4} \\
 y^{3/4} &= 8 \\
 (y^{3/4})^{4/3} &= 8^{4/3} \\
 y &= (8^{1/3})^4 \\
 y &= 2^4 = 16
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \log_r 5 &= \frac{1}{2} \\
 r^{1/2} &= 5 \\
 (r^{1/2})^2 &= 5^2 \\
 r &= 25
 \end{aligned}$$

$$\begin{aligned}
 46. \quad \log_4 (5x + 1) &= 2 \\
 4^2 &= 5x + 1 \\
 16 &= 5x + 1 \\
 5x &= 15 \\
 x &= 3
 \end{aligned}$$

$$\begin{aligned}
 47. \quad \log_5 (9x - 4) &= 1 \\
 5^1 &= 9x - 4 \\
 9 &= 9x \\
 1 &= x
 \end{aligned}$$

48. $\log_4 x - \log_4 (x + 3) = -1$

$$\begin{aligned}\log_4 \frac{x}{x+3} &= -1 \\ 4^{-1} &= \frac{x}{x+3} \\ \frac{1}{4} &= \frac{x}{x+3} \\ 4x &= x+3 \\ 3x &= 3 \\ x &= 1\end{aligned}$$

49. $\log_9 m - \log_9 (m - 4) = -2$

$$\begin{aligned}\log_9 \frac{m}{m-4} &= -2 \\ 9^{-2} &= \frac{m}{m-4} \\ \frac{1}{81} &= \frac{m}{m-4} \\ m-4 &= 81m \\ -4 &= 80m \\ -0.05 &= m\end{aligned}$$

This value is not possible since $\log_9 (-0.05)$ does not exist.

Thus, there is no solution to the original equation.

50. $\log (x + 5) + \log (x + 2) = 1$

$$\begin{aligned}\log [(x+5)(x+2)] &= 1 \\ (x+5)(x+2) &= 10^1 \\ x^2 + 7x + 10 &= 10 \\ x^2 + 7x &= 0 \\ x(x+7) &= 0 \\ x &= 0 \text{ or } x = -7\end{aligned}$$

$x = -7$ is not a solution of the original equation because if $x = -7$, $x + 5$ and $x + 2$ would be negative, and the domain of $y = \log x$ is $(0, \infty)$. Therefore, $x = 0$.

51. $\log_3 (x - 2) + \log_3 (x + 6) = 2$

$$\begin{aligned}\log_3 [(x-2)(x+6)] &= 2 \\ (x-2)(x+6) &= 3^2 \\ x^2 + 4x - 12 &= 9 \\ x^2 + 4x - 21 &= 0 \\ (x+7)(x-3) &= 0 \\ x &= -7 \text{ or } x = 3\end{aligned}$$

$x = -7$ does not check in the original equation. The only solution is 3.

52. $\log_3 (x^2 + 17) - \log_3 (x + 5) = 1$

$$\begin{aligned}\log_3 \frac{x^2 + 17}{x+5} &= 1 \\ 3^1 &= \frac{x^2 + 17}{x+5} \\ 3x + 15 &= x^2 + 17 \\ 0 &= x^2 - 3x + 2 \\ 0 &= (x-1)(x-2) \\ x &= 1 \text{ or } x = 2\end{aligned}$$

53. $\log_2 (x^2 - 1) - \log_2 (x + 1) = 2$

$$\begin{aligned}\log_2 \frac{x^2 - 1}{x+1} &= 2 \\ 2^2 &= \frac{x^2 - 1}{x+1} \\ 4 &= \frac{(x-1)(x+1)}{x+1} \\ 4 &= x-1 \\ x &= 5\end{aligned}$$

54. $\ln(5x + 4) = 2$

$$\begin{aligned}5x + 4 &= e^2 \\ 5x &= e^2 - 4 \\ x &= \frac{e^2 - 4}{5} \approx 0.6778\end{aligned}$$

55. $\ln x + \ln 3x = -1$

$$\begin{aligned}\ln 3x^2 &= -1 \\ 3x^2 &= e^{-1} \\ x^2 &= \frac{e^{-1}}{3} \\ x &= \sqrt{\frac{e^{-1}}{3}} = \frac{1}{\sqrt{3e}} \approx 0.3502\end{aligned}$$

$$56. \ln(x+1) - \ln x = 1$$

$$\ln \frac{x+1}{x} = 1$$

$$\frac{x+1}{x} = e^1$$

$$x+1 = ex$$

$$ex - x = 1$$

$$x(e-1) = 1$$

$$x = \frac{1}{e-1} \approx 0.5820$$

$$57. 2^x = 6$$

$$\ln 2^x = \ln 6$$

$$x \ln 2 = \ln 6$$

$$x = \frac{\ln 6}{\ln 2} \approx 2.5850$$

$$58. 5^x = 12$$

$$x \log 5 = \log 12$$

$$x = \frac{\log 12}{\log 5}$$

$$\approx 1.5440$$

$$59. e^{k-1} = 6$$

$$\ln e^{k-1} = \ln 6$$

$$(k-1) \ln e = \ln 6$$

$$k-1 = \frac{\ln 6}{\ln e}$$

$$k-1 = \frac{\ln 6}{1}$$

$$k = 1 + \ln 6$$

$$\approx 2.7918$$

$$60. e^{2y} = 15$$

$$\ln e^{2y} = \ln 15$$

$$2y \ln e = \ln 15$$

$$2y(1) = \ln 15$$

$$y = \frac{\ln 15}{2}$$

$$\approx 1.3540$$

$$61. 3^{x+1} = 5^x$$

$$\ln 3^{x+1} = \ln 5^x$$

$$(x+1) \ln 3 = x \ln 5$$

$$x \ln 3 + \ln 3 = x \ln 5$$

$$x \ln 5 - x \ln 3 = \ln 3$$

$$x(\ln 5 - \ln 3) = \ln 3$$

$$x = \frac{\ln 3}{\ln(5/3)} \approx 2.1507$$

$$62. 2^{x+1} = 6^{x-1}$$

$$\ln 2^{x+1} = \ln 6^{x-1}$$

$$(x+1) \ln 2 = (x-1) \ln 6$$

$$x \ln 2 + \ln 2 = x \ln 6 - \ln 6$$

$$x \ln 6 - x \ln 2 = \ln 2 + \ln 6$$

$$x(\ln 6 - \ln 2) = \ln 2 + \ln 6$$

$$x \ln 3 = \ln 12$$

$$x = \frac{\ln 12}{\ln 3} \approx 2.2619$$

$$63. 5(0.10)^x = 4(0.12)^x$$

$$\ln[5(0.10)^x] = \ln[4(0.12)^x]$$

$$\ln 5 + x \ln 0.10 = \ln 4 + x \ln 0.12$$

$$x(\ln 0.12 - \ln 0.10) = \ln 5 - \ln 4$$

$$x = \frac{\ln 5 - \ln 4}{\ln 0.12 - \ln 0.10}$$

$$= \frac{\ln 1.25}{\ln 1.2}$$

$$\approx 1.2239$$

$$64. 1.5(1.05)^x = 2(1.01)^x$$

$$\ln[1.5(1.05)^x] = \ln[2(1.01)^x]$$

$$\ln 1.5 + x \ln 1.05 = \ln 2 + x \ln 1.01$$

$$x(\ln 1.01 - \ln 1.05) = \ln 1.5 - \ln 2$$

$$x = \frac{\ln 0.75}{\ln(1.01/1.05)}$$

$$\approx 7.4069$$

$$65. 10^{x+1} = e^{(\ln 20)(x+1)}$$

$$66. 10^{x^2} = e^{(\ln 10)x^2}$$

$$67. e^{3x} = (e^3)^x \approx 20.09^x$$

68. $e^{-4x} = (e^{-4})^x \approx 0.0183^x$

69. $f(x) = \log(5 - x)$

$$5 - x > 0$$

$$-x > -5$$

$$x < 5$$

The domain of f is $x < 5$.

70. $f(x) = \ln(x^2 - 9)$

Since the domain of $f(x) = \ln x$ is $(0, \infty)$,

the domain of $f(x) = \ln(x^2 - 9)$ is the set of all real numbers x for which

$$x^2 - 9 > 0.$$

To solve this quadratic inequality, first solve the corresponding quadratic equation.

$$x^2 - 9 = 0$$

$$(x + 3)(x - 3) = 0$$

$$x + 3 = 0 \text{ or } x - 3 = 0$$

$$x = -3 \text{ or } x = 3$$

These two solutions determine three intervals on the number line: $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$.

If $x = -4$, $(-4 + 2)(-4 - 2) > 0$.

If $x = 0$, $(0 + 2)(0 - 2) \not> 0$.

If $x = 4$, $(4 + 2)(4 - 2) > 0$.

The domain is $x < -3$ or $x > 3$, which is written in interval notation as $(-\infty, -3) \cup (3, \infty)$.

71. $\log A - \log B = 0$

$$\log \frac{A}{B} = 0$$

$$\frac{A}{B} = 10^0 = 1$$

$$A = B$$

$$A - B = 0$$

Thus, solving $\log A - \log B = 0$ is equivalent to solving $A - B = 0$

72. $(\log(x + 2))^2 \neq 2\log(x + 2)$

$$(\log(x + 2))^2 = (\log(x + 2))(\log(x + 2))$$

$$2\log(x + 2) \neq 2(\log x + \log 2)$$

$$2\log(x + 2) = \log(x + 2)^2$$

$$\log 2 \neq 100$$

$$\log 2 \approx 0.30103 \text{ because } 10^{0.30103} = 2$$

73. Let $m = \log_a \frac{x}{y}$, $n = \log_a x$, and $p = \log_a y$.

$$\text{Then } a^m = \frac{x}{y}, a^n = x, \text{ and } a^p = y.$$

Substituting gives

$$a^m = \frac{x}{y} = \frac{a^n}{a^p} = a^{n-p}.$$

$$\text{So } m = n - p.$$

Therefore,

$$\log_a \frac{x}{y} = \log_a x - \log_a y.$$

74. Let $m = \log_a x^r$ and $n = \log_a x$.

$$\text{Then, } a^m = x^r \text{ and } a^n = x.$$

Substituting gives

$$a^m = x^r = (a^n)^r = a^{nr}.$$

Therefore, $m = nr$, or

$$\log_a x^r = r \log_a x.$$

75. From Example 8, the doubling time t in years when $m = 1$ is given by

$$t = \frac{\ln 2}{\ln(1 + r)}.$$

(a) Let $r = 0.03$.

$$\begin{aligned} t &= \frac{\ln 2}{\ln(1.03)} \\ &= 23.4 \text{ years} \end{aligned}$$

(b) Let $r = 0.06$.

$$\begin{aligned} t &= \frac{\ln 2}{\ln 1.06} \\ &= 11.9 \text{ years} \end{aligned}$$

(c) Let $r = 0.08$.

$$t = \frac{\ln 2}{\ln 1.08} \\ = 9.0 \text{ years}$$

(d) Since $0.001 \leq 0.03 \leq 0.05$, for $r = 0.03$, we use the rule of 70.

$$\frac{70}{100r} = \frac{70}{100(0.03)} = 23.3 \text{ years}$$

Since $0.05 \leq 0.06 \leq 0.12$, for $r = 0.06$, we use the rule of 72.

$$\frac{72}{100r} = \frac{72}{100(0.06)} = 12 \text{ years}$$

For $r = 0.08$, we use the rule of 72.

$$\frac{72}{100(0.08)} = 9 \text{ years}$$

76. (a) $t = \frac{\ln 2}{\ln(1+r)}$

$$t = \frac{\ln 2}{\ln(1+0.07)}$$

$$t \approx 10.24$$

It will take 11 years for the compound amount to be at least double.

(b) $t = \frac{\ln 3}{\ln(1+0.07)}$

$$t \approx 16.24$$

It will take 17 years for the compound amount to be at least triple.

(c) The rule of 72 gives

$$\frac{72}{100(0.07)} = 10.29$$

years as the doubling time, which is close to the answer found in part (a).

77. $A = Pe^{rt}$

$$1200 = 500e^{r \cdot 14}$$

$$2.4 = e^{14r}$$

$$\ln(2.4) = \ln e^{14r}$$

$$\ln(2.4) = 14r$$

$$\frac{\ln(2.4)}{14} = r$$

$$0.0625 \approx r$$

The interest rate should be 6.25%.

78.

r (sec)	0.001	0.02	0.05	0.08	0.12
$\frac{\ln 2}{\ln(1+r)}$	693.5	35	14.2	9.01	6.12
$\frac{70}{100r}$	700	35	14	8.75	5.83
$\frac{72}{100r}$	720	36	14.4	9	6

For $0.001 \leq r < 0.05$, the Rule of 70 is more accurate. For $0.05 < r \leq 0.12$, the Rule of 72 is more accurate. At $r = 0.05$, the two are equally accurate.

79. After x years at Humongous Enterprises, your salary would be $45,000(1 + 0.04)^x$ or $45,000(1.04)^x$. After x years at Crabapple Inc., your salary would be $30,000(1 + 0.06)^x$ or $30,000(1.06)^x$.

First we find when the salaries would be equal.

$$45,000(1.04)^x = 30,000(1.06)^x$$

$$\frac{(1.04)^x}{(1.06)^x} = \frac{30,000}{45,000}$$

$$\left(\frac{1.04}{1.06}\right)^x = \frac{2}{3}$$

$$\log\left(\frac{1.04}{1.06}\right)^x = \log\left(\frac{2}{3}\right)$$

$$x \log\left(\frac{1.04}{1.06}\right) = \log\left(\frac{2}{3}\right)$$

$$x = \frac{\log\left(\frac{2}{3}\right)}{\log\left(\frac{1.04}{1.06}\right)}$$

$$x \approx 21.29$$

$$2013 + 21.29 = 2034.29$$

Therefore, on July 1, 2035, the job at Crabapple, Inc., will pay more.

80. If the number N is proportional to $m^{-0.6}$, where m is the mass, then $N = km^{-0.6}$, for some constant of proportionality k .

Taking the common log of both sides, we have

$$\log N = \log(km^{-0.6})$$

$$= \log k + \log m^{-0.6}$$

$$= \log k - 0.6 \log m.$$

This is a linear equation in $\log m$. Its graph is a straight line with slope -0.6 and vertical intercept $\log k$.

81. (a) The total number of individuals in the community is $50 + 50$, or 100.

$$\text{Let } P_1 = \frac{50}{100} = 0.5, P_2 = 0.5.$$

$$\begin{aligned} H &= -[P_1 \ln P_1 + P_2 \ln P_2] \\ &= -[0.5 \ln 0.5 + 0.5 \ln 0.5] \\ &\approx 0.693 \end{aligned}$$

- (b) For 2 species, the maximum diversity is $\ln 2$.

- (c) Yes, $\ln 2 \approx 0.693$.

$$\begin{aligned} 82. \quad H &= -[P_1 \ln P_1 + P_2 \ln P_2 \\ &\quad + P_3 \ln P_3 + P_4 \ln P_4] \\ H &= -[0.521 \ln 0.521 + 0.324 \ln 0.324 \\ &\quad + 0.081 \ln 0.081 + 0.074 \ln 0.074] \\ H &= 1.101 \end{aligned}$$

83. (a) 3 species, $\frac{1}{3}$ each:

$$\begin{aligned} P_1 &= P_2 = P_3 = \frac{1}{3} \\ H &= -(P_1 \ln P_1 + P_2 \ln P_2 + P_3 \ln P_3) \\ &= -3 \left(\frac{1}{3} \ln \frac{1}{3} \right) \\ &= -\ln \frac{1}{3} \\ &\approx 1.099 \end{aligned}$$

- (b) 4 species, $\frac{1}{4}$ each:

$$\begin{aligned} P_1 &= P_2 = P_3 = P_4 = \frac{1}{4} \\ H &= -(P_1 \ln P_1 + P_2 \ln P_2 + P_3 \ln P_3 + P_4 \ln P_4) \\ &= -4 \left(\frac{1}{4} \ln \frac{1}{4} \right) \\ &= -\ln \frac{1}{4} \\ &\approx 1.386 \end{aligned}$$

- (c) Notice that

$$-\ln \frac{1}{3} = \ln (3^{-1})^{-1} = \ln 3 \approx 1.099$$

and

$$-\ln \frac{1}{4} = \ln (4^{-1})^{-1} = \ln 4 \approx 1.386$$

by Property c of logarithms, so the populations are at a maximum index of diversity.

$$\begin{aligned} 84. \quad mX + N &= m \log_b x + \log_b n \\ &= \log_b x^m + \log_b n \\ &= \log_b nx^m \\ &= \log_b y \\ &= Y \end{aligned}$$

Thus, $Y = mX + N$.

$$85. \quad C(t) = C_0 e^{-kt}$$

When $t = 0$, $C(t) = 2$, and when $t = 3$, $C(t) = 1$.

$$\begin{aligned} 2 &= C_0 e^{-k(0)} \\ C_0 &= 2 \\ 1 &= 2e^{-3k} \\ \frac{1}{2} &= e^{-3k} \\ -3k &= \ln \frac{1}{2} = \ln 2^{-1} = -\ln 2 \\ k &= \frac{\ln 2}{3} \\ T &= \frac{1}{k} \ln \frac{C_2}{C_1} \\ T &= \frac{1}{\frac{\ln 2}{3}} \ln \frac{5C_1}{C_1} \\ T &= \frac{3 \ln 5}{\ln 2} \\ T &\approx 7.0 \end{aligned}$$

The drug should be given about every 7 hours.

86. (a) From the given graph, when $x = 10$ g, $y \approx 1.3 \text{ cm}^3/\text{g/hr}$, and when $x = 1000$ g, $y \approx 0.41 \text{ cm}^3/\text{g/hr}$.

- (b) If $y = ax^b$, then

$$\begin{aligned} \ln y &= \ln (ax^b) \\ &= \ln a + b \ln x. \end{aligned}$$

Thus, there is a linear relationship between $\ln y$ and $\ln x$.

$$\begin{aligned}
 \text{(c)} \quad & 1.3 = a(10)^b \\
 & 0.41 = a(1000)^b \\
 & \frac{1.3}{0.41} = \frac{a(10)^b}{a(1000)^b} \\
 & \frac{1.3}{0.41} = \left(\frac{10}{1000}\right)^b \\
 & \ln\left(\frac{1.3}{0.41}\right) = \ln(0.01)^b \\
 & \ln\left(\frac{1.3}{0.41}\right) = b \ln(0.01) \\
 & b = \frac{\ln\left(\frac{1.3}{0.41}\right)}{\ln(0.01)} \\
 & b \approx -0.25
 \end{aligned}$$

Substituting this value into $1.3 = a(10)^b$,

$$\begin{aligned}
 1.3 &= a(10)^{-0.25} \\
 a &= \frac{1.3}{(10)^{-0.25}} \approx 2.3.
 \end{aligned}$$

Therefore, $y = 2.3x^{-0.25}$.

(d) If $x = 100$,

$$\begin{aligned}
 y &= 2.3(100)^{-0.25} \\
 &\approx 0.73.
 \end{aligned}$$

We predict that the basal metabolism for a marsupial carnivore with a body mass of 100 g will be about $0.73 \text{ cm}^3/\text{g}/\text{hr}$.

$$87. \text{ (a)} \quad h(t) = 37.79(1.021)^t$$

Double the 2005 population is $2(42.69) = 85.38$ million

$$\begin{aligned}
 85.38 &= 37.79(1.021)^t \\
 \frac{85.38}{37.79} &= (1.021)^t \\
 \log_{1.021}\left(\frac{85.38}{37.79}\right) &= t \\
 t &= \frac{\ln\left(\frac{85.38}{37.79}\right)}{\ln 1.021} \\
 &\approx 39.22
 \end{aligned}$$

The Hispanic population is estimated to double their 2005 population in 2039.

$$\text{(b)} \quad h(t) = 11.14(1.023)^t$$

Double the 2005 population is $2(12.69) = 25.38$ million

$$\begin{aligned}
 25.38 &= 11.14(1.023)^t \\
 \frac{25.38}{11.14} &= (1.023)^t \\
 \log_{1.023}\left(\frac{25.38}{11.14}\right) &= t \\
 t &= \frac{\ln\left(\frac{25.38}{11.14}\right)}{\ln 1.023} \\
 &\approx 36.21
 \end{aligned}$$

The Asian population is estimated to double their 2005 population in 2036.

$$88. \quad N(r) = -5000 \ln r$$

$$\text{(a)} \quad N(0.9) = -5000 \ln(0.9) \approx 530$$

$$\text{(b)} \quad N(0.5) = -5000 \ln(0.5) \approx 3500$$

$$\text{(c)} \quad N(0.3) = -5000 \ln(0.3) \approx 6000$$

$$\text{(d)} \quad N(0.7) = -5000 \ln(0.7) \approx 1800$$

$$\text{(e)} \quad -5000 \ln r = 1000$$

$$\ln r = \frac{1000}{-5000}$$

$$\ln r = -\frac{1}{5}$$

$$r = e^{-1/5}$$

$$r \approx 0.8$$

$$89. \quad C = B \log_2 \left(\frac{s}{n} + 1 \right)$$

$$\frac{C}{B} = \log_2 \left(\frac{s}{n} + 1 \right)$$

$$2^{C/B} = \frac{s}{n} + 1$$

$$\frac{s}{n} = 2^{C/B} - 1$$

$$90. \quad \text{Decibel rating: } 10 \log \frac{I}{I_0}$$

$$\text{(a)} \quad \text{Intensity, } I = 115I_0$$

$$\begin{aligned}
 10 \log \left(\frac{115I_0}{I_0} \right) \\
 = 10 \log 115 \\
 \approx 21
 \end{aligned}$$

$$\text{(b)} \quad I = 9,500,000I_0$$

$$\begin{aligned}
 10 \log \left(\frac{9.5 \times 10^6 I_0}{I_0} \right) &= 10 \log 9.5 \times 10^6 \\
 &\approx 70
 \end{aligned}$$

$$(c) \quad I = 1,200,000,000I_0$$

$$10 \log \left(\frac{1.2 \times 10^9 I_0}{I_0} \right) = 10 \log 1.2 \times 10^9 \\ \approx 91$$

$$(d) \quad I = 895,000,000,000I_0$$

$$10 \log \left(\frac{8.95 \times 10^{11} I_0}{I_0} \right) = 10 \log 8.95 \times 10^{11} \\ \approx 120$$

$$(e) \quad I = 109,000,000,000,000I_0$$

$$10 \log \left(\frac{1.09 \times 10^{14} I_0}{I_0} \right) = 10 \log 1.09 \times 10^{14} \\ \approx 140$$

$$(f) \quad I_0 = 0.0002 \text{ microbars}$$

$$1,200,000,000I_0 = 1,200,000,000(0.0002) \\ = 240,000 \text{ microbars}$$

$$895,000,000,000I_0 = 895,000,000,000(0.0002) \\ = 179,000,000 \text{ microbars}$$

91. Let I_1 be the intensity of the sound whose decibel rating is 85.

$$(a) \quad 10 \log \frac{I_1}{I_0} = 85$$

$$\log \frac{I_1}{I_0} = 8.5$$

$$\log I_1 - \log I_0 = 8.5$$

$$\log I_1 = 8.5 + \log I_0$$

Let I_2 be the intensity of the sound whose decibel rating is 75.

$$10 \log \frac{I_2}{I_0} = 75$$

$$\log \frac{I_2}{I_0} = 7.5$$

$$\log I_2 - \log I_0 = 7.5$$

$$\log I_0 = \log I_2 - 7.5$$

Substitute for I_0 in the equation for $\log I_1$.

$$\log I_1 = 8.5 + \log I_0 \\ = 8.5 + \log I_2 - 7.5 \\ = 1 + \log I_2$$

$$\log I_1 - \log I_2 = 1$$

$$\log \frac{I_1}{I_2} = 1$$

Then $\frac{I_1}{I_2} = 10$, so $I_2 = \frac{1}{10} I_1$. This means the intensity of the sound that had a rating of 75 decibels is $\frac{1}{10}$ as intense as the sound that had a rating of 85 decibels.

$$92. \quad R(I) = \log \frac{I}{I_0}$$

$$(a) \quad R(1,000,000I_0)$$

$$= \log \frac{1,000,000I_0}{I_0}$$

$$= \log 1,000,000 = 6$$

$$(b) \quad R(1,000,000,000I_0)$$

$$= \log \frac{1,000,000,000I_0}{I_0}$$

$$= \log 100,000,000 = 8$$

$$(c) \quad R(I) = \log \frac{I}{I_0}$$

$$6.7 = \log \frac{I}{I_0}$$

$$10^{6.7} = \frac{I}{I_0}$$

$$I \approx 5,000,000I_0$$

$$(d) \quad R(I) = \log \frac{I}{I_0}$$

$$8.1 = \log \frac{I}{I_0}$$

$$10^{8.1} = \frac{I}{I_0}$$

$$I \approx 126,000,000I_0$$

$$(e) \quad \frac{1985 \text{ quake}}{1999 \text{ quake}} = \frac{126,000,000I_0}{5,000,000I_0} \approx 25$$

The 1985 earthquake had an amplitude more than 25 times that of the 1999 earthquake.

$$(f) \quad R(E) = \frac{2}{3} \log \frac{E}{E_0}$$

For the 1999 earthquake

$$6.7 = \frac{2}{3} \log \frac{E}{E_0}$$

$$10.05 = \log \frac{E}{E_0}$$

$$\frac{E}{E_0} = 10^{10.05}$$

$$E = 10^{10.05} E_0$$

For the 1985 earthquake,

$$8.1 = \frac{2}{3} \log \frac{E}{E_0}$$

$$12.15 = \log \frac{E}{E_0}$$

$$\frac{E}{E_0} = 10^{12.15}$$

$$E = 10^{12.15} E_0$$

The ratio of their energies is

$$\frac{10^{12.15} E_0}{10^{10.15} E_0} = 10^{2.1} \approx 126$$

The 1985 earthquake had an energy about 126 times that of the 1999 earthquake.

- (g) Find the energy of a magnitude 6.7 earthquake. Using the formula from part f,

$$6.7 = \frac{2}{3} \log \frac{E}{E_0}$$

$$\log \frac{E}{E_0} = 10.05$$

$$\frac{E}{E_0} = 10^{10.05}$$

$$E = E_0 10^{10.05}$$

For an earthquake that releases 15 times this much energy, $E = E_0(15)10^{10.05}$.

$$\begin{aligned} R(E_0(15)10^{10.05}) &= \frac{2}{3} \log \left(\frac{E_0(15)10^{10.05}}{E_0} \right) \\ &= \frac{2}{3} \log (15 \cdot 10^{10.05}) \\ &\approx 7.5 \end{aligned}$$

So, it's true that a magnitude 7.5 earthquake releases 15 times more energy than one of magnitude 6.7.

93. $\text{pH} = -\log[\text{H}^+]$

- (a) For pure water:

$$7 = -\log[\text{H}^+]$$

$$-7 = \log[\text{H}^+]$$

$$10^{-7} = [\text{H}^+]$$

For acid rain:

$$4 = -\log[\text{H}^+]$$

$$-4 = \log[\text{H}^+]$$

$$10^{-4} = [\text{H}^+]$$

$$\frac{10^{-4}}{10^{-7}} = 10^3 = 1000$$

The acid rain has a hydrogen ion concentration 1000 times greater than pure water.

- (b) For laundry solution:

$$11 = -\log[\text{H}^+]$$

$$10^{-11} = [\text{H}^+]$$

For black coffee:

$$5 = -\log[\text{H}^+]$$

$$10^{-5} = [\text{H}^+]$$

$$\frac{10^{-5}}{10^{-11}} = 10^6 = 1,000,000$$

The coffee has a hydrogen ion concentration 1,000,000 times greater than the laundry mixture.

94. For concert pitch A, $f = 440$.

$$\begin{aligned} P &= 69 + 12 \log_2(440/440) = 69 + 12 \log_2(1) \\ &= 69 + 12 \cdot 0 = 69 \end{aligned}$$

For one octave higher than concert pitch A, $f = 880$.

$$\begin{aligned} P &= 69 + 12 \log_2(880/440) = 69 + 12 \log_2(2) \\ &= 69 + 12 \cdot 1 = 69 + 12 = 81 \end{aligned}$$

2.6 Applications: Growth and Decay; Mathematics of Finance

Your Turn 1

$$y = y_0 e^{kt}$$

$$18 = 5e^{k(16)}$$

$$e^{16k} = \ln(18/5)$$

$$16k = \ln(18/5)$$

$$k = \frac{\ln(18/5)}{16} \approx 0.08$$

$$y = 5e^{0.08t}$$

Your Turn 2

$$\text{Let } A(t) = \frac{1}{10} A_0 \text{ and } k = -(\ln 2/5600)$$

$$\begin{aligned}
 A(t) &= A_0 e^{kt} \\
 \frac{1}{10} A_0 &= A_0 e^{-(\ln 2/5600)t} \\
 \frac{1}{10} &= e^{-(\ln 2/5600)t} \\
 \ln\left(\frac{1}{10}\right) &= \ln e^{-(\ln 2/5600)t} \\
 \ln\left(\frac{1}{10}\right) &= -(\ln 2/5600)t \\
 t &= -\frac{5600}{\ln 2} \ln\left(\frac{1}{10}\right) \\
 &\approx 18602.80
 \end{aligned}$$

The age of the sample is about 18,600 years.

Your Turn 3

- (a) 4.25% compounded monthly

$$\begin{aligned}
 \left(1 + \frac{0.0425}{12}\right)^{12} - 1 &= (1.0035417)^{12} - 1 \\
 &\approx 0.0433
 \end{aligned}$$

The effective rate is 4.33%.

- (b) 3.75% compounded continuously

$$e^{0.0375} - 1 \approx 0.0382$$

The effective rate is 3.82%.

Your Turn 4

$A = 50,000$, $r = 0.315$, and $m = 4$.

$$\begin{aligned}
 50,000 &= 30,000 \left(1 + \frac{0.0315}{4}\right)^{4t} \\
 \frac{5}{3} &= (1.007875)^{4t} \\
 t &= \frac{\ln(5/3)}{4 \ln(1.007875)} \approx 16.28
 \end{aligned}$$

We need to round up to the nearest quarter, so \$30,000 will grow to \$50,000 in 16.5 years.

Your Turn 5

$$\begin{aligned}
 A &= Pe^{rt} \\
 4500 &= 3200e^{7r} \\
 1.40625 &= e^{7r} \\
 \ln 1.40625 &= \ln e^{7r} \\
 7r &= \ln 1.40625 \\
 r &= \frac{\ln 1.40625}{7} \approx 0.0487
 \end{aligned}$$

The interest rate needed is 4.87%.

2.6 Exercises

- y_0 represents the initial quantity; k represents the rate of growth or decay.
- The half-life of a quantity is the time period for the quantity to decay to one-half of the initial amount.
- Assume that $y = y_0 e^{kt}$ represents the amount remaining of a radioactive substance decaying with a half-life of T . Since $y = y_0$ is the amount of the substance at time $t = 0$, then $y = \frac{y_0}{2}$ is the amount at time $t = T$. Therefore, $\frac{y_0}{2} = y_0 e^{kT}$, and solving for k yields

$$\begin{aligned}
 \frac{1}{2} &= e^{kT} \\
 \ln\left(\frac{1}{2}\right) &= kT \\
 k &= \frac{\ln\left(\frac{1}{2}\right)}{T} \\
 &= \frac{\ln(2^{-1})}{T} \\
 &= -\frac{\ln 2}{T}.
 \end{aligned}$$

- Assume that $y = y_0 e^{kt}$ is the amount left of a radioactive substance decaying with a half-life of T . From Exercise 5, we know $k = \frac{-\ln 2}{T}$, so

$$\begin{aligned}
 y &= y_0 e^{(-\ln 2/T)t} = y_0 e^{-(t/T) \ln 2} = y_0 e^{(-\ln 2^{-t/T})} \\
 &= y_0 2^{-t/T} = y_0 \left[\left(\frac{1}{2}\right)^{-1}\right]^{-t/T} = y_0 \left(\frac{1}{2}\right)^{t/T}
 \end{aligned}$$

- $r = 4\%$ compounded quarterly,
 $m = 4$

$$\begin{aligned}
 r_E &= \left(1 + \frac{r}{m}\right)^m - 1 \\
 &= \left(1 + \frac{0.04}{4}\right)^4 - 1 \\
 &\approx 0.0406 \\
 &\approx 4.06\%
 \end{aligned}$$

8. $r = 6\%$ compounded monthly, $m = 12$

$$\begin{aligned} r_E &= \left(1 + \frac{0.06}{12}\right)^{12} - 1 \\ &\approx 0.0617 = 6.17\% \end{aligned}$$

9. $r = 8\%$ compounded continuously

$$\begin{aligned} r_E &= e^r - 1 \\ &= e^{0.08} - 1 \\ &\approx 0.0833 = 8.33\% \end{aligned}$$

10. $r = 5\%$ compounded continuously

$$\begin{aligned} r_E &= e^r - 1 \\ &= e^{0.05} - 1 \\ &\approx 0.0513 = 5.13\% \end{aligned}$$

11. $A = \$10,000$, $r = 6\%$, $m = 4$, $t = 8$

$$\begin{aligned} P &= A \left(1 + \frac{r}{m}\right)^{-tm} \\ &= 10,000 \left(1 + \frac{0.06}{4}\right)^{-8(4)} \\ &\approx \$6209.93 \end{aligned}$$

12. $A = \$45,678.93$, $r = 7.2\%$, $m = 12$, $t = 11$ months

$$\begin{aligned} P &= A \left(1 + \frac{r}{m}\right)^{-tm} \\ &= 45,678.93 \left(1 + \frac{0.072}{12}\right)^{-(11/12)(12)} \\ &\approx \$42,769.89 \end{aligned}$$

13. $A = \$7300$, $r = 5\%$ compounded continuously, $t = 3$

$$\begin{aligned} A &= Pe^{rt} \\ P &= \frac{A}{e^{rt}} \\ &= \frac{7300}{e^{0.05(3)}} \approx \$6283.17 \end{aligned}$$

14. $A = \$25,000$, $r = 4.6\%$ compounded continuously, $t = 8$

$$\begin{aligned} A &= Pe^{rt} \\ P &= \frac{A}{e^{rt}} \\ &= \frac{25,000}{e^{0.046(8)}} \approx \$17,302.93 \end{aligned}$$

15. $r = 9\%$ compounded semiannually

$$\begin{aligned} r_E &= \left(1 + \frac{0.09}{2}\right)^2 - 1 \\ &\approx 0.0920 = 9.20\% \end{aligned}$$

16. $r = 6.2\%$, $m = 4$

$$\begin{aligned} r_E &= \left(1 + \frac{0.062}{4}\right)^4 - 1 \\ &\approx 0.0635 = 6.35\% \end{aligned}$$

17. $r = 6\%$ compounded monthly

$$\begin{aligned} r_E &= \left(1 + \frac{0.06}{12}\right)^{12} - 1 \\ &\approx 0.0617 \\ &\approx 6.17\% \end{aligned}$$

18. $A = \$20,000$, $t = 4$, $r = 6.5\%$, $m = 12$

$$\begin{aligned} A &= P \left(1 + \frac{r}{m}\right)^{mt} \\ 20,000 &= P \left(1 + \frac{0.065}{12}\right)^{12(4)} \\ \frac{20,000}{(1.05416)^{48}} &= P \\ \$15,431.86 &= P \end{aligned}$$

19. (a) $A = \$307,000$, $t = 3$, $r = 6\%$, $m = 2$

$$\begin{aligned} A &= P \left(1 + \frac{r}{m}\right)^{mt} \\ 307,000 &= P \left(1 + \frac{0.06}{2}\right)^{3(2)} \\ 307,000 &= P(1.03)^6 \\ \frac{307,000}{(1.03)^6} &= P \\ \$257,107.67 &= P \end{aligned}$$

$$\begin{aligned} \text{(b) Interest} &= 307,000 - 257,107.67 \\ &= \$49,892.33 \end{aligned}$$

$$\text{(c) } P = \$200,000$$

$$\begin{aligned} A &= 200,000(1.03)^6 \\ &= 238,810.46 \end{aligned}$$

The additional amount needed is

$$\begin{aligned} &307,000 - 238,810.46 \\ &= \$68,189.54. \end{aligned}$$

(d) $A = Pe^{rt}$

$$307,000 = 200,000e^{3r}$$

$$1.535 = e^{3r}$$

$$\ln 1.535 = \ln e^{3r}$$

$$3r = \ln 1.535$$

$$r = \frac{\ln 1.535}{3} \approx 0.1428$$

The interest rate needed is 14.28%.

20. $A = \$40,000, t = 5$

(a) $r = 0.064, m = 4$

$$A = P \left(1 + \frac{r}{m} \right)^{mt}$$

$$40,000 = P \left(1 + \frac{0.064}{4} \right)^{5(4)}$$

$$\frac{40,000}{(1.016)^{20}} = P$$

$$\$29,119.63 = P$$

(b) Interest = $\$40,000 - \$29,119.63$
 $= \$10,880.37$

(c) $P = \$20,000$

$$A = 20,000 \left(1 + \frac{0.064}{4} \right)^{5(4)}$$

$$= \$27,472.88$$

The amount needed will be

$$\$40,000 - \$27,472.88 = \$12,527.12.$$

(d) $A = Pe^{rt}$

$$40,000 = 20,000e^{5r}$$

$$2 = e^{5r}$$

$$\ln 2 = \ln e^{5r}$$

$$5r = \ln 2$$

$$r = \frac{\ln 2}{5} \approx 0.1386$$

The interest rate needed is 13.86%.

21. $P = \$60,000$

(a) $r = 8\%$ compounded quarterly:

$$A = P \left(1 + \frac{r}{m} \right)^{tm}$$

$$= 60,000 \left(1 + \frac{0.08}{4} \right)^{5(4)}$$

$$\approx \$89,156.84$$

$r = 7.75\%$ compounded continuously

$$A = Pe^{rt}$$

$$= 60,000e^{0.775(5)}$$

$$\approx \$88,397.58$$

Linda will earn more money at 8% compounded quarterly.

(b) She will earn \$759.26 more.

(c) $r = 8\%, m = 4$:

$$r_E = \left(1 + \frac{r}{m} \right)^m - 1$$

$$= \left(1 + \frac{0.08}{4} \right)^4 - 1$$

$$\approx 0.0824$$

$$= 8.24\%$$

$r = 7.75\%$ compounded continuously:

$$r_E = e^r - 1$$

$$= e^{0.0775} - 1$$

$$\approx 0.0806$$

$$= 8.06\%$$

(d) $A = \$80,000$

$$A = Pe^{rt}$$

$$80,000 = 60,000e^{0.0775t}$$

$$\frac{4}{3} = e^{0.0775t}$$

$$\ln \frac{4}{3} = \ln e^{0.0775t}$$

$$\ln 4 - \ln 3 = 0.0775t$$

$$\frac{\ln 4 - \ln 3}{0.0775} = t$$

$$3.71 = t$$

\$60,000 will grow to \$80,000 in about 3.71 years.

$$(e) \quad 60,000 \left(1 + \frac{0.08}{4} \right)^{4x} \geq 80,000$$

$$(1.02)^{4x} \geq \frac{80,000}{60,000}$$

$$(1.02)^{4x} \geq \frac{4}{3}$$

$$\log (1.02)^{4x} \geq \log \left(\frac{4}{3} \right)$$

$$4x \log (1.02) \geq \log \left(\frac{4}{3} \right)$$

$$x \geq \frac{\log \left(\frac{4}{3} \right)}{4 \log (1.02)} \approx 3.63$$

We need to round up to the nearest quarter, so it will take 3.75 years.

$$22. (a) \quad 11,000 = 5000 \left(1 + \frac{0.063}{4} \right)^{4t}$$

$$\frac{11}{5} = (1.01575)^{4t}$$

$$\ln \left(\frac{11}{5} \right) = 4t \ln 1.01575$$

$$t = \frac{\ln \left(\frac{11}{5} \right)}{4 \ln 1.01575}$$

$$t \approx 12.61$$

Since the interest is only added at the end of the quarter, it will take 12.75 years.

$$(b) \quad 11,000 = 5000e^{0.063t}$$

$$\frac{11}{5} = e^{0.063t}$$

$$0.063t = \ln \left(\frac{11}{5} \right)$$

$$t = \frac{\ln \left(\frac{11}{5} \right)}{0.063}$$

$$t \approx 12.52$$

It will take about 12.52 years.

$$23. \quad S(x) = 1000 - 800e^{-x}$$

$$(a) \quad \begin{aligned} S(0) &= 1000 - 800e^0 \\ &= 1000 - 800 \\ &= 200 \end{aligned}$$

$$(b) \quad S(x) = 500$$

$$500 = 1000 - 800e^{-x}$$

$$-500 = -800e^{-x}$$

$$\frac{5}{8} = e^{-x}$$

$$\ln \frac{5}{8} = \ln e^{-x}$$

$$-\ln \frac{5}{8} = x$$

$$0.47 \approx x$$

Sales reach 500 in about $\frac{1}{2}$ year.

(c) Since $800e^{-x}$ will never actually be zero, $S(x) = 1000 - 800e^{-x}$ will never be 1000.

(d) Graphing the function $y = S(x)$ on a graphing calculator will show that there is a horizontal asymptote at $y = 1000$. This indicates that the limit on sales is 1000 units.

$$24. \quad S(x) = 5000 - 4000e^{-x}$$

$$(a) \quad \begin{aligned} S(0) &= 5000 - 4000e^0 \\ &= 1000 \end{aligned}$$

Since $S(x)$ represents sales in thousands, in year 0 the sales are \$1,000,000.

$$(b) \quad \text{Let } S(x) = 4500.$$

$$4500 = 5000 - 4000e^{-x}$$

$$-500 = -4000e^{-x}$$

$$0.125 = e^{-x}$$

$$-\ln 0.125 = x$$

$$x \approx 2$$

It will take about 2 years for sales to reach \$4,500,000.

(c) Graphing the function $y = S(x)$ on a graphing calculator will show that there is a horizontal asymptote at $y = 5000$. Since this represents \$5000 thousand or \$5,000,000, the limit on sales is \$5,000,000.

$$25. (a) \quad P = P_0 e^{kt}$$

$$\text{When } t = 1650, P = 500.$$

$$\text{When } t = 2010, P = 6756.$$

$$500 = P_0 e^{1650k}$$

$$6756 = P_0 e^{2010k}$$

$$\frac{6756}{500} = \frac{P_0 e^{2010k}}{P_0 e^{1650k}}$$

$$\frac{6756}{500} = e^{360k}$$

$$360k = \ln\left(\frac{6756}{500}\right)$$

$$k = \frac{\ln\left(\frac{6756}{500}\right)}{360}$$

$$k = 0.007232$$

Substitute this value into $500 = P_0 e^{1650k}$ to find P_0 .

$$500 = P_0 e^{1650(0.007232)}$$

$$P_0 = \frac{500}{e^{1650(0.007232)}}$$

$$P_0 = 0.003286$$

Therefore, $P(t) = 0.003286e^{0.007232t}$.

$$\begin{aligned} \text{(b)} \quad P(1) &= 0.003286e^{0.007232} \\ &\approx 0.0033 \text{ million, or } 3300. \end{aligned}$$

The exponential equation gives a world population of only 3300 in the year 1.

- (c) No, the answer in part (b) is too small. Exponential growth does not accurately describe population growth for the world over a long period of time.

26. (a) $y = 2y_0$ after 12 hours.

$$y = y_0 e^{kt}$$

$$2y_0 = y_0 e^{12k}$$

$$2 = e^{12k}$$

$$\ln 2 = \ln e^{12k}$$

$$12k = \ln 2$$

$$k = \frac{\ln 2}{12} \approx 0.05776$$

$$y = y_0 e^{0.05776t}$$

$$\begin{aligned} \text{(b)} \quad y &= y_0 e^{(\ln 2/12)t} \\ &= y_0 e^{(\ln 2)(t/12)} \\ &= y_0 [e^{\ln 2}]^{t/12} \\ &= y_0 2^{t/12} \text{ since } e^{\ln 2} = 2 \end{aligned}$$

(c) For 10 days, $t = 10 \cdot 24$ or 240.

$$\begin{aligned} y &= (1)2^{240/12} \\ &= 2^{20} \\ &= 1,048,576 \end{aligned}$$

For 15 days, $t = 15 \cdot 24$ or 360.

$$\begin{aligned} y &= (1)2^{360/12} \\ &= 2^{30} \\ &= 1,073,741,824 \end{aligned}$$

27. $y = y_0 e^{kt}$
 $y = 40,000, y_0 = 25,000, t = 10$

(a) $40,000 = 25,000e^{k(10)}$

$$1.6 = e^{10k}$$

$$\ln 1.6 = 10k$$

$$0.047 = k$$

The equation is

$$y = 25,000e^{0.047t}$$

(b) $y = 25,000e^{0.047t}$

$$= 25,000(e^{0.047})^t$$

$$= 25,000(1.048)^t$$

(c) $y = 60,000$

$$60,000 = 25,000e^{0.047t}$$

$$2.4 = e^{0.047t}$$

$$\ln 2.4 = 0.047t$$

$$18.6 = t$$

There will be 60,000 bacteria in about 18.6 hours.

28. $y = y_0 e^{kt}$

(a) $y = 20,000, y_0 = 50,000, t = 9$

$$20,000 = 50,000e^{9k}$$

$$0.40 = e^{9k}$$

$$\ln 0.4 = 9k$$

$$-0.102 = k$$

The equation is

$$y = 50,000e^{-0.102t}$$

$$(b) \quad \frac{1}{2}(50,000) = 25,000$$

$$25,000 = 50,000e^{-0.102t}$$

$$0.5 = e^{-0.102t}$$

$$\ln 0.5 = -0.102t$$

$$6.8 = t$$

Half the bacteria remain after about 6.8 hours.

$$29. \quad f(t) = 500e^{0.1t}$$

$$(a) \quad f(t) = 3000$$

$$3000 = 500e^{0.1t}$$

$$6 = e^{0.1t}$$

$$\ln 6 = 0.1t$$

$$17.9 \approx t$$

It will take 17.9 days.

- (b) If $t = 0$ corresponds to January 1, the date January 17 should be placed on the product. January 18 would be more than 17.9 days.

$$30. \quad \text{Use } y = y_0e^{-kt}.$$

When $t = 5$, $y = 0.37y_0$.

$$0.37y_0 = y_0e^{-5k}$$

$$0.37 = e^{-5k}$$

$$-5k = \ln(0.37)$$

$$k = \frac{\ln(0.37)}{-5}$$

$$k \approx 0.1989$$

31. (a) From the graph, the risks of chromosomal abnormality per 1000 at ages 20, 35, 42, and 49 are 2, 5, 24, and 125, respectively.

(Note: It is difficult to read the graph accurately. If you read different values from the graph, your answers to parts (b)-(e) may differ from those given here.)

$$(b) \quad y = Ce^{kt}$$

When $t = 20$, $y = 2$, and when $t = 35$, $y = 5$.

$$2 = Ce^{20k}$$

$$5 = Ce^{35k}$$

$$\frac{5}{2} = \frac{Ce^{35k}}{Ce^{20k}}$$

$$2.5 = e^{15k}$$

$$15k = \ln 2.5$$

$$k = \frac{\ln 2.5}{15} \quad k \approx 0.061$$

$$(c) \quad y = Ce^{kt}$$

When $t = 42$, $y = 29$, and when $t = 49$, $y = 125$.

$$24 = Ce^{42k}$$

$$125 = Ce^{49k}$$

$$\frac{125}{24} = \frac{Ce^{49k}}{Ce^{42k}}$$

$$\frac{125}{24} = e^{7k}$$

$$7k = \ln\left(\frac{125}{24}\right)$$

$$k = \frac{\ln\left(\frac{125}{24}\right)}{7} \quad k \approx 0.24$$

- (d) Since the values of k are different, we cannot assume the graph is of the form $y = Ce^{kt}$.

- (e) The results are summarized in the following table.

n	Value of k for [20, 35]	Value of k for [42, 49]
2	0.00093	0.0017
3	2.3×10^{-5}	2.5×10^{-5}
4	6.3×10^{-7}	4.1×10^{-7}

The value of n should be somewhere between 3 and 4.

32.

$$A(t) = A_0e^{kt}$$

$$0.60A_0 = A_0e^{(-\ln 2/5600)t}$$

$$0.60 = e^{(-\ln 2/5600)t}$$

$$\ln 0.60 = -\frac{\ln 2}{5600}t$$

$$\frac{5600(\ln 0.60)}{-\ln 2} = t$$

$$4127 \approx t$$

The sample was about 4100 years old.

$$\begin{aligned}
 33. \quad \frac{1}{2}A_0 &= A_0e^{-0.053t} \\
 \frac{1}{2} &= e^{-0.053t} \\
 \ln \frac{1}{2} &= -0.053t \\
 -\ln 2 &= -0.053t \\
 t &= \frac{\ln 2}{0.52} \\
 t &\approx 13
 \end{aligned}$$

The half-life of plutonium 241 is about 13 years.

$$\begin{aligned}
 34. \quad \frac{1}{2}A_0 &= A_0e^{-0.00043t} \\
 \frac{1}{2} &= e^{-0.00043t} \\
 \ln \frac{1}{2} &= -0.00043t \\
 -\ln 2 &= -0.00043t \\
 t &= \frac{\ln 2}{0.00043} \\
 t &\approx 1612
 \end{aligned}$$

The half-life of radium 226 is about 1600 years.

$$\begin{aligned}
 35. \quad (a) \quad A(t) &= A_0\left(\frac{1}{2}\right)^{t/13} \\
 A(100) &= 4.0\left(\frac{1}{2}\right)^{100/13} \\
 A(100) &\approx 0.0193
 \end{aligned}$$

After 100 years, about 0.0193 gram will remain.

$$\begin{aligned}
 (b) \quad 0.1 &= 4.0\left(\frac{1}{2}\right)^{t/13} \\
 \frac{0.1}{4.0} &= \left(\frac{1}{2}\right)^{t/13} \\
 \ln 0.025 &= \frac{t}{13} \ln \left(\frac{1}{2}\right) \\
 t &= \frac{13 \ln 0.025}{\ln \left(\frac{1}{2}\right)} \\
 t &\approx 69.19
 \end{aligned}$$

It will take 69 years.

$$\begin{aligned}
 36. \quad (a) \quad A(t) &= A_0\left(\frac{1}{2}\right)^{t/1620} \\
 A(100) &= 4.0\left(\frac{1}{2}\right)^{100/1620} \\
 A(100) &= 3.8
 \end{aligned}$$

After 100 years, about 3.8 grams will remain.

$$\begin{aligned}
 (b) \quad 0.1 &= 4.0\left(\frac{1}{2}\right)^{t/1620} \\
 \frac{0.1}{4} &= \left(\frac{1}{2}\right)^{t/1620} \\
 \ln 0.025 &= \frac{t}{1620} \ln \frac{1}{2} \\
 t &= \frac{1600 \ln 0.025}{\ln \left(\frac{1}{2}\right)} \\
 t &= 8600
 \end{aligned}$$

The half-life is about 8600 years.

$$37. \quad (a) \quad y = y_0e^{kt}$$

When $t = 0$, $y = 500$, so $y_0 = 500$.

When $t = 3$, $y = 386$.

$$\begin{aligned}
 386 &= 500e^{3k} \\
 \frac{386}{500} &= e^{3k} \\
 e^{3k} &= 0.772 \\
 3k &= \ln 0.772 \\
 k &= \frac{\ln 0.772}{3} \\
 k &\approx -0.0863 \\
 y &= 500e^{-0.0863t}
 \end{aligned}$$

(b) From part (a), we have

$$\begin{aligned}
 k &= \frac{\ln \left(\frac{386}{500}\right)}{3} \\
 y &= 500e^{kt} \\
 &= 500e^{[\ln(386/500)/3]t} \\
 &= 500e^{\ln(386/500) \cdot (t/3)} \\
 &= 500[e^{\ln(386/500)}]^{t/3} \\
 &= 500(386/500)^{t/3} \\
 &= 500(0.772)^{t/3}
 \end{aligned}$$

$$(c) \quad \frac{1}{2}y_0 = y_0 e^{-0.0863t}$$

$$\ln \frac{1}{2} = -0.0863t$$

$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.0863}$$

$$t \approx 8.0$$

The half-life is about 8.0 days.

$$38. (a) \quad y = y_0 e^{kt}$$

When $t = 0$, $y = 25.0$, so $y_0 = 25.0$

When $t = 50$, $y = 19.5$

$$19.5 = 25.0e^{50k}$$

$$\frac{19.5}{25.0} = e^{50k}$$

$$50k = \ln\left(\frac{19.5}{25.0}\right)$$

$$k = \frac{\ln\left(\frac{19.5}{25.0}\right)}{50}$$

$$k = -0.00497$$

$$y = 25.0e^{-0.00497t}$$

(b) From part (a), we have

$$k = \frac{\ln\left(\frac{386}{500}\right)}{3}$$

$$\begin{aligned} y &= 25.0e^{kt} \\ &= 25.0e^{[(\ln(19.5/25.0)/50)]t} \\ &= 25.0e^{\ln(19.5/25.0) \cdot (t/50)} \\ &= 25.0[e^{\ln(19.5/25.0)}]^{t/50} \\ &= 25.0(19.5/25.0)^{t/50} \\ &= 25(0.78)^{t/50} \end{aligned}$$

$$(c) \quad \frac{1}{2}y_0 = y_0 e^{-0.00497t}$$

$$\frac{1}{2} = e^{-0.00497t}$$

$$-0.00497t = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.00497}$$

$$t \approx 139$$

The half-life is about 139 days.

$$39. \quad y = 40e^{-0.004t}$$

$$(a) \quad t = 180$$

$$\begin{aligned} y &= 40e^{-0.004(180)} = 40e^{-0.72} \\ &\approx 19.5 \text{ watts} \end{aligned}$$

$$(b) \quad 20 = 20e^{-0.0004t}$$

$$\frac{1}{2} = e^{-0.0004t}$$

$$\ln \frac{1}{2} = -0.0004t$$

$$\frac{\ln 2}{0.004} = t$$

$$173 \approx t$$

It will take about 173 days.

(c) The power will never be completely gone. The power will approach 0 watts but will never be exactly 0.

$$40. \quad A(t) = A_0 \left(\frac{1}{2}\right)^{t/5600}$$

$$\begin{aligned} A(43,000) &= A_0 \left(\frac{1}{2}\right)^{43,000/5600} \\ &\approx 0.005A_0 \end{aligned}$$

About 0.5% of the original carbon 14 was present.

$$41. \quad P(t) = 100e^{-0.1t}$$

$$(a) \quad P(4) = 100e^{-0.1(4)} \approx 67\%$$

$$(b) \quad P(10) = 100e^{-0.1(10)} \approx 37\%$$

$$(c) \quad 10 = 100e^{-0.1t}$$

$$0.1 = e^{-0.1t}$$

$$\ln(0.1) = -0.1t$$

$$\frac{-\ln(0.1)}{0.1} = t$$

$$23 \approx t$$

It would take about 23 days.

$$(d) \quad 1 = 100e^{-0.1t}$$

$$0.01 = e^{-0.1t}$$

$$\ln(0.01) = -0.1t$$

$$\frac{-\ln(0.01)}{0.1} = t$$

$$46 \approx t$$

It would take about 46 days.

42. (a) Let t = the number of degrees Celsius.

$$y = y_0 \cdot e^{kt}$$

$$y_0 = 10 \text{ When } t = 0^\circ.$$

To find k , let $y = 11$ when $t = 10^\circ$.

$$11 = 10e^{10k}$$

$$e^{10k} = \frac{11}{10}$$

$$10k = \ln 1.1$$

$$k = \frac{\ln 1.1}{10}$$

$$\approx 0.0095$$

The equation is

$$y = 10e^{0.0095t}.$$

- (b) Let $y = 1.5$; solve for t .

$$15 = 10e^{0.0095t}$$

$$\ln 1.5 = 0.0095t$$

$$t = \frac{\ln 1.5}{0.0095t}$$

$$\approx 42.7$$

15 grams will dissolve at 42.7°C .

43. $t = 9, T_0 = 18, C = 5, k = .6$

$$f(t) = T_0 + Ce^{-kt}$$

$$f(t) = 18 + 5e^{-0.6(9)}$$

$$= 18 + 5e^{-5.4}$$

$$\approx 18.02$$

The temperature is about 18.02° .

44. $f(t) = T_0 + Ce^{-kt}$
- $$25 = 20 + 100e^{-0.1t}$$
- $$5 = 100e^{-0.1t}$$
- $$e^{-0.1t} = 0.05$$
- $$-0.1t = \ln 0.05$$
- $$t = \frac{\ln 0.05}{-0.1}$$
- $$\approx 30$$

It will take about 30 min.

45. $C = -14.6, k = 0.6, T_0 = 18^\circ$,
- $$f(t) = 10^\circ$$
- $$f(t) = T_0 + Ce^{-kt}$$

$$f(t) = 18 + (-14.6)e^{-0.6t}$$

$$-8 = -14.6e^{-0.6t}$$

$$0.5479 = e^{-0.6t}$$

$$\ln 0.5479 = -0.6t$$

$$\frac{-\ln 0.5479}{0.6} = t$$

$$1 \approx t$$

It would take about 1 hour for the pizza to thaw.

Chapter 2 Review Exercises

1. True
2. False; for example $f(x) = \frac{x}{x+1}$ is a rational function but not an exponential function.
3. True
4. True
5. False; an exponential function has the form $f(x) = a^x$.
6. False; the vertical asymptote is at $x = 6$.
7. True
8. False; the domain includes all numbers except $x = 2$ and $x = -2$.
9. False; the amount is $A = 2000\left(1 + \frac{0.04}{12}\right)^{24}$.
10. False; the logarithmic function $f(x) = \log_a x$ is not defined for $a = 1$.
11. False; $\ln(5 + 7) = \ln 12 \neq \ln 5 + \ln 7$
12. False; $(\ln 3)^4 \neq 4 \ln 3$ since $(\ln 3)^4$ means $(\ln 3)(\ln 3)(\ln 3)(\ln 3)$.
13. False; $\log_{10} 0$ is undefined since $10^x = 0$ has no solution.
14. True
15. False; $\ln(-2)$ is undefined.

16. False; $\frac{\ln 4}{\ln 8} = 0.6667$ and
 $\ln 4 - \ln 8 = \ln(1/2) \approx -0.6931$.

17. True

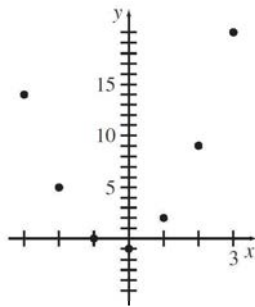
18. True

23. $y = (2x - 1)(x + 1)$
 $= 2x^2 + x - 1$

x	-3	-2	-1	0	1	2	3
y	14	5	0	-1	1	9	20

Pairs: $(-3, 14), (-2, 5), (-1, 0), (0, -1), (1, 2),$
 $(2, 9), (3, 20)$

Range: $\{-1, 0, 2, 5, 9, 14, 20\}$

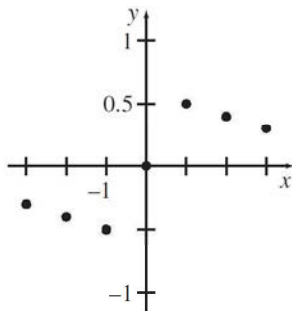


24. $y = \frac{x}{x^2 + 1}$

x	-3	-2	-1	0	1	2	3
y	$-\frac{3}{10}$	$-\frac{2}{5}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{3}{10}$

Pairs: $\left(-3, -\frac{3}{10}\right), \left(-2, -\frac{2}{5}\right), \left(-1, -\frac{1}{2}\right),$
 $(0, 0), \left(1, \frac{1}{2}\right), \left(2, \frac{2}{5}\right), \left(3, \frac{3}{10}\right)$

Range: $\left\{-\frac{1}{2}, -\frac{2}{5}, -\frac{3}{10}, 0, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}\right\}$



25. $f(x) = 5x^2 - 3$ and $g(x) = -x^2 + 4x + 1$

(a) $f(-2) = 5(-2)^2 - 3 = 17$

(b) $g(3) = -(3)^2 + 4(3) + 1 = 4$

(c) $f(-k) = 5(-k)^2 - 3 = 5k^2 - 3$

(d) $g(3m) = -(3m)^2 + 4(3m) + 1$
 $= -9m^2 + 12m + 1$

(e) $f(x + h) = 5(x + h)^2 - 3$
 $= 5(x^2 + 2xh + h^2) - 3$
 $= 5x^2 + 10xh + 5h^2 - 3$

(f)

$g(x + h) = -(x + h)^2 + 4(x + h) + 1$
 $= -(x^2 + 2xh + h^2) + 4x + 4h + 1$
 $= -x^2 - 2xh - h^2 + 4x + 4h + 1$

(g) $\frac{f(x + h) - f(x)}{h}$
 $= \frac{5(x + h)^2 - 3 - (5x^2 - 3)}{h}$
 $= \frac{5(x^2 + 2xh + h^2) - 3 - 5x^2 + 3}{h}$
 $= \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h}$
 $= \frac{10xh + 5h^2}{h} = 10x + 5h$

(h)

$\frac{g(x + h) - g(x)}{h}$
 $= \frac{-(x + h)^2 + 4(x + h) + 1 - (-x^2 + 4x + 1)}{h}$
 $= \frac{-(x^2 + 2xh + h^2) + 4x + 4h + 1 + x^2 - 4x - 1}{h}$
 $= \frac{-x^2 - 2xh - h^2 + 4h + x^2}{h}$
 $= \frac{-2xh - h^2 + 4h}{h}$
 $= -2x - h + 4$

26. $f(x) = 2x^2 + 5$ and $g(x) = 3x^2 + 4x - 1$

(a) $f(-3) = 2(-3)^2 + 5 = 23$

(b) $g(2) = 3(2)^2 + 4(2) - 1 = 19$

(c) $f(3m) = 2(3m)^2 + 5 = 18m^2 + 5$

$$\begin{aligned} \text{(d)} \quad g(-k) &= 3(-k)^2 + 4(-k) - 1 \\ &= 3k^2 - 4k - 1 \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad f(x+h) &= 2(x+h)^2 + 5 \\ &= 2(x^2 + 2xh + h^2) + 5 \\ &= 2x^2 + 4xh + 2h^2 + 5 \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad g(x+h) &= 3(x+h)^2 + 4(x+h) - 1 \\ &= 3(x^2 + 2xh + h^2) + 4x + 4h - 1 \\ &= 3x^2 + 6xh + 3h^2 + 4x + 4h - 1 \end{aligned}$$

$$\begin{aligned} \text{(g)} \quad \frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 + 5 - (2x^2 + 5)}{h} \\ &= \frac{2(x^2 + 2xh + h^2) + 5 - 2x^2 - 5}{h} \\ &= \frac{2x^2 + 4xh + 2h^2 + 5 - 2x^2 - 5}{h} \\ &= \frac{4xh + 2h^2}{h} \\ &= 4x + 2h \end{aligned}$$

$$\begin{aligned} \text{(h)} \quad \frac{g(x+h) - g(x)}{h} &= \frac{3(x+h)^2 + 4(x+h) - 1 - (3x^2 + 4x - 1)}{h} \\ &= \frac{3(x^2 + 2xh + h^2) + 4x + 4h - 1 - 3x^2 - 4x + 1}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 + 4x + 4h - 1 - 3x^2 - 4x + 1}{h} \\ &= \frac{6xh + 3h^2 + 4h}{h} \\ &= 6x + 3h + 4 \end{aligned}$$

$$\begin{aligned} 27. \quad y &= \frac{3x-4}{x} \\ x &\neq 0 \\ \text{Domain: } &(-\infty, 0) \cup (0, \infty) \end{aligned}$$

$$28. \quad y = \frac{\sqrt{x-2}}{2x+3}$$

$$\begin{aligned} x-2 &\geq 0 \quad \text{and} \quad 2x+3 \neq 0 \\ x &\geq 2 \quad \quad \quad 2x \neq -3 \\ &\quad \quad \quad x \neq -\frac{3}{2} \end{aligned}$$

Domain: $[2, \infty)$

$$\begin{aligned} 29. \quad y &= \ln(x+7) \\ x+7 &> 0 \\ x &> -7 \end{aligned}$$

Domain: $(-7, \infty)$.

$$\begin{aligned} 30. \quad y &= \ln(x^2 - 16) \\ x^2 - 16 &> 0 \\ x^2 &> 16 \\ x &> 4 \quad \text{or} \quad x < -4 \end{aligned}$$

Domain: $(-\infty, -4) \cup (4, \infty)$

$$\begin{aligned} 31. \quad y &= 2x^2 + 3x - 1 \\ \text{The graph is a parabola.} \\ \text{Let } y &= 0. \end{aligned}$$

$$\begin{aligned} 0 &= 2x^2 + 3x - 1 \\ x &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9+8}}{4} \\ &= \frac{-3 \pm 17}{4} \end{aligned}$$

The x -intercepts are $\frac{-3+\sqrt{17}}{4} \approx 0.28$ and $\frac{-3-\sqrt{17}}{4} \approx -1.48$.

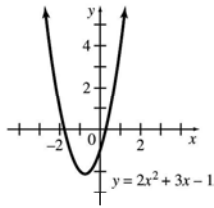
Let $x = 0$.

$$\begin{aligned} y &= 2(0)^2 + 3(0) - 1 \\ -1 &\text{ is the } y\text{-intercept.} \end{aligned}$$

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-3}{2(2)} = -\frac{3}{4}$$

$$\begin{aligned} y &= 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) - 1 \\ &= \frac{9}{8} - \frac{9}{4} - 1 \\ &= -\frac{17}{8} \end{aligned}$$

The vertex is $(-\frac{3}{4}, -\frac{17}{8})$.



32. $y = -\frac{1}{4}x^2 + x + 2$

The graph is a parabola.

Let $y = 0$.

$$0 = -\frac{1}{4}x^2 + x + 2$$

Multiply by 4.

$$\begin{aligned} 0 &= -x^2 + 4x + 8 \\ x &= \frac{-4 \pm \sqrt{4^2 - 4(-1)(8)}}{2(-1)} \\ &= \frac{-4 \pm \sqrt{48}}{-2} = 2 \pm 2\sqrt{3} \end{aligned}$$

The x -intercepts are $2 + 2\sqrt{3} \approx 5.46$ and $2 - 2\sqrt{3} \approx -1.46$.

Let $x = 0$.

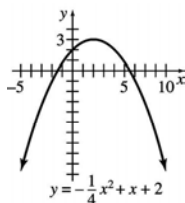
$$y = -\frac{1}{4}(0)^2 + 0 + 2$$

$y = 2$ is the y -intercept.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-1}{2(-\frac{1}{4})} = 2$$

$$\begin{aligned} y &= -\frac{1}{4}(2)^2 + 2 + 2 \\ &= -1 + 4 \\ &= 3 \end{aligned}$$

The vertex is $(2, 3)$.



33. $y = -x^2 + 4x + 2$

Let $y = 0$.

$$\begin{aligned} 0 &= -x^2 + 4x + 2 \\ x &= \frac{-4 \pm \sqrt{4^2 - 4(-1)(2)}}{2(-1)} \\ &= \frac{-4 \pm \sqrt{24}}{-2} \\ &= 2 \pm \sqrt{6} \end{aligned}$$

The x -intercepts are $2 + \sqrt{6} \approx 4.45$ and $2 - \sqrt{6} \approx -0.45$.

Let $x = 0$.

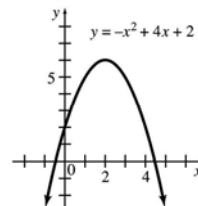
$$y = -0^2 + 4(0) + 2$$

2 is the y -intercept.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-4}{2(-1)} = \frac{-4}{-2} = 2$$

$$y = -2^2 + 4(2) + 2 = 6$$

The vertex is $(2, 6)$.

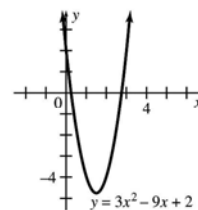


34. $y = 3x^2 - 9x + 2$

x -intercepts: 0.24 and 2.76

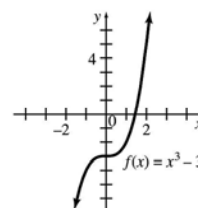
y -intercept: 2

$$\text{Vertex: } (\frac{3}{2}, -\frac{19}{4})$$



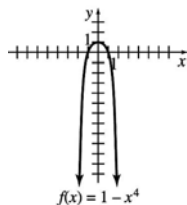
35. $f(x) = x^3 - 3$

Translate the graph of $f(x) = x^3$ 3 units down.



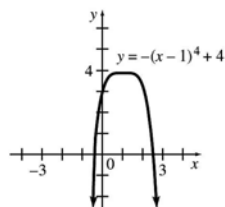
36. $f(x) = 1 - x^4$
 $= -x^4 + 1$

Reflect the graph of $y = x^4$ vertically then translate 1 unit upward.



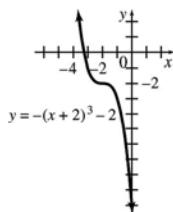
37. $y = -(x - 1)^4 + 4$

Translate the graph of $y = x^4$ 1 unit to the right and reflect vertically. Translate 4 units upward.



38. $y = -(x + 2)^3 - 2$

Translate the graph of $y = x^3$ 2 units to the left and reflect vertically. Translate 2 units downward.



39. $f(x) = \frac{8}{x}$

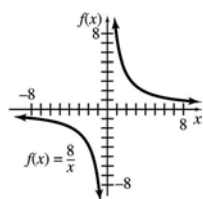
Vertical asymptote: $x = 0$

Horizontal asymptote:

$\frac{8}{x}$ approaches zero as x gets larger.

$y = 0$ is an asymptote.

x	-4	-3	-2	-1	1	2	3	4
y	-2	-2.7	-4	-8	8	4	2.7	2

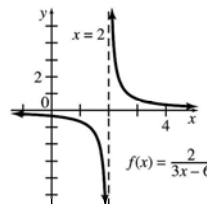


40. $f(x) = \frac{2}{3x - 6}$

Vertical asymptote: $3x - 6 = 0$ or $x = 2$

Horizontal asymptote: $y = 0$, since $\frac{2}{3x-6}$ approaches zero as x gets larger.

x	0	1	3	4	5
y	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{9}$



41. $f(x) = \frac{4x - 2}{3x + 1}$

Vertical asymptote:

$$3x + 1 = 0$$

$$x = -\frac{1}{3}$$

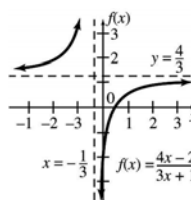
Horizontal asymptote:

As x gets larger,

$$\frac{4x - 2}{3x + 1} \approx \frac{4x}{3x} = \frac{4}{3}.$$

$y = \frac{4}{3}$ is an asymptote.

x	-3	-2	-1	0	1	2	3
y	1.75	2	3	-2	0.5	0.86	1

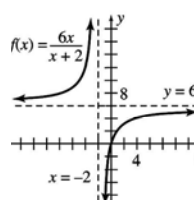


42. $f(x) = \frac{6x}{x + 2}$

Vertical asymptote: $x = -2$

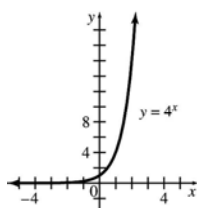
Horizontal asymptote: $y = 6$

x	-5	-4	-3	-1	0	1	2
y	10	12	18	-6	0	2	3



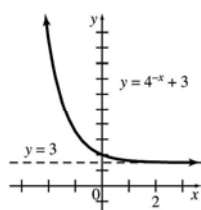
43. $y = 4^x$

x	-2	-1	0	1	2
y	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16



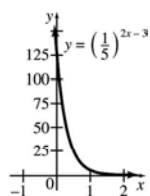
44. $y = 4^{-x} + 3$

x	-2	-1	0	1	2
y	19	7	4	$\frac{13}{4}$	$\frac{49}{16}$



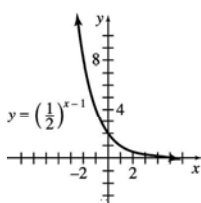
45. $y = \left(\frac{1}{5}\right)^{2x-3}$

x	0	1	2
y	125	5	$\frac{1}{5}$



46. $y = \left(\frac{1}{2}\right)^{x-1}$

x	-2	-1	0	1	2
y	8	4	2	1	$\frac{1}{2}$

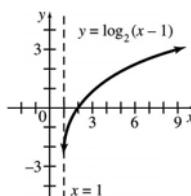


47. $y = \log_2(x-1)$

$2^y = x - 1$

$x = 1 + 2^y$

x	2	3	5	9
y	0	1	2	3

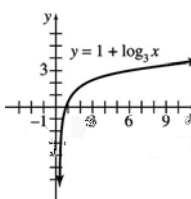


48. $y = 1 + \log_3 x$

$y - 1 = \log_3 x$

$3^{y-1} = x$

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
y	-1	0	1	2	3



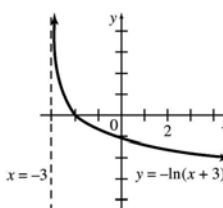
49. $y = -\ln(x+3)$

$-y = \ln(x+3)$

$e^{-y} = x+3$

$e^{-y} - 3 = x$

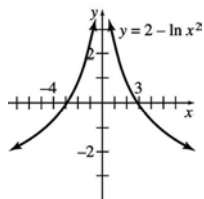
x	-2.63	-2	-0.28	4.39
y	1	0	-1	-2



50. $y = 2 - \ln x^2$

x	-4	-3	-2	-1
y	-0.8	-0.2	0.6	2

x	1	2	3	4
y	2	0.6	-0.2	-0.8



$$\begin{aligned}
 51. \quad 2^{x+2} &= \frac{1}{8} \\
 2^{x+2} &= \frac{1}{2^3} \\
 2^{x+2} &= 2^{-3} \\
 x + 2 &= -3 \\
 x &= -5
 \end{aligned}$$

$$\begin{aligned}
 52. \quad \left(\frac{9}{16}\right)^x &= \frac{3}{4} \\
 \left(\frac{3}{4}\right)^{2x} &= \left(\frac{3}{4}\right)^1 \\
 2x &= 1 \\
 x &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 53. \quad 9^{2y+3} &= 27^y \\
 (3^2)^{2y+3} &= (3^3)^y \\
 3^{4y+6} &= 3^{3y} \\
 4y + 6 &= 3y \\
 y &= -6
 \end{aligned}$$

$$\begin{aligned}
 54. \quad \frac{1}{2} &= \left(\frac{b}{4}\right)^{1/4} \\
 \left(\frac{1}{2}\right)^4 &= \frac{b}{4} \\
 4\left(\frac{1}{2}\right)^4 &= b \\
 4\left(\frac{1}{16}\right) &= b \\
 \frac{1}{4} &= b
 \end{aligned}$$

$$\begin{aligned}
 55. \quad 3^5 &= 243 \\
 \text{The equation in logarithmic form is} \\
 \log_3 243 &= 5.
 \end{aligned}$$

$$\begin{aligned}
 56. \quad 5^{1/2} &= \sqrt{5} \\
 \text{The equation in logarithmic form is} \\
 \log_5 \sqrt{5} &= \frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 57. \quad e^{0.8} &= 2.22554 \\
 \text{The equation in logarithmic form is} \\
 \ln 2.22554 &= 0.8.
 \end{aligned}$$

$$\begin{aligned}
 58. \quad 10^{1.07918} &= 12 \\
 \text{The equation in logarithmic form is} \\
 \log_{10} 12 &= 1.07918.
 \end{aligned}$$

$$\begin{aligned}
 59. \quad \log_2 32 &= 5 \\
 \text{The equation in exponential form is} \\
 2^5 &= 32.
 \end{aligned}$$

$$\begin{aligned}
 60. \quad \log_9 3 &= \frac{1}{2} \\
 \text{The equation in exponential form is} \\
 9^{1/2} &= 3.
 \end{aligned}$$

$$\begin{aligned}
 61. \quad \ln 82.9 &= 4.41763 \\
 \text{The equation in exponential form is} \\
 e^{4.41763} &= 82.9.
 \end{aligned}$$

$$\begin{aligned}
 62. \quad \log 3.21 &= 0.50651 \\
 \text{The equation in exponential form is} \\
 10^{0.50651} &= 3.21. \\
 \text{Recall that } \log x &\text{ means } \log_{10} x.
 \end{aligned}$$

$$\begin{aligned}
 63. \quad \log_3 81 &= x \\
 3^x &= 81 \\
 3^x &= 3^4 \\
 x &= 4
 \end{aligned}$$

$$\begin{aligned}
 64. \quad \log_{32} 16 &= x \\
 32^x &= 16 \\
 2^{5x} &= 2^4 \\
 5x &= 4 \\
 x &= \frac{4}{5}
 \end{aligned}$$

$$65. \log_4 8 = x$$

$$4^x = 8$$

$$(2^2)^x = 2^3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

$$66. \log_{100} 1000 = x$$

$$100^x = 1000$$

$$(10^2)^x = 10^3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

$$67. \log_5 3k + \log_5 7k^3$$

$$= \log_5 3k(7k^3)$$

$$= \log_5 (21k^4)$$

$$68. \log_3 2y^3 - \log_3 8y^2$$

$$= \log_3 \frac{2y^3}{8y^2}$$

$$= \log_3 \frac{y}{4}$$

$$69. 4\log_3 y - 2\log_3 x$$

$$= \log_3 y^4 - \log_3 x^2$$

$$= \log_3 \left(\frac{y^4}{x^2} \right)$$

$$70. 3\log_4 r^2 - 2\log_4 r$$

$$= \log_4 (r^2)^3 - \log_4 r^2$$

$$= \log_4 \left(\frac{r^6}{r^2} \right)$$

$$= \log_4 (r^4)$$

$$71. 6^p = 17$$

$$\ln 6^p = \ln 17$$

$$p \ln 6 = \ln 17$$

$$p = \frac{\ln 17}{\ln 6}$$

$$\approx 1.581$$

$$72. 3^{z-2} = 17$$

$$\ln 3^{z-2} = \ln 17$$

$$(z-2) \ln 3 = \ln 17$$

$$z-2 = \frac{\ln 17}{\ln 3}$$

$$z = \frac{\ln 17}{\ln 3} + 2$$

$$\approx 4.183$$

$$73. 2^{1-m} = 7$$

$$\ln 2^{1-m} = \ln 7$$

$$(1-m) \ln 2 = \ln 7$$

$$1-m = \frac{\ln 7}{\ln 2}$$

$$-m = \frac{\ln 7}{\ln 2} - 1$$

$$m = 1 - \frac{\ln 7}{\ln 2}$$

$$\approx -1.807$$

$$74. 12^{-k} = 9$$

$$\ln 12^{-k} = \ln 9$$

$$-k \ln 12 = \ln 9$$

$$k = -\frac{\ln 9}{\ln 12}$$

$$\approx -0.884$$

$$75. e^{-5-2x} = 5$$

$$\ln e^{-5-2x} = \ln 5$$

$$-5-2x = \ln 5$$

$$-2x = \ln 5 + 5$$

$$x = \frac{\ln 5 + 5}{-2}$$

$$\approx -3.305$$

$$76. e^{3x-1} = 14$$

$$\ln (e^{3x-1}) = \ln 14$$

$$3x-1 = \ln 14$$

$$3x = 1 + \ln 14$$

$$x = \frac{1 + \ln 14}{3}$$

$$\approx 1.213$$

$$77. \quad \left(1 + \frac{m}{3}\right)^5 = 15$$

$$\left[\left(1 + \frac{m}{3}\right)^5\right]^{1/5} = 15^{1/5}$$

$$1 + \frac{m}{3} = 15^{1/5}$$

$$\frac{m}{3} = 15^{1/5} - 1$$

$$m = 3(15^{1/5} - 1) \\ \approx 2.156$$

$$78. \quad \left(1 + \frac{2p}{5}\right)^2 = 3$$

$$1 + \frac{2p}{5} = \pm\sqrt{3}$$

$$5 + 2p = \pm 5\sqrt{3}$$

$$2p = -5 \pm 5\sqrt{3}$$

$$p = \frac{-5 \pm 5\sqrt{3}}{2}$$

$$p = \frac{-5 + 5\sqrt{3}}{2} \approx 1.830$$

$$\text{or } p = \frac{-5 - 5\sqrt{3}}{2} \approx -6.830$$

$$79. \quad \log_k 64 = 6$$

$$k^6 = 64$$

$$k^6 = 2^6$$

$$k = 2$$

$$80. \quad \log_3(2x + 5) = 5$$

$$3^5 = 2x + 5$$

$$243 = 2x + 5$$

$$238 = 2x$$

$$x = 119$$

$$81. \quad \log(4p + 1) + \log p = \log 3$$

$$\log[p(4p + 1)] = \log 3$$

$$\log(4p^2 + p) = \log 3$$

$$4p^2 + p = 3$$

$$4p^2 + p - 3 = 0$$

$$(4p - 3)(p + 1) = 0$$

$$4p - 3 = 0 \quad \text{or} \quad p + 1 = 0$$

$$p = \frac{3}{4} \quad p = -1$$

p cannot be negative, so $p = \frac{3}{4}$.

$$82. \quad \log_2(5m - 2) - \log_2(m + 3) = 2$$

$$\log_2 \frac{5m - 2}{m + 3} = 2$$

$$\frac{5m - 2}{m + 3} = 2^2$$

$$5m - 2 = 4(m + 3)$$

$$5m - 2 = 4m + 12$$

$$m = 14$$

$$83. \quad f(x) = a^x; a > 0, a \neq 1$$

(a) The domain is $(-\infty, \infty)$.

(b) The range is $(0, \infty)$.

(c) The y-intercept is 1.

(d) The x-axis, $y = 0$, is a horizontal asymptote.

(e) The function is increasing if $a > 1$.

(f) The function is decreasing if $0 < a < 1$.

$$84. \quad f(x) = \log_a x; a > 0, a \neq 1$$

(a) The domain is $(0, \infty)$.

(b) The range is $(-\infty, \infty)$.

(c) The x-intercept is 1.

(d) The y-axis, $x = 0$, is a vertical asymptote.

(e) f is increasing if $a > 1$.

(f) f is decreasing if $0 < a < 1$.

86. (a) For x in the interval $0 < x \leq 1$, the renter is charged the fixed cost of \$60 and 1 day's rent of \$60 so

$$C\left(\frac{3}{4}\right) = \$40 + \$60(1) \\ = \$40 + \$60 \\ = \$100.$$

$$\begin{aligned} \text{(b)} \quad C\left(\frac{9}{10}\right) &= \$40 + \$60(1) \\ &= \$40 + \$60 \\ &= \$100. \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad C(1) &= \$40 + \$60(1) \\ &= \$40 + \$60 \\ &= \$100. \end{aligned}$$

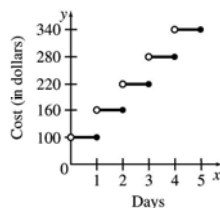
- (d) For x in the interval $1 < x \leq 2$, the renter is charged the fixed cost of \$40 and 2 days rent of \$120, so

$$\begin{aligned} C\left(1\frac{5}{8}\right) &= \$40 + \$60(2) \\ &= \$40 + \$120 \\ &= \$160. \end{aligned}$$

- (e) For x in the interval $2 < x \leq 3$ the renter is charged the fixed cost of \$40 and 3 days rent of \$180. So

$$\begin{aligned} C\left(2\frac{1}{9}\right) &= \$40 + \$60(3) \\ &= \$40 + \$180 \\ &= \$220. \end{aligned}$$

(f)



- (g) The independent variable is the number of days, or x .

- (h) The dependent variable is the cost, or $C(x)$.

$$87. \quad y = \frac{7x}{100 - x}$$

$$\text{(a)} \quad y = \frac{7(80)}{100 - 80} = \frac{560}{20} = 28$$

The cost is \$28,000.

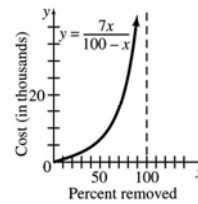
$$\text{(b)} \quad y = \frac{7(50)}{100 - 50} = \frac{350}{50} = 7$$

The cost is \$7000.

$$\text{(c)} \quad \frac{7(90)}{100 - 90} = \frac{630}{10} = 63$$

The cost is \$63,000.

- (d) Plot the points (80, 28), (50, 7), and (90, 63).



- (e) No, because all of the pollutant would be removed when $x = 100$, at which point the denominator of the function would be zero.

$$88. \quad p = \$6902, r = 6\%, t = 8, m = 2$$

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

$$\begin{aligned} A &= 6902\left(1 + \frac{0.06}{2}\right)^{8(2)} \\ &= 6902(1.03)^{16} \\ &= \$11,075.68 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= A - P \\ &= \$11,075.68 - \$6902 \\ &= \$4173.68 \end{aligned}$$

$$89. \quad P = \$2781.36, r = 4.8\%, t = 6, m = 4$$

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

$$\begin{aligned} A &= 2781.36\left(1 + \frac{0.048}{4}\right)^{(6)(4)} \\ &= 2781.36(1.012)^{24} \\ &= \$3703.31 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= \$3703.31 - \$2781.36 \\ &= \$921.95 \end{aligned}$$

$$90. \quad P = \$12,104, r = 6.2\%, t = 2$$

$$\begin{aligned} A &= Pe^{rt} \\ &= 12,104e^{0.062(2)} \\ &= \$13,701.92 \end{aligned}$$

$$91. \quad P = \$12,104, r = 6.2\%, t = 4$$

$$\begin{aligned} A &= Pe^{rt} \\ A &= 12,104e^{0.062(4)} \\ &= 12,104e^{0.248} \\ &= \$15,510.79 \end{aligned}$$

92. $A = \$1500, r = 0.06, t = 9$

$$\begin{aligned} A &= Pe^{rt} \\ &= 1500e^{0.06(9)} \\ &= 1500e^{0.54} \\ &= \$2574.01 \end{aligned}$$

93. $P = \$12,000, r = 0.05, t = 8$

$$\begin{aligned} A &= 12,000e^{0.05(8)} \\ &= 12,000e^{0.40} \\ &= \$17,901.90 \end{aligned}$$

94. \$1000 deposited at 6% compounded semiannually.

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

To double:

$$\begin{aligned} 2(1000) &= 1000\left(1 + \frac{0.06}{2}\right)^{t \cdot 2} \\ 2 &= 1.03^{2t} \\ \ln 2 &= 2t \ln 1.03 \\ t &= \frac{\ln 2}{2 \ln 1.03} \\ &\approx 12 \text{ years} \end{aligned}$$

To triple:

$$\begin{aligned} 3(1000) &= 1000\left(1 + \frac{0.06}{2}\right)^{t \cdot 2} \\ 3 &= 1.03^{2t} \\ \ln 3 &= 2t \ln 1.03 \\ t &= \frac{\ln 3}{2 \ln 1.03} \\ &\approx 19 \text{ years} \end{aligned}$$

95. \$2100 deposited at 4% compounded quarterly.

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

To double:

$$\begin{aligned} 2(2100) &= 2100\left(1 + \frac{0.04}{4}\right)^{t \cdot 4} \\ 2 &= 1.01^{4t} \\ \ln 2 &= 4t \ln 1.01 \\ t &= \frac{\ln 2}{4 \ln 1.01} \\ &\approx 17.4 \end{aligned}$$

Because interest is compounded quarterly, round the result up to the nearest quarter, which is 17.5 years or 70 quarters.

To triple:

$$\begin{aligned} 3(2100) &= 2100\left(1 + \frac{0.04}{4}\right)^{t \cdot 4} \\ 3 &= 1.01^{4t} \\ \ln 3 &= 4t \ln 1.01 \\ t &= \frac{\ln 3}{4 \ln 1.01} \\ &\approx 27.6 \end{aligned}$$

Because interest is compounded quarterly, round the result up to the nearest quarter, which is 27.75 years or 111 quarters.

96. $r = 7\%, m = 4$

$$\begin{aligned} r_E &= \left(1 + \frac{r}{m}\right)^m - 1 \\ &= \left(1 + \frac{0.07}{4}\right)^4 - 1 \\ &= 0.0719 = 7.19\% \end{aligned}$$

97. $r = 6\%, m = 12$

$$\begin{aligned} r_E &= \left(1 + \frac{r}{m}\right)^m - 1 \\ &= \left(1 + \frac{0.06}{12}\right)^{12} - 1 \\ &= 0.0617 = 6.17\% \end{aligned}$$

98. $r = 5\%$ compounded continuously

$$\begin{aligned} r_E &= e^r - 1 \\ &= e^{0.05} - 1 \\ &= 0.0513 = 5.13\% \end{aligned}$$

99. $A = \$2000, r = 6\%, t = 5, m = 1$

$$\begin{aligned} P &= A\left(1 + \frac{r}{m}\right)^{-tm} \\ &= 2000\left(1 + \frac{0.06}{1}\right)^{-5(1)} \\ &= 2000(1.06)^{-5} \\ &= \$1494.52 \end{aligned}$$

100. $A = \$10,000, r = 8\%, m = 2, t = 6$

$$\begin{aligned} P &= A \left(1 + \frac{r}{m} \right)^{-tm} \\ &= 10,000 \left(1 + \frac{0.08}{2} \right)^{-2(6)} \\ &= 10,000(1.04)^{-12} \\ &= \$6245.97 \end{aligned}$$

101. $r = 7\%, t = 8, m = 2, P = 10,000$

$$\begin{aligned} A &= P \left(1 + \frac{r}{m} \right)^{tm} \\ &= 10,000 \left(1 + \frac{0.07}{2} \right)^{8(2)} \\ &= 10,000(1.035)^{16} \\ &= \$17,339.86 \end{aligned}$$

102. $P = \$1, r = 0.08$

$$\begin{aligned} A &= Pe^{rt}, A = 3(1) \\ 3 &= 1e^{0.08t} \\ \ln 3 &= 0.08t \\ \frac{\ln 3}{0.08} &= t \\ 13.7 &= t \end{aligned}$$

It would take about 13.7 years.

103. $P = \$6000, A = \$8000, t = 3$

$$\begin{aligned} A &= Pe^{rt} \\ 8000 &= 6000e^{3r} \\ \frac{4}{3} &= e^{3r} \\ \ln 4 - \ln 3 &= 3r \\ r &= \frac{\ln 4 - \ln 3}{3} \\ r &\approx 0.0959 \text{ or about } 9.59\% \end{aligned}$$

104. $P = A \left(1 + \frac{r}{m} \right)^{-tm}$

$$\begin{aligned} P &= 25,000 \left(1 + \frac{0.06}{12} \right)^{-3(12)} \\ &= 25,000(1.005)^{-36} \\ &= \$20,891.12 \end{aligned}$$

105. (a) $n = 1000 - (p - 50)(10), p \geq 50$
 $= 1000 - 10p + 500$
 $= 1500 - 10p$

(b) $R = pn$
 $R = p(1500 - 10p)$

(c) $p \geq 50$

Since n cannot be negative,

$$\begin{aligned} 1500 - 10p &\geq 0 \\ -10p &\geq -1500 \\ p &\leq 150. \end{aligned}$$

Therefore, $50 \leq p \leq 150$.

(d) Since $n = 1500 - 10p$,
 $10p = 1500 - n$
 $p = 150 - \frac{n}{10}$.

$$\begin{aligned} R &= pn \\ R &= \left(150 - \frac{n}{10} \right) n \end{aligned}$$

(e) Since she can sell at most 1000 tickets,
 $0 \leq n \leq 1000$.

(f) $R = -10p^2 + 1500p$
 $\frac{-b}{2a} = \frac{-1500}{2(-10)} = 75$

The price producing maximum revenue is \$75.

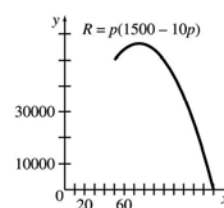
(g) $R = -\frac{1}{10}n^2 + 150n$
 $\frac{-b}{2a} = \frac{-150}{2(-\frac{1}{10})} = 750$

The number of tickets producing maximum revenue is 750.

(h) $R(p) = -10p^2 + 1500p$
 $R(75) = -10(75)^2 + 1500(75)$
 $= -56,250 + 112,500$
 $= 56,250$

The maximum revenue is \$56,250.

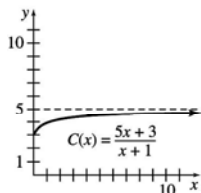
(i)



- (j) The revenue starts at \$50,000 when the price is \$50, rises to a maximum of \$56,250 when the price is \$75, and falls to 0 when the price is \$150.

106. $C(x) = \frac{5x + 3}{x + 1}$

(a)



(b) $C(x + 1) - C(x)$

$$\begin{aligned} &= \frac{5(x + 1) + 3}{(x + 1) + 1} - \frac{5x + 3}{x + 1} \\ &= \frac{5x + 8}{x + 2} - \frac{5x + 3}{x + 1} \\ &= \frac{(5x + 8)(x + 1) - (5x + 3)(x + 2)}{(x + 2)(x + 1)} \\ &= \frac{5x^2 + 13x + 8 - 5x^2 - 13x - 6}{(x + 2)(x + 1)} \\ &= \frac{2}{(x + 2)(x + 1)} \end{aligned}$$

(c) $A(x) = \frac{C(x)}{x} = \frac{\frac{5x+3}{x+1}}{x}$

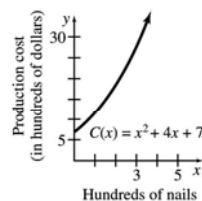
$$= \frac{5x + 3}{x(x + 1)}$$

(d) $A(x + 1) - A(x)$

$$\begin{aligned} &= \frac{5(x + 1) + 3}{(x + 1)[(x + 1) + 1]} - \frac{5x + 3}{x(x + 1)} \\ &= \frac{5x + 8}{(x + 1)(x + 2)} - \frac{5x + 3}{x(x + 1)} \\ &= \frac{x(5x + 8) - (5x + 3)(x + 2)}{x(x + 1)(x + 2)} \\ &= \frac{5x^2 + 8x - 5x^2 - 13x - 6}{x(x + 1)(x + 2)} \\ &= \frac{-5x - 6}{x(x + 1)(x + 2)} \end{aligned}$$

107. $C(x) = x^2 + 4x + 7$

(a)



(b)

$$\begin{aligned} &C(x + 1) - C(x) \\ &= (x + 1)^2 + 4(x + 1) + 7 - (x^2 + 4x + 7) \\ &= x^2 + 2x + 1 + 4x + 4 + 7 - x^2 - 4x - 7 \\ &= 2x + 5 \end{aligned}$$

(c) $A(x) = \frac{C(x)}{x} = \frac{x^2 + 4x + 7}{x}$

$$= x + 4 + \frac{7}{x}$$

(d) $A(x + 1) - A(x)$

$$\begin{aligned} &= (x + 1) + 4 + \frac{7}{x + 1} - \left(x + 4 + \frac{7}{x}\right) \\ &= x + 1 + 4 + \frac{7}{x + 1} - x - 4 - \frac{7}{x} \\ &= 1 + \frac{7}{x + 1} - \frac{7}{x} \\ &= 1 + \frac{7x - 7(x + 1)}{x(x + 1)} \\ &= 1 + \frac{7x - 7x - 7}{x(x + 1)} \\ &= 1 - \frac{7}{x(x + 1)} \end{aligned}$$

108. (a) $y = a^t$

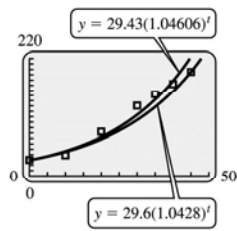
Let $y_0 = 29.6$, the value of y at $t = 0$.

Use the point (45, 195.3) to find a .

$$\begin{aligned} 195.3 &= 29.6a^{45} \\ a^{45} &= \frac{195.3}{29.6} \\ a &= \sqrt[45]{\frac{195.3}{29.6}} \\ &\approx 1.0428 \\ y &= 29.6(1.0428)^t \end{aligned}$$

(b) $y = 29.43(1.04606)^t$

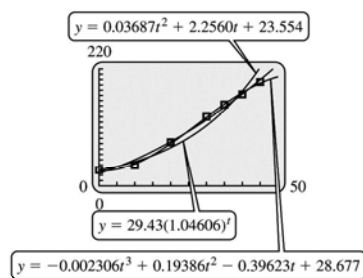
(c)



Yes, the answers are close to each other.

(d) $y = 0.03687t^2 + 2.2560t + 23.554$
 $y = -0.002306t^3 + 0.19386t^2$
 $-0.39623t + 28.677$

(e)



109. $F(x) = -\frac{2}{3}x^2 + \frac{14}{3}x + 96$

The maximum fever occurs at the vertex of the parabola.

$$x = \frac{-b}{2a} = \frac{-\frac{14}{3}}{-\frac{4}{3}} = \frac{7}{2}$$

$$\begin{aligned} y &= -\frac{2}{3}\left(\frac{7}{2}\right)^2 + \frac{14}{3}\left(\frac{7}{2}\right) + 96 \\ &= -\frac{2}{3}\left(\frac{49}{4}\right) + \frac{49}{3} + 96 \\ &= -\frac{49}{6} + \frac{49}{3} + 96 \\ &= -\frac{49}{6} + \frac{98}{6} + \frac{576}{6} = \frac{625}{6} \approx 104.2 \end{aligned}$$

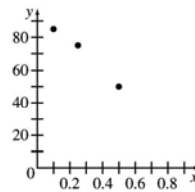
The maximum fever occurs on the third day. It is about 104.2°F.

110. (a) $100\% - 87.5\% = 12.5\%$

$$12.5\% = 0.125 = \frac{125}{1000} = \frac{1}{8}$$

The fraction let in is 1 over the SPF rating.

(b)



(c) $UVB = 1 - \frac{1}{SPF}$

(d) $1 - \frac{1}{8} = 87.5\%$

$$1 - \frac{1}{4} = 75.0\%$$

The increase is 12.5%.

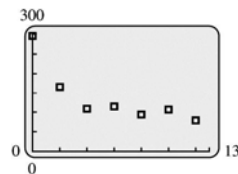
(e) $1 - \frac{1}{30} = 96.\bar{6}\%$

$$1 - \frac{1}{15} = 93.\bar{3}\%$$

The increase is 3. $\bar{3}\%$ or about 3.3%.

(f) The increase in percent protection decreases to zero.

111. (a)

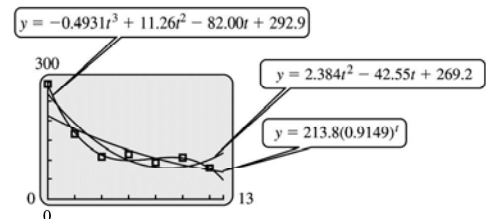


(b) $y = 2.384t^2 - 42.55t + 269.2$

$$y = -0.4931t^3 + 11.26t^2 - 82.00t + 292.9$$

$$y = 213.8(0.9149)^t$$

(c)



The cubic function seems to best capture the behavior of the data.

(d) $x = 20$ corresponds to 2015.

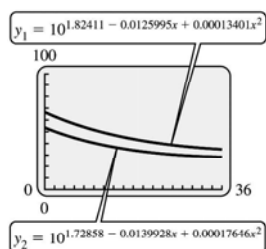
$$\begin{aligned}
 y &= 2.384(20)^2 - 42.55(20) \\
 &\quad + 269.2 \approx 372 \\
 y &= -0.4931(20)^3 + 11.26(20)^2 \\
 &\quad - 82.00(20) + 292.9 \\
 &\approx -788 \\
 y &= 213.8(0.9149)^{20} \approx 36
 \end{aligned}$$

The only realistic value is given by the exponential function because the pattern of the data suggests that the number of cases decrease over time.

- 112. (a)** The first three years of infancy corresponds to 0 months to 36 months, so the domain is $[0, 36]$.

- (b)** In both cases, the graph of the quadratic function in the exponent opens upward and the x coordinate of the vertex is greater than 36 ($x \approx 47$ for the awake infants and $x \approx 40$ for the sleeping infants). So the quadratic functions are both decreasing over this time. Therefore, both respiratory rates are decreasing.

(c)



- (d)** When $x = 12$, the waking respiratory rate is $y = 49.23$ breaths per minute, and the sleeping respiratory rate is $y = 38.55$. Therefore, for a 1-year-old infant in the 95th percentile, the waking respiratory rate is approximately $49.23 - 38.55 \approx 10.7$ breaths per minute higher.

- 113.** This function has a maximum value at $x \approx 187.9$. At $x \approx 187.9$, $y \approx 345$. The largest girth for which this formula gives a reasonable answer is 187.9 cm. The predicted mass of a polar bear with this girth is 345 kg.

114. $y = 17,000$, $y_0 = 15,000$, $t = 4$

(a) $y = y_0 e^{kt}$
 $17,000 = 15,000 e^{4k}$
 $\frac{17}{15} = e^{4k}$
 $\ln\left(\frac{17}{15}\right) = 4k$
 $\frac{0.125}{4} = k$
 $0.0313 = k$

So, $y = 15,000 e^{0.0313t}$.

(b) $45,000 = 15,000 e^{0.0313t}$
 $3 = e^{0.0313t}$
 $\ln 3 = 0.0313t$
 $\frac{\ln 3}{0.0313} = k$
 $35 = k$

It would take about 35 years.

115. $p(t) = \frac{1.79 \cdot 10^{11}}{(2026.87 - t)^{0.99}}$

(a) $p(2010) \approx 10.915$ billion

This is about 4.006 billion more than the estimate of 6.909 billion.

(b) $p(2020) \approx 26.56$ billion
 $p(2025) \approx 96.32$ billion

116. $I(x) \geq 1$

$I(x) = 10e^{-0.3x}$

$$\begin{aligned}
 10e^{-0.3x} &\geq 1 \\
 e^{-0.3x} &\geq 0.1 \\
 -0.3x &\geq \ln 0.1 \\
 x &\leq \frac{\ln 0.1}{-0.3} \approx 7.7
 \end{aligned}$$

The greatest depth is about 7.7 m.

117. Graph

$y = c(t) = e^{-t} - e^{-2t}$

on a graphing calculator and locate the maximum point. A calculator shows that the x -coordinate of the maximum point is about 0.69, and the y -coordinate is exactly 0.25. Thus, the maximum concentration of 0.25 occurs at about 0.69 minutes.

118. $g(t) = \frac{c}{a} + \left(g_0 - \frac{c}{a}\right)e^{-at}$

- (a) If $g_0 = 0.08$, $c = 0.1$, and $a = 1.3$, the function becomes

$$g(t) = \frac{0.1}{1.3} + \left(0.08 - \frac{0.1}{1.3}\right)e^{-1.3t}.$$

Graph this function on a graphing calculator. Use a window with $X_{\min} = 0$, since this represents the time when the drug is first injected. A good choice for the viewing window is $[0, 5]$ by $[0.07, 0.11]$, $X_{\text{scl}} = 0.5$, $Y_{\text{scl}} = 0.01$.

From the graph, we see that the maximum value of g for $t \geq 0$ occurs at $t = 0$, the time when the drug is first injected.

The maximum amount of glucose in the bloodstream, given by $G(0)$, is 0.08 gram.

- (b) From the graph, we see that the amount of glucose in the bloodstream decreases from the initial value of 0.08 gram, so it will never increase to 0.1 gram. We can also reach this conclusion by graphing $y_1 = G(t)$ and $y_2 = 0.1$ on the same screen with the window given in (a) and observing that the graphs of y_1 and y_2 do not intersect.
- (c) From the graph, we see that as t increases, the graph of $y_1 = G(t)$ becomes almost horizontal, and $G(t)$ approaches approximately 0.0769.

Note that

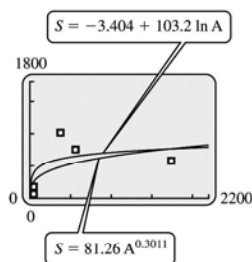
$$\frac{c}{a} = \frac{0.1}{1.3} \approx 0.0769.$$

The amount of glucose in the bloodstream after a long time approaches 0.0769 grams.

119. (a) $S = -3.404 + 103.2 \ln A$

(b) $S = 81.26A^{0.3011}$

(c)



(d) $S \approx 234$

$S \approx 163$

Neither number is close to the actual number of 421.

120. $y = y_0 e^{-kt}$

(a) $100,000 = 128,000e^{-k(5)}$

$$128,000 = 100,000e^{5k}$$

$$\frac{128}{100} = e^{5k}$$

$$\ln\left(\frac{128}{100}\right) = 5k$$

$$0.05 \approx k$$

$$y = 100,000e^{-0.05t}$$

(b) $70,000 = 100,000e^{-0.05t}$

$$\frac{7}{10} = e^{-0.05t}$$

$$\ln \frac{7}{10} = -0.05t$$

$$7.1 \approx t$$

It will take about 7.1 years.

121. $t = (1.26 \times 10^9) \frac{\ln\left[1 + 8.33\left(\frac{A}{K}\right)\right]}{\ln 2}$

(a) $A = 0, K > 0$

$$t = (1.26 \times 10^9) \frac{\ln[1 + 8.33(0)]}{\ln 2}$$

$$= (1.26 \times 10^9)(0) = 0 \text{ years}$$

(b) $t = (1.26 \times 10^9) \frac{\ln[1 + 8.33(0.212)]}{\ln 2}$

$$= (1.26 \times 10^9) \frac{\ln 2.76596}{\ln 2}$$

$$= 1,849,403,169$$

or about 1.85×10^9 years

- (c) As r increases, t increases, but at a slower and slower rate. As r decreases, t decreases at a faster and faster rate

122. (a) $P = kD^1$

$$164.8 = k(30.1)$$

$$k = \frac{164.8}{30.1} \approx 5.48$$

For $n = 1$, $P = 5.48D$.

$$P = kD^{1.5}$$

$$164.8 = k(30.1)^{1.5}$$

$$k = \frac{164.8}{(30.1)^{1.5}} \approx 1.00$$

For $n = 1.5$, $P = 1.00D^{1.5}$.

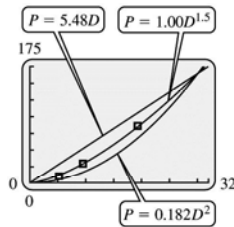
$$P = kD^2$$

$$164.8 = k(30.1)^2$$

$$k = \frac{164.8}{(30.1)^2} \approx 0.182$$

For $n = 2$, $P = 0.182D^2$.

(b)



$P = 1.00D^{1.5}$ appears to be the best fit.

(c) $P = 1.00(39.5)^{1.5} \approx 248.3$ years

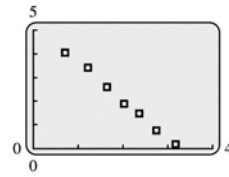
(d) We obtain

$$P = 1.00D^{1.5}.$$

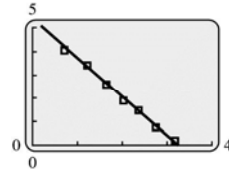
This is the same as the function found in part (b).

Extended Application: Power Functions

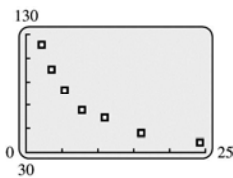
1. a.



b. Yes; $Y = 5.065 - 0.3289X$

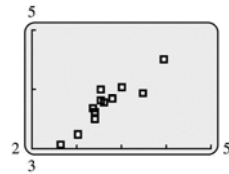


c. It is decreasing; the exponent is negative. As the price increases the demand decreases.

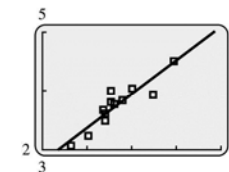


d. $y = 158.4x^{-0.3289x}$

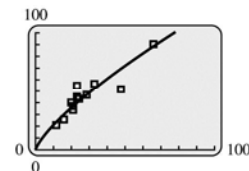
2. a.



b. Yes; $Y = 0.7617X + 1.2874$



c. $y = 3.623x^{0.7617}$



d. An approximate value for the power is 0.76.

Chapter 3

THE DERIVATIVE

3.1 Limits

Your Turn 1

$$f(x) = x^2 + 2$$

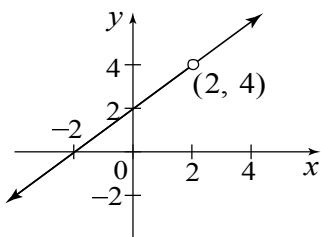
x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$	2.81	2.9801	2.998001	2.99980001	3	3.00020001	3.002001	3.0201	3.21

The table suggests that, as x get closer and closer to 1 from either side, $f(x)$ gets closer and closer to 3.

So, $\lim_{x \rightarrow 1} (x^2 + 2) = 3$.

Your Turn 2

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(\cancel{x - 2})}{(\cancel{x - 2})} = x + 2, \text{ provided } x \neq 2$$



The graph of $y = \frac{x^2 - 4}{x - 2}$ is the graph of $y = x + 2$, except there is a hole at (2, 4).

Looking at the graph, we see that as x is close to, but not equal to 2, $f(x)$ approaches 4.

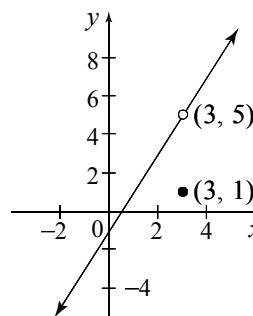
$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$$

Your Turn 3

$$\text{Find } \lim_{x \rightarrow 3} f(x) \text{ if } f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$$

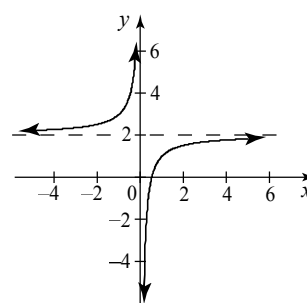
The graph of f is shown in the next column.

$$\lim_{x \rightarrow 3} f(x) = 5$$



Your Turn 4

$$\text{Find } \lim_{x \rightarrow 0} \frac{2x - 1}{x}.$$



$$\lim_{x \rightarrow 0^-} f(x) = \infty$$

$$\lim_{x \rightarrow 0^+} f(x) = -\infty$$

Since there is no real number that $f(x)$ approaches as x approaches 0 from either side, nor does $f(x)$ approach either ∞ or $-\infty$, $\lim_{x \rightarrow 0} \frac{2x - 1}{x}$ does not exist.

Your Turn 5

Let $\lim_{x \rightarrow 2} f(x) = 3$ and $\lim_{x \rightarrow 2} g(x) = 4$.

$$\begin{aligned}\lim_{x \rightarrow 2} [f(x) + g(x)]^2 &= \left[\lim_{x \rightarrow 2} [f(x) + g(x)] \right]^2 \\ &= \left[\lim_{x \rightarrow 2} f(x) + \lim_{x \rightarrow 2} g(x) \right]^2 \\ &= [3 + 4]^2 \\ &= 7^2 \\ &= 49\end{aligned}$$

Your Turn 6

$$\begin{aligned}\lim_{x \rightarrow -3} \frac{x^2 - x - 12}{x + 3} &= \lim_{x \rightarrow -3} \frac{(x - 4)(\cancel{x + 3})}{\cancel{x + 3}} \quad (x \neq -3) \\ &= \lim_{x \rightarrow -3} x - 4 \\ &= (-3) - 4 \\ &= -7\end{aligned}$$

Your Turn 7

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt{x})^2 - 1}{(x - 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{\cancel{x} - 1}{(\cancel{x} - 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{1 + 1} = \frac{1}{2}\end{aligned}$$

Your Turn 8

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x^2 + 3x - 4}{6x^2 - 5x + 7} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{3x}{x^2} - \frac{4}{x^2}}{\frac{6x^2}{x^2} - \frac{5x}{x^2} + \frac{7}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x} - \frac{4}{x^2}}{6 - \frac{5}{x} + \frac{7}{x^2}} \\ &= \frac{2 + 0 - 0}{6 - 0 + 0} = \frac{2}{6} = \frac{1}{3}\end{aligned}$$

3.1 Exercises

1. Since $\lim_{x \rightarrow 2^-} f(x)$ does not equal $\lim_{x \rightarrow 2^+} f(x)$, $\lim_{x \rightarrow 2} f(x)$ does not exist. The answer is c.

2. Since $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = -1$, $\lim_{x \rightarrow 2} f(x) = -1$. The answer is a.

3. Since $\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x) = 6$, $\lim_{x \rightarrow 4} f(x) = 6$. The answer is b.

4. Since $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = -\infty$, $\lim_{x \rightarrow 1} f(x) = -\infty$. The answer is b.

5. (a) By reading the graph, as x gets closer to 3 from the left or right, $f(x)$ gets closer to 3.

$$\lim_{x \rightarrow 3} f(x) = 3$$

(b) By reading the graph, as x gets closer to 0 from the left or right, $f(x)$ gets closer to 1.

$$\lim_{x \rightarrow 0} f(x) = 1.$$

6. (a) By reading the graph, as x gets closer to 2 from the left or right, $F(x)$ gets closer to 4.

$$\lim_{x \rightarrow 2} F(x) = 4$$

(b) By reading the graph, as x gets closer to -1 from left or right, $F(x)$ gets closer to 4.

$$\lim_{x \rightarrow -1} F(x) = 4$$

7. (a) By reading the graph, as x gets closer to 0 from the left or right, $f(x)$ gets closer to 0.

$$\lim_{x \rightarrow 0} f(x) = 0$$

(b) By reading the graph, as x gets closer to 2 from the left, $f(x)$ gets closer to -2 , but as x gets closer to 2 from the right, $f(x)$ gets closer to 1.

$$\lim_{x \rightarrow 2} f(x) \text{ does not exist.}$$

8. (a) By reading the graph, as x gets closer to 3 from the left or right, $g(x)$ gets closer to 2.

$$\lim_{x \rightarrow 3} g(x) = 2$$

- (b) By reading the graph, as x gets closer to 5 from the left, $g(x)$ gets closer to -2 , but as x gets closer to 5 from the right, $g(x)$ gets closer to 1.

$$\lim_{x \rightarrow 5} g(x) \text{ does not exist.}$$

9. (a) (i) By reading the graph, as x gets closer to -2 from the left, $f(x)$ gets closer to -1 .

$$\lim_{x \rightarrow -2^-} f(x) = -1$$

- (ii) By reading the graph, as x gets closer to -2 from the right, $f(x)$ gets closer to $-\frac{1}{2}$.

$$\lim_{x \rightarrow -2^+} f(x) = -\frac{1}{2}$$

- (iii) Since $\lim_{x \rightarrow -2^-} f(x) = -1$ and $\lim_{x \rightarrow -2^+} f(x) = -\frac{1}{2}$, $\lim_{x \rightarrow -2} f(x)$ does not exist.

- (iv) $f(-2)$ does not exist since there is no point on the graph with an x -coordinate of -2 .

- (b) (i) By reading the graph, as x gets closer to -1 from the left, $f(x)$ gets closer to $-\frac{1}{2}$.

$$\lim_{x \rightarrow -1^-} f(x) = -\frac{1}{2}$$

- (ii) By reading the graph, as x gets closer to -1 from the right, $f(x)$ gets closer to $-\frac{1}{2}$.

$$\lim_{x \rightarrow -1^+} f(x) = -\frac{1}{2}$$

- (iii) Since $\lim_{x \rightarrow -1^-} f(x) = -\frac{1}{2}$ and

$$\lim_{x \rightarrow -1^+} f(x) = -\frac{1}{2}, \quad \lim_{x \rightarrow -1} f(x) = -\frac{1}{2}.$$

- (iv) $f(-1) = -\frac{1}{2}$ since $(-1, -\frac{1}{2})$ is a point of the graph.

10. (a) (i) By reading the graph, as x gets closer to 1 from the left, $f(x)$ gets closer to 1.

$$\lim_{x \rightarrow 1^-} f(x) = 1$$

- (ii) By reading the graph, as x gets closer to 1 from the right, $f(x)$ gets closer to 1.

$$\lim_{x \rightarrow 1^+} f(x) = 1$$

- (iii) Since $\lim_{x \rightarrow 1^-} f(x) = 1$ and

$$\lim_{x \rightarrow 1^+} f(x) = 1, \quad \lim_{x \rightarrow 1} f(x) = 1.$$

- (iv) $f(1) = 2$ since $(1, 2)$ is part of the graph.

- (b) (i) By reading the graph, as x gets closer to 2 from the left, $f(x)$ gets closer to 0.

$$\lim_{x \rightarrow 2^-} f(x) = 0$$

- (ii) By reading the graph, as x gets closer to 2 from the right, $f(x)$ gets closer to 0.

$$\lim_{x \rightarrow 2^+} f(x) = 0$$

- (iii) Since $\lim_{x \rightarrow 2^-} f(x) = 0$ and $\lim_{x \rightarrow 2^+} f(x) = 0$, $\lim_{x \rightarrow 2} f(x) = 0$.

- (iv) $f(2) = 0$ since $(2, 0)$ is point of the graph.

11. By reading the graph, as x moves further to the right, $f(x)$ gets closer to 3.

$$\text{Therefore, } \lim_{x \rightarrow \infty} f(x) = 3.$$

12. By reading the graph, as x moves further to the left, $g(x)$ gets larger and larger. Therefore, $\lim_{x \rightarrow -\infty} g(x) = \infty$ (does not exist).

13. $\lim_{x \rightarrow 2} F(x)$ in Exercise 6 exists because

$$\lim_{x \rightarrow 2^-} F(x) = 4 \quad \text{and} \quad \lim_{x \rightarrow 2^+} F(x) = 4.$$

$\lim_{x \rightarrow -2} f(x)$ in Exercise 9 does not exist since

$$\lim_{x \rightarrow -2^-} f(x) = -1, \quad \text{but} \quad \lim_{x \rightarrow -2^+} f(x) = -\frac{1}{2}.$$

15. From the table, as x approaches 1 from the left or the right, $f(x)$ approaches 4.

$$\lim_{x \rightarrow 1} f(x) = 4$$

16. $f(x) = 2x^2 - 4x + 7$; find $\lim_{x \rightarrow 1} f(x)$.

Substitute 0.9 for x in the expression at the right to get $f(0.9) = 5.02$.

Continue substituting to complete the table.

x	0.9	0.99	0.999
$f(x)$	5.02	5.0002	5.000002
x	1.001	1.01	1.1
$f(x)$	5.000002	5.0002	5.02

As x approaches 1 from the left or the right, $f(x)$ approaches 5.

$$\lim_{x \rightarrow 1} f(x) = 5$$

17. $k(x) = \frac{x^3 - 2x - 4}{x - 2}$; find $\lim_{x \rightarrow 2} k(x)$.

x	1.9	1.99	1.999
$k(x)$	9.41	9.9401	9.9941
x	2.001	2.01	2.1
$k(x)$	10.006	10.0601	10.61

As x approaches 2 from the left or the right, $k(x)$ approaches 10.

$$\lim_{x \rightarrow 2} k(x) = 10$$

18. $f(x) = \frac{2x^3 + 3x^2 - 4x - 5}{x + 1}$; find $\lim_{x \rightarrow -1} f(x)$.

x	-1.1	-1.01	-1.001
$f(x)$	-3.68	-3.969	-3.996
x	-0.999	-0.99	-0.9
$f(x)$	-4.002	-4.02	-4.28

As x approaches -1 from the left or the right, $f(x)$ approaches -4 .

$$\lim_{x \rightarrow -1} f(x) = -4$$

19. $h(x) = \frac{\sqrt{x} - 2}{x - 1}$; find $\lim_{x \rightarrow 1} h(x)$.

x	0.9	0.99	0.999
$h(x)$	10.51317	100.50126	1000.50013
x	1.001	1.01	1.1
$h(x)$	-999.50012	-99.50124	-9.51191

$$\lim_{x \rightarrow 1^-} = \infty$$

$$\lim_{x \rightarrow 1^+} = -\infty$$

Thus, $\lim_{x \rightarrow 1} h(x)$ does not exist.

20. $f(x) = \frac{\sqrt{x} - 3}{x - 3}$; find $\lim_{x \rightarrow 3} f(x)$.

x	2.9	2.99	2.999
$f(x)$	12.9706	127.0838	1268.237
x	3.001	3.01	3.1
$f(x)$	-1267.66	-125.506	-12.6795

$$\lim_{x \rightarrow 3^-} f(x) = \infty$$

$$\lim_{x \rightarrow 3^+} f(x) = -\infty$$

Thus, $\lim_{x \rightarrow 3} f(x)$ does not exist.

21. $\lim_{x \rightarrow 4} [f(x) - g(x)] = \lim_{x \rightarrow 4} f(x) - \lim_{x \rightarrow 4} g(x)$
 $= 9 - 27 = -18$

22. $\lim_{x \rightarrow 4} [(g(x) \cdot f(x))]$
 $= \left[\lim_{x \rightarrow 4} g(x) \right] \cdot \left[\lim_{x \rightarrow 4} f(x) \right]$
 $= 27.9 = 243$

23. $\lim_{x \rightarrow 4} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow 4} f(x)}{\lim_{x \rightarrow 4} g(x)} = \frac{9}{27} = \frac{1}{3}$

24. $\lim_{x \rightarrow 4} \log_3 f(x) = \log_3 \lim_{x \rightarrow 4} f(x)$
 $= \log_3 9 = 2$

25. $\lim_{x \rightarrow 4} \sqrt{f(x)} = \lim_{x \rightarrow 4} [f(x)^{1/2}]$
 $= \left[\lim_{x \rightarrow 4} f(x) \right]^{1/2}$
 $= 9^{1/2} = 3$

26. $\lim_{x \rightarrow 4} \sqrt[3]{g(x)} = \lim_{x \rightarrow 4} [g(x)^{1/3}]$
 $= \left[\lim_{x \rightarrow 4} g(x) \right]^{1/3}$
 $= 27^{1/3} = 3$

$$\begin{aligned}
 27. \quad \lim_{x \rightarrow 4} 2^{f(x)} &= 2^{\lim_{x \rightarrow 4} f(x)} \\
 &= 2^9 \\
 &= 512
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \lim_{x \rightarrow 4} [1 + f(x)]^2 &= \left[\lim_{x \rightarrow 4} (1 + f(x)) \right]^2 \\
 &= \left[\lim_{x \rightarrow 4} 1 + \lim_{x \rightarrow 4} f(x) \right]^2 \\
 &= (1 + 9)^2 = 10^2 \\
 &= 100
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \lim_{x \rightarrow 4} \frac{f(x) + g(x)}{2g(x)} &= \frac{\lim_{x \rightarrow 4} [f(x) + g(x)]}{\lim_{x \rightarrow 4} 2g(x)} \\
 &= \frac{\lim_{x \rightarrow 4} f(x) + \lim_{x \rightarrow 4} g(x)}{2 \lim_{x \rightarrow 4} g(x)} \\
 &= \frac{9 + 27}{2(27)} = \frac{36}{54} = \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \lim_{x \rightarrow 4} \frac{5g(x) + 2}{1 - f(x)} &= \frac{\lim_{x \rightarrow 4} [5g(x) + 2]}{\lim_{x \rightarrow 4} [1 - f(x)]} \\
 &= \frac{5 \lim_{x \rightarrow 4} g(x) + \lim_{x \rightarrow 4} 2}{\lim_{x \rightarrow 4} 1 - \lim_{x \rightarrow 4} f(x)} \\
 &= \frac{5 \cdot 27 + 2}{1 - 9} = \frac{137}{-8}
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} \\
 &= \lim_{x \rightarrow 3} (x + 3) \\
 &= \lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 3 \\
 &= 3 + 3 \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} &= \lim_{x \rightarrow -2} \frac{(x + 2)(x - 2)}{(x + 2)} \\
 &= \lim_{x \rightarrow -2} (x - 2) \\
 &= -2 - 2 = -4
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \lim_{x \rightarrow 1} \frac{5x^2 - 7x + 2}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{(5x - 2)(x - 1)}{(x + 1)(x - 1)} \\
 &= \lim_{x \rightarrow 1} \frac{5x - 2}{x + 1} \\
 &= \frac{5 - 2}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + x - 6} &= \lim_{x \rightarrow -3} \frac{(x - 3)(x + 3)}{(x - 2)(x + 3)} \\
 &= \lim_{x \rightarrow -3} \frac{x - 3}{x - 2} \\
 &= \frac{-3 - 3}{-3 - 2} \\
 &= \frac{-6}{-5} = \frac{6}{5}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \lim_{x \rightarrow -2} \frac{x^2 - x - 6}{x + 2} &= \lim_{x \rightarrow -2} \frac{(x - 3)(x + 2)}{x + 2} \\
 &= \lim_{x \rightarrow -2} (x - 3) \\
 &= \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} (-3) \\
 &= -2 - 3 \\
 &= -5
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} &= \lim_{x \rightarrow 5} \frac{(x - 5)(x + 2)}{(x - 5)} \\
 &= \lim_{x \rightarrow 5} (x + 2) \\
 &= 5 + 2 = 7
 \end{aligned}$$

$$\begin{aligned}
 37. \quad \lim_{x \rightarrow 0} \frac{\frac{1}{x+3} - \frac{1}{3}}{x} &= \lim_{x \rightarrow 0} \left(\frac{1}{x+3} - \frac{1}{3} \right) \left(\frac{1}{x} \right) \\
 &= \lim_{x \rightarrow 0} \left[\frac{3}{3(x+3)} - \frac{x+3}{3(x+3)} \right] \left(\frac{1}{x} \right) \\
 &= \lim_{x \rightarrow 0} \frac{3 - x - 3}{3(x+3)(x)} \\
 &= \lim_{x \rightarrow 0} \frac{-x}{3(x+3)x} \\
 &= \lim_{x \rightarrow 0} \frac{-1}{3(x+3)} \\
 &= \frac{-1}{3(0+3)} \\
 &= -\frac{1}{9}
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \lim_{x \rightarrow 0} \frac{\frac{-1}{x+2} + \frac{1}{2}}{x} \\
 &= \lim_{x \rightarrow 0} \left(\frac{-1}{x+2} + \frac{1}{2} \right) \left(\frac{1}{x} \right) \\
 &= \lim_{x \rightarrow 0} \left[\frac{-2}{2(x+2)} + \frac{x+2}{2(x+2)} \right] \left(\frac{1}{x} \right) \\
 &= \lim_{x \rightarrow 0} \frac{-2 + x + 2}{2(x+2)(x)} \\
 &= \lim_{x \rightarrow 0} \frac{x}{2x(x+2)} \\
 &= \lim_{x \rightarrow 0} \frac{1}{2(x+2)} \\
 &= \frac{1}{2(0+2)} = \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 39. \quad \lim_{x \rightarrow 25} \frac{\sqrt{x} - 5}{x - 25} \\
 &= \lim_{x \rightarrow 25} \frac{\sqrt{x} - 5}{x - 25} \cdot \frac{\sqrt{x} + 5}{\sqrt{x} + 5} \\
 &= \lim_{x \rightarrow 25} \frac{x - 25}{(x - 25)(\sqrt{x} + 5)} \\
 &= \lim_{x \rightarrow 25} \frac{1}{\sqrt{x} + 5} \\
 &= \frac{1}{\sqrt{25} + 5} \\
 &= \frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \lim_{x \rightarrow 36} \frac{\sqrt{x} - 6}{x - 36} \\
 &= \lim_{x \rightarrow 36} \frac{\sqrt{x} - 6}{x - 36} \cdot \frac{\sqrt{x} + 6}{\sqrt{x} + 6} \\
 &= \lim_{x \rightarrow 36} \frac{(x - 36)}{(x - 36)(\sqrt{x} + 6)} \\
 &= \lim_{x \rightarrow 36} \frac{1}{\sqrt{x} + 6} \\
 &= \frac{1}{\sqrt{36} + 6} \\
 &= \frac{1}{6 + 6} \\
 &= \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} \\
 &= \lim_{h \rightarrow 0} (2x + h) \\
 &= 2x + 0 = 2x
 \end{aligned}$$

$$\begin{aligned}
 42. \quad \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \\
 &= 3x^2 + 3x(0) + (0)^2 = 3x^2
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \lim_{x \rightarrow \infty} \frac{3x}{7x - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{3x}{x}}{\frac{7x}{x} - \frac{1}{x}} \\
 &= \lim_{x \rightarrow \infty} \frac{3}{7 - \frac{1}{x}} \\
 &= \frac{3}{7 - 0} = \frac{3}{7}
 \end{aligned}$$

$$\begin{aligned}
 44. \quad \lim_{x \rightarrow -\infty} \frac{8x + 2}{4x - 5} &= \lim_{x \rightarrow -\infty} \frac{\frac{8x}{x} + \frac{2}{x}}{\frac{4x}{x} - \frac{5}{x}} \\
 &= \lim_{x \rightarrow -\infty} \frac{8 + \frac{2}{x}}{4 - \frac{5}{x}} \\
 &= \frac{8 + 0}{4 - 0} = 2
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \lim_{x \rightarrow -\infty} \frac{-3x^2 + 2x}{2x^2 - 2x + 1} &= \lim_{x \rightarrow -\infty} \frac{\frac{3x^2}{x^2} + \frac{2x}{x^2}}{\frac{2x^2}{x^2} - \frac{2x}{x^2} + \frac{1}{x^2}} \\
 &= \lim_{x \rightarrow -\infty} \frac{3 + \frac{2}{x}}{2 - \frac{2}{x} + \frac{1}{x^2}} \\
 &= \frac{3 - 0}{2 + 0 + 0} = \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 46. \quad \lim_{x \rightarrow \infty} \frac{x^2 + 2x - 5}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2} + \frac{2x}{x^2} - \frac{5}{x^2}}{\frac{3x^2}{x^2} + \frac{2}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{1 + \frac{2}{x} - \frac{5}{x^2}}{3 + \frac{2}{x^2}} \\
 &= \frac{1 + 0 - 0}{3 + 0} = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 47. \quad \lim_{x \rightarrow \infty} \frac{3x^3 + 2x - 1}{2x^4 - 3x^3 - 2} &= \lim_{x \rightarrow \infty} \frac{\frac{3x^3}{x^4} + \frac{2x}{x^4} - \frac{1}{x^4}}{\frac{2x^4}{x^4} - \frac{3x^3}{x^4} - \frac{2}{x^4}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{3}{x} + \frac{2}{x^3} - \frac{1}{x^4}}{2 - \frac{3}{x} - \frac{2}{x^4}} \\
 &= \frac{0 + 0 - 0}{2 - 0 - 0} = 0
 \end{aligned}$$

$$\begin{aligned}
 48. \quad \lim_{x \rightarrow \infty} \frac{2x^2 - 1}{3x^4 + 2} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^4} - \frac{1}{x^4}}{\frac{3x^4}{x^4} + \frac{2}{x^4}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2} - \frac{1}{x^4}}{3 + \frac{2}{x^4}} \\
 &= \frac{0 - 0}{3 + 0} = \frac{0}{3} = 0
 \end{aligned}$$

$$\begin{aligned}
 49. \quad \lim_{x \rightarrow \infty} \frac{2x^3 - x - 3}{6x^2 - x - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^3}{x^2} - \frac{x}{x^2} - \frac{3}{x^2}}{\frac{6x^2}{x^2} - \frac{x}{x^2} - \frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{2x - \frac{1}{x} - \frac{3}{x^2}}{6 - \frac{1}{x} - \frac{1}{x^2}} = \infty
 \end{aligned}$$

The limit does not exist.

$$\begin{aligned}
 50. \quad \lim_{x \rightarrow \infty} \frac{x^4 - x^3 - 3x}{7x^2 + 9} &= \lim_{x \rightarrow \infty} \frac{\frac{x^4}{x^2} - \frac{x^3}{x^2} - \frac{3x}{x^2}}{\frac{7x^2}{x^2} + \frac{9}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{x^2 - x - \frac{3}{x}}{7 + \frac{9}{x^2}} \\
 &= \infty
 \end{aligned}$$

The limit does not exist.

$$\begin{aligned}
 51. \quad \lim_{x \rightarrow \infty} \frac{2x^2 - 7x^4}{9x^2 + 5x - 6} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} - \frac{7x^4}{x^2}}{\frac{9x^2}{x^2} + \frac{5x}{x^2} - \frac{6}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{2 - 7x^2}{9 + \frac{5}{x} - \frac{6}{x^2}}
 \end{aligned}$$

The denominator approaches 9, while the numerator becomes a negative number that is larger and larger in magnitude, so

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 7x^4}{9x^2 + 5x - 6} = -\infty \text{ (does not exist).}$$

$$\begin{aligned}
 52. \quad \lim_{x \rightarrow \infty} \frac{-5x^3 - 4x^2 + 8}{6x^2 + 3x + 2} &= \lim_{x \rightarrow \infty} \frac{\frac{-5x^3}{x^2} - \frac{4x^2}{x^2} + \frac{8}{x^2}}{\frac{6x^2}{x^2} + \frac{3x}{x^2} + \frac{2}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{-5x - 4 + \frac{8}{x^2}}{6 + \frac{3}{x} + \frac{2}{x^2}}
 \end{aligned}$$

The denominator approaches 6, while the numerator becomes a negative number that is larger and larger in magnitude, so

$$\lim_{x \rightarrow \infty} \frac{-5x^3 - 4x^2 + 8}{6x^2 + 3x + 2} = -\infty \text{ (does not exist).}$$

$$53. \quad \lim_{x \rightarrow -1^-} f(x) = 1 \text{ and } \lim_{x \rightarrow -1^+} f(x) = 1.$$

Therefore $\lim_{x \rightarrow -1} f(x) = 1$.

$$54. \quad \lim_{x \rightarrow -2^-} g(x) = -1 \text{ and } \lim_{x \rightarrow -2^+} g(x) = -1.$$

Therefore $\lim_{x \rightarrow -2} g(x) = -1$.

$$55. \quad (a) \lim_{x \rightarrow 3} f(x) = 2.$$

$$(b) \lim_{x \rightarrow 5} f(x) \text{ does not exist since } \lim_{x \rightarrow 5} f(x) = 2 \text{ and } \lim_{x \rightarrow 5^+} f(x) = 8.$$

56. (a) $\lim_{x \rightarrow 0} g(x)$ does not exist since $\lim_{x \rightarrow 0^-} g(x) = 5$
and $\lim_{x \rightarrow 0^+} g(x) = -2$.
(b) $\lim_{x \rightarrow 3} g(x) = 7$.

57. Find $\lim_{x \rightarrow 3} f(x)$, where $f(x) = \frac{x^2 - 9}{x - 3}$.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	5.9	5.99	5.999	6.001	6.01	6.1

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6.$$

58. Find $\lim_{x \rightarrow -2} g(x)$, where $g(x) = \frac{x^2 - 4}{x + 2}$.

x	-2.1	-2.01	-2.001	-1.999	-1.99	-1.9
$g(x)$	-4.1	-4.01	-4.001	-3.999	-3.99	-3.9

$$\lim_{x \rightarrow -2} g(x) = \lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2} = -4.$$

59. Find $\lim_{x \rightarrow 1} f(x)$, where $f(x) = \frac{5x^2 - 7x + 2}{x^2 - 1}$.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.316	1.482	1.498	1.502	1.517	1.667

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{5x^2 - 7x + 2}{x^2 - 1} = 1.5 = \frac{3}{2}.$$

60. Find $\lim_{x \rightarrow 3} g(x)$, where $g(x) = \frac{x^2 - 9}{x^2 + x - 6}$.

x	-3.1	-3.01	-3.001
$g(x)$	1.196	1.1996	1.19996

x	-2.999	-2.99	-2.9
$g(x)$	1.20004	1.2004	1.204

$$\lim_{x \rightarrow -3} g(x) = \lim_{x \rightarrow -3} \frac{x^2 - 9}{x^2 + x - 6} = 1.2 = \frac{6}{5}.$$

61. (a) $\lim_{x \rightarrow -2} \frac{3x}{(x + 2)^3}$ does not exist since

$$\lim_{x \rightarrow -2^+} \frac{3x}{(x + 2)^3} = -\infty$$

$$\text{and } \lim_{x \rightarrow -2^-} \frac{3x}{(x + 2)^3} = \infty.$$

- (b) Since $(x + 2)^3 = 0$ when $x = -2$, $x = -2$ is the vertical asymptote of the graph of $F(x)$.
(c) The two answers are related. Since $x = -2$ is a vertical asymptote, we know that $\lim_{x \rightarrow -2} F(x)$ does not exist.

62. $G(x) = \frac{-6}{(x - 4)^2}$

- (a) $\lim_{x \rightarrow 4} G(x) = -\infty$ (does not exist), since by looking at the graph of $G(x)$, we see that as x gets closer to 4 from either the right or the left, $g(x)$ gets smaller.
(b) Since $(x - 4)^2 = 0$ when $x = 4$, $x = 4$ is the vertical asymptote of the graph of $G(x)$.
(c) The two answers are related. Since $x = 4$ is a vertical asymptote, we know the $\lim_{x \rightarrow 4} G(x)$ does not exist.

65. (a) $\lim_{x \rightarrow -\infty} e^x = 0$ since, as the graph goes

further to the left, e^x gets closer to 0.

- (b) The graph of e^x has a horizontal asymptote at $y = 0$ since $\lim_{x \rightarrow -\infty} e^x = 0$.

66. (a) $y = xe^{-x}$

From the graph, it appears that

$$\lim_{x \rightarrow \infty} xe^{-x} = 0.$$

x	1	10	50
y	0.37	0.00045	9.64×10^{-21}

- (b) $y = x^2 e^{-x}$

From the graph, it appears that

$$\lim_{x \rightarrow \infty} x^2 e^{-x} = 0.$$

x	1	10	50
y	0.37	0.0045	4.82×10^{-19}

- (c) $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$. As n gets larger, y is still very small but larger than for smaller values of n .

67. (a) $\lim_{x \rightarrow 0^+} \ln x = -\infty$ (does not exist) since, as the graph gets closer to $x = 0$, the value of $\ln x$ get smaller.
- (b) The graph of $y = \ln x$ has a vertical asymptote at $x = 0$ since $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

68. (a) $y = x \ln x$

From the graph, it appears that

$$\lim_{x \rightarrow 0^+} x \ln x = 0.$$

x	1	0.5	0.1	0.01	0.001
y	0	-0.347	-0.230	-0.0461	-0.0069

- (b) $y = x(\ln x)^2$

From the graph it appears that

$$\lim_{x \rightarrow 0^+} x(\ln x)^2 = 0.$$

x	1	0.5	0.1	0.01	0.001
y	0	0.240	0.53	0.212	0.048

- (c) $\lim_{x \rightarrow 0^+} x(\ln x)^n = 0$. As n gets larger, y is still very small but bigger than for smaller values of n .

71. $\lim_{x \rightarrow 1} \frac{x^4 + 4x^3 - 9x^2 + 7x - 3}{x - 1}$

(a)

x	1.01	1.001	1.0001	0.99	0.999	0.9999
$f(x)$	5.0908	5.009	5.0009	4.9108	4.991	4.9991

As $x \rightarrow 1^-$ and as $x \rightarrow 1^+$, we see that $f(x) \rightarrow 5$.

(b) Graph

$$y = \frac{x^4 + 4x^3 - 9x^2 + 7x - 3}{x - 1}$$

on a graphing calculator. One suitable choice for the viewing window is $[-6, 6]$ by $[-10, 40]$ with $Xscl = 1$, $Yscl = 10$.

Because $x - 1 = 0$ when $x = 1$, we know that the function is undefined at this x -value. The graph does not show an asymptote at $x = 1$. This indicates that the rational expression that defines this function is not written in lowest terms, and that the graph should have an open circle to show a “hole” in the graph at $x = 1$. The graphing calculator doesn’t show the hole, but if we try to find the value of the function at $x = 1$, we see that it is undefined. (Using the TABLE feature on a TI-84 Plus, we see that for $x = 1$, the y -value is listed as “ERROR.”)

By viewing the function near $x = 1$, and using the ZOOM feature, we see that as x gets close to 1 from the left or the right, y gets close to 5, suggesting that

$$\lim_{x \rightarrow 1} \frac{x^4 + 4x^3 - 9x^2 + 7x - 3}{x - 1} = 5.$$

72. $\lim_{x \rightarrow 2} \frac{x^4 + x - 18}{x^2 - 4}$

(a)

x	2.01	2.001	2.0001	1.99	1.999	1.9999
$f(x)$	8.29	8.25	8.25	8.21	8.25	8.25

(b) Graph

$$y = \frac{x^4 + x - 18}{x^2 - 4}.$$

One suitable choice for the viewing window is $[-5, 5]$ by $[0, 20]$. Because $x^2 - 4 = 0$ when $x = -2$ or $x = 2$, we know that the function is undefined at these two x -values. The graph shows an asymptote at $x = -2$. There should be open circle to show a “hole” in the graph at $x = 2$. The graphing calculator doesn’t show the hole, but if we try to find the value of the function at $x = 2$, we see that it is undefined. (Using the TABLE feature on a TI-84 Plus, we see that for $x = 2$, the y -value is listed as “ERROR.”)

By viewing the function near $x = 2$ and using the ZOOM feature, we verify that the required limit is 8.25. (We may not be able to get this value exactly.)

73. $\lim_{x \rightarrow -1} \frac{x^{1/3} + 1}{x + 1}$

(a)

x	-1.01	-1.001	-1.0001
$f(x)$	0.33223	0.33322	0.33332
x	-0.99	-0.999	-0.9999
$f(x)$	0.33445	0.33344	0.33334

We see that as $x \rightarrow -1^-$ and as $x \rightarrow -1^+$,
 $f(x) \rightarrow 0.3333$ or $\frac{1}{3}$.

(b) Graph $y = \frac{x^{1/3} + 1}{x + 1}$.

One suitable choice for the viewing window is $[-5, 5]$ by $[-2, 2]$

Because $x + 1 = 0$ when $x = -1$, we know that the function is undefined at this x -value. The graph does not show an asymptote at $x = -1$. This indicates that the rational expression that defined this function is not written lowest terms, and that the graph should have an open circle to show a “hole” in the graph at $x = -1$. The graphing calculator doesn’t show the hole, but if we try to find the value of the function at $x = -1$, we see that it is undefined. (Using the TABLE feature on a TI-83, we see that for $x = -1$, the y -value is listed as “ERROR.”)

By viewing the function near $x = -1$ and using the ZOOM feature, we see that as x gets close to -1 from the left or right, y gets close to 0.3333 , suggesting that

$$\lim_{x \rightarrow -1} \frac{x^{1/3} + 1}{x + 1} = 0.3333 \text{ or } \frac{1}{3}.$$

74. $\lim_{x \rightarrow 4} \frac{x^{3/2} - 8}{x + x^{1/2} - 6}$

(a)

x	4.1	4.01	4.001	4.0001
$f(x)$	2.4179	2.4018	2.4002	2.4
x	3.9	3.99	3.999	3.9999
$f(x)$	2.3819	2.3982	2.3998	2.4

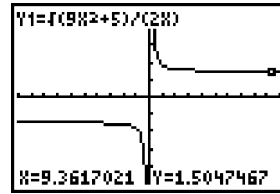
(b) Graph

$$y = \frac{x^{3/2} - 8}{x + x^{1/2} - 6}.$$

This function is undefined at $x = 4$ because this value would make the denominator equal to 0. However, by viewing the function near $x = 4$ and using the ZOOM feature, we verify that the required limit is 2.4.

75. $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^2 + 5}}{2x}$

Graph the functions on a graphing calculator. A good choice for the viewing window is $[-10, 10]$ by $[-5, 5]$.



(a) The graph appears to have horizontal asymptotes at $y = \pm 1.5$. We see that as $x \rightarrow \infty$, $y \rightarrow 1.5$, so we determine that

$$\lim_{x \rightarrow \infty} \frac{\sqrt{9x^2 + 5}}{2x} = 1.5.$$

(b) As $x \rightarrow \infty$,

$$\frac{\sqrt{9x^2 + 5}}{2x} \rightarrow \frac{3|x|}{2x}.$$

Since $x > 0$, $|x| = x$, so

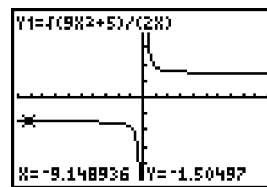
$$\frac{3|x|}{2x} = \frac{3x}{2x} = \frac{3}{2}.$$

Thus,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{9x^2 + 5}}{2x} = \frac{3}{2} \text{ or } 1.5.$$

76. $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 5}}{2x}$

Graph this function on a graphing calculator. A good choice for the viewing window is $[-10, 10]$ by $[-5, 5]$.



(a) The graph appears to have horizontal asymptotes at $y = \pm 1.5$. We see that as $x \rightarrow -\infty$, $y \rightarrow -1.5$, so we determine that

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 5}}{2x} = -1.5.$$

(b) As $x \rightarrow -\infty$,

$$\frac{\sqrt{9x^2 + 5}}{2x} \rightarrow \frac{3|x|}{2x}.$$

Since $x < 0$, $|x| = -x$, so

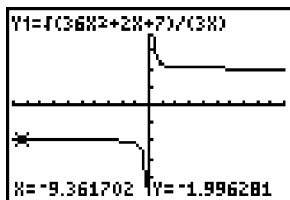
$$\frac{3|x|}{2x} = \frac{3(-x)}{2x} = -\frac{3}{2}.$$

Thus,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 5}}{2x} = -\frac{3}{2} \text{ or } -1.5.$$

77.
$$\lim_{x \rightarrow -\infty} \frac{\sqrt{36x^2 + 2x + 7}}{3x}$$

Graph this function on a graphing calculator.
A good choice for the viewing window is $[-10, 10]$ by $[-5, 5]$.



(a) The graph appears to have horizontal asymptotes at $y = \pm 2$. We see that as $x \rightarrow -\infty$, $y \rightarrow -2$, so we determine that

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{36x^2 + 2x + 7}}{3x} = -2.$$

(b) As $x \rightarrow -\infty$,

$$\frac{\sqrt{36x^2 + 2x + 7}}{3x} \rightarrow \frac{6|x|}{3x}.$$

Since $x < 0$, $|x| = -x$, so

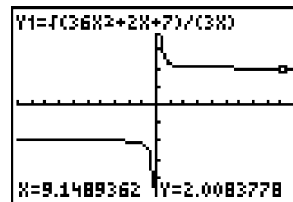
$$\frac{6|x|}{3x} = \frac{6(-x)}{3x} = -2.$$

Thus,

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{36x^2 + 2x + 7}}{3x} = -2.$$

78.
$$\lim_{x \rightarrow \infty} \frac{\sqrt{36x^2 + 2x + 7}}{3x}$$

Graph this function on a graphing calculator. A good choice for the viewing window is $[-10, 10]$ by $[-5, 5]$.



(a) The graph appears to have horizontal asymptotes at $y = \pm 2$. We see that as $x \rightarrow \infty$, $y \rightarrow 2$, so we determine that

$$\lim_{x \rightarrow \infty} \frac{\sqrt{36x^2 + 2x + 7}}{3x} = 2.$$

(b) As $x \rightarrow \infty$,

$$\frac{\sqrt{36x^2 + 2x + 7}}{3x} \rightarrow \frac{6|x|}{3x}.$$

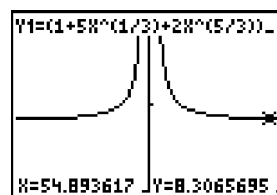
Since $x > 0$, $|x| = x$, so

$$\frac{6|x|}{3x} = \frac{6x}{3x} = 2.$$

Thus,
$$\lim_{x \rightarrow \infty} \frac{\sqrt{36x^2 + 2x + 7}}{3x} = 2.$$

79.
$$\lim_{x \rightarrow \infty} \frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5}$$

Graph this function on a graphing calculator. A good choice for the viewing window is $[-20, 20]$ by $[0, 20]$ with Xscl = 5, Yscl = 5.



(a) The graph appears to have a horizontal asymptote at $y = 8$. We see that as $x \rightarrow \infty$, $y \rightarrow 8$, so we determine that

$$\lim_{x \rightarrow \infty} \frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5} = 8.$$

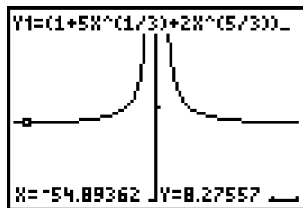
(b) As $x \rightarrow \infty$,

$$\frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5} \rightarrow \frac{8x^5}{x^5} = 8.$$

$$\text{Thus, } \lim_{x \rightarrow \infty} \frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5} = 8.$$

80.
$$\lim_{x \rightarrow -\infty} \frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5}$$

Graph this function on a graphing calculator. A good choice for the viewing window is $[-60, 60]$ by $[0, 20]$ with Xscl = 10, Yscl = 10.



(a) The graph appears to have a horizontal asymptote at $y = 8$. We see that as $x \rightarrow -\infty$, $y \rightarrow 8$, so we determine that

$$\lim_{x \rightarrow -\infty} \frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5} = 8.$$

(b) As
$$\frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5} \rightarrow \frac{8x^5}{x^5} = 8.$$

$$\text{Thus, } \lim_{x \rightarrow -\infty} \frac{(1 + 5x^{1/3} + 2x^{5/3})^3}{x^5} = 8.$$

82. (a)
$$\lim_{x \rightarrow 12} G(t)$$

As t approaches 12 from either direction, the value of $G(t)$ for the corresponding point on the graph approaches 3.

Thus, $\lim_{x \rightarrow 12} G(t) = 3$ which represents 3 million gallons.

(b)
$$\lim_{x \rightarrow 16^+} G(t) = 1.5$$

$$\lim_{x \rightarrow 16^-} G(t) = 2$$

Since $\lim_{x \rightarrow 16^+} G(t) \neq \lim_{x \rightarrow 16^-} G(t)$,

$\lim_{x \rightarrow 16} G(t)$ does not exist.

(c) $G(16)$ is the value of function $G(t)$ when $t = 16$. This value occurs at the solid dot on the graph $G(16) = 2$ which represents 2 million gallons.

(d) The tipping point occurs at the break in the graph, when $t = 16$ months.

83. (a)
$$\lim_{x \rightarrow 53} T(x) = 3$$
 cents because $T(x)$ is constant at 3 cents as x approaches 53 from the left or the right.

(b)
$$\lim_{x \rightarrow 09^-} T(x) = 7.25$$
 cents because as x approaches 09 from the left, $T(x)$ is constant at 7.25 cents.

(c)
$$\lim_{x \rightarrow 09^+} T(x) = 8.25$$
 cents because as x approaches 09 from the right, $T(x)$ is constant at 8.25 cents.

(d) $\lim_{x \rightarrow 09} T(x)$ does not exist because

$$\lim_{x \rightarrow 09^-} T(x) \neq \lim_{x \rightarrow 09^+} T(x).$$

(e) $T(09) = 8.25$ cents since $(09, 8.25)$ is a point on the graph.

84. (a)
$$\lim_{t \rightarrow 2009^-} C(t) = 42$$
 cents because as

x approaches 09 from the left, $C(t)$ is constant at 42 cents.

(b)
$$\lim_{t \rightarrow 2009^+} C(t) = 44$$
 cents because as x approaches 09 from the right, $C(t)$ is constant at 44 cents.

(c) $\lim_{t \rightarrow 2009} C(t)$ does not exist because

$$\lim_{t \rightarrow 2009^-} C(t) \neq \lim_{t \rightarrow 2009^+} C(t).$$

(d) $C(2009) = 44$ cents since $(2009, 44)$ is a point on the graph.

85.
$$C(x) = 15,000 + 6x$$

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{15,000 + 6x}{x} = \frac{15,000}{x} + 6$$

$$\lim_{x \rightarrow \infty} \bar{C}(x) = \lim_{x \rightarrow \infty} \frac{15,000}{x} + 6 = 0 + 6 = 6$$

This means that the average cost approaches \$6 as the number of tapes produced becomes very large.

$$\begin{aligned}
 86. \quad \bar{C}(x) &= \frac{C(x)}{x} \\
 &= \frac{0.0738x + 111.83}{x} \\
 &= 0.0738 + \frac{111.83}{x}
 \end{aligned}$$

$$\lim_{x \rightarrow \infty} \bar{C}(x) = 0.0738$$

The average cost approaches \$0.0738 per mile as the number of miles becomes very large.

$$\begin{aligned}
 87. \quad P(s) &= \frac{63s}{s+8} \\
 \lim_{s \rightarrow \infty} \frac{63s}{s+8} &= \lim_{s \rightarrow \infty} \frac{\frac{63s}{s}}{\frac{s}{s} + \frac{8}{s}} \\
 &= \lim_{s \rightarrow \infty} \frac{63}{1 + \frac{8}{s}} \\
 &= \frac{63}{1+0} \\
 &= 63
 \end{aligned}$$

The number of items of work a new employee produces gets closer and closer to 63 as the number of days of training increases.

$$\begin{aligned}
 88. \quad \lim_{n \rightarrow \infty} \left[R \left[\frac{1 - (1+i)^{-n}}{i} \right] \right] \\
 &= \frac{R}{i} \lim_{n \rightarrow \infty} [1 - (1+i)^{-n}] \\
 &= \frac{R}{i} \left[\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} (1+i)^{-n} \right] \\
 &= \frac{R}{i} [1 - 0] = \frac{R}{i}
 \end{aligned}$$

$$\begin{aligned}
 89. \quad \lim_{n \rightarrow \infty} \left[\frac{R}{i-g} \left[1 - \left(\frac{1+g}{1+i} \right)^n \right] \right] \\
 &= \frac{R}{i-g} \lim_{n \rightarrow \infty} \left[1 - \left(\frac{1+g}{1+i} \right)^n \right] \\
 &= \frac{R}{i-g} \left[\lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \left(\frac{1+g}{1+i} \right)^n \right]
 \end{aligned}$$

assuming $i > g$,

$$\begin{aligned}
 &= \frac{R}{i-g} [1 - 0] \\
 &= \frac{R}{i-g}
 \end{aligned}$$

$$\begin{aligned}
 90. \quad (a) \quad N(65) &= 71.8e^{-8.96e^{(-0.0685(65))}} \\
 &\approx 64.68
 \end{aligned}$$

To the nearest whole number, this species of alligator has approximately 65 teeth after 65 days of incubation by this formula.

(b) Since

$$\lim_{t \rightarrow \infty} (-8.96e^{-0.0685t}) = -8.96 \cdot 0 = 0, \text{ it}$$

follows that

$$\begin{aligned}
 \lim_{t \rightarrow \infty} 71.8e^{-8.96e^{(-0.0685t)}} &= 71.8e^0 \\
 &= 71.8 \cdot 1 \\
 &= 71.8
 \end{aligned}$$

So, to the nearest whole number,

$$\lim_{t \rightarrow \infty} N(t) \approx 72. \text{ Therefore, by this model a}$$

newborn alligator of this species will have about 72 teeth.

$$\begin{aligned}
 91. \quad (a) \quad D(t) &= 155(1 - e^{-0.0133t}) \\
 D(20) &= 155(1 - e^{-0.0133(20)}) \\
 &= 155(1 - e^{-0.266}) \\
 &\approx 36.2
 \end{aligned}$$

The depth of the sediment layer deposited below the bottom of the lake in 1970 was 36.2 cm.

$$\begin{aligned}
 (b) \quad \lim_{t \rightarrow \infty} D(t) &= \lim_{t \rightarrow \infty} 155(1 - e^{-0.0133t}) \\
 &= 155 \lim_{t \rightarrow \infty} (1 - e^{-0.0133t}) \\
 &= 155 \left(\lim_{t \rightarrow \infty} 1 - \lim_{t \rightarrow \infty} e^{-0.0133t} \right) \\
 &= 155(1) - 155 \lim_{t \rightarrow \infty} e^{-0.0133t} \\
 &= 155 - 155(0) = 155
 \end{aligned}$$

Thus,

$$\lim_{t \rightarrow \infty} D(t) = 155.$$

Going back in time (t is years before 1990), the depth of the sediment approaches 155 cm.

$$\begin{aligned}
 92. \quad A(h) &= \frac{0.17h}{h^2 + 2} \\
 \lim_{x \rightarrow \infty} A(h) &= \lim_{x \rightarrow \infty} \frac{0.17h}{h^2 + 2} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{0.17h}{h^2}}{\frac{h^2}{h^2} + \frac{2}{h^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{0.17}{h}}{1 + \frac{2}{h^2}} \\
 &= \frac{0}{1 + 0} = 0
 \end{aligned}$$

This means that the concentration of the drug in the bloodstream approaches 0 as the number of hours after injection increases.

$$93. \quad (a) \quad p_2 = \frac{1}{2} + \left(0.7 - \frac{1}{2}\right)[1 - 2(0.2)]^2 = 0.572$$

$$(b) \quad p_4 = \frac{1}{2} + \left(0.7 - \frac{1}{2}\right)[1 - 2(0.2)]^4 = 0.526$$

$$(c) \quad p_8 = \frac{1}{2} + \left(0.7 - \frac{1}{2}\right)[1 - 2(0.2)]^8 = 0.503$$

$$\begin{aligned}
 (d) \quad \lim_{n \rightarrow \infty} P_n &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} + \left(p_0 - \frac{1}{2}\right)(1 - 2p)^n \right] \\
 &= \frac{1}{2} + \lim_{n \rightarrow \infty} \left(p_0 - \frac{1}{2}\right)(1 - 2p)^n \\
 &= \frac{1}{2} + \left(p_0 - \frac{1}{2}\right) \lim_{n \rightarrow \infty} (1 - 2p)^n \\
 &= \frac{1}{2} + \left(p_0 - \frac{1}{2}\right) \cdot 0 = \frac{1}{2}
 \end{aligned}$$

The number in parts (a), (b), and (c) represent the probability that the legislator will vote yes on the second, fourth, and eighth votes. In (d), as the number of roll calls increases, the probability gets close to 0.5, but is never less than 0.5.

3.2 Continuity

Your Turn 1

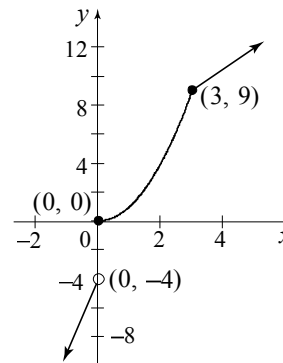
$$f(x) = \sqrt{5x + 3}$$

The square root function is discontinuous wherever $5x + 3 < 0$. There is a discontinuity when

$$5a + 3 < 0, \text{ or } a < -\frac{3}{5}.$$

Your Turn 2

$$f(x) = \begin{cases} 5x - 4 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 3 \\ x + 6 & \text{if } x > 3 \end{cases}$$



$$\lim_{x \rightarrow 0^-} f(x) = -4$$

$$\lim_{x \rightarrow 0^+} f(x) = 0$$

Because $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$, the limit doesn't exist, so f is discontinuous at $x = 0$.

3.2 Exercises

1. Discontinuous at $x = -1$

(a) $f(-1)$ does not exist.

$$(b) \quad \lim_{x \rightarrow -1^-} f(x) = \frac{1}{2}$$

$$(c) \quad \lim_{x \rightarrow -1^+} f(x) = \frac{1}{2}$$

(d) $\lim_{x \rightarrow -1} f(x) = \frac{1}{2}$ (since (a) and (b) have the same answers)

(e) $f(-1)$ does not exist.

2. Discontinuous at $x = -1$

$$(a) \quad f(-1) = 2$$

$$(b) \quad \lim_{x \rightarrow -1^-} f(x) = 2$$

$$(c) \quad \lim_{x \rightarrow -1^+} f(x) = 4$$

(d) $\lim_{x \rightarrow -1} f(x)$ does not exist (since parts (a) and (b) have different answers).

(e) $\lim_{x \rightarrow -1} f(x)$ does not exist.

3. Discontinuous at $x = 1$
- (a) $f(1) = 2$
 - (b) $\lim_{x \rightarrow 1^-} f(x) = -2$
 - (c) $\lim_{x \rightarrow 1^+} f(x) = -2$
 - (d) $\lim_{x \rightarrow 1} f(x) = -2$ (since (a) and (b) have the same answers)
 - (e) $\lim_{x \rightarrow 1} f(x) \neq f(1)$
4. Discontinuous at $x = -2$ and $x = 3$.
- (a) $f(-2) = 1$ $f(3) = 1$
 - (b) $\lim_{x \rightarrow -2^-} f(x) = -1$ $\lim_{x \rightarrow 3^-} f(x) = -1$
 - (c) $\lim_{x \rightarrow -2^+} f(x) = -1$ $\lim_{x \rightarrow 3^+} f(x) = -1$
 - (d) $\lim_{x \rightarrow -2} f(x) = -1$ (since parts (a) and (b) have the same answer)
 $\lim_{x \rightarrow 3} f(x) = -1$ (since parts (a) and (b) have the same answer)
 - (e) $\lim_{x \rightarrow -2} f(x) \neq f(-2)$ $\lim_{x \rightarrow 3} f(x) \neq f(3)$
5. Discontinuous at $x = -5$ and $x = 0$
- (a) $f(-5)$ does not exist. $f(0)$ does not exist.
 - (b) $\lim_{x \rightarrow -5^-} f(x) = \infty$ (limit does not exist)
 $\lim_{x \rightarrow 0^-} f(x) = 0$
 - (c) $\lim_{x \rightarrow -5^+} f(x) = -\infty$ (limit does not exist)
 $\lim_{x \rightarrow 0^+} f(x) = 0$
 - (d) $\lim_{x \rightarrow -5} f(x)$ does not exist, since the answers to (a) and (b) are different. $\lim_{x \rightarrow 0} f(x) = 0$, since the answers to (a) and (b) are the same.
 - (e) $f(-5)$ does not exist and $\lim_{x \rightarrow -5} f(x)$ does not exist. $f(0)$ does not exist.
6. Discontinuous at $x = 0$ and $x = 2$
- (a) $f(0)$ does not exist. $f(2)$ does not exist.
 - (b) $\lim_{x \rightarrow 0^-} f(x) = -\infty$ (limit does not exist)
 $\lim_{x \rightarrow 2^-} f(x) = -2$
- (c) $\lim_{x \rightarrow 0^+} f(x) = -\infty$ (limit does not exist)
- (d) $\lim_{x \rightarrow 2^+} f(x) = -2$
- (e) $\lim_{x \rightarrow 0} f(x) = -\infty$ (limit does not exist)
- (f) $\lim_{x \rightarrow 2} f(x) = -2$ (since parts (a) and (b) have the same answer)
- (g) $f(0)$ does not exist and $\lim_{x \rightarrow 0} f(x)$ does not exist. $f(2)$ does not exist.
7. $f(x) = \frac{5+x}{x(x-2)}$
- $f(x)$ is discontinuous at $x = 0$ and $x = 2$ since the denominator equals 0 at these two values.
- $\lim_{x \rightarrow 0} f(x)$ does not exist since $\lim_{x \rightarrow 0^-} f(x) = \infty$ and $\lim_{x \rightarrow 0^+} f(x) = -\infty$.
- $\lim_{x \rightarrow 2} f(x)$ does not exist since $\lim_{x \rightarrow 2^-} f(x) = -\infty$ and $\lim_{x \rightarrow 2^+} f(x) = \infty$.
8. $f(x) = \frac{-2x}{(2x+1)(3x+6)}$
- $f(x)$ is discontinuous at $x = -\frac{1}{2}$ and $x = -2$ since the denominator equals 0 at these two values.
- $\lim_{x \rightarrow -2} f(x)$ does not exist since $\lim_{x \rightarrow -2^-} f(x) = +\infty$ and $\lim_{x \rightarrow -2^+} f(x) = -\infty$.
- $\lim_{x \rightarrow -\frac{1}{2}} f(x)$ does not exist since $\lim_{x \rightarrow -\frac{1}{2}^-} f(x) = -\infty$ and $\lim_{x \rightarrow -\frac{1}{2}^+} f(x) = +\infty$.
9. $f(x) = \frac{x^2 - 4}{x - 2}$
- $f(x)$ is discontinuous at $x = 2$ since the denominator equals zero at that value.
- Since for $x \neq 2$
- $$\frac{x^2 - 4}{x - 2} = \frac{(x+2)(x-2)}{x-2} = x+2,$$
- $$\lim_{x \rightarrow 2} f(x) = 2+2 = 4.$$

10. $f(x) = \frac{x^2 - 25}{x + 5}$

$f(x)$ is discontinuous at $x = -5$ since the denominator equals zero at that value.

Since for $x \neq -5$

$$\frac{x^2 - 25}{x + 5} = \frac{(x + 5)(x - 5)}{x + 5} = x - 5,$$

$$\lim_{x \rightarrow -5} f(x) = -5 - 5 = -10.$$

11. $p(x) = x^2 - 4x + 11$

Since $p(x)$ is a polynomial function, it is continuous everywhere and thus discontinuous nowhere.

12. $q(x) = -3x^3 + 2x^2 - 4x + 1$

Since $q(x)$ is a polynomial function, it is continuous everywhere and thus discontinuous nowhere.

13. $p(x) = \frac{|x + 2|}{x + 2}$

$p(x)$ is discontinuous at $x = -2$ since the denominator is zero at that value.

$$\text{since } \lim_{x \rightarrow -2^-} p(x) = -1 \text{ and } \lim_{x \rightarrow -2^+} p(x) = 1,$$

$$\lim_{x \rightarrow -2} p(x) \text{ does not exist.}$$

14. $r(x) = \frac{|5 - x|}{x - 5}$

$r(x)$ is discontinuous at $x = 5$ since the denominator is zero at that value.

$$\text{Since } \lim_{x \rightarrow 5^+} r(x) = 1 \text{ and } \lim_{x \rightarrow 5^-} r(x) = -1,$$

$$\lim_{x \rightarrow 5} r(x) \text{ does not exist.}$$

15. $k(x) = e^{\sqrt{x-1}}$

The function is undefined for $x < 1$, so the function is discontinuous for $a < 1$. The limit as x approaches any $a < 1$ does not exist because the function is undefined for $x < 1$.

16. $j(x) = e^{1/x}$

$j(x)$ is discontinuous at $x = 0$ since the function is undefined there.

$$\lim_{x \rightarrow 0} j(x) \text{ does not exist since } \lim_{x \rightarrow 0^-} j(x) = 0 \text{ and}$$

$$\lim_{x \rightarrow 0^+} j(x) = \infty.$$

17. As x approaches 0 from the left or the right, $\left| \frac{x}{x-1} \right|$ approaches 0 and $r(x) = \ln \left| \frac{x}{x-1} \right|$ goes to $-\infty$.

So $\lim_{x \rightarrow 0} r(x)$ does not exist. As x approaches 1

from the left or the right, $\left| \frac{x}{x-1} \right|$ goes to ∞ and so

does $r(x) = \ln \left| \frac{x}{x-1} \right|$. So $\lim_{x \rightarrow 1} r(x)$ does not exist.

18. As x approaches -2 from the left or the right,

$$\left| \frac{x+2}{x-3} \right| \text{ approaches 0 and } r(x) = \ln \left| \frac{x+2}{x-3} \right| \text{ goes to}$$

$-\infty$. So $\lim_{x \rightarrow -2} r(x)$ does not exist. As x

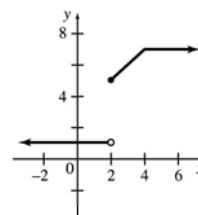
approaches 3 from the left or the right, $\left| \frac{x+2}{x-3} \right|$ goes

to ∞ and so does $r(x) = \ln \left| \frac{x+2}{x-3} \right|$. So

$\lim_{x \rightarrow 3} r(x)$ does not exist.

19. $f(x) = \begin{cases} 1 & \text{if } x < 2 \\ x + 3 & \text{if } 2 \leq x \leq 4 \\ 7 & \text{if } x > 4 \end{cases}$

(a)



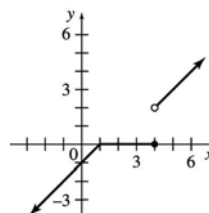
(b) $f(x)$ is discontinuous at $x = 2$.

(c) $\lim_{x \rightarrow 2^-} f(x) = 1$

$$\lim_{x \rightarrow 2^+} f(x) = 5$$

20. $f(x) = \begin{cases} x - 1 & \text{if } x < 2 \\ 0 & \text{if } 1 \leq x \leq 4 \\ x - 2 & \text{if } x > 4 \end{cases}$

(a)



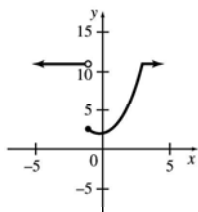
(b) $f(x)$ is discontinuous at $x = 4$.

(c) $\lim_{x \rightarrow 4^-} f(x) = 0$

$$\lim_{x \rightarrow 4^+} f(x) = 2$$

$$21. \quad g(x) = \begin{cases} 11 & \text{if } x < -1 \\ x^2 + 2 & \text{if } -1 \leq x \leq 3 \\ 11 & \text{if } x > 3 \end{cases}$$

(a)

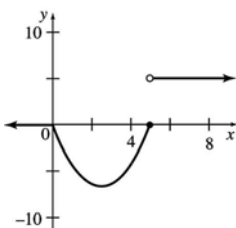
(b) $g(x)$ is discontinuous at $x = -1$.

$$(c) \quad \lim_{x \rightarrow -1^-} g(x) = 11$$

$$\lim_{x \rightarrow -1^+} g(x) = (-1)^2 + 2 = 3$$

$$22. \quad g(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 - 5x & \text{if } 0 \leq x \leq 5 \\ 5 & \text{if } x > 5 \end{cases}$$

(a)

(b) $g(x)$ is discontinuous at $x = 5$.

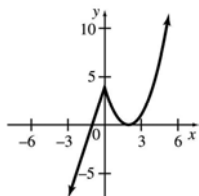
$$(c) \quad \lim_{x \rightarrow 5^-} g(x) = 5^2 - 5(5)$$

$$= 0$$

$$\lim_{x \rightarrow 5^+} g(x) = 5$$

$$23. \quad h(x) = \begin{cases} 4x + 4 & \text{if } x \leq 0 \\ x^2 - 4x + 4 & \text{if } x > 0 \end{cases}$$

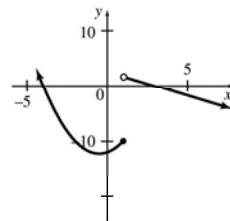
(a)



(b) There are no points of discontinuity.

$$24. \quad h(x) = \begin{cases} x^2 + x - 12 & \text{if } x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

(a)

(b) $h(x)$ is discontinuous at $x = 1$.

$$(c) \quad \lim_{x \rightarrow 1^-} h(x) = 1^2 + 1 - 12 = -10$$

$$\lim_{x \rightarrow 1^+} h(x) = 3 - 1 = 2$$

25. Find k so that $kx^2 = x + k$ for $x = 2$.

$$k(2)^2 = 2 + k$$

$$4k = 2 + k$$

$$3k = 2$$

$$k = \frac{2}{3}$$

26. Find k so that $x^3 + k = kx - 5$ for $x = 3$.

$$3^3 + k = 3k - 5$$

$$27 + k = 3k - 5$$

$$32 = 2k$$

$$16 = k$$

$$27. \quad \frac{2x^2 - x - 15}{x - 3} = \frac{(2x + 5)(x - 3)}{x - 3} = 2x + 5$$

Find k so that $2x + 5 = kx - 1$ for $x = 3$.

$$2(3) + 5 = k(3) - 1$$

$$6 + 5 = 3k - 1$$

$$11 = 3k - 1$$

$$12 = 3k$$

$$4 = k$$

$$28. \quad \frac{3x^2 + 2x - 8}{x + 2} = \frac{(3x - 4)(x + 2)}{x + 2} = 3x - 4$$

Find k so that $3x - 4 = 3x + k$ for $x = -2$.

$$3(-2) - 4 = 3(-2) + k$$

$$-6 - 4 = -6 + k$$

$$-10 = -6 + k$$

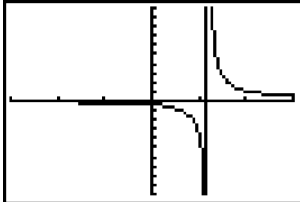
$$-4 = k$$

$$31. f(x) = \frac{x^2 + x + 2}{x^3 - 0.9x^2 + 4.14x - 5.4} = \frac{P(x)}{Q(x)}$$

(a) Graph

$$Y_1 = \frac{P(x)}{Q(x)} = \frac{x^2 + x + 2}{x^3 - 0.9x^2 + 4.14x - 5.4}$$

on a graphing calculator. A good choice for the viewing window is $[-3, 3]$ by $[-10, 10]$.

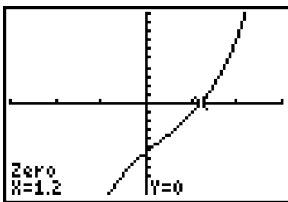


The graph has a vertical asymptote at $x = 1.2$, which indicates that f is discontinuous at $x = 1.2$.

(b) Graph

$$Y_2 = Q(x) = x^3 - 0.9x^2 + 4.14x - 5.4$$

using the same viewing window.



We see that this graph has one x -intercept, 1.2. This indicates that 1.2 is the only real solution of the equation $Q(x) = 0$.

This result verifies our answer from part (a) because a rational function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

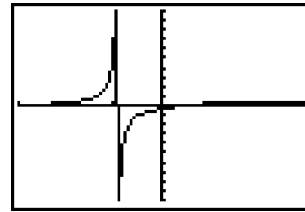
will be discontinuous wherever $Q(x) = 0$.

$$32. f(x) = \frac{x^2 + 3x - 2}{x^3 - 0.9x^2 + 4.14x + 5.4} = \frac{P(x)}{Q(x)}$$

(a) Graph

$$Y_1 = \frac{P(x)}{Q(x)} = \frac{x^2 + 3x - 2}{x^3 - 0.9x^2 + 4.14x + 5.4}$$

on a graphing calculator. A good choice for the viewing window is $[-3, 3]$ by $[-10, 10]$.

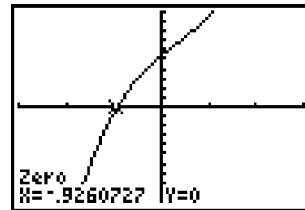


The graph has a vertical asymptote at $x \approx -0.9$. It is difficult to read this value accurately from the graph.

(b) Graph

$$Y_2 = Q(x) = x^3 - 0.9x^2 + 4.14x + 5.4$$

using the same viewing window.



We see that this graph has one x -intercept, ≈ -0.926 . This indicates that -0.926 is the only real solution of the equation $Q(x) = 0$.

A rational function of the form $f(x) = \frac{P(x)}{Q(x)}$

will be discontinuous wherever $Q(x) = 0$, so we see that f is discontinuous at $x \approx -0.926$.

The result in part (b) is consistent with the result in part (a), but the result in part (b) is more accurate.

$$\begin{aligned} 33. g(x) &= \frac{x+4}{x^2+2x-8} \\ &= \frac{x+4}{(x-2)(x+4)} \\ &= \frac{1}{x-2}, x \neq -4 \end{aligned}$$

If $g(x)$ is defined so that $g(-4) = \frac{1}{-4-2} = -\frac{1}{6}$, then the function becomes continuous at -4 . It cannot be made continuous at 2. The correct answer is (a).

34. f is discontinuous at $x = -6$, -4 , and 3 because the limit does not exist there. Also, $f(-6)$ and $f(3)$ are undefined.

f is discontinuous at $x = 4$ because even though $\lim_{x \rightarrow 4} f(x) = 1$, $f(4)$ is undefined.

f is discontinuous at $x = 0$ because $\lim_{x \rightarrow 0} f(x) \neq f(0)$.

35. (a) $\lim_{x \rightarrow 6} P(x)$

As x approaches 6 from the left or the right, the value of $P(x)$ for the corresponding point on the graph approaches 500.

Thus, $\lim_{x \rightarrow 6} P(x) = \500 .

(b) $\lim_{x \rightarrow 10^-} P(x) = \1500

because, as x approaches 10 from the left, $P(x)$ approaches \$1500.

(c) $\lim_{x \rightarrow 10^+} P(x) = \1000 because, as x approaches 10 from the right, $P(x)$ approaches \$1000.

(d) Since $\lim_{x \rightarrow 10^+} P(x) \neq \lim_{x \rightarrow 10^-} P(x)$, $\lim_{x \rightarrow 10} P(x)$ does not exist.

(e) From the graph, the function is discontinuous at $x = 10$. This may be the result of a change of shifts.

(f) From the graph, the second shift will be as profitable as the first shift when 15 units are produced.

36. In dollars,

$$C(x) = 4x \text{ if } 0 < x \leq 150$$

$$C(x) = 3x \text{ if } 150 < x \leq 400$$

$$C(x) = 2.5x \text{ if } 400 < x.$$

(a) $C(130) = 4(130) = \$520$

(b) $C(150) = 4(150) = \$600$

(c) $C(210) = 3(210) = \$630$

(d) $C(400) = 3(400) = \$1200$

(e) $C(500) = 2.5(500) = \$1250$

(f) C is discontinuous at $x = 150$ and $x = 400$ because those represent points of price change.

37. In dollars,

$$F(x) = \begin{cases} 1.25x & \text{if } 0 < x \leq 100 \\ 1.00x & \text{if } x > 100. \end{cases}$$

(a) $F(80) = 1.25(80) = \$100$

(b) $F(150) = 1.00(150) = \$150$

(c) $F(100) = 1.25(100) = \$125$

(d) F is discontinuous at $x = 100$.

38. $C(t)$ is a step function. The average cost per day is $A(t) = \frac{C(t)}{t}$ for integer t . Thus, $A(t)$ is also a step function.

(a) $A(4) = \frac{36(4)}{4} = \36

(b) $A(5) = \frac{36(5)}{5} = \36

(c) $A(6) = \frac{180}{6} = \$30$

(d) $A(7) = \frac{180}{7} \approx \25.71

(e) $A(8) = \frac{180 + 36}{8} = \27

(f) $\lim_{t \rightarrow 5^-} A(t) = 36$ because as t approaches 5 from the left, $A(t)$ is equal to 36.

(g) $\lim_{t \rightarrow 5^+} A(t) = 36$ because just to the right of 5, $A(t)$ is equal to 36.

(h) $A(t)$ is discontinuous at 1, 2, 3, 4, 7, 8, 9, 10, 11.

39. $C(x)$ is a step function.

(a) $\lim_{x \rightarrow 3^-} C(x) = \2.92

(b) $\lim_{x \rightarrow 3^+} C(x) = \3.76

(c) $\lim_{x \rightarrow 3} C(x)$ does not exist.

(d) $C(3) = \$2.92$

(e) $\lim_{x \rightarrow 14^+} C(x) = \10.56

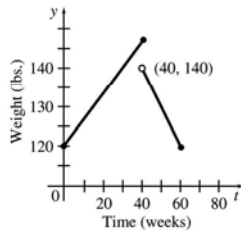
(f) $\lim_{x \rightarrow 14^-} C(x) = \10.56

(g) $\lim_{x \rightarrow 14} C(x) = \10.56

(h) $C(14) = \$10.56$

(i) $C(t)$ is discontinuous at 1, 2, 3, 4, 5, 6, 7, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, and 60.

40. (a) Since $t = 0$ weeks the woman weighs 120 lbs. and at $t = 40$ weeks she weighs 147 lbs. graph the line beginning at coordinate (0, 120) and ending at (40, 147), with closed circles at these points. Since immediately after giving birth, she loses 14 lbs. and continues to lose 13 more lbs. over the following 20 weeks, graph the line between the points (40, 133) and (60, 120) with an open circle at (40, 133) and a closed circle at (60, 120).



(b) From the graph, we see that

$$\begin{aligned}\lim_{t \rightarrow 40^-} w(t) &= 147 \neq 133 \\ &= \lim_{t \rightarrow 40^+} w(t),\end{aligned}$$

where $w(t)$ is the weight in pounds t weeks after conception. Therefore, w is discontinuous at $t = 40$.

41.

$$W(t) = \begin{cases} 48 + 3.64t + 0.6363t^2 + 0.00963t^3, & 1 \leq t \leq 28 \\ -1,004 + 65.8t, & 28 < t \leq 56 \end{cases}$$

$$\begin{aligned}\text{(a)} \quad W(25) &= 48 + 3.64(25) + 0.6363(25)^2 \\ &\quad + 0.00963(25)^3 \\ &\approx 687.156\end{aligned}$$

A male broiler at 25 days weighs about 687 grams.

(b) $W(t)$ is not a continuous function.

At $t = 28$

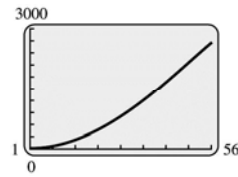
$$\begin{aligned}\lim_{t \rightarrow 28^-} W(t) &= \lim_{t \rightarrow 28^-} 48 + 3.64t + 0.6363t^2 + 0.00963t^3 \\ &= 48 + 3.64(28) + 0.6363(28)^2 + 0.00963(28)^3 \\ &\approx 860.18\end{aligned}$$

$$\begin{aligned}\text{and} \quad \lim_{t \rightarrow 28^-} W(t) &\neq \lim_{t \rightarrow 28^+} (-1004 + 65.8t) \\ &= -1004 + 65.8(28) \\ &= 838.4\end{aligned}$$

$$\text{so} \quad \lim_{t \rightarrow 28^-} W(t) \neq \lim_{t \rightarrow 28^+} W(t)$$

Thus $W(t)$ is discontinuous.

(c)



3.3 Rates of Change

Your Turn 1

$$A(t) = 11.14(1.023)^t$$

Average rate of change from $t = 0$ (2000) to $t = 10$ (2010) is

$$\begin{aligned}\frac{A(10) - A(0)}{10 - 0} &= \frac{11.14(1.023)^{10} - 11.14(1.023)^0}{10} \\ &= \frac{13.98 - 11.14}{10} = \frac{2.84}{10} \approx 0.284,\end{aligned}$$

or 0.0284 million.

The U.S. Asian population increased, on average, by 284,000 people per year.

Your Turn 2

Average rate of change in percent of households between 2007 and 2009 is

$$\begin{aligned}\frac{L(2009) - L(2007)}{9 - 7} &= \frac{73.5 - 82.0}{2} \\ &= \frac{-8.5}{2} = -4.25\end{aligned}$$

On average, the percent of U.S. households with landline telephones decreased by 4.25% per year.

Your Turn 3

For $t = 2$, the instantaneous velocity is

$$\lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h} \text{ feet per second.}$$

$$\begin{aligned}s(2+h) &= 2(2+h)^2 - 5(2+h) + 40 \\ &= 2(4 + 4h + h^2) - 10 - 5h + 40 \\ &= 8 + 8h + h^2 - 10 - 5h + 40 \\ &= h^2 + 3h + 38 \\ s(2) &= 2(2)^2 - 5(2) + 40 \\ &= 2(4) - 10 + 40 \\ &= 38\end{aligned}$$

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h} &= \lim_{h \rightarrow 0} \frac{h^2 + 3h + 38 - 38}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 + 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(h+3)}{h} \\
 &= \lim_{h \rightarrow 0} h + 3 = 3,
 \end{aligned}$$

or 3 feet per second.

Your Turn 4

$$C(x) = x^2 - 2x + 12$$

The instantaneous rate of change of cost when $x = 4$ is

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{C(h+4) - C(4)}{h} &= \lim_{h \rightarrow 0} \frac{[(h+4)^2 - 2(h+4) + 12] - [4^2 - 2(4) + 12]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 + 8h + 16 - 2h - 8 + 12 - 16 + 8 - 12}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 + 6h}{h} = \lim_{h \rightarrow 0} \frac{h(h+6)}{h} \\
 &= \lim_{h \rightarrow 0} h + 6 = 0 + 6 = 6
 \end{aligned}$$

When $x = 4$, the cost increases at a rate of \$6 per unit.

Your Turn 5

$$A(t) = 11.14(1.023)^t, \quad t = 10 \text{ corresponds to 2010}$$

$$\lim_{h \rightarrow 0} \frac{11.14(1.023)^{10+h} - 11.14(1.023)^{10}}{h}$$

Use the TABLE feature on a TI-84 Plus calculator.

h	$\frac{11.14(1.023)^{10+h} - 11.14(1.023)^{10}}{h}$
1	0.32164
0.1	0.31836
0.01	0.31803
0.001	0.318
0.0001	0.318
0.00001	0.318

The limit seems to be approaching 0.318 million.
The instantaneous rate of change in the U.S. Asian population is about 318,000 people per year in 2010.

3.3 Exercises

1. $y = x^2 + 2x = f(x)$ between $x = 1$ and $x = 3$

Average rate of change

$$\begin{aligned}
 &= \frac{f(3) - f(1)}{3 - 1} \\
 &= \frac{15 - 3}{2} \\
 &= 6
 \end{aligned}$$

2. $y = -4x^2 - 6 = f(x)$ between $x = 2$ and $x = 6$

Average rate of change

$$\begin{aligned}
 &= \frac{f(6) - f(2)}{6 - 2} \\
 &= \frac{(-150) - (-22)}{6 - 2} \\
 &= \frac{-150 + 22}{4} \\
 &= \frac{-128}{4} = -32
 \end{aligned}$$

3. $y = -3x^3 + 2x^2 - 4x + 1 = f(x)$ between $x = -2$ and $x = 1$

Average rate of change

$$\begin{aligned}
 &= \frac{f(1) - f(-2)}{1 - (-2)} \\
 &= \frac{(-4) - (-41)}{1 - (-2)} = \frac{-45}{3} = -15
 \end{aligned}$$

4. $y = 2x^3 - 4x^2 + 6x = f(x)$ between $x = -1$ and $x = 4$

Average rate of change

$$\begin{aligned}
 &= \frac{f(4) - f(-1)}{4 - (-1)} \\
 &= \frac{88 - (-12)}{5} \\
 &= \frac{100}{5} \\
 &= 20
 \end{aligned}$$

5. $y = \sqrt{x} = f(x)$ between $x = 1$ and $x = 4$
Average rate of change

$$\begin{aligned} &= \frac{f(4) - f(1)}{4 - 1} \\ &= \frac{2 - 1}{3} \\ &= \frac{1}{3} \end{aligned}$$

6. $y = \sqrt{3x - 2} = f(x)$ between $x = 1$ and $x = 2$

$$\begin{aligned} \text{Average rate of change} &= \frac{f(2) - f(1)}{2 - 1} \\ &= \frac{2 - 1}{2 - 1} \\ &= \frac{1}{1} = 1 \end{aligned}$$

7. $y = e^x = f(x)$ between $x = -2$ and $x = 0$

$$\begin{aligned} \text{Average rate of change} &= \frac{f(0) - f(-2)}{0 - (-2)} \\ &= \frac{1 - e^{-2}}{2} \\ &\approx 0.4323 \end{aligned}$$

8. $y = \ln x = f(x)$ between $x = 2$ and $x = 4$

$$\begin{aligned} \text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{\ln 4 - \ln 2}{2} \\ &\approx 0.3466 \end{aligned}$$

9.

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{s(6 + h) - s(6)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(6 + h)^2 + 5(6 + h) + 2 - [6^2 + 5(6) + 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 + 17h + 68 - 68}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 17h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h + 17)}{h} = \lim_{h \rightarrow 0} (h + 17) = 17 \end{aligned}$$

The instantaneous velocity at $t = 6$ is 17.

10. $s(t) = t^2 + 5t + 2$

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{s(1 + h) - s(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(1 + h)^2 + 5(1 + h) + 2] - [(1)^2 + 5(1) + 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[1 + 2h + h^2 + 5 + 5h + 2] - [1 + 5 + 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{8 + 7h + h^2 - 8}{h} = \lim_{h \rightarrow 0} \frac{7h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(7 + h)}{h} = \lim_{h \rightarrow 0} (7 + h) = 7 \end{aligned}$$

The instantaneous velocity at $t = 1$ is 7.

11. $s(t) = 5t^2 - 2t - 7$

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{s(2 + h) - s(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[5(2 + h)^2 - 2(2 + h) - 7] - [5(2)^2 - 2(2) - 7]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[20 + 20h + 5h^2 - 4 - 2h - 7] - [20 - 4 - 7]}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 + 18h + 5h^2 - 9}{h} = \lim_{h \rightarrow 0} \frac{18h + 5h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(18 + 5h)}{h} = \lim_{h \rightarrow 0} (18 + 5h) = 18 \end{aligned}$$

The instantaneous velocity at $t = 2$ is 18.

12. $s(t) = 5t^2 - 2t - 7$

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{s(3 + h) - s(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[5(3 + h)^2 - 2(3 + h) - 7] - [5(3)^2 - 2(3) - 7]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[45 + 30h + 5h^2 - 6 - 2h - 7] - [45 - 6 - 7]}{h} \\ &= \lim_{h \rightarrow 0} \frac{32 + 28h + 5h^2 - 32}{h} = \lim_{h \rightarrow 0} \frac{28h + 5h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(28 + 5h)}{h} = \lim_{h \rightarrow 0} (28 + 5h) = 28 \end{aligned}$$

The instantaneous velocity at $t = 3$ is 28.

13. $s(t) = t^3 + 2t + 9$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(1+h)^3 + 2(1+h) + 9] - [(1)^3 + 2(1) + 9]}{h} \\ &= \lim_{h \rightarrow 0} \frac{[1 + 3h + 3h^2 + h^3 + 2 + 2h + 9] - [1 + 2 + 9]}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3 + 3h^2 + 5h + 12 - 12}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3 + 3h^2 + 5h}{h} = \lim_{h \rightarrow 0} \frac{h(h^2 + 3h + 5)}{h} \\ &= \lim_{h \rightarrow 0} (h^2 + 3h + 5) = 5 \end{aligned}$$

The instantaneous velocity at $t = 1$ is 5.

14.

$$\begin{aligned} & s(t) = t^3 + 2t + 9 \\ & \lim_{h \rightarrow 0} \frac{s(4+h) - s(4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(4+h)^3 + 2(4+h) + 9 - [4^3 + 2(4) + 9]}{h} \\ &= \lim_{h \rightarrow 0} \frac{4^3 + 3(4^2)h + 3(4h^2) + h^3 + 8 + 2h + 9 - (4^3 + 8 + 9)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3 + 12h^2 + 48h + 2h}{h} = \lim_{h \rightarrow 0} \frac{h^3 + 12h^2 + 50h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h^2 + 12h + 50)}{h} = \lim_{h \rightarrow 0} (h^2 + 12h + 50) = 50 \end{aligned}$$

The instantaneous velocity at $t = 4$ is 50.

15. $f(x) = x^2 + 2x$ at $x = 0$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(0+h)^2 + 2(0+h) - [0^2 + 2(0)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 + 2h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h+2)}{h} \\ &= \lim_{h \rightarrow 0} h + 2 = 2 \end{aligned}$$

The instantaneous rate of change at $x = 0$ is 2.

16. $s(t) = -4t^2 - 6$ at $t = 2$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-4(2+h)^2 - 6 - [-4(2)^2 - 6]}{h} \\ &= \lim_{h \rightarrow 0} \frac{-4(4 + 4h + h^2) - 6 + 16 + 6}{h} \\ &= \lim_{h \rightarrow 0} \frac{-16h - 4h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(-16 - 4h)}{h} \\ &= \lim_{h \rightarrow 0} (-16 - 4h) = -16 \end{aligned}$$

The instantaneous rate of change at $t = 2$ is -16 .

17. $g(t) = 1 - t^2$ at $t = -1$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{g(-1+h) - g(-1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - (-1+h)^2 - [1 - (-1)^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - (1 - 2h + h^2) - 1 + 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h - h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2 - h)}{h} \\ &= \lim_{h \rightarrow 0} (2 - h) = 2 \end{aligned}$$

The instantaneous rate of change at $t = -1$ is 2.

18. $F(x) = x^2 + 2$ at $x = 0$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{F(0+h) - F(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(0+h)^2 + 2 - [0^2 + 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2}{h} \\ &= \lim_{h \rightarrow 0} h \\ &= 0 \end{aligned}$$

The instantaneous rate of change at $x = 0$ is 0.

19. $f(x) = x^x$ at $x = 2$

h	
0.01	$\frac{f(2 + 0.01) - f(2)}{0.01}$ $= \frac{2.01^{2.01} - 2^2}{0.01}$ $= 6.84$
0.001	$\frac{f(2 + 0.001) - f(2)}{0.001}$ $= \frac{2.001^{2.001} - 2^2}{0.001}$ $= 6.779$
0.0001	$\frac{f(2 + 0.0001) - f(2)}{0.0001}$ $= \frac{2.0001^{2.0001} - 2^2}{0.0001}$ $= 6.773$
0.00001	$\frac{f(2 + 0.00001) - f(2)}{0.00001}$ $= \frac{2.00001^{2.00001} - 2^2}{0.00001}$ $= 6.7727$
0.000001	$\frac{f(2 + 0.000001) - f(2)}{0.000001}$ $= \frac{2.000001^{2.000001} - 2^2}{0.000001}$ $= 6.7726$

The instantaneous rate of change at $x = 2$ is 6.773.

20. $f(x) = x^x$ at $x = 3$

h	
0.01	$\frac{f(3 + 0.01) - f(3)}{0.01}$ $= \frac{3.01^{3.01} - 3^3}{0.01}$ $= 57.3072$
0.001	$\frac{f(3 + 0.001) - f(3)}{0.001}$ $= \frac{3.001^{3.001} - 3^3}{0.001}$ $= 56.7265$
0.00001	$\frac{f(3 + 0.00001) - f(3)}{0.00001}$ $= \frac{3.00001^{3.00001} - 3^3}{0.00001}$ $= 56.6632$
0.000001	$\frac{f(3 + 0.000001) - f(3)}{0.000001}$ $= \frac{3.000001^{3.000001} - 3^3}{0.000001}$ $= 56.6626$
0.0000001	$\frac{f(3 + 0.0000001) - f(3)}{0.0000001}$ $= \frac{3.0000001^{3.0000001} - 3^3}{0.0000001}$ $= 56.6625$

The instantaneous rate of change at $x = 3$ is 56.66.

21. $f(x) = x^{\ln x}$ at $x = 2$

h	
0.01	$\frac{f(2 + 0.01) - f(2)}{0.01}$ $= \frac{2.01^{\ln 2.01} - 2^{\ln 2}}{0.01}$ $= 1.1258$
0.001	$\frac{f(2 + 0.001) - f(2)}{0.001}$ $= \frac{2.001^{\ln 2.001} - 2^{\ln 2}}{0.001}$ $= 1.1212$
h	
0.0001	$\frac{f(2 + 0.0001) - f(2)}{0.0001}$ $= \frac{2.0001^{\ln 2.0001} - 2^{\ln 2}}{0.0001}$ $= 1.1207$
0.00001	$\frac{f(2 + 0.00001) - f(2)}{0.00001}$ $= \frac{2.00001^{\ln 2.00001} - 2^{\ln 2}}{0.00001}$ $= 1.1207$

The instantaneous rate of change at $x = 2$ is 1.121.

22. $f(x) = x^{\ln x}$ at $x = 3$

h	
0.01	$\frac{f(3 + 0.01) - f(3)}{0.01}$ $= \frac{3.01^{\ln 3.01} - 3^{\ln 3}}{0.01}$ $= 2.4573$
0.001	$\frac{f(3 + 0.001) - f(3)}{0.001}$ $= \frac{3.001^{\ln 3.001} - 3^{\ln 3}}{0.001}$ $= 2.4495$
0.00001	$\frac{f(3 + 0.00001) - f(3)}{0.00001}$ $= \frac{3.00001^{\ln 3.00001} - 3^{\ln 3}}{0.00001}$ $= 2.4486$

The instantaneous rate of change at $x = 3$ is 2.449.

24. If the instantaneous rate of change of $f(x)$ with respect to x is positive when $x = 1$, the function would be increasing.

25. $P(x) = 2x^2 - 5x + 6$

(a) $P(4) = 18$

$P(2) = 4$

Average rate of change of profit

$$= \frac{P(4) - P(2)}{4 - 2}$$

$$= \frac{18 - 4}{2} = \frac{14}{2} = 7,$$

which is \$700 per item.

(b) $P(3) = 9$

$P(2) = 4$

Average rate of change of profit

$$= \frac{P(3) - P(2)}{3 - 2} = \frac{9 - 4}{1} = 5$$

which is \$500 per item.

(c) $\lim_{h \rightarrow 0} \frac{P(2 + h) - P(2)}{h}$

$$= \lim_{h \rightarrow 0} \frac{2(2 + h)^2 - 5(2 + h) + 6 - 4}{h}$$

$$= \lim_{h \rightarrow 0} \frac{8 + 8h + 2h^2 - 10 - 5h + 2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h^2 + 3h}{h} = \lim_{h \rightarrow 0} \frac{h(2h + 3)}{h}$$

$$= \lim_{h \rightarrow 0} (2h + 3) = 3,$$

which is \$300 per item.

(d) $\lim_{h \rightarrow 0} \frac{P(4 + h) - P(4)}{h}$

$$= \lim_{h \rightarrow 0} \frac{2(4 + h)^2 - 5(4 + h) + 6 - 18}{h}$$

$$= \lim_{h \rightarrow 0} \frac{32 + 16h + 2h^2 - 20 - 5h - 12}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h^2 + 11h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(2h + 11)}{h}$$

$$= \lim_{h \rightarrow 0} 2h + 11 = 11,$$

which is \$1100 per item.

26. $R = 10x - 0.002x^2$

(a) Average rate of change $= \frac{R(1001) - R(1000)}{1001 - 1000} = \frac{8005.998 - 8000}{1} = 5.998$

The average rate of change is \$5998.

(b) Marginal revenue $= \lim_{h \rightarrow 0} \frac{R(1000 + h) - R(1000)}{h}$

$$= \lim_{h \rightarrow 0} \frac{[10(1000 + h) - 0.002(1000 + h)^2] - [10(1000) - 0.002(1000)^2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[10,000 + 10h - 0.002(1,000,000 + 2000h + h^2)] - 8000}{h}$$

$$= \lim_{h \rightarrow 0} \frac{10,000 + 10h - 2000 - 4h - 0.002h^2 - 8000}{h}$$

$$= \lim_{h \rightarrow 0} \frac{6h - 0.002h^2}{h} = \lim_{h \rightarrow 0} \frac{h(6 - 0.002h)}{h} = \lim_{h \rightarrow 0} (6 - 0.002h) = 6$$

The marginal revenue is \$6000. This is approximately the revenue generated by the 1000th unit produced.

(c) Additional revenue $= R(1001) - R(1000) = [10(1001) - 0.002(1001)^2] - [10(1000) - 0.002(1000)^2]$

$$= 8005.998 - 8000$$

The additional revenue is \$5998.

(d) The answers to parts (a) and (c) are the same and are approximately equal to the answer to part (b).

27. $N(p) = 80 - 5p^2, 1 \leq p \leq 4$

(a) Average rate of change of demand is

$$\frac{N(3) - N(2)}{3 - 2} = \frac{35 - 60}{1}$$

$$= -25 \text{ boxes per dollar.}$$

(b) Instantaneous rate of change when p is 2 is

$$\lim_{h \rightarrow 0} \frac{N(2 + h) - N(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{80 - 5(2 + h)^2 - [80 - 5(2)^2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{80 - 20 - 20h - 5h^2 - (80 - 20)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-5h^2 - 20h}{h} = -20 \text{ boxes per dollar.}$$

Around the \$2 point, a \$1 price increase (say, from \$1.50 to \$2.50) causes a drop in demand of about 20 boxes.

(c) Instantaneous rate of change when p is 3 is

$$\lim_{h \rightarrow 0} \frac{80 - 5(3 + h)^2 - [80 - 5(3)^2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{80 - 45 - 30h - 5h^2 - 80 + 45}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-30h - 5h^2}{h}$$

$$= -30 \text{ boxes per dollar.}$$

(d) As the price increases, the demand decreases; this is an expected change.

28. $A(t) = 1000(1.05)^t$

(a) Average rate of change in the total amount from $t = 0$ to $t = 5$:

$$\frac{A(5) - A(0)}{5 - 0} = \frac{1000(1.05)^5 - 1000(1.05)^0}{5}$$

$$= 55.2563,$$

which is \$55.26 per year.

(b) Average rate of change in the total amount from $t = 5$ to $t = 10$:

$$\frac{A(10) - A(5)}{10 - 5} = \frac{1000(1.05)^{10} - 1000(1.05)^5}{5}$$

$$= 70.5226,$$

which is \$70.52 per year.

- (c) Instantaneous rate of change for $t = 5$:

$$\lim_{h \rightarrow 0} \frac{1000(1.05)^{5+h} - 1000(1.05)^5}{h}$$

Use the TABLE feature on a TI-84 Plus calculator to estimate the limit.

h	$\frac{1000(1.05)^{5+h} - 1000(1.05)^5}{h}$
1	63.814
0.1	62.422
0.01	62.285
0.001	62.272
0.0001	62.27
0.00001	62.27

The limit seems to be approaching 62.27. So, the instantaneous rate of change for $t = 5$ is about \$62.27 per year.

29. $A(t) = 1000e^{0.05t}$

- (a) Average rate of change in the total amount from $t = 0$ to $t = 5$:

$$\frac{A(5) - A(0)}{5 - 0} = \frac{1000e^{0.05(5)} - 1000e^{0.05(0)}}{5}$$

$$= 56.8051,$$

which is \$56.81 per year.

- (b) Average rate of change in the total amount from $t = 5$ to $t = 10$:

$$\frac{A(10) - A(5)}{10 - 5} = \frac{1000e^{0.05(10)} - 1000e^{0.05(5)}}{5}$$

$$= 72.93917,$$

which is \$72.94 per year.

- (c) Instantaneous rate of change for $t = 5$:

$$\lim_{h \rightarrow 0} \frac{1000e^{0.05(5+h)} - 1000e^{0.05(5)}}{h}$$

Use the TABLE feature on a TI-84 Plus calculator to estimate the limit.

h	$\frac{1000e^{0.05(5+h)} - 1000e^{0.05(5)}}{h}$
1	65.833
0.1	64.362
0.01	64.217
0.001	64.203
0.0001	64.201
0.00001	64.201

The limit seems to be approaching 64.201. So, the instantaneous rate of change for $t = 5$ is about \$64.20 per year.

30. (a) $s(4) = 10$

$$s(1) = 1$$

$$\text{Average rate of change} = \frac{s(4) - s(1)}{4 - 1}$$

$$= \frac{10 - 1}{4 - 1} = \frac{9}{3}$$

which is \$3000 per year.

(b) $s(7) = 10$

$$s(4) = 10$$

$$\text{Average rate of change} = \frac{s(7) - s(4)}{7 - 4}$$

$$= \frac{10 - 10}{7 - 4} = \frac{0}{3} = 0$$

which is \$0 per year.

(c) $s(12) = 1$

$$s(7) = 10$$

$$\text{Average rate of change} = \frac{s(12) - s(7)}{12 - 7}$$

$$= \frac{1 - 10}{12 - 7} = -1.8$$

which is -\$1800 per year.

- (d) Sales increase during the first four years, then stay constant until year 7, then decrease.

- (e) Many answers are possible; usually this trend occurs with high-tech products or products that mirror trends, such as CDs.

31. Let $P(t)$ = the price per gallon of gasoline for the month t , where $t = 1$ represents January, $t = 2$ represents February, and so on.

(a) $P(1) = 329$ (cents)

$$P(7) = 435$$
 (cents)

Average change in price from January to July:

$$\frac{P(7) - P(1)}{7 - 1} = \frac{435 - 329}{6} = \frac{106}{6} = 17.6667$$

On average, the price of gasoline increased about 17.7 cents per gallon per month.

(b) $P(7) = 435$ (cents), $P(12) = 195$ (cents)

Average change in price from July to December:

$$\frac{P(12) - P(7)}{12 - 7} = \frac{195 - 435}{5} = \frac{-240}{5} = -48$$

On average, the price of gasoline decreased about 48.0 cents per gallon per month.

(c) $P(1) = 329$ (cents), $P(12) = 195$ (cents)

Average change in price from January to December:

$$\begin{aligned} \frac{P(12) - P(1)}{12 - 1} &= \frac{195 - 329}{11} \\ &= \frac{-134}{11} = -12.1818 \end{aligned}$$

On average, the price of gasoline decreased about 12.2 cents per gallon per month.

32. (a) From 2000 to 2008,
 $\frac{378.1 - 214.0}{8} = 20.5$;
 \$20.5 billion per year.

(b) From 2008 to 2018,
 $\frac{93.1 - 378.1}{10} = -28.5$;
 -\$28.5 billion per year.

33. $p(t) = t^2 + t$

(a) $p(1) = 1^2 + 1 = 2$

$p(4) = 4^2 + 4 = 20$

Average rate of change

$$= \frac{p(4) - p(1)}{4 - 1} = \frac{20 - 2}{3} = 6$$

The average rate of change is 6% per day.

(b) $\lim_{h \rightarrow 0} \frac{p(3+h) - p(3)}{h}$
 $= \lim_{h \rightarrow 0} \frac{(3+h)^2 + (3+h) - (3^2 + 3)}{h}$
 $= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 + 3 + h - 12}{h}$
 $= \lim_{h \rightarrow 0} \frac{h^2 + 7h}{h}$
 $= \lim_{h \rightarrow 0} \frac{h(h+7)}{h}$
 $= \lim_{h \rightarrow 0} (h+7) = 7$

The instantaneous rate of change is 7% per day.
 The number of people newly infected on day 3 is about 7% of the total population.

34. Let $P(t)$ = world population estimated in billions for year t .

(a) $P(1990) = 5.3$

If replacement-level fertility is reached in 2010, $P(2050) = 8.6$.

Average rate of change

$$\begin{aligned} &= \frac{P(2050) - P(1990)}{2050 - 1990} \\ &= \frac{8.6 - 5.3}{60} = 0.055 \end{aligned}$$

On average, the population will increase 55 million per year.

If replacement-level fertility is reached in 2030, $P(2050) = 9.2$.

Average rate of change

$$\begin{aligned} &= \frac{P(2050) - P(1990)}{2050 - 1990} \\ &= \frac{9.2 - 5.3}{60} = 0.065 \end{aligned}$$

On average, the population will increase 65 million per year.

If replacement-level fertility is reached in 2050, $P(2050) = 9.8$.

Average rate of change

$$\begin{aligned} &= \frac{P(2050) - P(1990)}{2050 - 1990} \\ &= \frac{9.8 - 5.3}{60} = 0.075 \end{aligned}$$

On average, the population will increase 75 million per year. The projection for replacement-level fertility by 2010 predicts the smallest rate of change in world population.

- (b) If replacement-level fertility is reached in 2010

$$P(2090) = 9.3$$

$$P(2130) = 9.6$$

Average rate of change

$$\begin{aligned} &= \frac{P(2130) - P(2090)}{2130 - 2090} \\ &= \frac{9.6 - 9.3}{40} \\ &= 0.0075 \end{aligned}$$

On average, the population will increase 7.5 million per year.

If replacement-level fertility is reached in 2030,

$$P(2090) = 10.3$$

$$P(2130) = 10.6$$

Average rate of change

$$\begin{aligned} &= \frac{P(2130) - P(2090)}{2130 - 2090} \\ &= \frac{10.6 - 10.3}{40} \\ &= 0.0075 \end{aligned}$$

On average, the population will increase 7.5 million per year.

If replacement-level fertility is reached in 2050,

$$P(2090) = 11.35$$

$$P(2130) = 11.75$$

Average rate of change

$$\begin{aligned} &= \frac{P(2130) - P(2090)}{2130 - 2090} \\ &= \frac{11.75 - 11.35}{40} \\ &= 0.01 \end{aligned}$$

On average, the population will increase 10 million per year. From 2090 – 2130 the three projections show almost the same rate of change in world population.

35. (a)
- $P(1) = 3, P(2) = 5$

$$\begin{aligned} \text{Average rate of change} &= \frac{P(2) - P(1)}{2 - 1} \\ &= \frac{5 - 3}{2 - 1} \\ &= \frac{2}{1} \\ &= 2 \end{aligned}$$

From 1 min to 2 min, the population of bacteria increases, on the average, 2 million per min.

- (b)
- $P(2) = 5, P(3) = 4.2$

$$\begin{aligned} \text{Average rate of change} &= \frac{P(3) - P(2)}{3 - 2} \\ &= \frac{4.2 - 5}{3 - 2} \\ &= \frac{-0.8}{1} \\ &= -0.8 \end{aligned}$$

From 2 min to 3 min, the population of bacteria decreases, on the average, 0.8 million or 800,000 per min.

- (c)
- $P(3) = 4.2, P(4) = 2$

$$\begin{aligned} \text{Average rate of change} &= \frac{P(4) - P(3)}{4 - 3} \\ &= \frac{2 - 4.2}{4 - 3} \\ &= \frac{-2.2}{1} \\ &= -2.2 \end{aligned}$$

From 3 min to 4 min, the population of bacteria decreases, on the average, 2.2 million per min.

- (d)
- $P(4) = 2, P(5) = 1$

$$\begin{aligned} \text{Average rate of change} &= \frac{P(5) - P(4)}{5 - 4} \\ &= \frac{1 - 2}{5 - 4} \\ &= \frac{-1}{1} \\ &= -1 \end{aligned}$$

From 4 min to 5 min, the population decreases, on the average, –1 million per min.

- (e) The population increased up to 2 min after the bactericide was introduced, but decreased after 2 min.
- (f) The rate of decrease of the population slows down at about 3 min. The graph becomes less and less steep after that point.

- 36.
- $L(t) = -0.01t^2 + 0.788t - 7.048$

$$\begin{aligned} \text{(a)} \quad \frac{L(28) - L(22)}{28 - 22} &= \frac{7.176 - 5.448}{6} \\ &= 0.288 \end{aligned}$$

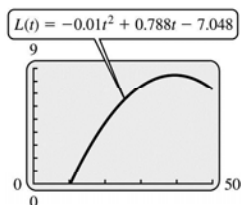
The average rate of growth during weeks 22 through 28 is 0.288 mm per week.

(b)

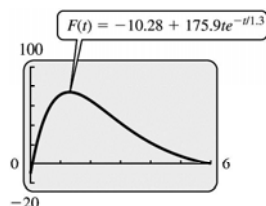
$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{L(t+h) - L(t)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{L(22+h) - L(22)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[-0.01(22+h)^2 + 0.788(22+h) - 7.048] - 5.448}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-0.01(h^2 + 44h + 484) + 17.336 + 0.788h - 12.496}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-0.01h^2 + 0.348h}{h} \\
 &= \lim_{h \rightarrow 0} (-0.01h + 0.348) \\
 &= 0.348
 \end{aligned}$$

The instantaneous rate of growth at exactly 22 weeks is 0.348 mm per week.

(c)



37. (a)



(b) The average rate of change during the first hour is

$$\frac{F(1) - F(0)}{1 - 0} \approx 81.51$$

kilojoules per hour per hour.

(c) Store $F(t)$ in a function menu of a graphing calculator. Store $\frac{Y_1(1+X) - Y_1(1)}{X}$ as Y_2 in the function menu, where Y_1 represents $F(t)$. Substitute small values for X in Y_2 perhaps with use of a table feature of the graphing calculator. As X is allowed to get smaller, Y_2 approaches 18.81 kilojoules per hour per hour.

(d) Through use of a MAX/feature program of a graphing calculator, the maximum point seen in part (a) is estimated to occur at approximately $t = 1.3$ hours.

38. (a) The average rate of change of $M(t)$ on the interval $[105, 115]$ is

$$\frac{M(115) - M(105)}{115 - 105} = \frac{0.8}{10} = 0.08$$

kilograms per day.

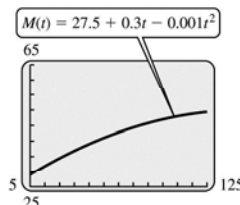
(b) Calculate $\lim_{h \rightarrow 0} \frac{M(105+h) - M(105)}{h}$

$$\begin{aligned}
 M(105+h) &= 27.5 + 0.3(105+h) \\
 &\quad - 0.001(105+h)^2 \\
 &= 27.5 + 31.5 + 0.3h \\
 &\quad - (11.025 + 0.21h + 0.001h^2) \\
 &= 47.975 + 0.09h - 0.001h^2 \\
 M(105) &= 47.975
 \end{aligned}$$

So, the instantaneous rate of change of $M(t)$ at $t = 105$ is

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \left(\frac{47.975 + 0.09h - 0.001h^2 - 47.975}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left(\frac{0.09h - 0.001h^2}{h} \right) \\
 &= \lim_{h \rightarrow 0} (0.09 - 0.001h) \\
 &= 0.09 \text{ kilograms per day.}
 \end{aligned}$$

(c)



39. Let $I(t)$ represent immigration (in thousands) in year t .

$$(a) \frac{I(1955) - I(1905)}{1955 - 1905} = \frac{238 - 1026}{50} = \frac{-788}{50} = -15.76$$

The average rate of change is $-15,760$ immigrants per year.

$$(b) \frac{I(2005) - I(1955)}{2005 - 1955} = \frac{1122 - 238}{50} = \frac{884}{50} = 17.68$$

The average rate of change is 17,680 immigrants per year.

$$\begin{aligned}
 \text{(c)} \quad \frac{I(2005) - I(1905)}{2005 - 1905} &= \frac{1122 - 1026}{100} \\
 &= \frac{96}{100} \\
 &= 0.96
 \end{aligned}$$

The average rate of change is 960 immigrants per year.

$$\text{(d)} \quad \frac{-15,760 + 17,680}{2} = \frac{1920}{2} = 960$$

They are equal. This will not be true for all time periods. (It is true for only time periods of equal length.)

(e) 2009 is 4 years after 2005.

$$1,122,000 + 4(960/\text{year}) = 1,125,840$$

The predicted number of immigrants in 2009 is about 1,126,000 immigrants. The predicted value is about 5,000 less than the actual number of 1,130,818.

40. Let $D(t)$ represent the percent of students (8th, 10th, or 12th graders) who have used marijuana by the year t .

(a) 8th graders:

$$\begin{aligned}
 \frac{D(2006) - D(2003)}{2006 - 2003} &= \frac{15.7 - 17.5}{3} \\
 &= -0.60 \text{ percent per year}
 \end{aligned}$$

$$\begin{aligned}
 \frac{D(2009) - D(2006)}{2009 - 2006} &= \frac{15.7 - 15.7}{3} \\
 &= 0 \text{ percent per year}
 \end{aligned}$$

$$\begin{aligned}
 \frac{D(2009) - D(2003)}{2009 - 2003} &= \frac{15.7 - 17.5}{6} \\
 &= -0.30 \text{ percent per year}
 \end{aligned}$$

(b) 10th graders:

$$\begin{aligned}
 \frac{D(2006) - D(2003)}{2006 - 2003} &= \frac{31.8 - 36.4}{3} \\
 &= -1.53 \text{ percent per year}
 \end{aligned}$$

$$\begin{aligned}
 \frac{D(2009) - D(2006)}{2009 - 2006} &= \frac{32.3 - 31.8}{3} \\
 &= 0.17 \text{ percent per year}
 \end{aligned}$$

$$\begin{aligned}
 \frac{D(2009) - D(2003)}{2009 - 2003} &= \frac{32.3 - 36.4}{6} \\
 &= -0.68 \text{ percent per year}
 \end{aligned}$$

(c) 12th graders:

$$\begin{aligned}
 \frac{D(2006) - D(2003)}{2006 - 2003} &= \frac{42.3 - 46.1}{3} \\
 &= -1.27 \text{ percent per year}
 \end{aligned}$$

$$\begin{aligned}
 \frac{D(2009) - D(2006)}{2009 - 2006} &= \frac{42.0 - 42.3}{3} \\
 &= -0.10 \text{ percent per year}
 \end{aligned}$$

$$\begin{aligned}
 \frac{D(2009) - D(2003)}{2009 - 2003} &= \frac{42.0 - 46.1}{6} \\
 &= -0.68 \text{ percent per year}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \text{(a)} \quad \frac{T(3000) - T(1000)}{3000 - 1000} &= \frac{23 - 15}{2000} \\
 &= \frac{8}{2000} \\
 &= \frac{4}{1000}
 \end{aligned}$$

From 1000 to 3000 ft, the temperature changes about 4° per 1000 ft; the temperature rises (on the average).

$$\begin{aligned}
 \text{(b)} \quad \frac{T(5000) - T(1000)}{5000 - 1000} &= \frac{22 - 15}{4000} \\
 &= \frac{7}{4000} \\
 &= \frac{1.75}{1000}
 \end{aligned}$$

From 1000 to 5000 ft, the temperature changes about 1.75° per 1000 ft; the temperature rises (on the average).

$$\begin{aligned}
 \text{(c)} \quad \frac{T(9000) - T(3000)}{9000 - 3000} &= \frac{15 - 23}{6000} \\
 &= \frac{-8}{6000} \\
 &= \frac{-\frac{4}{3}}{1000}
 \end{aligned}$$

From 3000 to 9000 ft, the temperature changes about $-\frac{4}{3}^\circ$ per 1000 ft; the temperature falls (on the average).

$$\text{(d)} \quad \frac{T(9000) - T(1000)}{9000 - 1000} = \frac{15 - 15}{8000} = 0$$

From 1000 to 9000 ft, the temperature changes about 0° per 1000 ft; the temperature stays constant (on the average).

(e) The temperature is highest at 3000 ft and lowest at 1000 ft. If 7000 ft is changed to 10,000 ft, the lowest temperature would be at 10,000 ft.

(f) The temperature at 9000 ft is the same as 1000 ft.

$$42. \quad (a) \quad \frac{s(2) - s(0)}{2 - 0} = \frac{10 - 0}{2} = 5 \text{ ft/sec}$$

$$(b) \quad \frac{s(4) - s(2)}{4 - 2} = \frac{14 - 10}{2} = 2 \text{ ft/sec}$$

$$(c) \quad \frac{s(6) - s(4)}{6 - 4} = \frac{20 - 14}{2} = 3 \text{ ft/sec}$$

$$(d) \quad \frac{s(8) - s(6)}{8 - 6} = \frac{30 - 20}{2} = 5 \text{ ft/sec}$$

$$(e) \quad (i) \quad \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

$$= \frac{f(4 + 2) - f(4 - 2)}{(2)(2)}$$

$$= \frac{f(6) - f(2)}{4}$$

$$= \frac{20 - 10}{4}$$

$$= \frac{10}{4} = 2.5 \text{ ft/sec}$$

$$(ii) \quad \frac{2 + 3}{2} = 2.5 \text{ ft/sec}$$

$$(f) \quad (i) \quad \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

$$= \frac{f(6 + 2) - f(6 - 2)}{(2)(2)}$$

$$= \frac{f(8) - f(4)}{4}$$

$$= \frac{30 - 14}{4}$$

$$= \frac{16}{4} = 4 \text{ ft/sec}$$

$$(ii) \quad \frac{3 + 5}{2} = 4 \text{ ft/sec}$$

43. (a) Average rate of change from 0.5 to 1:

$$\frac{f(1) - f(0.5)}{1 - 0.5} = \frac{55 - 30}{0.5} = 50 \text{ mph}$$

Average rate of change from 1 to 1.5:

$$\frac{f(1.5) - f(1)}{1.5 - 1} = \frac{80 - 55}{0.5} = 50 \text{ mph}$$

Estimate of instantaneous velocity is

$$\frac{50 + 50}{2} = 50 \text{ mph.}$$

- (b) Average rate of change from 1.5 to 2:

$$\frac{f(2) - f(1.5)}{2 - 1.5} = \frac{104 - 80}{0.5} = 48 \text{ mph}$$

Average rate of change from 2 to 2.5

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{124 - 104}{0.5} = 40 \text{ mph}$$

Estimate of instantaneous velocity is

$$\frac{48 + 40}{2} = 44 \text{ mph.}$$

$$44. \quad s(t) = t^2 + 5t + 2$$

- (a) Average velocity

$$= \frac{s(6) - s(4)}{6 - 4}$$

$$= \frac{68 - 38}{6 - 4}$$

$$= \frac{30}{2} = 15 \text{ ft/sec}$$

- (b) Average velocity

$$= \frac{s(5) - s(4)}{5 - 4}$$

$$= \frac{52 - 38}{5 - 4}$$

$$\frac{14}{1} = 14 \text{ ft/sec}$$

- (c) $\lim_{h \rightarrow 0} \frac{s(4 + h) - s(4)}{h}$

$$= \lim_{h \rightarrow 0} \frac{(4 + h)^2 + 5(4 + h) + 2 - 38}{h}$$

$$= \lim_{h \rightarrow 0} \frac{16 + 8h + h^2 + 20 + 5h + 2 - 38}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 + 13h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h(h + 13)}{h}$$

$$= \lim_{h \rightarrow 0} (h + 13)$$

$$= 13 \text{ ft/sec}$$

3.4 Definition of the Derivative

Your Turn 1

$$f(x) = x^2 - x, \quad x = -2 \text{ and } x = 1.$$

$$\text{Slope of secant line} = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{0 - 6}{3} = -2$$

Use the point-slope form and the point $(1, f(1))$, or $(1, 0)$.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 1)$$

$$y = -2x + 2$$

Your Turn 2

$$f(x) = x^2 - x$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 - (x+h)] - [x^2 - x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh - h + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x - 1 + h)}{h} \\ &= \lim_{h \rightarrow 0} (2x - 1 + h) = 2x - 1 + 0 \\ &= 2x - 1 \end{aligned}$$

$$f'(-2) = 2(-2) - 1 = -5$$

Your Turn 3

$$f(x) = x^3 - 1$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 1] - [x^3 - 1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 1 - x^3 + 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \\ &= 3x^2 + 0 + 0 \\ &= 3x^2 \end{aligned}$$

$$f'(-1) = 3(-1)^2 = 3$$

Your Turn 4

$$f(x) = -\frac{2}{x}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(-\frac{2}{x+h}\right) - \left(-\frac{2}{x}\right)}{h} \\ &= \lim_{h \rightarrow 0} \left[-\frac{2}{x+h} + \frac{2}{x} \right] \cdot \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2x + 2x + 2h}{x(x+h)} \cdot \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h \rightarrow 0} \frac{2}{x(x+h)} \\ &= \frac{2}{x(x+0)} \\ &= \frac{2}{x^2} \end{aligned}$$

Your Turn 5

$$f(x) = 2\sqrt{x}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} \cdot \frac{2\sqrt{x+h} + 2\sqrt{x}}{2\sqrt{x+h} + 2\sqrt{x}} \\ &= \lim_{h \rightarrow 0} \frac{4(x+h) - 4x}{h(2\sqrt{x+h} + 2\sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{4x + 4h - 4x}{h(2\sqrt{x+h} + 2\sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{4h}{h(2\sqrt{x+h} + 2\sqrt{x})} \\ &= \lim_{h \rightarrow 0} \frac{4}{2\sqrt{x+h} + 2\sqrt{x}} \\ &= \frac{4}{2\sqrt{x} + 2\sqrt{x}} = \frac{4}{4\sqrt{x}} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

Your Turn 6

$$\begin{aligned}
C(x) &= 10x - 0.002x^2 \\
C'(x) &= \lim_{h \rightarrow 0} \frac{C(x+h) - C(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{10(x+h) - 0.002(x+h)^2 - 10x + 0.002x^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{10x + 10h - 0.002x^2 - 0.004xh - 0.002h^2 - 10x + 0.002x^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{10h - 0.004xh - 0.002h^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{h(10 - 0.004x - 0.002h)}{h} \\
&= \lim_{h \rightarrow 0} (10 - 0.004x - 0.002h) \\
&= 10 - 0.004x + 0 \\
&= 10 - 0.004x \\
C'(100) &= 10 - 0.004(100) = 10 - 0.4 = 9.60
\end{aligned}$$

The rate of change of the cost when $x = 100$ is \$9.60.

Your Turn 7

$$f(x) = 2\sqrt{x} \text{ at } x = 4$$

From Your Turn 5, we have

$$f'(x) = \frac{1}{\sqrt{x}}.$$

$$\text{At } x = 4: f'(4) = \frac{1}{\sqrt{4}} = \frac{1}{2} \text{ and } f(4) = 2\sqrt{4} = 2(2) = 4$$

Slope of the tangent line at $(4, f(4))$, or $(4, 4)$ is $\frac{1}{2}$.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{1}{2}(x - 4)$$

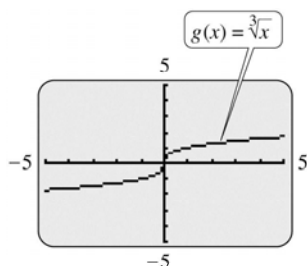
$$y - 4 = \frac{1}{2}x - 2$$

$$y = \frac{1}{2}x + 2$$

3.4 Exercises

1. (a) $f(x) = 5$ is a horizontal line and has slope 0; the derivative is 0.
- (b) $f(x) = x$ has slope 1; the derivative is 1.
- (c) $f(x) = -x$ has slope of -1 , the derivative is -1 .
- (d) $x = 3$ is vertical and has undefined slope; the derivative does not exist.
- (e) $y = mx + b$ has slope m ; the derivative is m .

2. (a)



The line tangent to $g(x) = \sqrt[3]{x}$ at $x = 0$ is a vertical line. Since the slope of a vertical line is undefined, $g'(0)$ does not exist.

3. $f(x) = \frac{x^2-1}{x+2}$ is not differentiable when $x+2 = 0$ or $x = -2$ because the function is undefined and a vertical asymptote occurs there.
4. If the rate of change of $f(x)$ is zero when $x = a$, the tangent line at that point must have a slope of zero. Thus, the tangent line is horizontal at that point.
5. Using the points $(5,3)$ and $(6,5)$, we have

$$m = \frac{5-3}{6-5} = \frac{2}{1} = 2.$$

6. Using the points $(2,2)$ and $(-2,6)$, we have

$$m = \frac{6-2}{-2-2} = \frac{4}{-4} = -1.$$

7. Using the points $(-2,2)$ and $(2,3)$, we have

$$m = \frac{3-2}{2-(-2)} = \frac{1}{4}.$$

8. Using the points $(3,-1)$ and $(-2,3)$, we have

$$m = \frac{3-(-1)}{-2-3} = \frac{4}{-5} = -\frac{4}{5}.$$

9. Using the points $(-3,-3)$ and $(0,-3)$, we have

$$m = \frac{-3-(-3)}{0-3} = \frac{0}{-3} = 0.$$

10. The line tangent to the curve at $(4,2)$ is a vertical line. A vertical line has undefined slope.

11. $f(x) = 3x - 7$

$$\begin{aligned} \text{Step 1 } f(x+h) &= 3(x+h) - 7 \\ &= 3x + 3h - 7 \end{aligned}$$

$$\begin{aligned} \text{Step 2 } f(x+h) - f(x) &= 3x + 3h - 7 - (3x - 7) \\ &= 3x + 3h - 7 - 3x + 7 \\ &= 3h \end{aligned}$$

$$\text{Step 3 } \frac{f(x+h) - f(x)}{h} = \frac{3h}{h} = 3$$

$$\begin{aligned} \text{Step 4 } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} 3 = 3 \end{aligned}$$

$$f'(-2) = 3, f'(0) = 3, f'(3) = 3$$

12. $f(x) = -2x + 5$

$$\begin{aligned} \text{Step 1 } f(x+h) &= -2(x+h) + 5 \\ &= -2x - 2h + 5 \end{aligned}$$

$$\begin{aligned} \text{Step 2 } f(x+h) - f(x) &= -2x - 2h + 5 - (-2x + 5) \\ &= -2x - 2h + 5 + 2x - 5 \\ &= -2h \end{aligned}$$

$$\text{Step 3 } \frac{f(x+h) - f(x)}{h} = \frac{-2h}{h} = -2$$

$$\begin{aligned} \text{Step 4 } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} -2 = -2 \end{aligned}$$

$$f'(-2) = -2, f'(0) = -2, f'(3) = -2$$

13. $f(x) = -4x^2 + 9x + 2$

Step 1 $f(x + h)$

$$\begin{aligned} &= -4(x + h)^2 + 9(x + h) + 2 \\ &= -4(x^2 + 2xh + h^2) + 9x + 9h + 2 \\ &= -4x^2 - 8xh - 4h^2 + 9x + 9h + 2 \end{aligned}$$

Step 2 $f(x + h) - f(x)$

$$\begin{aligned} &= -4x^2 - 8xh - 4h^2 + 9x + 9h + 2 \\ &\quad - (-4x^2 + 9x + 2) \\ &= -8xh - 4h^2 + 9h \\ &= h(-8x - 4h + 9) \end{aligned}$$

Step 3 $\frac{f(x + h) - f(x)}{h}$

$$\begin{aligned} &= \frac{h(-8x - 4h + 9)}{h} \\ &= -8x - 4h + 9 \end{aligned}$$

Step 4 $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} (-8x - 4h + 9) \\ &= -8x + 9 \\ f'(-2) &= -8(-2) + 9 = 25 \\ f'(0) &= -8(0) + 9 = 9 \\ f'(3) &= -8(3) + 9 = -15 \end{aligned}$$

14. $f(x) = 6x^2 - 5x - 1$

Step 1 $f(x + h)$

$$\begin{aligned} &= 6(x + h)^2 - 5(x + h) - 1 \\ &= 6(x^2 + 2xh + h^2) - 5x - 5h - 1 \\ &= 6x^2 + 12xh + 6h^2 - 5x - 5h - 1 \end{aligned}$$

Step 2 $f(x + h) - f(x)$

$$\begin{aligned} &= 6x^2 + 12xh + 6h^2 - 5x - 5h - 1 \\ &\quad - (6x^2 - 5x - 1) \\ &= 6h^2 + 12xh - 5h \\ &= h(6h + 12x - 5) \end{aligned}$$

Step 3 $\frac{f(x + h) - f(x)}{h} = \frac{h(6h + 12x - 5)}{h}$

$$= 6h + 12x - 5$$

Step 4 $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} (6h + 12x - 5) \\ &= 12x - 5 \end{aligned}$$

$$f'(-2) = 12(-2) - 5 = -29$$

$$f'(0) = 12(0) - 5 = -5$$

$$f'(3) = 12(3) - 5 = 31$$

15. $f(x) = \frac{12}{x}$

$$f(x + h) = \frac{12}{x + h}$$

$$\begin{aligned} f(x + h) - f(x) &= \frac{12}{x + h} - \frac{12}{x} \\ &= \frac{12x - 12(x + h)}{x(x + h)} \\ &= \frac{12x - 12x - 12h}{x(x + h)} \\ &= \frac{-12h}{x(x + h)} \end{aligned}$$

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{-12h}{hx(x + h)} \\ &= \frac{-12}{x(x + h)} \\ &= \frac{-12}{x^2 + xh} \end{aligned}$$

$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{-12}{x^2 + xh} \\ &= \frac{-12}{x^2} \end{aligned}$$

$$f'(-2) = \frac{-12}{(-2)^2} = \frac{-12}{4} = -3$$

$f'(0) = \frac{-12}{0^2}$ which is undefined so $f'(0)$ does not exist.

$$\begin{aligned} f'(3) &= \frac{-12}{3^2} \\ &= \frac{-12}{9} = -\frac{4}{3} \end{aligned}$$

16. $f(x) = \frac{3}{x}$

$$f(x+h) = \frac{3}{x+h}$$

$$\begin{aligned} f(x+h) - f(x) &= \frac{3}{x+h} - \frac{3}{x} \\ &= \frac{3x - 3(x+h)}{x(x+h)} \\ &= \frac{-3h}{x(x+h)} \end{aligned}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{-3h}{hx(x+h)} \\ &= \frac{-3}{x(x+h)} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-3}{x(x+h)} \\ &= \frac{-3}{x^2} \end{aligned}$$

$$f'(-2) = -\frac{3}{4}$$

$f'(0) = -\frac{3}{0}$ which is undefined, so $f'(0)$ does not exist.

$$f'(3) = -\frac{3}{3^2} = -\frac{1}{3}$$

17. $f(x) = \sqrt{x}$

Steps 1-3 are combined.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

$f'(-2) = \frac{1}{2\sqrt{-2}}$ which is undefined so $f'(-2)$ does not exist.

$f'(0) = \frac{1}{2\sqrt{0}} = \frac{1}{0}$ which is undefined so $f'(0)$ does not exist.

$$f'(3) = \frac{1}{2\sqrt{3}}$$

18. $f(x) = -3\sqrt{x}$

Steps 1-3 are combined.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{-3\sqrt{x+h} + 3\sqrt{x}}{h} \end{aligned}$$

Rationalize the numerator.

$$\begin{aligned} &= \frac{-3\sqrt{x+h} + 3\sqrt{x}}{h} \cdot \frac{-3\sqrt{x+h} - 3\sqrt{x}}{-3\sqrt{x+h} - 3\sqrt{x}} \\ &= \frac{9(x+h) - 9x}{h(-3\sqrt{x+h} - 3\sqrt{x})} \\ &= \frac{9x + 9h - 9x}{h(-3\sqrt{x+h} - 3\sqrt{x})} \\ &= \frac{9}{-3\sqrt{x+h} - 3\sqrt{x}} = \frac{3}{-\sqrt{x+h} - \sqrt{x}} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{3}{-\sqrt{x+h} - \sqrt{x}} \\ &= \frac{3}{-\sqrt{x} - \sqrt{x}} = \frac{3}{-2\sqrt{x}} \end{aligned}$$

$f'(-2) = \frac{3}{-2\sqrt{-2}}$ which is undefined so $f'(-2)$ does not exist.

$f'(0) = \frac{3}{-2\sqrt{0}} = \frac{3}{0}$ which is undefined so

$f'(0)$ does not exist.

$$f'(3) = \frac{3}{-2\sqrt{3}} = -\frac{3}{2\sqrt{3}}$$

19. $f(x) = 2x^3 + 5$

Steps 1-3 are combined.

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{2(x+h)^3 + 5 - (2x^3 + 5)}{h} \\ &= \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) + 5 - 2x^3 - 5}{h} \\ &= \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 + 5 - 2x^3 - 5}{h} \\ &= \frac{6x^2h + 6xh^2 + 2h^3}{h} \\ &= \frac{h(6x^2 + 6xh + 2h^2)}{h} \\ &= 6x^2 + 6xh + 2h^2 \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} (6x^2 + 6xh + 2h^2) = 6x^2$$

$$f'(-2) = 6(-2)^2 = 24$$

$$f'(0) = 6(0)^2 = 0$$

$$f'(3) = 6(3)^2 = 54$$

20. $f(x) = 4x^3 - 3$

Steps 1-3 are combined.

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{4(x+h)^3 - 3 - (4x^3 - 3)}{h} \\ &= \frac{4(x^3 + 3x^2h + 3xh^2 + h^3) - 3 - 4x^3 + 3}{h} \\ &= \frac{4x^3 + 12x^2h + 12xh^2 + 4h^3 - 3 - 4x^3 + 3}{h} \\ &= \frac{12x^2h + 12xh^2 + 4h^3}{h} \\ &= \frac{h(12x^2 + 12xh + 4h^2)}{h} \\ &= 12x^2 + 12xh + 4h^2 \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} (12x^2 + 12xh + 4h^2) = 12x^2$$

$$f'(-2) = 12(-2)^2 = 48$$

$$f'(0) = 12(0)^2 = 0$$

$$f'(3) = 12(3)^2 = 108$$

21. (a) $f(x) = x^2 + 2x$; $x = 3, x = 5$

$$\begin{aligned} \text{Slope of secant line} &= \frac{f(5) - f(3)}{5 - 3} \\ &= \frac{(5)^2 + 2(5) - [(3)^2 + 2(3)]}{2} \\ &= \frac{35 - 15}{2} \\ &= 10 \end{aligned}$$

Now use $m = 10$ and $(3, f(3)) = (3, 15)$ in the point-slope form.

$$y - 15 = 10(x - 3)$$

$$y - 15 = 10x - 30$$

$$y = 10x - 30 + 15$$

$$y = 10x - 15$$

(b) $f(x) = x^2 + 2x$; $x = 3$

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{[(x+h)^2 + 2(x+h)] - (x^2 + 2x)}{h} \\ &= \frac{(x^2 + 2hx + h^2 + 2x + 2h) - (x^2 + 2x)}{h} \\ &= \frac{2hx + h^2 + 2h}{h} = 2x + h + 2 \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} (2x + h + 2) = 2x + 2$$

$f'(3) = 2(3) + 2 = 8$ is the slope of the tangent line at $x = 3$.

Use $m = 8$ and $(3, 15)$ in the point-slope form.

$$y - 15 = 8(x - 3)$$

$$y = 8x - 9$$

22. (a) $f(x) = 6 - x^2$; $x = -1, x = 3$

$$\begin{aligned} \text{Slope of secant line} &= \frac{f(3) - f(-1)}{3 - (-1)} \\ &= \frac{6 - (3)^2 - [6 - (-1)^2]}{4} \\ &= \frac{-3 - 5}{4} \\ &= -2 \end{aligned}$$

Now use $m = -2$ and $(-1, f(-1)) = (-1, 5)$ in the point-slope form.

$$\begin{aligned}
 y - 5 &= -2[x - (-1)] \\
 y - 5 &= -2x - 2 \\
 y &= -2x - 2 + 5 \\
 y &= -2x + 3
 \end{aligned}$$

(b) $f(x) = 6 - x^2$; $x = -1$

$$\begin{aligned}
 \frac{f(x+h) - f(x)}{h} &= \frac{[6 - (x+h)^2] - [6 - (x)^2]}{h} \\
 &= \frac{[6 - (x^2 + 2xh + h^2)] - [6 - x^2]}{h} \\
 &= \frac{6 - x^2 - 2xh - h^2 - 6 + x^2}{h} \\
 &= \frac{-2xh - h^2}{h} = \frac{h(-2x - h)}{h} = -2x - h
 \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} (-2x - h) = -2x$$

$f'(-1) = -2(-1) = 2$ is the slope of the tangent line at $x = -1$. Use $m = 2$ and $(-1, 5)$ in the point-slope form.

$$\begin{aligned}
 y - 5 &= 2(x + 1) \\
 y - 5 &= 2x + 2 \\
 y &= 2x + 7
 \end{aligned}$$

23. (a) $f(x) = \frac{5}{x}$; $x = 2, x = 5$

$$\begin{aligned}
 \text{Slope of secant line} &= \frac{f(5) - f(2)}{5 - 2} \\
 &= \frac{\frac{5}{5} - \frac{5}{2}}{3} = \frac{1 - \frac{5}{2}}{3} \\
 &= -\frac{1}{2}
 \end{aligned}$$

Now use $m = -\frac{1}{2}$ and $(5, f(5)) = (5, 1)$ in the point-slope form.

$$\begin{aligned}
 y - 1 &= -\frac{1}{2}[x - 5] \\
 y - 1 &= -\frac{1}{2}x + \frac{5}{2} \\
 y &= -\frac{1}{2}x + \frac{5}{2} + 1 \\
 y &= -\frac{1}{2}x + \frac{7}{2}
 \end{aligned}$$

(b) $f(x) = \frac{5}{x}$; $x = 2$

$$\begin{aligned}
 \frac{f(x+h) - f(x)}{h} &= \frac{\frac{5}{x+h} - \frac{5}{x}}{h} \\
 &= \frac{\frac{5x - 5(x+h)}{(x+h)x}}{h} \\
 &= \frac{5x - 5x - 5h}{h(x+h)x} \\
 &= \frac{-5h}{h(x+h)x} \\
 &= \frac{-5}{(x+h)x}
 \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{-5}{(x+h)x} = -\frac{5}{x^2}$$

$f'(2) = \frac{-5}{2^2} = -\frac{5}{4}$ is the slope of the tangent line at $x = 2$.

Now use $m = -\frac{5}{4}$ and $(2, \frac{5}{2})$ in the point-slope form.

$$\begin{aligned}
 y - \frac{5}{2} &= -\frac{5}{4}(x - 2) \\
 y - \frac{5}{2} &= -\frac{5}{4}x + \frac{10}{4} \\
 y &= -\frac{5}{4}x + 5 \\
 5x + 4y &= 20
 \end{aligned}$$

24. (a) $f(x) = -\frac{3}{x+1}$; $x = 1, x = 5$

$$\begin{aligned}
 \text{Slope of secant line} &= \frac{f(5) - f(1)}{5 - 1} \\
 &= \frac{-\frac{3}{(5+1)} - \left[-\frac{3}{(1+1)}\right]}{4} \\
 &= \frac{-\frac{1}{2} + \frac{3}{2}}{4} \\
 &= \frac{1}{4}
 \end{aligned}$$

Now use $m = \frac{1}{4}$ and $(1, f(1)) = (1, -\frac{3}{2})$ in the point-slope form.

$$\begin{aligned}
 y - \left(-\frac{3}{2}\right) &= \frac{1}{4}(x - 1) \\
 y + \frac{3}{2} &= \frac{1}{4}x - \frac{1}{4} \\
 y &= \frac{1}{4}x - \frac{1}{4} - \frac{3}{2} \\
 y &= \frac{1}{4}x - \frac{7}{4}
 \end{aligned}$$

(b) $f(x) = \frac{-3}{x+1}; x = 1$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\frac{-3}{(x+h)+1} - \frac{-3}{x+1}}{h} \\ &= \frac{\frac{-3(x+1) + 3(x+h+1)}{(x+1)(x+h+1)}}{h} \\ &= \frac{\frac{-3x-3+3x+3h+3}{(x+1)(x+h+1)}}{h} \\ &= \frac{3h}{h(x+1)(x+h+1)}\end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{3}{(x+1)(x+h+1)} = \frac{3}{(x+1)^2}$$

$$f'(1) = \frac{3}{(1+1)^2} = \frac{3}{4} \text{ is the slope of the tangent}$$

line at $x = 1$. Use $m = \frac{3}{4}$ and $(1, -\frac{3}{2})$ in the point-slope form.

$$y - \left(-\frac{3}{2}\right) = \frac{3}{4}(x - 1)$$

$$y + \frac{3}{2} = \frac{3}{4}x - \frac{3}{4}$$

$$y = \frac{3}{4}x - \frac{9}{4}$$

25. (a) $f(x) = 4\sqrt{x}; x = 9, x = 16$

$$\begin{aligned}\text{Slope of secant line} &= \frac{f(16) - f(9)}{16 - 9} \\ &= \frac{4\sqrt{16} - 4\sqrt{9}}{7} \\ &= \frac{16 - 12}{7} = \frac{4}{7}\end{aligned}$$

Now use $m = \frac{4}{7}$ and $(9, f(9)) = (9, 12)$ in the point-slope form.

$$y - 12 = \frac{4}{7}(x - 9)$$

$$y - 12 = \frac{4}{7}x - \frac{36}{7}$$

$$y = \frac{4}{7}x - \frac{36}{7} + 12$$

$$y = \frac{4}{7}x + \frac{48}{7}$$

(b) $f(x) = 4\sqrt{x}; x = 9$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{4\sqrt{x+h} - 4\sqrt{x}}{h} \cdot \frac{4\sqrt{x+h} + 4\sqrt{x}}{4\sqrt{x+h} + 4\sqrt{x}} \\ &= \frac{16(x+h) - 16x}{h(4\sqrt{x+h} + 4\sqrt{x})}\end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{16(x+h) - 16x}{h(4\sqrt{x+h} + 4\sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{16h}{h(4\sqrt{x+h} + 4\sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{4}{(\sqrt{x+h} + \sqrt{x})} = \frac{4}{2\sqrt{x}}$$

$$= \frac{2}{\sqrt{x}}$$

$f'(9) = \frac{2}{\sqrt{9}} = \frac{2}{3}$ is the slope of the tangent line at $x = 9$.

Use $m = \frac{2}{3}$ and $(9, 12)$ in the point-slope form.

$$y - 12 = \frac{2}{3}(x - 9)$$

$$y = \frac{2}{3}x + 6$$

$$3y = 2x + 18$$

26. (a) $f(x) = \sqrt{x}; x = 25, x = 36$

$$\begin{aligned}\text{Slope of secant line} &= \frac{f(36) - f(25)}{36 - 25} \\ &= \frac{\sqrt{36} - \sqrt{25}}{11} \\ &= \frac{6 - 5}{11} = \frac{1}{11}\end{aligned}$$

Now use $m = \frac{1}{11}$ and $(25, f(25)) = (25, 5)$ in the point-slope form.

$$y - 5 = \frac{1}{11}(x - 25)$$

$$y - 5 = \frac{1}{11}x - \frac{25}{11}$$

$$y = \frac{1}{11}x - \frac{25}{11} + 5$$

$$y = \frac{1}{11}x + \frac{30}{11}$$

(b) $f(x) = \sqrt{x}; x = 25$

$$\begin{aligned}
& \frac{f(x+h) - f(x)}{h} \\
&= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
f'(x) &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \\
f'(25) &= \frac{1}{2\sqrt{25}} = \frac{1}{2 \cdot 5} = \frac{1}{10}
\end{aligned}$$

Use $m = \frac{1}{10}$ and $(25, 5)$ in the point-slope form.

$$\begin{aligned}
y - 5 &= \frac{1}{10}(x - 25) \\
y - 5 &= \frac{1}{10}x - \frac{25}{10} \\
y &= \frac{1}{10}x + \frac{5}{2}
\end{aligned}$$

27. $f(x) = -4x^2 + 11x$

$$\begin{aligned}
& \frac{f(x+h) - f(x)}{h} \\
&= \frac{-4(x+h)^2 + 11(x+h) - (-4x^2 + 11x)}{h} \\
&= \frac{-8xh - 4h^2 + 11h}{h} \\
f'(x) &= \lim_{h \rightarrow 0} (-8x - 4h + 11) = -8x + 11 \\
f'(2) &= -8(2) + 11 = -5 \\
f'(16) &= -8(16) + 11 = -117 \\
f'(-3) &= -8(-3) + 11 = 35
\end{aligned}$$

28. $f(x) = 6x^2 - 4x$

$$\begin{aligned}
& \frac{f(x+h) - f(x)}{h} \\
&= \frac{6(x+h)^2 - 4(x+h) - (6x^2 - 4x)}{h} \\
&= \frac{12xh + 6h^2 - 4h}{h} = 12x + 6h - 4
\end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} (12x + 6h - 4) = 12x - 4$$

$$f'(2) = 12(2) - 4 = 24 - 4 = 20$$

$$f'(16) = 12(16) - 4 = 192 - 4 = 188$$

$$f'(-3) = 12(-3) - 4 = -36 - 4 = -40$$

29. $f(x) = e^x$

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

$$f'(2) \approx 7.3891; f'(16) \approx 8,886,111; f'(-3) \approx 0.0498$$

30. $f(x) = \ln|x|$

$$\begin{aligned}
& \frac{f(x+h) - f(x)}{h} \\
&= \frac{\ln|x+h| - \ln|x|}{h}
\end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\ln|x+h| - \ln|x|}{h}$$

$$f'(2) = 0.5$$

$$f'(16) = 0.0625$$

$$f'(-3) = -0.3$$

31. $f(x) = -\frac{2}{x}$

$$\begin{aligned}
& \frac{f(x+h) - f(x)}{h} = \frac{\frac{-2}{x+h} - \left(\frac{-2}{x}\right)}{h} \\
&= \frac{\frac{-2x+2(x+h)}{(x+h)x}}{h} \\
&= \frac{2h}{h(x+h)x} = \frac{2}{(x+h)x}
\end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2}{(x+h)x} = \frac{2}{x^2}$$

$$f'(2) = \frac{2}{2^2} = \frac{1}{2}$$

$$f'(16) = \frac{2}{16^2} = \frac{2}{256} = \frac{1}{128}$$

$$f'(-3) = \frac{2}{(-3)^2} = \frac{2}{9}$$

$$32. f(x) = \frac{6}{x}$$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\frac{6}{x+h} - \frac{6}{x}}{h} \\ &= \frac{6x - 6(x+h)}{hx(x+h)} = \frac{-6}{x(x+h)}\end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{-6}{x(x+h)} = \frac{-6}{x^2}$$

$$f'(2) = \frac{-6}{(2)^2} = \frac{-6}{4} = -\frac{3}{2}$$

$$f'(16) = \frac{-6}{16^2} = \frac{-6}{256} = -\frac{3}{128}$$

$$f'(-3) = \frac{-6}{(-3)^2} = \frac{-6}{9} = -\frac{2}{3}$$

$$33. f(x) = \sqrt{x}$$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}\end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$f'(2) = \frac{1}{2\sqrt{2}}$$

$$f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$$

$$f'(-3) = \frac{1}{2\sqrt{-3}} \text{ is not a real number, so}$$

$f'(-3)$ does not exist.

$$34. f(x) = -3\sqrt{x}$$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-3\sqrt{x+h} + 3\sqrt{x}}{h} \\ &= \frac{-3\sqrt{x+h} + 3\sqrt{x}}{h} \cdot \frac{-3\sqrt{x+h} - 3\sqrt{x}}{-3\sqrt{x+h} - 3\sqrt{x}} \\ &= \frac{9(x+h) - 9x}{h(-3\sqrt{x+h} - 3\sqrt{x})} \\ &= \frac{9}{-3\sqrt{x+h} - 3\sqrt{x}} = \frac{3}{-\sqrt{x+h} - \sqrt{x}}\end{aligned}$$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{3}{-\sqrt{x+h} - \sqrt{x}} \\ &= \frac{3}{-\sqrt{x} - \sqrt{x}} = -\frac{3}{2\sqrt{x}}\end{aligned}$$

$$f'(2) = -\frac{3}{2\sqrt{2}}$$

$$f'(16) = \frac{-3}{2\sqrt{16}} = -\frac{3}{8}$$

$$f'(-3) = \frac{-3}{2\sqrt{-3}}, \text{ which is not a real number, so}$$

$f'(-3)$ does not exist.

35. At $x = 0$, the graph of $f(x)$ has a sharp point. Therefore, there is no derivative for $x = 0$.

36. No derivative exists at $x = -6$ because the function is not defined at $x = -6$.

37. For $x = -3$ and $x = 0$, the tangent to the graph of $f(x)$ is vertical. For $x = -1$, there is a gap in the graph of $f(x)$. For $x = 2$, the function $f(x)$ does not exist. For $x = 3$ and $x = 5$, the graph of $f(x)$ has sharp points. Therefore, no derivative exists for $x = -3$, $x = -1$, $x = 0$, $x = 2$, $x = 3$, and $x = 5$.

38. For $x = -5$ and $x = 0$, the function $f(x)$ is not defined. For $x = -3$ and $x = 2$ the graph of $f(x)$ has sharp points. For $x = 4$, the tangent to the graph is vertical. Therefore, no derivative exists for $x = -5$, $x = -3$, $x = 0$, $x = 2$, or $x = 4$.

39. (a) The rate of change of $f(x)$ is positive when $f(x)$ is increasing, that is, on $(a, 0)$ and (b, c) .

(b) The rate of change of $f(x)$ is negative when $f(x)$ is decreasing, that is, on $(0, b)$.

(c) The rate of change is zero when the tangent to the graph is horizontal, that is, at $x = 0$ and $x = b$.

40. The zeros of graph (b) correspond to the turning points of graph (a), the points where the derivative is zero. Graph (a) gives the distance, while graph (b) gives the velocity.

41. The zeros of graph (b) correspond to the turning points of graph (a), the points where the derivative is zero. Graph (a) gives the distance, while graph (b) gives the velocity.

$$42. f(x) = x^x, a = 2$$

(a)	h	
	0.01	$\frac{f(2 + 0.01) - f(2)}{0.01}$ $= \frac{2.01^{2.01} - 2^2}{0.01}$ $= 6.84$
	0.001	$\frac{f(2 + 0.001) - f(2)}{0.001}$ $= \frac{2.001^{2.001} - 2^2}{0.001}$ $= 6.779$
	0.0001	$\frac{f(2 + 0.0001) - f(2)}{0.0001}$ $= \frac{2.0001^{2.0001} - 2^2}{0.0001}$ $= 6.773$
	0.00001	$\frac{f(2 + 0.00001) - f(2)}{0.00001}$ $= \frac{2.00001^{2.00001} - 2^2}{0.00001}$ $= 6.7727$
	0.000001	$\frac{f(2 + 0.000001) - f(2)}{0.000001}$ $= \frac{2.000001^{2.000001} - 2^2}{0.000001}$ $= 6.7726$

It appears that $f'(2) \approx 6.773$.

- (b) Graph the function on a graphing calculator and move the cursor to an x -value near $x = 2$. A good choice for the initial viewing window is $[0, 3]$ by $[0, 10]$.

Now zoom in on the function several times. Each time you zoom in, the graph will look less like a curve and more like a straight line. When the graph appears to be a straight line, use the TRACE feature to select two points on the graph, and record their coordinates. Use these two points to compute the slope. The result will be very close to the most accurate value found in part (a), which is 6.773.

43. $f(x) = x^x$, $a = 3$

(a)	h	
	0.01	$\frac{f(3 + 0.01) - f(3)}{0.01}$ $= \frac{3.01^{3.01} - 3^3}{0.01}$ $= 57.3072$
	0.001	$\frac{f(3 + 0.001) - f(3)}{0.001}$ $= \frac{3.001^{3.001} - 3^3}{0.001}$ $= 56.7265$
	0.00001	$\frac{f(3 + 0.00001) - f(3)}{0.00001}$ $= \frac{3.00001^{3.00001} - 3^3}{0.00001}$ $= 56.6632$
	0.000001	$\frac{f(3 + 0.000001) - f(3)}{0.000001}$ $= \frac{3.000001^{3.000001} - 3^3}{0.000001}$ $= 56.6626$
	0.0000001	$\frac{f(3 + 0.0000001) - f(3)}{0.0000001}$ $= \frac{3.0000001^{3.0000001} - 3^3}{0.0000001}$ $= 56.6625$

It appears that $f'(3) \approx 56.66$.

- (b) Graph the function on a graphing calculator and move the cursor to an x -value near $x = 3$. A good choice for the initial viewing window is $[0, 4]$ by $[0, 60]$ with $Xscl = 1$, $Yscl = 10$.

Now zoom in on the function several times. Each time you zoom in, the graph will look less like a curve and more like a straight line. Use the TRACE feature to select two points on the graph, and record their coordinates. Use these two points to compute the slope. The result will be close to the most accurate value found in part (a), which is 56.66.

Note: In this exercise, the method used in part (a) gives more accurate results than the method used in part (b).

44. $f(x) = x^{1/x}$, $a = 2$

(a)

h	
0.01	$\frac{f(2 + .01) - f(2)}{0.01}$ $= \frac{2.01^{1/2.01} - 2^{1/2}}{0.01}$ $= 0.1071$
0.001	$\frac{f(2 + 0.001) - f(2)}{0.001}$ $= \frac{2.001^{1/2.001} - 2^{1/2}}{0.001}$ $= 0.1084$
0.0001	$\frac{f(2 + 0.0001) - f(2)}{0.0001}$ $= \frac{2.0001^{1/2.0001} - 2^{1/2}}{0.0001}$ $= 0.1085$
0.00001	$\frac{f(2 + 0.00001) - f(2)}{0.00001}$ $= \frac{2.00001^{1/2.00001} - 2^{1/2}}{0.00001}$ $= 0.1085$
0.000001	$\frac{f(2 + 0.000001) - f(2)}{0.000001}$ $= \frac{2.000001^{1/2.000001} - 2^{1/2}}{0.000001}$ $= 0.1085$

It appears that $f'(2) = 0.1085$.

- (b) Graph this function on a graphing calculator and move the cursor to an x -value near $x = 2$. A good choice for the initial viewing window is $[0, 5]$ by $[0, 3]$.

Follow the procedure outlined in the solution for Exercise 42, part (b). The final result will be close to the value found in part (a) of this exercise, which is 0.1085.

45. $f(x) = x^{1/x}$, $a = 3$

(a)

h	
0.01	$\frac{f(3 + 0.01) - f(3)}{0.01}$ $= \frac{3.01^{1/3.01} - 3^{1/3}}{0.01}$ $= -0.0160$
0.001	$\frac{f(3 + 0.001) - f(3)}{0.001}$ $= \frac{3.001^{1/3.001} - 3^{1/3}}{0.001}$ $= -0.0158$
0.0001	$\frac{f(3 + 0.0001) - f(3)}{0.0001}$ $= \frac{3.0001^{1/3.0001} - 3^{1/3}}{0.0001}$ $= -0.0158$

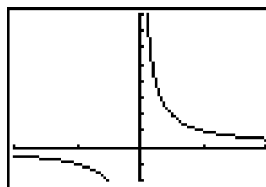
It appears that $f'(3) = -0.0158$.

- (b) Graph the function on a graphing calculator and move the cursor to an x -value near $x = 3$. A good choice for the initial viewing window is $[0, 5]$ by $[0, 3]$.

Follow the procedure outlined in the solution for Exercise 43, part (b). Note that near $x = 3$, the graph is very close to a horizontal line, so we expect that its slope will be close to 0. The final result will be close to the value found in part (a) of this exercise, which is -0.0158 .

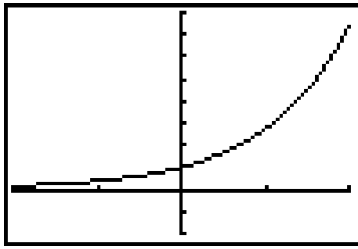
46. For Column A, let $h = 0.01$ $f(x) = \ln|x|$

Graph $y = \frac{\ln|x + 0.01| - \ln|x|}{0.01}$.



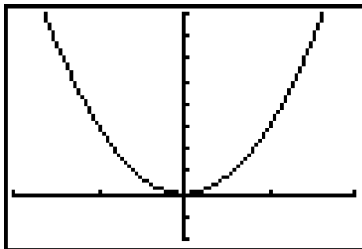
$$f(x) = e^x$$

$$\text{Graph } y = \frac{e^{x+0.01} - e^x}{0.01}.$$



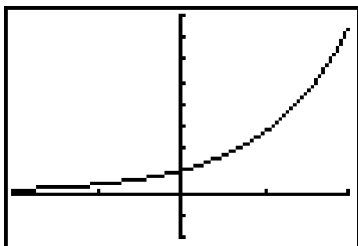
$$f(x) = x^3$$

$$\text{Graph } y = \frac{(x+0.01)^3 - x^3}{0.01}.$$

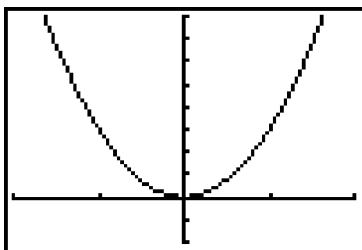


Column B

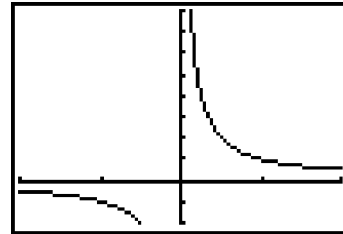
$$\text{Graph } y = e^x$$



$$\text{Graph } y = 3x^2$$



$$\text{Graph } y = \frac{1}{x}$$



We observe that the graph of

$$y = \frac{\ln|x+0.01| - \ln|x|}{0.01}$$

is very similar to the graph of

$$y = \frac{1}{x},$$

the graph of

$$y = \frac{e^{x+0.01} - e^x}{0.01}$$

is very similar to the graph of $y = e^x$, and the graph of

$$y = \frac{(x+0.01)^3 - x^3}{0.01}$$

is very similar to the graph of

$$y = 3x^2.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, the derivative of e^x is e^x , and the derivative of x^3 is $3x^2$.

48. (a)

$$f(x) = -4x^2 + 11x$$

$$f(x+h) = -4(x+h)^2 + 11(x+h)$$

$$= -4(x^2 + 2xh + h^2) + 11(x+h)$$

$$= -4x^2 - 8xh - 4h^2 + 11x + 11h$$

$$f(x+h) - f(x)$$

$$= -4x^2 - 8xh - 4h^2 + 11x + 11h + 4x^2 - 11x$$

$$= -8xh - 4h^2 + 11h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-8xh - 4h^2 + 11h}{h}$$

$$= -8x - 4h + 11$$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} (-8x + 4h + 11) \\
 &= -8x + 11
 \end{aligned}$$

$$f'(3) = -8(3) + 11 = -13$$

$$\begin{aligned}
 &\frac{f(3+0.1) - f(3)}{0.1} \\
 &= \frac{f(3.1) - f(3)}{0.1} \\
 &= \frac{-4(3.1)^2 + 11(3.1) - (-4(3)^2 + 11(3))}{0.1} \\
 &= \frac{-4(9.61) + 11(3.1) + 4(9) - 11(3)}{0.1} \\
 &= \frac{-38.44 + 34.1 + 36 - 33}{0.1} \\
 &= \frac{-1.34}{0.1} = -13.4
 \end{aligned}$$

$$\begin{aligned}
 &\frac{f(3+0.1) - f(3-0.1)}{2(0.1)} \\
 &= \frac{f(3.1) - f(2.9)}{0.2} \\
 &= \frac{-4(3.1)^2 + 11(3.1) - (-4(2.9)^2 + 11(2.9))}{0.2} \\
 &= \frac{-4(9.61) + 11(3.1) + 4(8.41) - 11(2.9)}{0.2} \\
 &= \frac{-38.44 + 34.1 + 33.64 - 31.9}{0.2} \\
 &= \frac{-2.6}{0.2} = -13
 \end{aligned}$$

(b) $f'(3) = -8(3) + 11 = -13$

$$\begin{aligned}
 &\frac{f(3+0.01) - f(3)}{0.01} \\
 &= \frac{f(3.01) - f(3)}{0.01} \\
 &= \frac{-4(3.01)^2 + 11(3.01) - (-4(3)^2 + 11(3))}{0.01} \\
 &= \frac{-4(9.0601) + 11(3.01) + 4(9) - 11(3)}{0.01} \\
 &= \frac{-36.2404 + 33.11 + 36 - 33}{0.01} \\
 &= \frac{-0.1304}{0.01} = -13.04
 \end{aligned}$$

$$\begin{aligned}
 &\frac{f(3+0.01) - f(3-0.01)}{2(0.01)} \\
 &= \frac{f(3.01) - f(2.99)}{0.02} \\
 &= \frac{-4(3.01)^2 + 11(3.01) - (-4(2.99)^2 + 11(2.99))}{0.02} \\
 &= \frac{-4(9.0601) + 11(3.01) + 4(8.9401) - 11(2.99)}{0.02} \\
 &= \frac{-36.2404 + 33.11 + 35.7604 - 32.89}{0.02} \\
 &= \frac{-0.26}{0.02} = -13
 \end{aligned}$$

(c) $f(x) = \frac{-2}{x}$

$$\begin{aligned}
 f(x+h) &= \frac{-2}{x+h} \\
 f(x+h) - f(x) &= \frac{-2}{x+h} - \left(\frac{-2}{x} \right) \\
 &= \frac{-2}{x+h} + \frac{2}{x} \\
 &= \frac{-2x + 2(x+h)}{x(x+h)} \\
 &= \frac{2h}{x(x+h)} \\
 \frac{f(x+h) - f(x)}{h} &= \frac{\frac{2h}{x(x+h)}}{h} \\
 &= \frac{2h}{x(x+h)} \cdot \frac{1}{h} \\
 &= \frac{2}{x(x+h)}
 \end{aligned}$$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2}{x(x+h)} = \frac{2}{x^2}
 \end{aligned}$$

$$f'(3) = \frac{2}{3^2} = \frac{2}{9} \approx 0.222222$$

$$\begin{aligned}
 \frac{f(3+0.1) - f(3)}{0.1} &= \frac{f(3.1) - f(3)}{0.1} \\
 &= \frac{\frac{-2}{3.1} - \frac{-2}{3}}{0.1} \\
 &\approx .215054
 \end{aligned}$$

$$\begin{aligned}
 \frac{f(3+0.1) - f(3-0.1)}{2(0.1)} &= \frac{f(3.1) - f(2.9)}{0.2} \\
 &= \frac{\frac{-2}{3.1} - \frac{-2}{2.9}}{0.2} \\
 &= 0.222469
 \end{aligned}$$

$$(d) f'(3) = \frac{2}{3^2} = \frac{2}{9} \approx 0.222222$$

$$\begin{aligned} \frac{f(3 + .01) - f(3)}{0.01} &= \frac{f(3.01) - f(3)}{0.01} \\ &= \frac{\frac{-2}{3.01} - \frac{-2}{3}}{0.01} \\ &\approx 0.221484 \end{aligned}$$

$$\begin{aligned} \frac{f(3 + 0.01) - f(3 - 0.01)}{2(0.01)} &= \frac{f(3.01) - f(2.99)}{0.02} \\ &= \frac{\frac{-2}{3.01} - \frac{-2}{2.99}}{0.02} \\ &\approx 0.222225 \end{aligned}$$

$$(e) f(x) = \sqrt{x}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

$$f'(3) = \frac{1}{2\sqrt{3}} \approx 0.288675$$

$$\begin{aligned} \frac{f(3 + 0.1) - f(3)}{0.1} &= \frac{f(3.1) - f(3)}{0.1} \\ &= \frac{\sqrt{3.1} - \sqrt{3}}{0.1} \\ &\approx 0.286309 \end{aligned}$$

$$\begin{aligned} \frac{f(3 + 0.1) - f(3 - 0.1)}{2(0.1)} &= \frac{f(3.1) - f(2.9)}{0.2} \\ &= \frac{\sqrt{3.1} - \sqrt{2.9}}{0.2} \\ &\approx 0.288715 \end{aligned}$$

$$(f) f'(3) = \frac{1}{2\sqrt{3}} \approx 0.288675$$

$$\begin{aligned} \frac{f(3 + 0.01) - f(3)}{0.01} &= \frac{f(3.01) - f(3)}{0.01} \\ &= \frac{\sqrt{3.01} - \sqrt{3}}{0.01} \\ &\approx 0.288435 \end{aligned}$$

$$\begin{aligned} \frac{f(3 + 0.01) - f(3 - 0.01)}{2(0.01)} &= \frac{f(3.01) - f(2.99)}{0.02} \\ &= \frac{\sqrt{3.01} - \sqrt{2.99}}{0.02} \\ &\approx 0.288676 \end{aligned}$$

$$49. D(p) = -2p^2 - 4p + 300$$

D is demand; p is price.

(a) Given that $D'(p) = -4p - 4$, the rate of change of demand with respect to price is $-4p - 4$, the derivative of the function $D(p)$.

$$(b) D'(10) = -4(10) - 4 = -44$$

The demand is decreasing at the rate of about 44 items for each increase in price of \$1.

$$50. P(x) = 1000 + 32x - 2x^2$$

(a) \$8000 is 8 thousands, so $x = 8$.

$$P'(8) = 32 - 4(8) = 32 - 32 = 0$$

No, the firm should not increase production, since the marginal profit is 0.

(b) \$6000, $x = 6$

$$P'(6) = 32 - 4(6) = 32 - 24 = 8$$

Yes, the first should increase production, since the marginal profit is positive, \$8000.

(c) \$12,000, $x = 12$

$$\begin{aligned} P'(12) &= 32 - 4(12) \\ &= 32 - 48 = -16 \end{aligned}$$

No, because the marginal profit is negative, -\$16,000.

(d) \$20,000, $x = 20$

$$\begin{aligned} P'(20) &= 32 - 4(20) \\ &= 32 - 80 = -48 \end{aligned}$$

No, because the marginal profit is negative, -\$48,000.

$$51. R(x) = 20x - \frac{x^2}{500}$$

$$(a) R'(x) = 20 - \frac{1}{250}x$$

At $y = 1000$,

$$\begin{aligned} R'(1000) &= 20 - \frac{1}{250}(1000) \\ &= \$16 \text{ per table.} \end{aligned}$$

- (b) The marginal revenue for the 1001st table is approximately $R'(1000)$. From (a), this is about \$16.

- (c) The actual revenue is

$$\begin{aligned} R(1001) - R(1000) &= 20(1001) - \frac{1001^2}{500} \\ &\quad - \left[20(1000) - \frac{1000^2}{500} \right] \\ &= 18,015.998 - 18,000 \\ &= \$15.998 \text{ or } \$16. \end{aligned}$$

- (d) The marginal revenue gives a good approximation of the actual revenue from the sale of the 1001st table.

52. $C(x) = -0.00375x^2 + 1.5x + 1000, 0 \leq x \leq 180$

- (a) The marginal cost is given by

$$C'(x) = -0.0075x + 1.5, 0 \leq x \leq 180$$

- (b) $C'(100) = -0.0075(100) + 1.5 = 0.75$

This represents the fact that the cost of producing the next (101st) taco is approximately \$0.75.

- (c) The exact cost to produce the 101st taco is given by

$$\begin{aligned} C(101) - C(100) &= [-0.00375(101)^2 + 1.5(101) + 1000] \\ &\quad - [-0.00375(100)^2 + 1.5(100) + 1000] \\ &= (-38.25375 + 151.5 + 1000) \\ &\quad - (-37.5 + 150 + 1000) \\ &= 0.74625 \end{aligned}$$

That is, the exact cost of producing the 101st taco is \$0.74625.

- (d) The exact cost of producing the 101st taco is \$0.00375 less than the approximate cost. They are very close.

- (e) $C(x) = ax^2 + bx + c$
 $C'(x) = 2ax + b$

$$\begin{aligned} [C(x+1) - C(x)] - C'(x) &= [a(x+1)^2 + b(x+1) + c] \\ &\quad - [ax^2 + bx + c] - (2ax + b) \\ &= ax^2 + 2ax + a + bx + b + c \\ &\quad - ax^2 - bx - c - 2ax - b \\ &= a \end{aligned}$$

- (f) $C(x) = ax^2 + bx + c$
 $C'(x) = 2ax + b$

$$\begin{aligned} C(x+1) - C(x) &= [a(x+1)^2 + b(x+1) + c] \\ &\quad - [ax^2 + bx + c] \\ &= ax^2 + 2ax + a + bx \\ &\quad + b + c - ax^2 - bx - c \\ &= 2ax + a + b \\ C'\left(x + \frac{1}{2}\right) &= 2a\left(x + \frac{1}{2}\right) + b \\ &= 2ax + a + b \end{aligned}$$

$$\text{Thus, } C(x+1) - C(x) = C'\left(x + \frac{1}{2}\right).$$

53. (a) $f(x) = 0.0000329x^3 - 0.00405x^2$
 $+ 0.0613x + 2.34$

$$f(10) = 2.54$$

$$f(20) = 2.03$$

$$f(30) = 1.02$$

- (b)

$$Y_1 = 0.0000329x^3 - 0.00405x^2 + 0.0613x + 2.34$$

$$nDeriv(Y_1, x, 0) \approx 0.061$$

$$nDeriv(Y_1, x, 10) \approx -0.019$$

$$nDeriv(Y_1, x, 20) \approx -0.079$$

$$nDeriv(Y_1, x, 30) \approx -0.120$$

$$nDeriv(Y_1, x, 35) \approx -0.133$$

54. (a) From the graph, V_{mp} is just about at the turning point of the curve. Thus, the slope of the tangent line is approximately zero. The power expenditure is not changing.

- (b) From the graph, the slope of the tangent line at V_{mr} is approximately 0.1. The power expended is increasing 0.1 unit per unit increase in speed.

- (c) The slope of the tangent line at V_{opt} is a bit greater than that at V_{mr} , about 0.12. The power expended increases 0.12 units for each unit increase in speed.

- (d) The power level first decreases to V_{mp} , then increases at greater rates.

- (e) V_{mr} is the point which produces the smallest slope of a line.

55. The derivative at (2, 4000) can be approximated by the slope of the line through (0, 2000) and (2, 4000).

The derivative is approximately

$$\frac{4000 - 2000}{2 - 0} = \frac{2000}{2} = 1000.$$

Thus the shellfish population is increasing at a rate of 1000 shellfish per unit time.

The derivative at about (10,10,300) can be approximated by the slope of the line through (10,10,300) and (13,12,000). The derivative is approximately

$$\frac{12,000 - 10,300}{13 - 10} = \frac{1700}{3} \approx 570.$$

The shellfish population is increasing at a rate of about 583 shellfish per unit time. The derivative at about (13, 11,250) can be approximated by the slope of the line through (13, 11,250) and (16, 12,000). The derivative is approximately

$$\frac{12,000 - 11,250}{16 - 13} = \frac{750}{3} \approx 250.$$

The shellfish population is increasing at a rate of 250 shellfish per unit time.

56. $I(t) = 27 + 72t - 1.5t^2$

$$\begin{aligned} \text{(a)} \quad I'(t) &= \lim_{h \rightarrow 0} \frac{I(t+h) - I(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{27 + 72t + 72h - 1.5t^2 - 3th - 1.5h^2 - 27 - 72t + 1.5t^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{72h - 3th - 1.5h^2}{h} \\ &= \lim_{h \rightarrow 0} 72 - 3t - 1.5h \\ &= 72 - 3t \\ I'(5) &= 72 - 3(5) \\ &= 72 - 15 \\ &= 57 \end{aligned}$$

The rate of change of the intake of food 5 minutes into a meal is 57 grams per minute.

$$\begin{aligned} \text{(b)} \quad I'(24) &\stackrel{?}{=} 0 \\ 72 - 3(24) &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

24 minutes after the meal starts the rate of food consumption is 0.

- (c) After 24 minutes the rate of food consumption is negative according to the function where a rate of zero is more accurate. A logical range for this function is

$$0 \leq t \leq 24.$$

57. (a) Set $M(v) = 150$ and solve for v .

$$0.0312443v^2 - 101.39v + 82,264 = 150$$

$$0.0312443v^2 - 101.39v + 82,114 = 0$$

Solve using the quadratic formula.

Let D equal the discriminant.

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-101.39)^2 - 4(0.0312443)(82,114) \\ &\approx 17.55 \end{aligned}$$

$$v = \frac{101.39 \pm \sqrt{D}}{2(0.0312443)}$$

$v \approx 1690$ meter per second or

$v \approx 1560$ meters per second.

Since the functions is defined only for $v \geq 1620$, the only solution is 1690 meters per second.

- (b) Calculate $\lim_{h \rightarrow 0} \frac{M(1700 + h) - M(1700)}{h}$

$$\begin{aligned} M(1700 + h) &= 0.0312443(1700 + h)^2 - 101.39(1700 + h) + 82,264 \\ &= 90,296.027 + 106.23062h + 0.0312443h^2 - 172,363 - 101.39h + 82,264 \\ &= 0.01312443h^2 + 4.84062h + 197.027 \end{aligned}$$

$M(1700) = 197.027$, so the derivative of $M(v)$ at $v = 1700$ is

$$\begin{aligned} &\lim_{h \rightarrow 0} \left(\frac{0.0312443h^2 + 4.84062h + 197.027 - 197.027}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{0.0312443h^2 + 4.84062h}{h} \right) \\ &= \lim_{h \rightarrow 0} (0.0312443h + 4.84062) \\ &= 4.84062 \\ &\approx 4.84 \text{ days per meter per second} \end{aligned}$$

The increase in velocity for this cheese from 1700 m/s to 1701 m/s indicates that the approximate age of the cheese has increased by 4.84 days.

58. The slope of the tangent line to the graph at the first point is found by finding two points on the tangent line.

$$(x_1, y_1) = (1000, 13.5)$$

$$(x_2, y_2) = (0, 18.5)$$

$$m = \frac{18.5 - 13.5}{0 - 1000} = \frac{5}{-1000} = -0.005$$

At the second point, we have

$$\begin{aligned}
 (x_1, y_1) &= (1000, 13.5) \\
 (x_2, y_2) &= (2000, 21.5) \\
 m &= \frac{21.5 - 13.5}{2000 - 1000} \\
 &= \frac{8}{1000} \\
 &= 0.008
 \end{aligned}$$

At the third point, we have

$$\begin{aligned}
 (x_1, y_1) &= (5000, 20) \\
 (x_2, y_2) &= (3000, 22.5) \\
 m &= \frac{22.5 - 20}{3000 - 5000} \\
 &= \frac{2.5}{-2000} \\
 &= -0.00125
 \end{aligned}$$

At 500 ft, the temperature decreases 0.005° per foot. At about 1500 ft, the temperature increases 0.008° per foot. At 5000 ft, the temperature decreases 0.00125° per foot.

- 59. (a)** The derivative does not exist at the two “corners” or “sharp points.” The x -values of these points are 0.75 and 3.
- (b)** To find $T'(0.5)$, calculate the slope of the line segment with positive slope, since this portion of the graph includes the value $x = 0.5$ (marked as $\frac{1}{2}$ hr on the graph.) To find this slope, use the points (0, 100) (the starting point) and (0.75, 875) (the beginning point of the cleaning cycle).

$$\begin{aligned}
 m = T'(0.5) &= \frac{875 - 100}{0.75 - 0} \\
 &= \frac{775}{0.75} \approx 1033
 \end{aligned}$$

The oven temperature is increasing at 1033° per hour.

- (c)** To find $T'(2)$, find the slope of the line segment containing the value $x = 2$. This segment is horizontal, so

$$m = T'(2) = 0.$$

The oven temperature is not changing.

- (d)** To find $T'(3.5)$, calculate the slope of the line segment with negative slope, since this portion of the graph includes the value $x = 3.5$. To find this slope, use the points (3, 875) (the end of the cleaning cycle) and (3.75, 100) (the stopping point).

$$\begin{aligned}
 m = T'(3.5) &= \frac{100 - 875}{3.75 - 3} \\
 &= \frac{-775}{0.75} \approx -1033
 \end{aligned}$$

The oven temperature is decreasing at 1033° per hour.

60. (a) The slope of the graph at 16 looks horizontal. Thus, the derivative for a 16 ounce bat is about 0 mph per oz.

The slope of the graph at $x = 25$ can be estimated using the points (25, 63.4) and (26, 62.8).

$$\text{slope} = \frac{62.8 - 63.4}{26 - 25} = -0.6$$

Thus the derivative is for a 25-ounce bat is about -0.6 mph per oz.

- (b) The optimal bat is 16 oz.

61. (a) At 40 oz the tangent looks horizontal; thus the derivative for a 40-ounce bat is about 0 mph per oz. The slope of the graph at $x = 30$ can be estimated using the points (30, 79) and (32, 80).

$$\text{slope} = \frac{80 - 79}{32 - 30} = 0.5$$

Thus, the derivative for a 30 ounce bat is about 0.5 mph per oz.

- (b) The optimal bat is 40 oz.

Note also that Y_1 is positive whenever Y_2 is increasing, and that Y_1 is negative whenever Y_2 is decreasing.

4. Since the x -intercepts of the graph f' occur whenever the graph of f has a horizontal tangent line, Y_2 is the derivative of Y_1 . Notice that Y_2 has 2 x -intercepts; each occurs at an x -value where the tangent line to Y_1 is horizontal.

Note also that Y_2 is negative whenever Y_1 is decreasing, and Y_2 is positive whenever Y_1 is increasing.

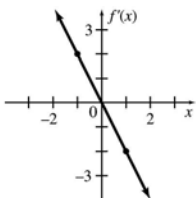
5. Since the x -intercepts of the graph of f' occur whenever the graph of f has a horizontal tangent line, Y_2 is the derivative of Y_1 . Notice that Y_2 has 1 x -intercept which occurs at the x -value where the tangent line to Y_1 is horizontal. Also notice that the range on which Y_1 is increasing, Y_2 is positive and the range on which it is decreasing, Y_2 is negative.

6. Since the x -intercepts of the graph f' occur whenever the graph of f has a horizontal tangent line, Y_1 is the derivative of Y_2 . Notice that Y_1 has 4 x -intercepts; each occurs at an x -value where the tangent line to Y_2 is horizontal.

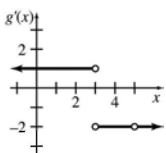
Note also that Y_1 is negative whenever Y_2 is decreasing and Y_1 is positive whenever Y_2 is increasing.

3.5 Graphical Differentiation

Your Turn 1



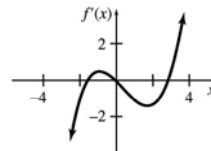
Your Turn 2



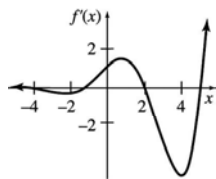
3.5 Exercises

3. Since the x -intercepts of the graph of f' occur whenever the graph of f has a horizontal tangent line, Y_1 is the derivative of Y_2 . Notice that Y_1 has 2 x -intercepts; each occurs at an x -value where the tangent line to Y_2 is horizontal.

8. To graph f' , observe the intervals where the slopes of lines are positive and where they are negative to determine where the derivative is positive and where it is negative. Also, whenever f has a horizontal tangent, f' will be 0.



Estimate the magnitude of the slope at several points by drawing tangents to the graph of f .



9. On the interval $(-\infty, -2)$, the graph of f is a horizontal line, so its slope is 0. Thus, on this interval, the graph of f' is $y = 0$ on $(-\infty, -2)$. On the interval $(-2, 0)$, the graph of f is a straight line, so its slope is constant. To find this slope, use the points $(-2, 2)$ and $(0, 0)$.

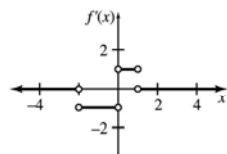
$$m = \frac{2 - 0}{-2 - 0} = \frac{2}{-2} = -1$$

On the interval $(0, 1)$, the slope is also constant. To find this slope, use the points $(0, 0)$ and $(1, 1)$.

$$m = \frac{1 - 0}{1 - 0} = 1$$

On the interval $(1, \infty)$, the graph is again a horizontal line, so $m = 0$. The graph of f' will be made up of portions of the y -axis and the lines $y = -1$ and $y = 1$.

Because the graph of f has “sharp points” or “corners” at $x = -2$, $x = 0$, and $x = 1$, we know that $f'(-2)$, $f'(0)$, and $f'(1)$ do not exist. We show this on the graph of f' by using open circles at the endpoints of the portions of the graph.



10. On the interval $(-\infty, -3)$, the graph of f is a straight line, so its slope is constant. To find this slope, use the points $(-6, -3)$ and $(-3, 0)$.

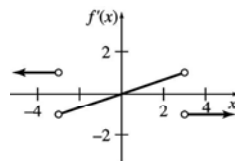
$$m = \frac{0 - (-3)}{-3 - (-6)} = \frac{3}{3} = 1$$

On the interval $(3, \infty)$, the slope of f is also constant. To find this slope, use the points $(3, 0)$ and $(6, -3)$.

$$m = \frac{-3 - 0}{6 - 3} = \frac{-3}{3} = -1$$

Thus, we have $f'(x) = 1$ on $(-\infty, -3)$ and $f'(x) = -1$ on $(3, \infty)$. Because the graph of f has sharp points at $x = -3$ and $x = 3$, we know that $f'(-3)$ and $f'(3)$ do not exist. We show this on the graph of $f'(x)$ by using open circles.

We also observe that the slopes of tangent lines are negative on $(-3, 0)$, that the graph has a horizontal tangent at $x = 0$, and that the slopes of tangent lines are positive on $(0, 3)$. Thus, f' is negative on $(-3, 0)$, 0 at $x = 0$, and positive on $(0, 3)$. Furthermore, by drawing tangents, we see that on $(-3, 3)$, f' increases from -1 to 1 .



11. On the interval $(-\infty, -2)$, the graph of f is a straight line, so its slope is constant. To find this slope, use the points $(-4, 2)$ and $(-2, 0)$.

$$m = \frac{0 - 2}{-2 - (-4)} = \frac{-2}{2} = -1$$

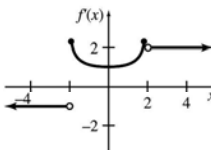
On the interval $(2, \infty)$, the slope of f is also constant. To find this slope, use the points $(2, 0)$ and $(3, 2)$.

$$m = \frac{2 - 0}{3 - 2} = \frac{2}{1} = 2$$

Thus, we have $f'(x) = -1$ on $(-\infty, -2)$ and $f'(x) = 2$ on $(2, \infty)$.

Because f is discontinuous at $x = -2$ and $x = 2$, we know that $f'(-2)$ and $f'(2)$ do not exist, which we indicate with open circles at $(-2, -1)$ and $(2, 2)$ on the graph of f' .

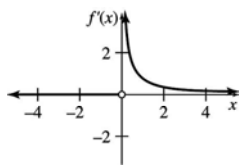
On the interval $(-2, 2)$ all tangent lines have positive slopes, so the graph of f' will be above the y -axis. Notice that the slope of f (and thus the y -value of f') decreases on $(-2, 0)$ and increases on $(0, 2)$ with a minimum value on this interval of about 1 at $x = 0$.



12. On the interval $(-\infty, 0)$, the graph of f is a horizontal line, so its slope is 0. Thus, the graph of f' is $y = 0$ (the x -axis) on $(-\infty, 0)$.

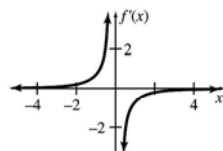
Since f is discontinuous at $x = 0$, $f'(0)$ does not exist. Thus, the graph of f' has an open circle at $x = 0$.

On the interval $(0, \infty)$, the slopes are positive but decreasing at a slower rate as x gets larger. Therefore, the value of f' will be positive but decreasing on this interval. This value approaches 0, but never becomes 0.

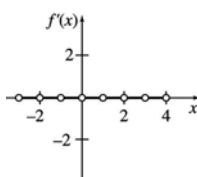


13. We observe that the slopes of tangent lines are positive on the interval $(-\infty, 0)$ and negative on the interval $(0, \infty)$, so the value of f' will be positive on $(-\infty, 0)$ and negative on $(0, \infty)$. Since f is undefined at $x = 0$, $f'(0)$ does not exist. Notice that the graph of f becomes very flat when $|x| \rightarrow \infty$. The value of f approaches 0 and also the slope approaches 0. Thus, $y = 0$ (the x -axis) is a horizontal asymptote for both the graph of f and the graph of f' .

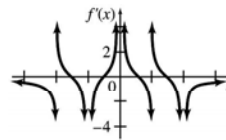
As $x \rightarrow 0^-$ and $x \rightarrow 0^+$, the graph of f gets very steep, so $|f'(x)| \rightarrow \infty$. Thus, $x = 0$ (the y -axis) is a vertical asymptote for both the graph of f and the graph of f' .



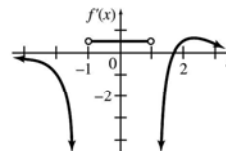
14. The graph of f is a step function. (This is the greatest integer function, $f(x) = \lfloor x \rfloor$.) The graph is made up of an infinite series of horizontal line segments. Thus, the derivative will be 0 everywhere it is defined. However, since f is discontinuous wherever x is an integer, $f'(x)$ does not exist at any integer.



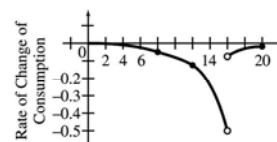
15. The slope of $f(x)$ is undefined at $x = -2, -1, 0, 1$, and 2, and the graph approaches vertical (unbounded slope) as x approaches those values. Accordingly, the graph of $f'(x)$ has vertical asymptotes at $x = -2, -1, 0, 1$, and 2. $f(x)$ has turning points (zero slope) at $x = -1.5, -0.5, 0.5$, and 1.5, so the graph of $f'(x)$ crosses the x -axis at those values. Elsewhere, the graph of $f'(x)$ is negative where $f(x)$ is decreasing and positive where $f(x)$ is increasing.



16. The slope of $f(x)$ is undefined at $x = -1$ and 1. The graph approaches vertical (unbounded slope) as x approaches -1 from the left and 1 from the right. Accordingly, the graph of $f'(x)$ has vertical asymptotes for $x = -1$ approached from the left and for $x = 1$ approached from the right. $f(x)$ has a turning point (zero slope) near $x = 1.7$, so the graph of $f'(x)$ crosses the x -axis near $x = 1.7$. Elsewhere, the graph of $f'(x)$ is negative where $f(x)$ is decreasing and positive where $f(x)$ is increasing. Between $x = -1$ and $x = 1$, the slope of $f(x)$ is 0.5; therefore, $f'(x) = 0.5$ between -1 and 1.



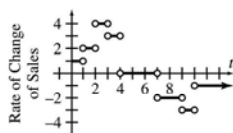
17. The graph of G decreases steadily with varying degrees of steepness. The steepness increases (that is, the slopes of the tangent lines becomes more negative) between $t = 12$ and $t = 16$. Since G is discontinuous at $t = 16$, $G'(16)$ doesn't exist. The graph continues to decrease after $t = 16$, but the slopes of the tangent lines become less negative as the curve gets flatter. So, the derivative values are increasing toward 0.



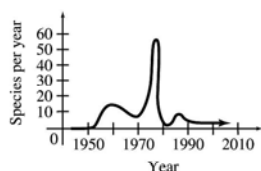
18. Let S' be the derivative function showing the rate of change of sales. On the interval $[0, 1]$, the slope of the line segment between points $(0, 0)$ and $(1, 1)$ is 1. Thus, on $[0, 1]$ the graph of S' is $y = 1$. On the interval $[1, 2]$, the slope of the line segment between points $(1, 1)$ and $(2, 3)$ is 2. Thus on $[1, 2]$, the graph of S' is $y = 2$. On $[2, 3]$, the slope of the line segment between points $(2, 3)$ and $(3, 7)$ is 4. Thus on $[2, 3]$, the graph of S' is $y = 4$. Continuing in this fashion, the table gives the slopes of the line segments (which make up the graph of S') on the remaining intervals.

Interval	Graph of S'
$[0, 1]$	$y = 1$
$[1, 2]$	$y = 2$
$[2, 3]$	$y = 4$
$[3, 4]$	$y = 3$
$[4, 7]$	$y = 0$
$[7, 9]$	$y = -2$
$[9, 10]$	$y = -3$
$[10, 12]$	$y = -1$

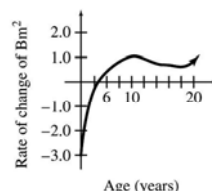
Because the graph of the annual sales function has “sharp points” or “corners” where the slopes of the line segments change, we know that S' does not exist at these points. We show this on the graph of S' by using open circles.



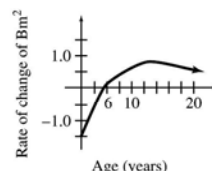
19. The graph rises steadily, with varying degrees of steepness. The graph is steepest around 1976 and nearly flat around 1950 and 1980. Accordingly, the rate of change is always positive, with a maximum value around 1976 and values near zero around 1950 and 1980.



20. (a) The curve slants downward up to about age 4.25, where it turns and begins to rise. There is a slight decline in steepness between ages 10 and 18. Correspondingly, the graph of the rate of change lies below the horizontal axis to the left of 4.25 years and above the horizontal axis to the right of that point.



- (b) The curve slants downward up to about age 5.75, where it turns and begins to rise. There is a slight decline in steepness between ages 13 and 20. Correspondingly, the graph of the rate of change lies below the horizontal axis to the left of 5.75 years and above the horizontal axis to the right of that point.

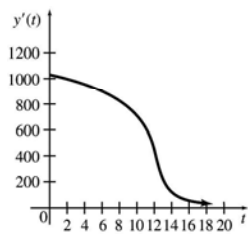


21. The growth rate of the function $y = f(x)$ is given by the derivative of this function $y' = f'(x)$. We use the graph of f to sketch the graph of f' . First, notice as x increases, y increases throughout the domain of f , but at a slower and slower rate. The slope of f is positive but always decreasing, and approaches 0 as t gets large. Thus, y' will always be positive and decreasing. It will approach but never reach 0.

To plot point on the graph of f' , we need to estimate the slope of f at several points. From the graph of f , we obtain the values given in the following table.

t	y'
2	1000
10	700
13	250

Use these points to sketch the graph.

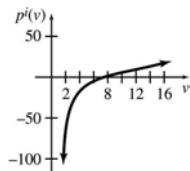


22. Let $P(V)$ represent the power corresponding to a given value of the tern's speed, V .

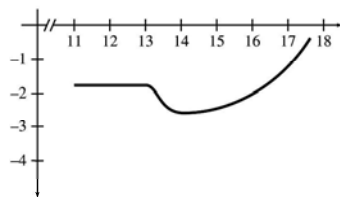
The rate of change of power as a function of time is given by the derivative of this function, $P'(V)$. We use the graph of P to sketch the graph of P' .

First, we observe that the graph of P has one turning point, at $V = V_{mp} \approx 5.5$. At this value of V , the graph has a horizontal tangent, so the graph of P' has an x -intercept at this value of V . Since the slopes of tangent lines are negative on the interval $(0, V_{mp})$ and positive when $V > V_{mp}$, the value of P' is negative on $(0, V_{mp})$ and positive when $V > V_{mp}$.

Use the tangents drawn on the graph and additional tangent as needed to estimate the slope at several points on the graph of V to improve the accuracy of the graph of P' .

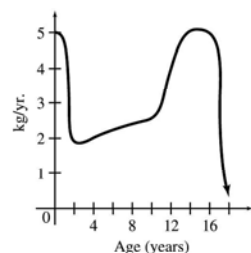


- 23.



About 9 cm; about 2.6 cm less per year

- 24.



Chapter 3 Review Exercises

1. True
2. True
3. True
4. False; for example, if $f(x) = \frac{x^2-4}{x+2}$,
 $\lim_{x \rightarrow -2} f(x) = -4$, but the graph of $f(x) = \frac{x^2-4}{x+2}$ has a hole at the point $(-2, -4)$.
5. True
6. False; for example, the rational function $f(x) = \frac{5}{x+1}$ is discontinuous at $x = -1$.
7. False; the derivative gives the instantaneous rate of change of a function.
8. True
9. True
10. True
11. False; the slope of the tangent line gives the instantaneous rate of change.
12. False; for example, the function $f(x) = |x|$ is continuous at $x = 0$, but $f'(0)$ does not exist. The graph of $f(x) = |x|$ has a "corner" at $x = 0$.
16. The derivative can be used to find the instantaneous rate of change at a point on a function, and the slope of a tangent line at a point on a function.
17. (a) $\lim_{x \rightarrow -3^-} = 4$
 (b) $\lim_{x \rightarrow -3^+} = 4$
 (c) $\lim_{x \rightarrow -3} = 4$ (since parts (a) and (b) have the same answer)
 (d) $f(-3) = 4$, since $(-3, 4)$ is a point of the graph.

18. (a) $\lim_{x \rightarrow -1^-} g(x) = -2$
 (b) $\lim_{x \rightarrow -1^+} g(x) = 2$
 (c) $\lim_{x \rightarrow -1} g(x)$ does not exist since parts (a) and (b) have different answers.
 (d) $g(-1) = -2$, since $(-1, -2)$ is a point on the graph.
19. (a) $\lim_{x \rightarrow 4^-} f(x) = \infty$
 (b) $\lim_{x \rightarrow 4^+} f(x) = -\infty$
 (c) $\lim_{x \rightarrow 4} f(x)$ does not exist since limits in (a) and (b) do not exist.
 (d) $f(4)$ does not exist since the graph has no point with an x -value of 4.
20. (a) $\lim_{x \rightarrow 2^-} h(x) = 1$
 (b) $\lim_{x \rightarrow 2^+} h(x) = 1$
 (c) $\lim_{x \rightarrow 2} h(x) = 1$
 (d) $h(2)$ does not exist since the graph has no point with an x -value of 2.
21. $\lim_{x \rightarrow -\infty} g(x) = \infty$ since the y -value gets very large as the x -value gets very small.
22. $\lim_{x \rightarrow \infty} f(x) = -3$ since the line $y = -3$ is a horizontal asymptote for the graph.
23. $\lim_{x \rightarrow 6} \frac{2x + 7}{x + 3} = \frac{2(6) + 7}{6 + 3} = \frac{19}{9}$

24. Let $f(x) = \frac{2x + 5}{x + 3}$.

x	-3.1	-3.01	-3.001
$f(x)$	12	102	1002
x	-2.9	-2.99	-2.999
$f(x)$	-8	-98	-998

As x approaches -3 from the left, $f(x)$ gets infinitely larger. As x approaches -3 from the right, $f(x)$ gets infinitely smaller.

Therefore, $\lim_{x \rightarrow -3} \frac{2x+5}{x+3}$ does not exist.

25. $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4} \frac{(x - 4)(x + 4)}{x - 4}$
 $= \lim_{x \rightarrow 4} (x + 4)$
 $= 4 + 4$
 $= 8$
26. $\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x - 2} = \lim_{x \rightarrow 2} \frac{(x + 5)(x - 2)}{x - 2}$
 $= \lim_{x \rightarrow 2} (x + 5) = 2 + 5 = 7$
27. $\lim_{x \rightarrow -4} \frac{2x^2 + 3x - 20}{x + 4} = \lim_{x \rightarrow -4} \frac{(2x - 5)(x + 4)}{x + 4}$
 $= \lim_{x \rightarrow -4} (2x - 5)$
 $= 2(-4) - 5$
 $= -13$
28. $\lim_{x \rightarrow 3} \frac{3x^2 - 2x - 21}{x - 3}$
 $= \lim_{x \rightarrow 3} \frac{(3x + 7)(x - 3)}{x - 3}$
 $= \lim_{x \rightarrow 3} (3x + 7) = 9 + 7 = 16$
29. $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3}$
 $= \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$
 $= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3}$
 $= \frac{1}{\sqrt{9} + 3}$
 $= \frac{1}{6}$
30. $\lim_{x \rightarrow 16} \frac{\sqrt{x} - 4}{x - 16}$
 $= \lim_{x \rightarrow 16} \frac{\sqrt{x} - 4}{x - 16} \cdot \frac{\sqrt{x} + 4}{\sqrt{x} + 4}$
 $= \lim_{x \rightarrow 16} \frac{x - 16}{(x - 16)(\sqrt{x} + 4)}$
 $= \lim_{x \rightarrow 16} \frac{1}{\sqrt{x} + 4} = \frac{1}{\sqrt{16} + 4}$
 $= \frac{1}{4 + 4} = \frac{1}{8}$

$$\begin{aligned}
 31. \quad \lim_{x \rightarrow \infty} \frac{2x^2 + 5}{5x^2 - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{5}{x^2}}{\frac{5x^2}{x^2} - \frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{2 + \frac{5}{x^2}}{5 - \frac{1}{x^2}} \\
 &= \frac{2 + 0}{5 - 0} \\
 &= \frac{2}{5}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \lim_{x \rightarrow \infty} \frac{x^2 + 6x + 8}{x^3 + 2x + 1} &= \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^3} + \frac{6x}{x^3} + \frac{8}{x^3}}{\frac{x^3}{x^3} + \frac{2x}{x^3} + \frac{1}{x^3}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{6}{x^2} + \frac{8}{x^3}}{1 + \frac{2}{x^2} + \frac{1}{x^3}} \\
 &= \frac{0 + 0 + 0}{1 + 0 + 0} = 0
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \lim_{x \rightarrow -\infty} \left(\frac{3}{8} + \frac{3}{x} - \frac{6}{x^2} \right) \\
 &= \lim_{x \rightarrow -\infty} \frac{3}{8} + \lim_{x \rightarrow -\infty} \frac{3}{x} - \lim_{x \rightarrow -\infty} \frac{6}{x^2} \\
 &= \frac{3}{8} + 0 - 0 \\
 &= \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \lim_{x \rightarrow -\infty} \left(\frac{9}{x^4} + \frac{10}{x^2} - 6 \right) \\
 &= \lim_{x \rightarrow -\infty} \frac{9}{x^4} + \lim_{x \rightarrow -\infty} \frac{10}{x^2} - \lim_{x \rightarrow -\infty} 6 \\
 &= 0 + 0 - 6 = -6
 \end{aligned}$$

35. As shown on the graph, $f(x)$ is discontinuous at x_2 and x_4 .

36. As shown on the graph, $f(x)$ is discontinuous at x_1 and x_4 .

37. $f(x)$ is discontinuous at $x = 0$ and $x = -\frac{1}{3}$ since that is where the denominator of $f(x)$ equals 0. $f(0)$ and $f\left(-\frac{1}{3}\right)$ do not exist.

$\lim_{x \rightarrow 0} f(x)$ does not exist since $\lim_{x \rightarrow 0^+} f(x) = -\infty$,

but $\lim_{x \rightarrow 0^-} f(x) = \infty$. $\lim_{x \rightarrow -\frac{1}{3}} f(x)$ does not exist

since $\lim_{x \rightarrow -\frac{1}{3}^-} f(x) = -\infty$, but $\lim_{x \rightarrow -\frac{1}{3}^+} f(x) = \infty$.

$$38. \quad f(x) = \frac{7 - 3x}{(1 - x)(3 + x)}$$

The function is discontinuous at $x = -3$ and $x = 1$ because those values make the denominator of the fraction equal to zero.

$\lim_{x \rightarrow -3} f(x)$ does not exist since $\lim_{x \rightarrow -3^-} f(x) = -\infty$

and $\lim_{x \rightarrow -3^+} f(x) = \infty$.

$\lim_{x \rightarrow 1} f(x)$ does not exist since $\lim_{x \rightarrow 1^-} f(x) = \infty$ and

$\lim_{x \rightarrow 1^+} f(x) = -\infty$.

$f(-3)$ and $f(1)$ do not exist since there is no point of the graph that has an x -value of -3 or 1 .

39. $f(x)$ is discontinuous at $x = -5$ since that is where the denominator of $f(x)$ equals 0. $f(-5)$ does not exist.

$\lim_{x \rightarrow -5} f(x)$ does not exist since $\lim_{x \rightarrow -5^-} f(x) = \infty$,

but $\lim_{x \rightarrow -5^+} f(x) = -\infty$.

$$40. \quad f(x) = \frac{x^2 - 9}{x + 3}$$

The function is discontinuous at $x = -3$ since this value makes the denominator of the fraction equal to zero.

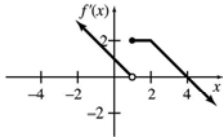
$$\begin{aligned}
 \lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} &= \lim_{x \rightarrow -3} \frac{(x + 3)(x - 3)}{x + 3} \\
 &= \lim_{x \rightarrow -3} (x - 3) \\
 &= -3 - 3 = -6
 \end{aligned}$$

$f(-3)$ does not exist since there is no point on the graph with an x -value of -3 .

41. $f(x) = x^2 + 3x - 4$ is continuous everywhere since f is a polynomial function.

42. $f(x) = 2x^2 - 5x - 3$ has no points of discontinuity since it is a polynomial function, which is continuous everywhere.

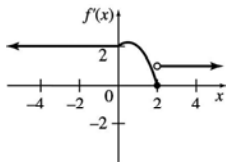
43. (a)

(b) The graph is discontinuous at $x = 1$.

$$(c) \lim_{x \rightarrow 1^-} f(x) = 0; \lim_{x \rightarrow 1^+} f(x) = 2$$

$$44. f(x) = \begin{cases} 2 & \text{if } x < 0 \\ -x^2 + x + 2 & \text{if } 0 \leq x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(a)

(b) The graph is discontinuous at $x = 2$.

$$(c) \lim_{x \rightarrow 2^-} f(x) = -4 + 2 + 2 = 0$$

$$\lim_{x \rightarrow 2^+} f(x) = 1$$

$$45. f(x) = \frac{x^4 + 2x^3 + 2x^2 - 10x + 5}{x^2 - 1}$$

(a) Find the values of $f(x)$ when x is close to 1.

x	y
1.1	2.6005
1.01	2.06
1.001	2.006
1.0001	2.0006
0.99	1.94
0.999	1.994
0.9999	1.9994

It appears that $\lim_{x \rightarrow 1} f(x) = 2$.

(b) Graph

$$y = \frac{x^4 + 2x^3 + 2x^2 - 10x + 5}{x^2 - 1}$$

on a graphing calculator. One suitable choice for the viewing window is $[-2, 6]$ by $[-10, 10]$.

Because $x^2 - 1 = 0$ when $x = -1$ or $x = 1$, this function is discontinuous at these two x -values. The graph shows a vertical asymptote at $x = -1$ but not at $x = 1$. The graph should have an open circle to show a “hole” in the graph at $x = 1$. The graphing calculator doesn’t show the hole, but trying to find the value of the function of $x = 1$ will show that this value is undefined.

By viewing the function near $x = 1$ and using the ZOOM feature, we see that as x gets close to 1 from the left or the right, y gets close to 2, suggesting that

$$\lim_{x \rightarrow 1} \frac{x^4 + 2x^3 + 2x^2 - 10x + 5}{x^2 - 1} = 2.$$

$$46. f(x) = \frac{x^4 + 3x^3 + 7x^2 + 11x + 2}{x^3 + 2x^2 - 3x - 6}$$

(a) Find values of $f(x)$ when x is close to -2 .

x	$f(x)$
-2.01	-12.62
-2.001	-12.96
-2.0001	-13
-1.99	-13.41
-1.999	-13.04
-1.9999	-13

It appears that $\lim_{x \rightarrow -2} f(x) = -13$.

(b) Graph

$$y = \frac{x^4 + 3x^3 + 7x^2 + 11x + 2}{x^3 + 2x^2 - 3x - 6}$$

on a graphing calculator. One suitable choice for the viewing window is $[-5, 5]$ by $[-10, 10]$. By viewing the function near $x = -2$, we see that as x gets close to -2 from the left on the right, y gets close to -13 , suggesting that

$$\lim_{x \rightarrow -2} \frac{x^4 + 3x^3 + 7x^2 + 11x + 2}{x^3 + 2x^2 - 3x - 6} = -13.$$

47. $y = 6x^3 + 2 = f(x)$; from $x = 1$ to $x = 4$

$$f(4) = 6(4)^3 + 2 = 386$$

$$f(1) = 6(1)^3 + 2 = 8$$

Average rate of change:

$$= \frac{386 - 8}{4 - 1} = \frac{378}{3} = 126$$

$$y' = 18x$$

Instantaneous rate of change at $x = 1$:

$$f'(1) = 18(1) = 18$$

48. $y = -2x^3 - 3x^2 + 8 = f(x)$

$$f(6) = -2(6)^3 - 3(6)^2 + 8 = -532$$

$$f(-2) = -2(-2)^3 - 3(-2)^2 + 8 = 12$$

Average rate of change:

$$\begin{aligned} &= \frac{f(6) - f(-2)}{6 - (-2)} \\ &= \frac{-532 - 12}{6 + 2} = \frac{-544}{8} = -68 \end{aligned}$$

$$y' = -6x^2 - 6x$$

Instantaneous rate of change at $x = -2$:

$$f'(-2) = -6(-2)^2 - 6(-2) = -6(4) + 12 = -12$$

49. $y = \frac{-6}{3x - 5} = f(x)$; from $x = 4$ to $x = 9$

$$f(9) = \frac{-6}{3(9) - 5} = \frac{-6}{22} = -\frac{3}{11}$$

$$f(4) = \frac{-6}{3(4) - 5} = -\frac{6}{7}$$

Average rate of change:

$$= \frac{\frac{-3}{11} - \left(-\frac{6}{7}\right)}{9 - 4} = \frac{\frac{-21 + 66}{77}}{5} = \frac{45}{5(77)} = \frac{9}{77}$$

$$y' = \frac{(3x - 5)(0) - (-6)(3)}{(3x - 5)^2} = \frac{18}{(3x - 5)^2}$$

Instantaneous rate of change at $x = 4$:

$$f'(4) = \frac{18}{(3 \cdot 4 - 5)^2} = \frac{18}{7^2} = \frac{18}{49}$$

50. $y = \frac{x + 4}{x - 1} = f(x)$

$$f(5) = \frac{5 + 4}{5 - 1} = \frac{9}{4}$$

$$f(2) = \frac{2 + 4}{2 - 1} = 6$$

Average rate of change:

$$\begin{aligned} &= \frac{\frac{9}{4} - 6}{5 - 2} = \frac{\frac{-15}{4}}{3} = -\frac{5}{4} \\ y' &= \frac{(x - 1)(1) - (x + 4)(1)}{(x - 1)^2} \\ &= \frac{x - 1 - x - 4}{(x - 1)^2} = \frac{-5}{(x - 1)^2} \end{aligned}$$

Instantaneous rate of change at $x = 2$:

$$f'(2) = \frac{-5}{(2 - 1)^2} = \frac{-5}{1} = -5$$

51. (a) $f(x) = 3x^2 - 5x + 7$; $x = 2$, $x = 4$

Slope of secant line

$$\begin{aligned} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{[3(4)^2 - 5(4) + 7] - [3(2)^2 - 5(2) + 7]}{2} \\ &= \frac{35 - 9}{2} \\ &= 13 \end{aligned}$$

Now use $m = 13$ and $2, f(2) = (2, 9)$ in the point-slope form.

$$y - 9 = 13(x - 2)$$

$$y - 9 = 13x - 26$$

$$y = 13x - 26 + 9$$

$$y = 13x - 17$$

(b) $f(x) = 3x^2 - 5x + 7; x = 2$

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{[3(x+h)^2 - 5(x+h) + 7] - [3x^2 - 5x + 7]}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 7 - 3x^2 + 5x - 7}{h} \\ &= \frac{6xh + 3h^2 - 5h}{h} \\ &= 6x + 3h - 5 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} 6x + 3h - 5 \\ &= 6x - 5 \\ f'(2) &= 6(2) - 5 \\ &= 7 \end{aligned}$$

Now use $m = 7$ and $(2, f(2)) = (2, 9)$ in the point-slope form.

$$\begin{aligned} y - 9 &= 7(x - 2) \\ y - 9 &= 7x - 14 \\ y &= 7x - 14 + 9 \\ y &= 7x - 5 \end{aligned}$$

52. (a) $f(x) = \frac{1}{x}; x = \frac{1}{2}, x = 3$

$$\begin{aligned} \text{Slope of secant line} &= \frac{f(3) - f\left(\frac{1}{2}\right)}{3 - \frac{1}{2}} \\ &= \frac{\frac{1}{3} - 2}{3 - \frac{1}{2}} = \frac{2 - 12}{18 - 3} \\ &= \frac{-10}{15} \\ &= -\frac{2}{3} \end{aligned}$$

Now use $m = -\frac{2}{3}$ and $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, 2\right)$ in the point-slope form.

$$\begin{aligned} y - 2 &= -\frac{2}{3}\left(x - \frac{1}{2}\right) \\ y - 2 &= -\frac{2}{3}x + \frac{1}{3} \\ y &= -\frac{2}{3}x + \frac{1}{3} + 2 \\ y &= -\frac{2}{3}x + \frac{7}{3} \end{aligned}$$

(b) $f(x) = \frac{1}{x}; x = \frac{1}{2}$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \frac{x - (x+h)}{xh(x+h)} = -\frac{h}{xh(x+h)} \\ &= -\frac{1}{x(x+h)} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} -\frac{1}{x(x+h)} \\ &= -\frac{1}{x^2} \\ f'\left(\frac{1}{2}\right) &= -\frac{1}{\left(\frac{1}{2}\right)^2} \\ &= -4 \end{aligned}$$

Now use $m = -4$ and $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, 2\right)$ in the point-slope form.

$$\begin{aligned} y - 2 &= -4\left(x - \frac{1}{2}\right) \\ y - 2 &= -4x + 2 \\ y &= -4x + 2 + 2 \\ y &= -4x + 4 \end{aligned}$$

53. (a) $f(x) = \frac{12}{x-1}; x = 3, x = 7$

$$\begin{aligned} \text{Slope of secant line} &= \frac{f(7) - f(3)}{7 - 3} \\ &= \frac{\frac{12}{7-1} - \frac{12}{3-1}}{4} \\ &= \frac{2 - 6}{4} \\ &= -1 \end{aligned}$$

Now use $m = -1$ and $(3, f(3)) = (3, 6)$ in the point-slope form.

$$\begin{aligned} y - 6 &= -1(x - 3) \\ y - 6 &= -x + 3 \\ y &= -x + 3 + 6 \\ y &= -x + 9 \end{aligned}$$

(b) $f(x) = \frac{12}{x-1}; x = 3$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\frac{12}{x+h-1} - \frac{12}{x-1}}{h} \\ &= \frac{12(x-1) - 12(x+h-1)}{h(x-1)(x+h-1)} \\ &= \frac{-12h}{h(x-1)(x+h-1)} \\ &= -\frac{12}{(x-1)(x+h-1)}\end{aligned}$$

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} -\frac{12}{(x-1)(x+h-1)} \\ &= -\frac{12}{(x-1)^2} \\ f'(3) &= -\frac{12}{(3-1)^2} \\ &= -3\end{aligned}$$

Now use $m = -3$ and $(3, f(x)) = (3, 6)$ in the point-slope form.

$$\begin{aligned}y - 6 &= -3(x - 3) \\ y - 6 &= -3x + 9 \\ y &= -3x + 9 + 6 \\ y &= -3x + 15\end{aligned}$$

54. (a) $f(x) = 2\sqrt{x-1}; x = 5, x = 10$

$$\begin{aligned}\text{Slope of secant line} &= \frac{f(10) - f(5)}{10 - 5} \\ &= \frac{2\sqrt{10-1} - 2\sqrt{5-1}}{5} \\ &= \frac{2(3) - 2(2)}{5} = \frac{2}{5}\end{aligned}$$

Now use $m = \frac{2}{5}$ and $(5, f(x)) = (5, 4)$ in the point-slope form.

$$\begin{aligned}y - 4 &= \frac{2}{5}(x - 5) \\ y - 4 &= \frac{2}{5}x - 2 \\ y &= \frac{2}{5}x - 2 + 4 \\ y &= \frac{2}{5}x + 2\end{aligned}$$

(b) $f(x) = 2\sqrt{x-1}; x = 5$

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{2\sqrt{x+h-1} - 2\sqrt{x-1}}{h} \\ &= \frac{2(\sqrt{x+h-1} - \sqrt{x-1})(\sqrt{x+h-1} + \sqrt{x-1})}{h(\sqrt{x+h-1} + \sqrt{x-1})} \\ &= \frac{2(x+h-1-x+1)}{h(\sqrt{x+h-1} + \sqrt{x-1})} = \frac{2h}{h(\sqrt{x+h-1} + \sqrt{x-1})} \\ &= \frac{2}{\sqrt{x+h-1} + \sqrt{x-1}} \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2}{\sqrt{x+h-1} + \sqrt{x-1}} = \frac{2}{2\sqrt{x-1}} = \frac{1}{\sqrt{x-1}}$$

$$f'(5) = \frac{1}{\sqrt{5-1}} = \frac{1}{2}$$

Now use $m = \frac{1}{2}$ and $(5, f(x)) = (5, 4)$ in the point-slope form.

$$y - 4 = \frac{1}{2}(x - 5)$$

$$y - 4 = \frac{1}{2}x - \frac{5}{2}$$

$$y = \frac{1}{2}x - \frac{5}{2} + 4$$

$$y = \frac{1}{2}x + \frac{3}{2}$$

55. $y = 4x^2 + 3x - 2 = f(x)$

$$\begin{aligned} y' &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[4(x+h)^2 + 3(x+h) - 2] - [4x^2 + 3x - 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(x^2 + 2xh + h^2) + 3x + 3h - 2 - 4x^2 - 3x + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^2 + 8xh + 4h^2 + 3x + 3h - 2 - 4x^2 - 3x + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{8xh + 4h^2 + 3h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(8x + 4h + 3)}{h} \\ &= \lim_{h \rightarrow 0} (8x + 4h + 3) \\ &= 8x + 3 \end{aligned}$$

56. $y = 5x^2 - 6x + 7 = f(x)$

$$\begin{aligned}
 y' &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[5(x+h)^2 - 6(x+h) + 7] - [5x^2 - 6x + 7]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5(x^2 + 2xh + h^2) - 6x - 6h + 7 - 5x^2 + 6x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5x^2 + 10xh + 5h^2 - 6x - 6h + 7 - 5x^2 + 6x + 7}{h} \\
 &= \lim_{h \rightarrow 0} \frac{10xh + 5h^2 - 6h}{h} = \lim_{h \rightarrow 0} \frac{h(10x + 5h - 6)}{h} \\
 &= \lim_{h \rightarrow 0} (10x + 5h - 6) = 10x - 6
 \end{aligned}$$

57. $f(x) = (\ln x)^x$, $x_0 = 3$

(a)

h	
0.01	$\frac{f(3 + 0.01) - f(3)}{0.01}$ $= \frac{(\ln 3.01)^{3.01} - (\ln 3)^3}{0.01} = 1.3385$
0.001	$\frac{f(3 + 0.001) - f(3)}{0.001}$ $= \frac{(\ln 3.001)^{3.001} - (\ln 3)^3}{0.001} = 1.3323$
0.0001	$\frac{f(3 + 0.0001) - f(3)}{0.0001}$ $= \frac{(\ln 3.0001)^{3.0001} - (\ln 3)^3}{0.0001} = 1.3317$
0.00001	$\frac{f(3 + 0.00001) - f(3)}{0.00001}$ $= \frac{(\ln 3.00001)^{3.00001} - (\ln 3)^3}{0.00001} = 1.3317$

It appears that $f'(3) \approx 1.332$.

(b) Using a graphing calculator will confirm this result.

58. $f(x) = x^{\ln x}$, $x_0 = 2$

(a)

h	
0.01	$\frac{f(2 + 0.01) - f(2)}{0.01}$ $= \frac{2.01^{\ln 2.01} - 2^{\ln 2}}{0.01} = 1.1258$
0.001	$\frac{f(2 + 0.001) - f(2)}{0.001}$ $= \frac{2.001^{\ln 2.001} - 2^{\ln 2}}{0.001} = 1.1212$
0.0001	$\frac{f(2 + 0.0001) - f(2)}{0.0001}$ $= \frac{2.0001^{\ln 2.0001} - 2^{\ln 2}}{0.0001} = 1.1207$
0.00001	$\frac{f(2 + 0.00001) - f(2)}{0.00001}$ $= \frac{2.00001^{\ln 2.00001} - 2^{\ln 2}}{0.00001} = 1.1207$

It appears that $f'(2) = 1.121$.

- (b) Graph the function on a graphing calculator and move the cursor to an x -value near $x = 2$. A good choice for the viewing window is $[0, 10]$ by $[0, 10]$.

Zoom in on the function until the graph looks like a straight line. Use the TRACE feature to select two points on the graph, and use these points to compute the slope. The result will be close to the most accurate value found in part (a), which is 1.121.

59. On the interval $(-\infty, 0)$, the graph of f is a straight line, so its slope is constant. To find this slope, use the points $(-2, 2)$ and $(0, 0)$.

$$m = \frac{0 - 2}{0 - (-2)} = \frac{-2}{2} = -1$$

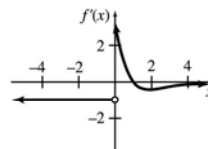
Thus, the value of f' will be -1 on this interval.

The graph of f has a sharp point at 0 , so $f'(0)$ does not exist. To show this, we use an open circle on the graph of f' at $(0, -1)$.

We also observe that the slope of f is positive but decreasing from $x = 0$ to about $x = 1$, and then

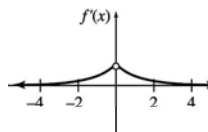
negative from there on. As $x \rightarrow \infty$, $f(x) \rightarrow 0$ and also $f'(x) = 0$.

Use this information to complete the graph of f' .



60. On the intervals $(-\infty, 0)$ and $(0, \infty)$, the slope of any tangent line will be positive, so the derivative will be positive. Thus, the graph of f' will lie above the y -axis. The slope of f and thus the value of f' approaches 0 when $x \rightarrow -\infty$ and $x \rightarrow \infty$ and approaches some particular but unknown positive value > 1 when $x \rightarrow 0^-$ and $x \rightarrow 0^+$.

Because f is discontinuous at $x = 0$, we know that $f'(0)$ does not exist, which we indicate with an open circle at $x = 0$ on the graph of f' .



$$\begin{aligned}
 61. \quad \lim_{x \rightarrow \infty} \frac{cf(x) - dg(x)}{f(x) - g(x)} &= \frac{\lim_{x \rightarrow \infty} [cf(x) - dg(x)]}{\lim_{x \rightarrow \infty} [f(x) - g(x)]} \\
 &= \frac{\lim_{x \rightarrow \infty} [cf(x)] - \lim_{x \rightarrow \infty} [dg(x)]}{\lim_{x \rightarrow \infty} [f(x)] - \lim_{x \rightarrow \infty} [g(x)]} \\
 &= \frac{c \lim_{x \rightarrow \infty} [f(x)] - d \lim_{x \rightarrow \infty} [g(x)]}{\lim_{x \rightarrow \infty} [f(x)] - \lim_{x \rightarrow \infty} [g(x)]} \\
 &= \frac{c \cdot c - d \cdot d}{c - d} = \frac{(c + d)(c - d)}{c - d} \\
 &= c + d
 \end{aligned}$$

The answer is (e).

$$62. \quad R(x) = 5000 + 16x - 3x^2$$

$$(a) \quad R'(x) = 16 - 6x$$

(b) Since x is in hundreds of dollars, \$1000 corresponds to $x = 10$.

$$\begin{aligned}
 R'(10) &= 16 - 6(10) \\
 &= 16 - 60 = -44
 \end{aligned}$$

An increase of \$100 spent on advertising when advertising expenditures are \$1000 will result in the revenue decreasing by \$44.

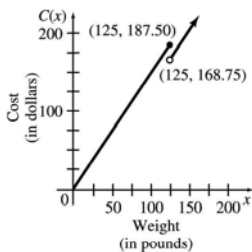
$$63. \quad C(x) = \begin{cases} 1.50x & \text{for } 0 < x \leq 125 \\ 1.35x & \text{for } x > 125 \end{cases}$$

$$(a) \quad C(100) = 1.50(100) = \$150$$

$$(b) \quad C(125) = 1.50(125) = \$187.50$$

$$(c) \quad C(140) = 1.35(140) = \$189$$

(d)



(e) By reading the graph, $C(x)$ is discontinuous at $x = 125$.

The average cost per pound is given

$$\text{by } \bar{C}(x) = \frac{C(x)}{x}.$$

$$\bar{C}(x) = \begin{cases} 1.50 & \text{for } 0 < x \leq 125 \\ 1.35 & \text{for } x > 125 \end{cases}$$

$$(f) \quad \bar{C}(100) = \$1.50$$

$$(g) \quad \bar{C}(125) = \$1.50$$

$$(h) \quad \bar{C}(140) = \$1.35$$

The marginal cost is given by

$$C(x) = \begin{cases} 1.50 & \text{for } 0 < x \leq 125 \\ 1.35 & \text{for } x > 125. \end{cases}$$

(i) $C'(100) = 1.50$; the 101st pound will cost \$1.50.

(j) $C'(140) = 1.35$; the 141st pound will cost \$1.35.

$$64. \quad P(x) = 15x + 25x^2$$

$$\begin{aligned}
 (a) \quad P(6) &= 15(6) + 25(6)^2 \\
 &= 90 + 900 = 990
 \end{aligned}$$

$$\begin{aligned}
 P(7) &= 15(7) + 25(7)^2 \\
 &= 105 + 1225 = 1330
 \end{aligned}$$

Average rate of change:

$$\begin{aligned}
 &= \frac{P(7) - P(6)}{7 - 6} = \frac{1330 - 990}{1} \\
 &= 340 \text{ cents or } \$3.40
 \end{aligned}$$

$$(b) \quad P(6) = 990$$

$$\begin{aligned}
 P(6.5) &= 15(6.5) + 25(6.5)^2 \\
 &= 97.5 + 1056.25 \\
 &= 1153.75
 \end{aligned}$$

Average rate of change:

$$\begin{aligned}
 &= \frac{P(6.5) - P(6)}{6.5 - 6} \\
 &= \frac{1153.75 - 990}{0.5} \\
 &= 327.5 \text{ cents or } \$3.28
 \end{aligned}$$

$$(c) \quad P(6) = 990$$

$$\begin{aligned}
 P(6.1) &= 15(6.1) + 25(6.1)^2 \\
 &= 91.5 + 930.25 \\
 &= 1021.75
 \end{aligned}$$

Average rate of

$$\begin{aligned}
 \text{change: } &= \frac{P(6.1) - P(6)}{6.1 - 6} \\
 &= \frac{1021.75 - 990}{0.1} \\
 &= 317.5 \text{ cents or } \$3.18
 \end{aligned}$$

$$(d) \quad P'(x) = 15 + 50x$$

$$\begin{aligned}
 P'(6) &= 15 + 50(6) \\
 &= 15 + 300 \\
 &= 315 \text{ cents or } \$3.15
 \end{aligned}$$

- (e) $P'(20) = 15 + 50(20) = 1015¢$ or \$10.15
- (f) $P'(30) = 15 + 50(30) = 1515¢$ or \$15.15
- (g) The domain of x is $[0, \infty)$ since pounds cannot be measured with negative numbers.
- (h) Since $P'(x) = 15 + 50x$ gives the marginal profit, and $x \geq 0$, $P'(x)$ can never be negative.

$$\begin{aligned} \text{(i)} \quad \bar{P}(x) &= \frac{P(x)}{x} \\ &= \frac{15x + 25x^2}{x} \\ &= 15 + 25x \end{aligned}$$

$$\text{(j)} \quad \bar{P}'(x) = 25$$

- (k) The marginal average profit cannot change since $\bar{P}'(x)$ is constant. The profit per pound never changes, no matter how many pounds are sold.

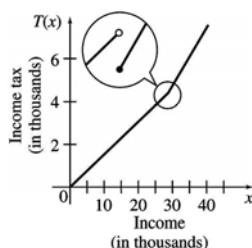
65. (b) The value of x for which the average cost is smallest is $x = 7.5$. This can be found by drawing a line from the origin to any point of $C(x)$. At $x = 7.5$, you will get a line with the smallest slope.
- (c) The marginal cost equals the average cost at the point where the average cost is smallest.

$$\begin{aligned} \text{66. (a)} \quad \lim_{x \rightarrow 29,300^-} T(x) &= (29,300)(0.15) \\ &= \$4395 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 29,300^+} T(x) &= 4350 + (0.27)(29,300 - 29,300) \\ &= \$4350 \end{aligned}$$

- (c) $\lim_{x \rightarrow 29,300} T(x)$ does not exist since parts (a) and (b) have different answers.

(d)



- (e) The graph is discontinuous at $x = 29,300$.
- (f) For $0 \leq x \leq 29,300$,

$$A(x) = \frac{T(x)}{x} = \frac{0.15x}{x} = 0.15.$$

For $x > 29,300$,

$$\begin{aligned} A(x) &= \frac{T(x)}{x} \\ &= \frac{4350 + (0.27)(x - 29,300)}{x} \\ &= \frac{0.27x - 3561}{x} \\ &= 0.27 - \frac{3561}{x}. \end{aligned}$$

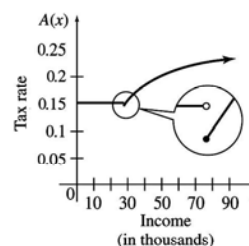
$$\text{(g)} \quad \lim_{x \rightarrow 29,300^-} A(x) = 0.15$$

$$\begin{aligned} \text{(h)} \quad \lim_{x \rightarrow 29,300^+} A(x) &= 0.27 - \frac{3561}{29,300} \\ &= 0.14846 \end{aligned}$$

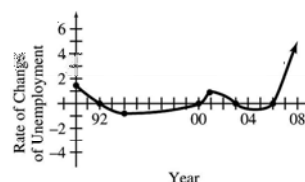
- (i) $\lim_{x \rightarrow 29,300} A(x)$ does not exist since parts (g) and (h) have different answers.

$$\text{(j)} \quad \lim_{x \rightarrow \infty} A(x) = 0.27 - 0 = 0.27$$

(k)



67.



In 2008, the unemployment rate is about 5.4%; the rate of change of the unemployment rate is about 2% per year.

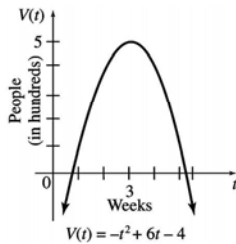
68. (a) The slope of the tangent line at $x = 2000$ is about 0.13; the number of people aged 65 and over with Alzheimer's Disease is going up at a rate of about 0.13 million per year.
- (b) The slope of the tangent line at $x = 2040$ is about 0.34; the number of people aged 65 and over with Alzheimer's Disease is going up at a rate of about 0.34 million per year.

$$(c) \frac{A(2040) - A(2000)}{2040 - 2000} = \frac{11.1 - 4.6}{40} \approx 0.16$$

The average rate of change in the number of people 65 and over with Alzheimer's Disease over this interval is about 0.16 million people per year.

$$69. V(t) = -t^2 + 6t - 4$$

(a)



(b) The x -intercepts of the parabola are 0.8 and 5.2, so a reasonable domain would be $[0.8, 5.2]$, which represents the time period from 0.8 to 5.2 weeks.

(c) The number of cases reaches a maximum at the vertex;

$$x = \frac{-b}{2a} = \frac{-6}{-2} = 3$$

$$V(3) = -3^2 + 6(3) - 4 = 5$$

The vertex of the parabola is $(3, 5)$. This represents a maximum at 3 weeks of 500 cases.

(d) The rate of change function is

$$V'(t) = -2t + 6.$$

(e) The rate of change in the number of cases at the maximum is

$$V'(3) = -2(3) + 6 = 0.$$

(f) The sign of the rate of change up to the maximum is $+$ because the function is increasing. The sign of the rate of change after the maximum is $-$ because the function is decreasing.

70. (a) The rate can be estimated by estimating the slope of the tangent line to the curve at the given time. Answers will vary depending on the points used to determine the slope of the tangent line.

(i) The tangent line to the curve at 17:37 can be found by using the endpoints at $(17:35.5, 0)$ and $(17:40, 400)$. The rate the whale is descending at 17:37 is about

$$\frac{400 - 0}{17:40 - 17:35.5} = \frac{400}{4.5} \approx 90 \text{ meters per minute.}$$

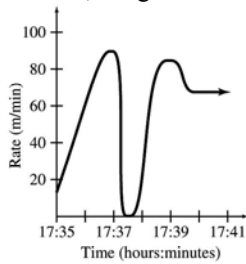
(ii) The tangent line to the curve at 17:39 can be found by using the endpoints at $(17:36.2, 0)$ and $(17:40.8, 400)$. The rate the whale is descending at 17:39 is about

$$\frac{400 - 0}{17:40.8 - 17:36.2} = \frac{400}{4.6} \approx 85 \text{ meters per minute.}$$

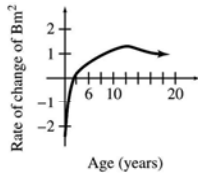
(b) The whale appears to have 5 distinct rates at which it is descending.

Interval	Rate (meters per minute)
17:35–17:35.3	$\frac{f(17:35.3) - f(17:35)}{17:35.3 - 17:35} = \frac{10 - 0}{0.3} \approx 13$
17:36–17:37	$\frac{f(17:37) - f(17:36)}{17:37 - 17:36} = \frac{130 - 40}{1} \approx 90$
17:37.3–17:37.7	$\frac{f(17:37.7) - f(17:37.3)}{17:37.7 - 17:37.3} = \frac{150 - 150}{0.4} = 0$
17:38.3–17:39.5	$\frac{f(17:39.5) - f(17:38.3)}{17:39.5 - 17:38.3} = \frac{300 - 200}{1.2} \approx 83$
17:40–17:41	$\frac{f(17:41) - f(17:40)}{17:41 - 17:40} = \frac{400 - 333}{1} = 67$

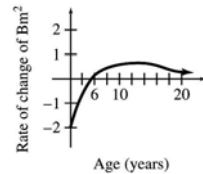
Making smooth transitions between each interval, we get



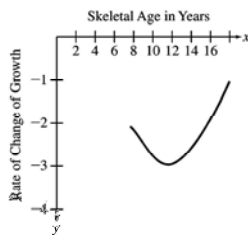
71. (a)



(b)



72.

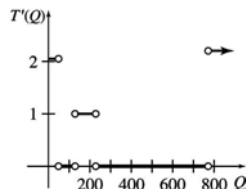


For a 10-year old girl, the remaining growth is about 14 cm and the rate of change is about -2.75 cm per year.

73. (a) The graph is discontinuous nowhere.

(b) The graph is not differentiable where the graph makes a sudden change, namely at $x = 50$, $x = 130$, $x = 230$, and $x = 770$.

(c)



Extended Application: A Model for Drugs Administered Intravenously

1. (a) Since the half-life of the drug is 9 hours, the exponential decay constant is

$$k = -\frac{\ln 2}{9} \approx -0.077.$$

Since the initial amount injected is 500 mg, the model for the amount in the bloodstream (in mg) after t hours is

$$A(t) = 500e^{-0.077t}$$

- (b) The average rate of change over the t -interval from 0 to 2 is

$$\begin{aligned} \frac{A(2) - A(0)}{2} &= \frac{500(e^{-0.077(2)} - e^{-0.077(0)})}{2} \\ &= -35.68 \text{ mg/hr} \end{aligned}$$

For the interval from $t = 9$ to $t = 11$, we get

$$\begin{aligned} \frac{A(11) - A(9)}{2} &= \frac{500(e^{-0.077(11)} - e^{-0.077(9)})}{2} \\ &= -17.84 \text{ mg/hr} \end{aligned}$$

2. (a) Since the half-life of the drug is 3 hours, the exponential decay constant is

$$k = -\frac{\ln 2}{3} \approx -0.23.$$

With an infusion rate of 350 mg/hr, the model is

$$A(t) = \frac{350}{0.23}(1 - e^{-0.23t}) \approx 1522(1 - e^{-0.23t})$$

- (b) The average rate of change over the t -interval from 0 to 3 is

$$\begin{aligned} \frac{A(3) - A(0)}{3} &= \frac{1522[(1 - e^{-0.23(3)}) - (1 - e^{-0.23(0)})]}{3} \\ &= \frac{1522(e^{-0.23(0)} - e^{-0.23(3)})}{3} \\ &= 252.9 \text{ mg/hr} \end{aligned}$$

For the interval from $t = 3$ to $t = 6$, we get

$$\begin{aligned} \frac{A(6) - A(3)}{3} &= \frac{1522(e^{-0.23(3)} - e^{-0.23(6)})}{3} \\ &= 126.8 \text{ mg/hr} \end{aligned}$$

3. The exponential decay constant for a half-life of 9 hours is as found in Exercise 1: $k = -0.077$.

Following Example 4, for a steady level of 240 mg, we want the infusion rate to satisfy

$$250 = \frac{r}{-k},$$

so, $r = 250(-k) = 240(0.077) = 19.25$ mg/hr. With this value of r , $\frac{r}{-k} = 250$, so the model is

$$A(t) = 250e^{-0.077t} + 250(1 - e^{-0.077t})$$

4.

t	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
$A(t)$	500	481.116	462.945	445.46	428.636	412.447	396.87	381.881	367.458	353.579
t	5	5.5	6	6.5	7	7.5	8	8.5	9	9.5
$A(t)$	340.225	327.376	315.011	303.114	291.666	280.65	270.05	259.851	250.037	240.593
t	10	10.5	11	11.5	12	12.5	13	13.5	14	14.5
$A(t)$	231.507	222.763	214.35	206.254	198.464	190.968	183.756	176.816	170.138	163.712
t	15	15.5	16	16.5	17	17.5	18	18.5	19	19.5
$A(t)$	157.529	151.579	145.854	140.346	135.045	129.945	125.037	120.314	115.77	111.398
t	20	20.5	21	21.5	22	22.5	23	23.5	24	
$A(t)$	107.191	103.142	99.247	95.498	91.891	88.421	85.081	81.868	78.776	

5.

t	0	0.5	1	1.5	2	2.5	3	3.5
$A(t)$	0	165.341	312.72	444.089	561.186	665.563	758.601	841.532
t	4	4.5	5	5.5	6	6.5	7	7.5
$A(t)$	915.454	981.345	1040.079	1092.432	1139.097	1180.694	1217.771	1250.821
t	8	8.5	9	9.5	10			
$A(t)$	1280.28	1306.539	1329.945	1350.809	1369.406			

6.

t	0	0.5	1	1.5	2	2.5	3	3.5
$A(t)$	250	250	250	250	250	250	250	250
t	4	4.5	5	5.5	6	6.5	7	7.5
$A(t)$	250	250	250	250	250	250	250	250
t	8	8.5	9	9.5	10			
$A(t)$	250	250	250	250	250			

The results show that the chosen infusion rate has the desired effect of keeping the drug at a constant level of 250 mg in the bloodstream.

Chapter 4

CALCULATING THE DERIVATIVE

4.1 Techniques for Finding Derivatives

Your Turn 1

$$\begin{aligned}f(t) &= \frac{1}{\sqrt{t}} = t^{-1/2} \\f'(t) &= -\frac{1}{2}t^{-1/2-1} \\&= -\frac{1}{2}t^{-3/2} \\&= -\frac{1}{2t^{3/2}} \text{ or } -\frac{1}{2\sqrt{t^3}}\end{aligned}$$

Your Turn 2

$$\begin{aligned}y &= 3\sqrt{x} = 3x^{1/2} \\ \frac{dy}{dx} &= 3 \cdot \frac{1}{2}x^{-1/2} \\&= \frac{3}{2}x^{-1/2} \\&= \frac{3}{2\sqrt{x}}\end{aligned}$$

Your Turn 3

$$\begin{aligned}h(t) &= -3t^2 + 2\sqrt{t} + \frac{5}{t^4} - 7 \\&= -3t^2 + 2t^{1/2} + 5t^{-4} - 7 \\h'(t) &= -6t + t^{-1/2} - 20t^{-5} \\&= -6t + \frac{1}{\sqrt{t}} - \frac{20}{t^5}\end{aligned}$$

Your Turn 4

Find the marginal cost of the cost function

$$C(x) = 5x^3 - 10x^2 + 75 \text{ when } x = 100.$$

$$\begin{aligned}C'(x) &= 15x^2 - 20x \\C'(100) &= 15(100)^2 - 20(100) \\&= 150,000 - 2000 \\&= 148,000\end{aligned}$$

The marginal cost when $x = 100$ is \$148,000.

Your Turn 5

Find the marginal revenue of the demand function
 $p = 16 - 1.25q$ when $q = 5$.

$$\begin{aligned}R(q) &= qp = q(16 - 1.25q) = 16q - 1.25q^2 \\R'(q) &= 16 - 2.5q \\R'(5) &= 16 - 2.5(5) = 3.5\end{aligned}$$

The marginal revenue for 5 units is \$3.50 per unit.

4.1 Exercises

- $y = 12x^3 - 8x^2 + 7x + 5$
$$\begin{aligned}\frac{dy}{dx} &= 12(3x^{3-1}) - 8(2x^{2-1}) + 7x^{1-1} + 0 \\&= 36x^2 - 16x + 7\end{aligned}$$
- $y = 8x^3 - 5x^2 - \frac{x}{12}$
$$\begin{aligned}\frac{dy}{dx} &= 8(3x^{3-1}) - 5(2x^{2-1}) - \frac{1}{12}x^{1-1} \\&= 24x^2 - 10x - \frac{1}{12}\end{aligned}$$
- $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$
$$\begin{aligned}\frac{dy}{dx} &= 3(4x^{4-1}) - 6(3x^{3-1}) + \frac{1}{8}(2x^{2-1}) + 0 \\&= 12x^3 - 18x^2 + \frac{1}{4}x\end{aligned}$$
- $y = 5x^4 + 9x^3 + 12x^2 - 7x$
$$\begin{aligned}\frac{dy}{dx} &= 5(4x^{4-1}) + 9(3x^{3-1}) \\&\quad + 12(2x^{2-1}) - 7(x^{1-1}) \\&= 20x^3 + 27x^2 + 24x - 7\end{aligned}$$
- $y = 6x^{3.5} - 10x^{0.5}$
$$\begin{aligned}\frac{dy}{dx} &= 6(3.5x^{3.5-1}) - 10(0.5x^{0.5-1}) \\&= 21x^{2.5} - 5x^{-0.5} \text{ or } 21x^{2.5} - \frac{5}{x^{0.5}}\end{aligned}$$

6. $f(x) = -2x^{1.5} + 12x^{0.5}$
 $f'(x) = -2(1.5x^{1.5-1}) + 12(0.5x^{0.5-1})$
 $= -3x^{0.5} + 6x^{-0.5}$ or $-3x^{0.5} + \frac{6}{x^{0.5}}$
7. $y = 8\sqrt{x} + 6x^{3/4} = 8x^{1/2} + 6x^{3/4}$
 $\frac{dy}{dx} = 8\left(\frac{1}{2}x^{1/2-1}\right) + 6\left(\frac{3}{4}x^{3/4-1}\right)$
 $= 4x^{-1/2} + \frac{9}{2}x^{-1/4}$ or $\frac{4}{x^{1/2}} + \frac{9}{2x^{1/4}}$
8. $y = -100\sqrt{x} - 11x^{2/3} = -100x^{1/2} - 11x^{2/3}$
 $\frac{dy}{dx} = -100\left(\frac{1}{2}x^{1/2-1}\right) - 11\left(\frac{2}{3}x^{2/3-1}\right)$
 $= -50x^{-1/2} - \frac{22x^{-1/3}}{3}$ or $\frac{-50}{x^{1/2}} - \frac{22}{3x^{1/3}}$
9. $y = 10x^{-3} + 5x^{-4} - 8x$
 $\frac{dy}{dx} = 10(-3x^{-3-1}) + 5(-4x^{-4-1}) - 8x^{1-1}$
 $= -30x^{-4} - 20x^{-5} - 8$ or $\frac{-30}{x^4} - \frac{20}{x^5} - 8$
10. $y = 5x^{-5} - 6x^{-2} + 13x^{-1}$
 $\frac{dy}{dx} = 5(-5x^{-5-1}) - 6(-2x^{-2-1}) + 13(-1x^{-1-1})$
 $= -25x^{-6} + 12x^{-3} - 13x^{-2}$
or $\frac{-25}{x^6} + \frac{12}{x^3} - \frac{13}{x^2}$
11. $f(t) = \frac{7}{t} - \frac{5}{t^3}$
 $= 7t^{-1} - 5t^{-3}$
 $f'(t) = 7(-1t^{-1-1}) - 5(-3t^{-3-1})$
 $= -7t^{-2} + 15t^{-4}$ or $\frac{-7}{t^2} + \frac{15}{t^4}$
12. $f(t) = \frac{14}{t} + \frac{12}{t^4} + \sqrt{2}$
 $= 14t^{-1} + 12t^{-4} + \sqrt{2}$
 $f'(t) = 14(-1t^{-1-1}) + 12(-4t^{-4-1}) + 0$
 $= -14t^{-2} - 48t^{-5}$ or $\frac{-14}{t^2} - \frac{48}{t^5}$
13. $y = \frac{6}{x^4} - \frac{7}{x^3} + \frac{3}{x} + \sqrt{5}$
 $= 6x^{-4} - 7x^{-3} + 3x^{-1} + \sqrt{5}$
 $\frac{dy}{dx} = 6(-4x^{-4-1}) - 7(-3x^{-3-1})$
 $+ 3(-1x^{-1-1}) + 0$
 $= -24x^{-5} + 21x^{-4} - 3x^{-2}$
or $\frac{-24}{x^5} + \frac{21}{x^4} - \frac{3}{x^2}$
14. $y = \frac{3}{x^6} + \frac{1}{x^5} - \frac{7}{x^2}$
 $= 3x^{-6} + x^{-5} - 7x^{-2}$
 $\frac{dy}{dx} = 3(-6x^{-6-1}) + (-5x^{-5-1}) - 7(-2x^{-2-1})$
 $= -18x^{-7} - 5x^{-6} + 14x^{-3}$
or $\frac{-18}{x^7} - \frac{5}{x^6} + \frac{14}{x^3}$
15. $p(x) = -10x^{-1/2} + 8x^{-3/2}$
 $p'(x) = -10\left(-\frac{1}{2}x^{-3/2}\right) + 8\left(-\frac{3}{2}x^{-5/2}\right)$
 $= 5x^{-3/2} - 12x^{-5/2}$ or $\frac{5}{x^{3/2}} - \frac{12}{x^{5/2}}$
16. $h(x) = x^{-1/2} - 14x^{-3/2}$
 $h'(x) = -\frac{1}{2}x^{-3/2} - 14\left(-\frac{3}{2}x^{-5/2}\right)$
 $= \frac{-x^{-3/2}}{2} + 21x^{-5/2}$ or $\frac{-1}{2x^{3/2}} + \frac{21}{x^{5/2}}$
17. $y = \frac{6}{4\sqrt{x}} = 6x^{-1/4}$
 $\frac{dy}{dx} = 6\left(-\frac{1}{4}\right)x^{-5/4}$
 $= -\frac{3}{2}x^{-5/4}$ or $\frac{-3}{2x^{5/4}}$
18. $y = \frac{-2}{\sqrt[3]{x}} = \frac{-2}{x^{1/3}} = -2x^{-1/3}$
 $\frac{dy}{dx} = -2\left(-\frac{1}{3}x^{-4/3}\right)$
 $= \frac{2x^{-4/3}}{3}$ or $\frac{2}{3x^{4/3}}$

$$\begin{aligned}
 19. \quad f(x) &= \frac{x^3 + 5}{x} = x^2 + 5x^{-1} \\
 f'(x) &= 2x^{2-1} + 5(-1x^{-1-1}) \\
 &= 2x - 5x^{-2} \text{ or } 2x - \frac{5}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad g(x) &= \frac{x^3 - 4x}{\sqrt{x}} = \frac{x^3 - 4x}{x^{1/2}} = x^{5/2} - 4x^{1/2} \\
 g'(x) &= \frac{5}{2}x^{5/2-1} - 4\left(\frac{1}{2}x^{1/2-1}\right) \\
 &= \frac{5}{2}x^{3/2} - 2x^{-1/2} \text{ or } \frac{5}{2}x^{3/2} - \frac{2}{\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad g(x) &= (8x^2 - 4x)^2 \\
 &= 64x^4 - 64x^3 + 16x^2 \\
 g'(x) &= 64(4x^{4-1}) - 64(3x^{3-1}) + 16(2x^{2-1}) \\
 &= 256x^3 - 192x^2 + 32x
 \end{aligned}$$

$$\begin{aligned}
 22. \quad h(x) &= (x^2 - 1)^3 \\
 &= x^6 - 3x^4 + 3x^2 - 1 \\
 h'(x) &= 6x^{6-1} - 3(4x^{4-1}) + 3(2x^{2-1}) - 0 \\
 &= 6x^5 - 12x^3 + 6x
 \end{aligned}$$

23. A quadratic function has degree 2. When the derivative is taken, the power will decrease by 1 and the derivative function will be linear, so the correct choice is (b).

24. A cubic function has degree 3, so its derivative function will be quadratic. The correct choice is (a).

$$\begin{aligned}
 26. \quad \frac{d}{dx}(4x^3 - 6x^{-2}) \\
 &= 4(3x^2) - 6(-2x^{-3}) \\
 &= 12x^2 + 12x^{-3} \\
 &= 12x^2 + \frac{12}{x^3} \quad \text{choice (c)} \\
 &= \frac{12x^2(x^3) + 12}{x^3} \\
 &= \frac{12x^5 + 12}{x^3} \quad \text{choice (b)}
 \end{aligned}$$

Neither choice (a) nor choice (d) equals

$$\frac{d}{dx}(4x^3 - 6x^{-2}).$$

$$\begin{aligned}
 27. \quad D_x \left[9x^{-1/2} + \frac{2}{x^{3/2}} \right] \\
 &= D_x[9x^{-1/2} + 2x^{-3/2}] \\
 &= 9\left(-\frac{1}{2}x^{-3/2}\right) + 2\left(-\frac{3}{2}x^{-5/2}\right) \\
 &= -\frac{9}{2}x^{-3/2} - 3x^{-5/2} \text{ or } \frac{-9}{2x^{3/2}} - \frac{3}{x^{5/2}}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad D_x \left[\frac{8}{\sqrt[4]{x}} - \frac{3}{\sqrt{x^3}} \right] \\
 &= D_x[8x^{-1/4} - 3x^{-3/2}] \\
 &= 8\left(-\frac{1}{4}x^{-5/4}\right) - 3\left(-\frac{3}{2}x^{-5/2}\right) \\
 &= -2x^{-5/4} + \frac{9x^{-5/2}}{2} \text{ or } \frac{-2}{x^{5/4}} + \frac{9}{2x^{5/2}}
 \end{aligned}$$

$$\begin{aligned}
 29. \quad f(x) &= \frac{x^4}{6} - 3x \\
 &= \frac{1}{6}x^4 - 3x \\
 f'(x) &= \frac{1}{6}(4x^3) - 3 \\
 &= \frac{2}{3}x^3 - 3 \\
 f'(-2) &= \frac{2}{3}(-2)^3 - 3 \\
 &= -\frac{16}{3} - 3 \\
 &= -\frac{25}{3}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad f(x) &= \frac{x^3}{9} - 7x^2 \\
 &= \frac{1}{9}x^3 - 7x^2 \\
 f'(x) &= \frac{1}{9}(3x^2) - 7(2x) \\
 &= \frac{1}{3}x^2 - 14x \\
 f'(3) &= \frac{1}{3}(3)^2 - 14(3) \\
 &= 3 - 42 \\
 &= -39
 \end{aligned}$$

31. $y = x^4 - 5x^3 + 2; x = 2$

$$y' = 4x^3 - 15x^2$$

$$y'(2) = 4(2)^3 - 15(2)^2 \\ = -28$$

The slope of tangent line at $x = 2$ is -28 .

Use $m = -28$ and $(x_1, y_1) = (2, -22)$ to obtain the equation.

$$y - (-22) = -28(x - 2) \\ y = -28x + 34$$

32. $y = -3x^5 - 8x^3 + 4x^2$

$$y' = -3(5x^4) - 8(3x^2) + 4(2x) \\ = -15x^4 - 24x^2 + 8x$$

$$y'(1) = -15(1)^4 - 24(1)^2 + 8(1) \\ = -15 - 24 + 8 \\ = -31$$

$$y(1) = -3(1)^5 - 8(1)^3 + 4(1)^2 = -7$$

The slope of the tangent line at $x = 1$ is -31 .

Use $m = -31$ and $(x_1, y_1) = (1, -7)$ to obtain the equation.

$$y - (-7) = -31(x - 1) \\ y + 7 = -31x + 31 \\ y = -31x + 24$$

33. $y = -2x^{1/2} + x^{3/2}$

$$y' = -2\left(\frac{1}{2}x^{-1/2}\right) + \frac{3}{2}x^{1/2} \\ = -x^{-1/2} + \frac{3}{2}x^{1/2}$$

$$= -\frac{1}{x^{1/2}} + \frac{3x^{1/2}}{2}$$

$$y'(9) = -\frac{1}{(9)^{1/2}} + \frac{3(9)^{1/2}}{2} \\ = -\frac{1}{3} + \frac{9}{2} \\ = \frac{25}{6}$$

The slope of the tangent line at $x = 9$ is $\frac{25}{6}$.

34. $y = -x^{-3} + x^{-2}$

$$y' = -(-3x^{-4}) + (-2x^{-3}) \\ = 3x^{-4} - 2x^{-3} \\ = \frac{3}{x^4} - \frac{2}{x^3}$$

$$y'(2) = \frac{3}{(2)^4} - \frac{2}{(2)^3} \\ = \frac{3}{16} - \frac{2}{8} \\ = -\frac{1}{16}$$

The slope of the tangent line at $x = 2$ is $-\frac{1}{16}$.

35. $f(x) = 9x^2 - 8x + 4$

$$f'(x) = 18x - 8$$

Let $f'(x) = 0$ to find the point where the slope of the tangent line is zero.

$$18x - 8 = 0$$

$$18x = 8$$

$$x = \frac{8}{18} = \frac{4}{9}$$

Find the y -coordinate.

$$f(x) = 9x^2 - 8x + 4$$

$$f\left(\frac{4}{9}\right) = 9\left(\frac{4}{9}\right)^2 - 8\left(\frac{4}{9}\right) + 4 \\ = 9\left(\frac{16}{81}\right) - \frac{32}{9} + 4 \\ = \frac{16}{9} - \frac{32}{9} + \frac{36}{9} = \frac{20}{9}$$

The slope of the tangent line is zero

at one point, $\left(\frac{4}{9}, \frac{20}{9}\right)$.

36. $f(x) = x^3 + 9x^2 + 19x - 10$

$$f'(x) = 3x^2 + 18x + 19$$

If the slope of the tangent line is -5 , then the $f'(x) = -5$.

$$3x^2 + 18x + 19 = -5$$

$$3x^2 + 18x + 24 = 0$$

$$3(x^2 + 6x + 8) = 0$$

$$3(x + 2)(x + 4) = 0$$

$$x = -2 \text{ or } x = -4$$

$$\begin{aligned} f(-2) &= (-2)^3 + 9(-2)^2 + 19(-2) - 10 \\ &= -8 + 36 - 38 - 10 \\ &= -20 \end{aligned}$$

$$\begin{aligned} f(-4) &= (-4)^3 + 9(-4)^2 + 19(-4) - 10 \\ &= -64 + 144 - 76 - 10 \\ &= -6 \end{aligned}$$

Thus, the points where the slope of the tangent line is -5 are $(-2, -20)$ and $(-4, -6)$.

37. $f(x) = 2x^3 + 9x^2 - 60x + 4$

$$f'(x) = 6x^2 + 18x - 60$$

If the tangent line is horizontal, then its slope is zero and $f'(x) = 0$.

$$6x^2 + 18x - 60 = 0$$

$$6(x^2 + 3x - 10) = 0$$

$$6(x + 5)(x - 2) = 0$$

$$x = -5 \text{ or } x = 2$$

Thus, the tangent line is horizontal at $x = -5$ and $x = 2$.

38. $f(x) = x^3 + 15x^2 + 63x - 10$

$$f'(x) = 3x^2 + 30x + 63$$

If the tangent line is horizontal, then its slope is zero and $f'(x) = 0$.

$$3x^2 + 30x + 63 = 0$$

$$3(x^2 + 10x + 21) = 0$$

$$3(x + 3)(x + 7) = 0$$

$$x = -3 \text{ or } x = -7$$

Therefore, the tangent line is horizontal at $x = -3$ or $x = -7$.

39. $f(x) = x^3 - 4x^2 - 7x + 8$

$$f'(x) = 3x^2 - 8x - 7$$

If the tangent line is horizontal, then its slope is zero and $f'(x) = 0$.

$$3x^2 - 8x - 7 = 0$$

$$x = \frac{8 \pm \sqrt{64 + 84}}{6}$$

$$x = \frac{8 \pm \sqrt{148}}{6}$$

$$x = \frac{8 \pm 2\sqrt{37}}{6}$$

$$x = \frac{2(4 \pm \sqrt{37})}{6}$$

$$x = \frac{4 \pm \sqrt{37}}{3}$$

Thus, the tangent line is horizontal

at $x = \frac{4 \pm \sqrt{37}}{3}$.

40. $f(x) = x^3 - 5x^2 + 6x + 3$

$$f'(x) = 3x^2 - 10x + 6$$

If the tangent line is horizontal, then its slope is zero and $f'(x) = 0$.

$$3x^2 - 10x + 6 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 72}}{6}$$

$$x = \frac{10 \pm \sqrt{28}}{6}$$

$$x = \frac{10 \pm 2\sqrt{7}}{6}$$

$$x = \frac{5 \pm \sqrt{7}}{3}$$

The tangent line is horizontal at $x = \frac{5 \pm \sqrt{7}}{3}$.

41. $f(x) = 6x^2 + 4x - 9$

$$f'(x) = 12x + 4$$

If the slope of the tangent line is -2 , $f'(x) = -2$.

$$12x + 4 = -2$$

$$12x = -6$$

$$x = -\frac{1}{2}$$

$$f\left(-\frac{1}{2}\right) = -\frac{19}{2}$$

The slope of the tangent line is -2 at $\left(-\frac{1}{2}, -\frac{19}{2}\right)$.

42. $f(x) = 2x^3 - 9x^2 - 12x + 5$

$$f'(x) = 6x^2 - 18x - 12$$

If the slope of the tangent line is 12 , $f'(x) = 12$.

$$6x^2 - 18x - 12 = 12$$

$$6x^2 - 18x - 24 = 0$$

$$6(x^2 - 3x - 4) = 0$$

$$6(x - 4)(x + 1) = 0$$

$$x = 4 \text{ or } x = -1$$

$$f(4) = -59 \text{ and } f(-1) = 6$$

The slope of the tangent line is -12 at $(4, -59)$ and $(-1, 6)$.

$$\begin{aligned} 43. \quad f(x) &= x^3 + 6x^2 + 21x + 2 \\ f'(x) &= 3x^2 + 12x + 21 \end{aligned}$$

If the slope of the tangent line is 9, $f'(x) = 9$.

$$\begin{aligned} 3x^2 + 12x + 21 &= 9 \\ 3x^2 + 12x + 12 &= 0 \\ 3(x^2 + 4x + 4) &= 0 \\ 3(x + 2)^2 &= 0 \\ x &= -2 \\ f(-2) &= -24 \end{aligned}$$

The slope of the tangent line is 9 at $(-2, -24)$.

$$\begin{aligned} 44. \quad f(x) &= 3g(x) - 2h(x) + 3 \\ f'(x) &= 3g'(x) - 2h'(x) \\ f'(5) &= 3g'(5) - 2h'(5) \\ &= 3(12) - 2(-3) = 42 \end{aligned}$$

$$\begin{aligned} 45. \quad f(x) &= \frac{1}{2}g(x) + \frac{1}{4}h(x) \\ f'(x) &= \frac{1}{2}g'(x) + \frac{1}{4}h'(x) \\ f'(2) &= \frac{1}{2}g'(2) + \frac{1}{4}h'(2) \\ &= \frac{1}{2}(7) + \frac{1}{4}(14) = 7 \end{aligned}$$

46. (a) From the graph, $f(1) = 2$, because the curve goes through $(1, 2)$.
 (b) $f'(1)$ gives the slope of the tangent line to f at 1. The line goes through $(-1, 1)$ and $(1, 2)$.

$$m = \frac{2 - 1}{1 - (-1)} = \frac{1}{2},$$

so $f'(1) = \frac{1}{2}.$

- (c) The domain of f is $[-1, \infty]$ because the x -coordinates of the points of f start at $x = -1$ and continue infinitely through the positive real numbers.
 (d) The range of f is $[0, \infty)$ because the y -coordinates of the points on f start at $y = 0$ and continue infinitely through the positive real numbers.

$$49. \quad \frac{f(x)}{k} = \frac{1}{k} \cdot f(x)$$

Use the rule for the derivative of a constant times a function.

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{k} \right] &= \frac{d}{dx} \left[\frac{1}{k} \cdot f(x) \right] \\ &= \frac{1}{k} f'(x) \\ &= \frac{f'(x)}{k} \end{aligned}$$

50. Graph the numerical derivative of

$$f(x) = 1.25x^3 + 0.01x^2 - 2.9x + 1$$

for x ranging from -5 to 5 .

- (a) When $x = 4$, the derivative equals 57.18.
 (b) The derivative crosses the x -axis at approximately -0.88 and 0.88 .
 51. The demand is given by $q = 5000 - 100p$. Solve for p .

$$p = \frac{5000 - q}{100}$$

$$\begin{aligned} R(q) &= q \left(\frac{5000 - q}{100} \right) \\ &= \frac{5000q - q^2}{100} \end{aligned}$$

$$R'(q) = \frac{5000 - 2q}{100}$$

$$\begin{aligned} \text{(a)} \quad R'(1000) &= \frac{5000 - 2(1000)}{100} \\ &= 30 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad R'(2500) &= \frac{5000 - 2(2500)}{100} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad R'(3000) &= \frac{5000 - 2(3000)}{100} \\ &= -10 \end{aligned}$$

$$52. \quad C(q) = 3000 - 20q + 0.03q^2$$

$$\begin{aligned} R(q) &= qp = q \left(\frac{5000 - q}{100} \right) \\ &= q \left(50 - \frac{q}{100} \right) \end{aligned}$$

(from Exercise 51)

The profit equation is found as follows.

$$\begin{aligned}
 P &= R - C \\
 &= q\left(50 - \frac{q}{100}\right) - (3000 - 20q + 0.03q^2) \\
 &= 50q - \frac{q^2}{100} - 3000 + 20q - 0.03q^2 \\
 &= 50q + 20q - \frac{q^2}{100} - 0.03q^2 - 3000 \\
 &= 70q - 0.01q^2 - 0.03q^2 - 3000
 \end{aligned}$$

Therefore, $P = 70q - 0.04q^2 - 3000$.

Now, the marginal profit is

$$\begin{aligned}
 P'(q) &= 70 - 0.04(2q) - 0 \\
 &= 70 - 0.08q.
 \end{aligned}$$

$$\begin{aligned}
 \text{(a)} \quad P'(500) &= 70 - 0.08(500) \\
 &= 70 - 40 \\
 &= 30
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P'(815) &= 70 - 0.08(815) \\
 &= 70 - 65.2 \\
 &= 4.8
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad P'(1000) &= 70 - 0.08(1000) \\
 &= 70 - 80 \\
 &= -10
 \end{aligned}$$

$$53. \quad p(q) = \frac{1000}{q^2} + 1000$$

If R is the revenue function, $R(q) = qp(q)$.

$$\begin{aligned}
 R(q) &= q\left(\frac{1000}{q^2} + 1000\right) \\
 R(q) &= 1000q^{-1} + 1000q \\
 R'(q) &= -1000q^{-2} + 1000 \\
 R'(q) &= 1000 - \frac{1000}{q^2} \\
 R'(q) &= 1000\left(1 - \frac{1}{q^2}\right) \\
 R'(10) &= 1000\left(1 - \frac{1}{10^2}\right) \\
 R'(10) &= 1000\left(\frac{99}{100}\right) \\
 &= 990
 \end{aligned}$$

The marginal revenue is \$990.

54. Profit = Revenue - Cost

$$\begin{aligned}
 P(q) &= qp(q) - C(q) \\
 P(q) &= q\left(\frac{1000}{q^2} + 1000\right) - (0.2q^2 + 6q + 50) \\
 &= \frac{1000}{q} + 1000q - 0.2q^2 - 6q - 50 \\
 &= 1000q^{-1} + 994q - 0.2q^2 - 50 \\
 P'(q) &= -1000q^{-2} + 994 - 0.4q \\
 &= 994 - 0.4q - \frac{1000}{q^2} \\
 P'(10) &= 994 - 0.4(10) - \frac{1000}{(10)^2} \\
 &= 994 - 4 - 10 \\
 &= 980
 \end{aligned}$$

The marginal profit is \$980.

$$\begin{aligned}
 55. \quad S(t) &= 100 - 100t^{-1} \\
 S'(t) &= -100(-1t^{-2}) \\
 &= 100t^{-2} \\
 &= \frac{100}{t^2}
 \end{aligned}$$

$$\text{(a)} \quad S'(1) = \frac{100}{(1)^2} = \frac{100}{1} = 100$$

$$\text{(b)} \quad S'(10) = \frac{100}{(10)^2} = \frac{100}{100} = 1$$

$$56. \quad C(x) = 2x; \quad R(x) = 6x - \frac{x^2}{1000}$$

$$\text{(a)} \quad C'(x) = 2$$

$$\text{(b)} \quad R'(x) = 6 - \frac{2x}{1000} = 6 - \frac{x}{500}$$

$$\begin{aligned}
 \text{(c)} \quad P(x) &= R(x) - C(x) \\
 &= \left(6x - \frac{x^2}{1000}\right) - 2x \\
 &= 4x - \frac{x^2}{1000} \\
 P'(x) &= 4 - \frac{2x}{1000} = 4 - \frac{x}{500}
 \end{aligned}$$

$$\text{(d)} \quad P'(x) = 4 - \frac{x}{500} = 0$$

$$\begin{aligned}
 4 &= \frac{x}{500} \\
 x &= 2000
 \end{aligned}$$

- (e) Using part (d), we can see that marginal profit is 0 when $x = 2000$ units.

$$\begin{aligned} P(2000) &= 4(2000) - \frac{(2000)^2}{1000} \\ &= 8000 - 4000 \\ &= 4000 \end{aligned}$$

The profit for 2000 units produced is \$4000.

57. (a) 1982 when $t = 50$:

$$\begin{aligned} C(50) &= 0.008446(50)^2 - 0.08924(50) + 1.254 \\ &= 17.907 \\ &\approx 18 \text{ cents} \end{aligned}$$

2002 when $t = 70$:

$$\begin{aligned} C(70) &= 0.008446(70)^2 - 0.08924(70) + 1.254 \\ &= 36.3926 \\ &\approx 36 \text{ cents} \end{aligned}$$

- (b) $C'(t) = 0.016892t - 0.08924$

1982 when $t = 50$:

$$\begin{aligned} C'(50) &= 0.016892(50) - 0.08924 \\ &= 0.75536 \\ &\approx 0.755 \text{ cent per year} \end{aligned}$$

2002 when $t = 70$:

$$\begin{aligned} C'(70) &= 0.016892(70) - 0.08924 \\ &= 1.0932 \\ &\approx 1.09 \text{ cents per year} \end{aligned}$$

- (c) Using a graphing calculator, a cubic function that models the postage cost data is

$$C(t) = -0.0001549t^3 + 0.02699t^2 - 0.6484t + 3.212$$

$$C'(t) = -0.0004657t^2 + 0.05398t - 0.6484$$

$$C'(50) \approx 0.889 \text{ cent per year}$$

$$C'(70) \approx 0.853 \text{ cent per year}$$

58. $M(t) = 0.005209t^3 - 0.04159t^2 - 0.3664t + 34.49$

$$M'(t) = 0.015627t^2 - 0.08318t - 0.3664$$

- (a) For 1960, $t = 0$:

$$\begin{aligned} M'(10) &= 0.015627(10)^2 \\ &\quad - 0.08318(10) - 0.3664 \\ &= \$0.3645 \text{ billion per year} \end{aligned}$$

- (b) For 1980, $t = 30$:

$$\begin{aligned} M'(30) &= 0.015627(30)^2 \\ &\quad - 0.08318(30) - 0.3664 \\ &= \$11.20 \text{ billion per year} \end{aligned}$$

- (c) For 1990, $t = 40$:

$$\begin{aligned} M'(40) &= 0.015627(40)^2 - 0.08318(40) - 0.3664 \\ &= \$21.31 \text{ billion per year} \end{aligned}$$

- (d) For 2000, $t = 50$:

$$\begin{aligned} M'(50) &= 0.015627(50)^2 - 0.08318(50) - 0.3664 \\ &= \$34.54 \text{ billion per year} \end{aligned}$$

59. $N(t) = 0.00437t^{3.2}$

$$N'(t) = 0.013984t^{2.2}$$

(a) $N'(5) \approx 0.4824$

(b) $N'(10) \approx 2.216$

60. $G(x) = -0.2x^2 + 450$

(a) $G(0) = -0.2(0)^2 + 450 = 450$

(b) $G(25) = -0.2(25)^2 + 450$
 $= -125 + 450$
 $= 325$

$$G'(x) = -2(0.2)x = -0.4x$$

(c) $G'(10) = -0.4(10) = -4$

After 10 units of insulin are injected, the blood sugar level is decreasing at a rate of 4 points per unit of insulin.

(d) $G'(25) = -0.4(25)$
 $= -10$

After 25 units of insulin are injected, the blood sugar level is decreasing at a rate of 10 points per unit of insulin.

61. $V(t) = -2159 + 1313t - 60.82t^2$

(a) $V(3) = -2159 + 1313(3) - 60.82(3)^2$
 $= 1232.62 \text{ cm}^3$

(b) $V'(t) = 1313 - 121.64t$
 $V'(3) = 1313 - 121.64(3)$
 $= 948.08 \text{ cm}^3/\text{yr}$

$$62. \quad w(c) = \frac{c^3}{100} - \frac{1500}{c}$$

(a) When $c = 30$, $w = 220$ g or 0.22 kg.

$$(b) \quad \frac{dw}{dc} = \frac{3c^2}{100} + \frac{1500}{c^2}$$

When $c = 30$, $\frac{dw}{dc} = 28\frac{2}{3}$ g/cm. When the circumference of the brain is 30 cm, it is increasing by $28\frac{2}{3}$ g with every centimeter the circumference increases.

$$63. \quad v = 2.69l^{1.86}$$

$$\frac{dv}{dl} = (1.86)2.69l^{1.86-1} \approx 5.00l^{0.86}$$

$$64. \quad l(x) = -2.318 + 0.2356x - 0.002674x^2$$

(a) The problem states that a fetus this formula concerns is at least 18 weeks old. So, the minimum x value should be 18. Considering the gestation time of a cow in general, a meaningful range for this function is $18 \leq x \leq 44$.

$$(b) \quad l'(x) = 0.2356 - (2)0.002674x \\ = 0.2356 - 0.005348x$$

$$(c) \quad l'(25) = 0.2356 - 0.005348(25) \\ = 0.1019 \text{ cm/week}$$

$$65. \quad t = 0.0588s^{1.125}$$

(a) When $s = 1609$, $t \approx 238.1$ seconds, or 3 minutes, 58.1 seconds.

$$(b) \quad \frac{dt}{ds} = 0.0588(1.125s^{1.125-1}) \\ = 0.06615s^{0.125}$$

When $s = 100$, $\frac{dt}{ds} \approx 0.118$ sec/m.

At 100 meters, the fastest possible time increases by 0.118 seconds for each additional meter.

(c) Yes, they have been surpassed. In 2000, the world record in the mile stood at 3:43.13. (Ref:www.runnersworld.com)

$$66. \quad V = C(R_0 - R)R^2 \\ = CR_0R^2 - CR^3$$

$$\frac{dV}{dR} = 2CR_0R - 3CR^2 = 0$$

$$CR(2R_0 - 3R) = 0$$

$$CR = 0 \text{ or } 2R_0 - 3R = 0$$

$$R = 0 \quad R = \frac{2}{3}R_0$$

Discard $R = 0$, since a closed windpipe produces no airflow. Velocity is maximized when $R = \frac{2}{3}R_0$.

$$67. \quad \text{BMI} = \frac{703w}{h^2}$$

$$(a) \quad 6'8'' = 80 \text{ in.}$$

$$\text{BMI} = \frac{703(250)}{80^2} \approx 27.5$$

$$(b) \quad \text{BMI} = \frac{703w}{80^2} = 24.9 \text{ implies}$$

$$w = \frac{24.9(80)^2}{703} \approx 227.$$

A 250-lb person needs to lose 23 pounds to get down to 227 lbs.

$$(c) \quad \text{If } f(h) = \frac{703(125)}{h^2} = 87,875h^{-2}, \text{ then}$$

$$f'(h) = 87,875(-2h^{-2-1}) \\ = -175,750h^{-3} = -\frac{175,750}{h^3}$$

$$(d) \quad f'(65) = -\frac{175,750}{65^3} \approx -0.64$$

For a 125-lb female with a height of 65 in. ($5'5''$), the BMI decreases by 0.64 for each additional inch of height.

(e) Sample Chart

ht/wt	140	160	180	200
60	27	31	35	39
65	23	27	30	33
70	20	23	26	29
75	17	20	22	25

$$68. \quad s(t) = 11t^2 + 4t + 2$$

$$(a) \quad v(t) = s'(t) = 22t + 4$$

$$(b) \quad v(0) = 22(0) + 4 = 4$$

$$v(5) = 22(5) + 4 = 114$$

$$v(10) = 22(10) + 4 = 224$$

69. $s(t) = 18t^2 - 13t + 8$

(a) $v(t) = s'(t) = 18(2t) - 13 + 0$
 $= 36t - 13$

(b) $v(0) = 36(0) - 13 = -13$
 $v(5) = 36(5) - 13 = 167$
 $v(10) = 36(10) - 13 = 347$

70. $s(t) = 4t^3 + 8t^2 + t$

(a) $v(t) = s'(t) = 4(3t^2) + 8(2t) + 1$
 $= 12t^2 + 16t + 1$

(b) $v(0) = 12(0)^2 + 16(0) + 1 = 1$
 $v(5) = 12(5)^2 + 16(5) + 1$
 $= 300 + 80 + 1 = 381$
 $v(10) = 12(10)^2 + 16(10) + 1$
 $= 1200 + 160 + 1 = 1361$

71. $s(t) = -3t^3 + 4t^2 - 10t + 5$

(a) $v(t) = s'(t) = -3(3t^2) + 4(2t) - 10 + 0$
 $= -9t^2 + 8t - 10$

(b) $v(0) = -9(0)^2 + 8(0) - 10 = -10$
 $v(5) = -9(5)^2 + 8(5) - 10$
 $= -225 + 40 - 10 = -195$
 $v(10) = -9(10)^2 + 8(10) - 10$
 $= -900 + 80 - 10 = -830$

72. $s(t) = -16t^2 + 144$

velocity $= s'(t) = -32t$

(a) $s'(1) = -32 \cdot 1 = -32$
 $s'(2) = -32 \cdot 2 = -64$

The rock's velocity is -32 ft/sec after 1 second and -64 ft/sec after 2 seconds.

(b) The rock will hit the ground when $s(t) = 0$.

$$-16t^2 + 144 = 0$$

$$t = \sqrt{\frac{144}{16}} = 3$$

The rock will hit the ground after 3 seconds.

(c) The velocity at impact is the velocity at 3 seconds.

$$s'(3) = -32 \cdot 3 = -96$$

Its velocity on impact is -96 ft/sec.

73. $s(t) = -16t^2 + 64t$

(a) $v(t) = s'(t) = -16(2t) + 64 = -32t + 64$
 $v(2) = -32(2) + 64 = -64 + 64 = 0$
 $v(3) = -32(3) + 64 = -96 + 64 = -32$

The ball's velocity is 0 ft/sec after 2 seconds and -32 ft/sec after 3 seconds.

(b) As the ball travels upward, its speed decreases because of the force of gravity until, at maximum height, its speed is 0 ft/sec.

In part (a), we found that $v(2) = 0$.

It takes 2 seconds for the ball to reach its maximum height.

(c) $s(2) = -16(2)^2 + 64(2)$
 $= -16(4) + 128$
 $= -64 + 128$
 $= 64$

It will go 64 ft high.

74. (a) $d(x) = 1.66 - 0.90x + 0.47x^2$

$$d(0.5) = 1.66 - 0.90(0.5) + 0.47(0.5)^2$$

$$= 1.3275 \text{ g/cm}^3$$

(b) $d'(x) = -0.90 + 0.47(2x)$
 $= -0.90 + 0.94x$
 $d'(0.5) = -0.90 + 0.94(0.5)$
 $= -0.43 \text{ g/cm}^3$

When the level of the Dead Sea decreases to 50% of the current level, the density of the brine is decreasing at the rate of 0.43 gram per cm^3 .

75. $y_1 = 4.13x + 14.63$

$$y_2 = -0.033x^2 + 4.647x + 13.347$$

(a) When $x = 5$, $y_1 \approx 35$ and $y_2 \approx 36$.

(b) $\frac{dy_1}{dx} = 4.13$
 $\frac{dy_2}{dx} = 0.033(2x) + 4.647$
 $= -0.066x + 4.647$

When $x = 5$, $\frac{dy_1}{dx} = 4.13$ and $\frac{dy_2}{dx} \approx 4.32$.

These values are fairly close and represent the rate of change of four years for a dog for one year of a human, for a dog that is actually 5 years old.

- (c) With the first two points eliminated, the dog age increases in 2-year steps and the human age increases in 8-year steps, for a slope of 4. The equation has the form $y = 4x + b$.

A value of 16 for b makes the numbers come out right. $y = 4x + b$. For a dog of age $x = 5$ years or more, the equivalent human age is given by $y = 4x + 16$.

4.2 Derivatives of Products and Quotients

Your Turn 1

$$\begin{aligned} y &= (x^3 + 7)(4 - x^2) \\ \frac{dy}{dx} &= (x^3 + 7)(-2x) + (4 - x^2)(3x^2) \\ &= -2x^4 - 14x + 12x^2 - 3x^4 \\ &= -5x^4 + 12x^2 - 14x \end{aligned}$$

Your Turn 2

$$\begin{aligned} f(x) &= \frac{3x + 2}{5 - 2x} \\ f'(x) &= \frac{(5 - 2x)(3) - (3x + 2)(-2)}{(5 - 2x)^2} \\ &= \frac{19}{(5 - 2x)^2} \end{aligned}$$

Your Turn 3

$$\begin{aligned} D_x \left[\frac{(5x - 3)(2x + 7)}{3x + 7} \right] &= \frac{[(3x + 7)D_x[(5x - 3)(2x + 7)] - [(5x - 3)(2x + 7)]D_x(3x + 7)]}{(3x + 7)^2} \\ &= \frac{[(3x + 7)[(5x - 3)(2) + (2x + 7)(5)] - (10x^2 + 29x - 21)(3)}{(3x + 7)^2} \\ &= \frac{[(3x + 7)(10x - 6 + 10x + 35) - (30x^2 + 87x - 63)]}{(3x + 7)^2} \\ &= \frac{60x^2 + 227x + 203 - 30x^2 - 87x + 63}{(3x + 7)^2} \\ &= \frac{30x^2 + 140x + 266}{(3x + 7)^2} \end{aligned}$$

Your Turn 4

$$C(x) = \frac{4x + 50}{x + 2}$$

The marginal average cost is

$$\begin{aligned} \frac{d}{dx} \left[\frac{C(x)}{x} \right] &= \frac{d}{dx} \left(\frac{4x + 50}{x^2 + 2x} \right) \\ &= \frac{(x^2 + 2x)(4) - (4x + 50)(2x + 2)}{(x^2 + 2x)^2} \\ &= \frac{4x^2 + 8x - (8x^2 + 108x + 100)}{(x^2 + 2x)^2} \\ &= \frac{4x^2 + 8x - 8x^2 - 108x - 100}{(x^2 + 2x)^2} \\ &= \frac{-4x^2 - 100x - 100}{(x^2 + 2x)^2} \end{aligned}$$

4.2 Exercises

- $y = (3x^2 + 2)(2x - 1)$
 $\frac{dy}{dx} = (3x^2 + 2)(2) + (2x - 1)(6x)$
 $= 6x^2 + 4 + 12x^2 - 6x$
 $= 18x^2 - 6x + 4$
- $y = (5x^2 - 1)(4x + 3)$
 $\frac{dy}{dx} = (5x^2 - 1)(4) + (10x)(4x + 3)$
 $= 20x^2 - 4 + 40x^2 + 30x$
 $= 60x^2 + 30x - 4$
- $y = (2x - 5)^2$
 $= (2x - 5)(2x - 5)$
 $\frac{dy}{dx} = (2x - 5)(2) + (2x - 5)(2)$
 $= 4x - 10 + 4x - 10$
 $= 8x - 20$
- $y = (7x - 6)^2$
 $= (7x - 6)(7x - 6)$
 $\frac{dy}{dx} = (7x - 6)(7) + (7)(7x - 6)$
 $= 49x - 42 + 49x - 42$
 $= 98x - 84$

$$\begin{aligned}
 5. \quad k(t) &= (t^2 - 1)^2 = (t^2 - 1)(t^2 - 1) \\
 k'(t) &= (t^2 - 1)(2t) + (t^2 - 1)(2t) \\
 &= 2t^3 - 2t + 2t^3 - 2t \\
 &= 4t^3 - 4t
 \end{aligned}$$

$$\begin{aligned}
 6. \quad g(t) &= (3t^2 + 2)^2 \\
 &= (3t^2 + 2)(3t^2 + 2) \\
 g'(t) &= (3t^2 + 2)(6t) + (6t)(3t^2 + 2) \\
 &= 18t^3 + 12t + 18t^3 + 12t \\
 &= 36t^3 + 24t
 \end{aligned}$$

$$\begin{aligned}
 7. \quad y &= (x + 1)(\sqrt{x} + 2) \\
 &= (x + 1)(x^{1/2} + 2) \\
 \frac{dy}{dx} &= (x + 1)\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 2)(1) \\
 &= \frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2} + x^{1/2} + 2 \\
 &= \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} + 2 \\
 &\text{or } \frac{3x^{1/2}}{2} + \frac{1}{2x^{1/2}} + 2
 \end{aligned}$$

$$\begin{aligned}
 8. \quad y &= (2x - 3)(\sqrt{x} - 1) \\
 &= (2x - 3)(x^{1/2} - 1)
 \end{aligned}$$

$$\begin{aligned}
 \frac{dy}{dx} &= (2x - 3)\left(\frac{1}{2}x^{-1/2}\right) + 2(x^{1/2} - 1) \\
 &= x^{1/2} - \frac{3}{2}x^{-1/2} + 2x^{1/2} - 2 \\
 &= 3x^{1/2} - \frac{3x^{-1/2}}{2} - 2 \\
 &\text{or } 3x^{1/2} - \frac{3}{2x^{1/2}} - 2
 \end{aligned}$$

$$\begin{aligned}
 9. \quad p(y) &= (y^{-1} + y^{-2})(2y^{-3} - 5y^{-4}) \\
 p'(y) &= (y^{-1} + y^{-2})(-6y^{-4} + 20y^{-5}) \\
 &\quad + (-y^2 - 2y^{-3})(2y^{-3} - 5y^{-4}) \\
 &= -6y^{-5} + 20y^{-6} - 6y^{-6} + 20y^{-7} \\
 &\quad - 2y^{-5} + 5y^{-6} - 4y^{-6} + 10y^{-7} \\
 &= -8y^{-5} + 15y^{-6} + 30y^{-7}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad q(x) &= (x^{-2} - x^{-3})(3x^{-1} + 4x^{-4}) \\
 q'(x) &= (x^{-2} - x^{-3})(-3x^{-2} - 16x^{-5}) \\
 &\quad + (-2x^{-3} + 3x^{-4})(3x^{-1} + 4x^{-4})
 \end{aligned}$$

$$\begin{aligned}
 q'(x) &= -3x^{-4} - 16x^{-7} + 3x^{-5} + 16x^{-8} \\
 &\quad - 6x^{-4} - 8x^{-7} + 9x^{-5} + 12x^{-8} \\
 q'(x) &= -9x^{-4} + 12x^{-5} - 24x^{-7} + 28x^{-8}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad f(x) &= \frac{6x + 1}{3x + 10} \\
 f'(x) &= \frac{(3x + 10)(6) - (6x + 1)(3)}{(3x + 10)^2} \\
 &= \frac{18x + 60 - 18x - 3}{(3x + 10)^2} \\
 &= \frac{57}{(3x + 10)^2}
 \end{aligned}$$

$$\begin{aligned}
 12. \quad f(x) &= \frac{8x - 11}{7x + 3} \\
 f'(x) &= \frac{(7x + 3)(8) - (8x - 11)(7)}{(7x + 3)^2} \\
 &= \frac{56x + 24 - 56x + 77}{(7x + 3)^2} \\
 &= \frac{101}{(7x + 3)^2}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad y &= \frac{5 - 3t}{4 + t} \\
 \frac{dy}{dx} &= \frac{(4 + t)(-3) - (5 - 3t)(1)}{(4 + t)^2} \\
 &= \frac{-12 - 3t - 5 + 3t}{(4 + t)^2} \\
 &= \frac{-17}{(4 + t)^2}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad y &= \frac{9 - 7t}{1 - t} \\
 \frac{dy}{dx} &= \frac{(1 - t)(-7) - (9 - 7t)(-1)}{(1 - t)^2} \\
 &= \frac{-7 + 7t + 9 - 7t}{(1 - t)^2} \\
 &= \frac{2}{(1 - t)^2}
 \end{aligned}$$

$$\begin{aligned}
 15. \quad y &= \frac{x^2 + x}{x - 1} \\
 \frac{dy}{dx} &= \frac{(x - 1)(2x + 1) - (x^2 + x)(1)}{(x - 1)^2} \\
 &= \frac{2x^2 + x - 2x - 1 - x^2 - x}{(x - 1)^2} \\
 &= \frac{x^2 - 2x - 1}{(x - 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad y &= \frac{x^2 - 4x}{x + 3} \\
 \frac{dy}{dx} &= \frac{(x + 3)(2x - 4) - (x^2 - 4x)(1)}{(x + 3)^2} \\
 &= \frac{2x^2 + 6x - 4x - 12 - x^2 + 4x}{(x + 3)^2} \\
 &= \frac{x^2 + 6x - 12}{(x + 3)^2}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad f(t) &= \frac{4t^2 + 11}{t^2 + 3} \\
 f'(t) &= \frac{(t^2 + 3)(8t) - (4t^2 + 11)(2t)}{(t^2 + 3)^2} \\
 &= \frac{8t^3 + 24t - 8t^3 - 22t}{(t^2 + 3)^2} \\
 &= \frac{2t}{(t^2 + 3)^2}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad y &= \frac{-x^2 + 8x}{4x^2 - 5} \\
 \frac{dy}{dx} &= \frac{(4x^2 - 5)(-2x + 8) - (-x^2 + 8x)(8x)}{(4x^2 - 5)^2} \\
 &= \frac{-8x^3 + 32x^2 + 10x - 40 + 8x^3 - 64x^2}{(4x^2 - 5)^2} \\
 &= \frac{-32x^2 + 10x - 40}{(4x^2 - 5)^2}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad g(x) &= \frac{x^2 - 4x + 2}{x^2 + 3} \\
 g'(x) &= \frac{(x^2 + 3)(2x - 4) - (x^2 - 4x + 2)(2x)}{(x^2 + 3)^2} \\
 &= \frac{2x^3 - 4x^2 + 6x - 12 - 2x^3 + 8x^2 - 4x}{(x^2 + 3)^2} \\
 &= \frac{4x^2 + 2x - 12}{(x^2 + 3)^2}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad k(x) &= \frac{x^2 + 7x - 2}{x^2 - 2} \\
 k'(x) &= \frac{(x^2 - 2)(2x + 7) - (x^2 + 7x - 2)(2x)}{(x^2 - 2)^2} \\
 &= \frac{2x^3 + 7x^2 - 4x - 14 - 2x^3 - 14x^2 + 4x}{(x^2 - 2)^2} \\
 &= \frac{-7x^2 - 14}{(x^2 - 2)^2}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad p(t) &= \frac{\sqrt{t}}{t - 1} \\
 &= \frac{t^{1/2}}{t - 1} \\
 p'(t) &= \frac{(t - 1)\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2}(1)}{(t - 1)^2} \\
 &= \frac{\frac{1}{2}t^{1/2} - \frac{1}{2}t^{-1/2} - t^{1/2}}{(t - 1)^2} \\
 &= \frac{-\frac{1}{2}t^{1/2} - \frac{1}{2}t^{-1/2}}{(t - 1)^2} \\
 &= \frac{-\frac{\sqrt{t}}{2} - \frac{1}{2\sqrt{t}}}{(t - 1)^2} \text{ or } \frac{-t - 1}{2\sqrt{t}(t - 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad r(t) &= \frac{\sqrt{t}}{2t+3} = \frac{t^{1/2}}{2t+3} \\
 r'(t) &= \frac{(2t+3)\left(\frac{1}{2}t^{-1/2}\right) - (t^{1/2})(2)}{(2t+3)^2} = \frac{t^{1/2} + \frac{3}{2}t^{-1/2} - 2t^{1/2}}{(2t+3)^2} = \frac{-t^{1/2} + \frac{3}{2t^{1/2}}}{(2t+3)^2} = \frac{-\sqrt{t} + \frac{3}{2\sqrt{t}}}{(2t+3)^2} \\
 &\text{or } \frac{-2t+3}{2\sqrt{t}(2t+3)^2}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad y &= \frac{5x+6}{\sqrt{x}} = \frac{5x+6}{x^{1/2}} = 5x^{1/2} + 6x^{-1/2} \\
 \frac{dy}{dx} &= \frac{5}{2}x^{-1/2} - 3x^{-3/2} \text{ or } \frac{5x-6}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad y &= \frac{4x-3}{\sqrt{x}} = \frac{4x-3}{x^{1/2}} = 4x^{1/2} - 3x^{-1/2} \\
 \frac{dy}{dx} &= 2x^{-1/2} + \frac{3}{2}x^{-3/2} \text{ or } \frac{4x+3}{2x\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad h(z) &= \frac{z^{2.2}}{z^{3.2} + 5} \\
 h'(z) &= \frac{(z^{3.2} + 5)(2.2z^{1.2}) - z^{2.2}(3.2z^{2.2})}{(z^{3.2} + 5)^2} = \frac{2.2z^{4.4} + 11z^{1.2} - 3.2z^{4.4}}{(z^{3.2} + 5)^2} = \frac{-z^{4.4} + 11z^{1.2}}{(z^{3.2} + 5)^2}
 \end{aligned}$$

$$\begin{aligned}
 26. \quad g(y) &= \frac{y^{1.4} + 1}{y^{2.5} + 2} \\
 g'(y) &= \frac{(y^{2.5} + 2)(1.4y^{0.4}) - (y^{1.4} + 1)(2.5y^{1.5})}{(y^{2.5} + 2)^2} = \frac{1.4y^{2.9} + 2.8y^{0.4} - 2.5y^{2.9} - 2.5y^{1.5}}{(y^{2.5} + 2)^2} \\
 &= \frac{-1.1y^{2.9} - 2.5y^{1.5} + 2.8y^{0.4}}{(y^{2.5} + 2)^2}
 \end{aligned}$$

$$\begin{aligned}
 27. \quad f(x) &= \frac{(3x^2 + 1)(2x - 1)}{5x + 4} \\
 f'(x) &= \frac{(5x + 4)[(3x^2 + 1)(2) + (6x)(2x - 1)] - (3x^2 + 1)(2x - 1)(5)}{(5x + 4)^2} \\
 &= \frac{(5x + 4)(18x^2 - 6x + 2) - (3x^2 + 1)(10x - 5)}{(5x + 4)^2} \\
 &= \frac{90x^3 - 30x^2 + 10x + 72x^2 - 24x + 8 - 30x^3 + 15x^2 - 10x + 5}{(5x + 4)^2} \\
 &= \frac{60x^3 + 57x^2 - 24x + 13}{(5x + 4)^2}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad g(x) &= \frac{(2x^2 + 3)(5x + 2)}{6x - 7} \\
 g'(x) &= \frac{(6x - 7)[(2x^2 + 3)(5) + (4x)(5x + 2)] - (2x^2 + 3)(5x + 2)(6)}{(6x - 7)^2} \\
 &= \frac{(6x - 7)(30x^2 + 8x + 15) - (2x^2 + 3)(30x + 12)}{(6x - 7)^2} \\
 &= \frac{180x^3 + 48x^2 + 90x - 210x^2 - 56x - 105 - 60x^3 - 24x^2 - 90x - 36}{(6x - 7)^2} \\
 &= \frac{120x^3 - 186x^2 - 56x - 141}{(6x - 7)^2}
 \end{aligned}$$

$$\begin{aligned}
 29. \quad h(x) &= f(x)g(x) \\
 h'(x) &= f(x)g'(x) + g(x)f'(x) \\
 h'(3) &= f(3)g'(3) + g(3)f'(3) = 9(5) + 4(8) = 77
 \end{aligned}$$

$$\begin{aligned}
 30. \quad h(x) &= \frac{f(x)}{g(x)} \\
 h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \\
 h'(3) &= \frac{g(3)f'(3) - f(3)g'(3)}{[g(3)]^2} = \frac{4(8) - 9(5)}{4^2} = -\frac{13}{16}
 \end{aligned}$$

31. In the first step, the two terms in the numerator are reversed. The correct work follows.

$$\begin{aligned}
 D_x \left(\frac{2x + 5}{x^2 - 1} \right) &= \frac{(x^2 - 1)(2) - (2x + 5)(2x)}{(x^2 - 1)^2} \\
 &= \frac{2x^2 - 2 - 4x^2 - 10x}{(x^2 - 1)^2} \\
 &= \frac{-2x^2 - 10x - 2}{(x^2 - 1)^2}
 \end{aligned}$$

32. In the first step, the denominator, $(x^3)^2 = x^6$, was omitted. The correct work follows.

$$\begin{aligned}
 D_x \left(\frac{x^2 - 4}{x^3} \right) &= \frac{x^3(2x) - (x^2 - 4)(3x^2)}{(x^3)^2} \\
 &= \frac{2x^4 - 3x^4 + 12x^2}{x^6} = \frac{-x^4 + 12x^2}{x^6} \\
 &= \frac{x^2(-x^2 + 12)}{x^2(x^4)} = \frac{-x^2 + 12}{x^4}
 \end{aligned}$$

33. $f(x) = \frac{x}{x-2}$, at (3,3)

$$m = f'(x) = \frac{(x-2)(1) - x(1)}{(x-2)^2} = -\frac{2}{(x-2)^2}$$

At (3,3),

$$m = -\frac{2}{(3-2)^2} = -2,$$

Use the point-slope from.

$$\begin{aligned} y - 3 &= -2(x - 3) \\ y &= -2x + 9 \end{aligned}$$

34. $f(x) = (2x-1)(x+4)$, at (1, 5)

$$\begin{aligned} m = f'(x) &= (2x-1)(1) + (x+4)(2) \\ &= 2x-1 + 2x+8 = 4x+7 \end{aligned}$$

At (1,5), $m = 4(1) + 7 = 11$.

Use point-slope form.

$$\begin{aligned} y - 5 &= 11(x - 1) \\ y - 5 &= 11x - 11 \\ y &= 11x - 6 \end{aligned}$$

35. (a) $f(x) = \frac{3x^3 + 6}{x^{2/3}}$

$$f'(x) = \frac{(x^{2/3})(9x^2) - (3x^3 + 6)(\frac{2}{3}x^{-1/3})}{(x^{2/3})^2} = \frac{9x^{8/3} - 2x^{8/3} - 4x^{-1/3}}{x^{4/3}} = \frac{7x^{8/3} - \frac{4}{x^{1/3}}}{x^{4/3}} = \frac{7x^3 - 4}{x^{5/3}}$$

(b) $f(x) = 3x^{7/3} + 6x^{-2/3}$

$$f'(x) = 3\left(\frac{7}{3}x^{4/3}\right) + 6\left(-\frac{2}{3}x^{-5/3}\right) = 7x^{4/3} - 4x^{-5/3}$$

(c) The derivatives are equivalent.

36. $f'(x) = kg'(x) + g(x) \cdot 0 = kg'(x)$

The result is the same as applying the rule for differentiating a constant times a function

37. $f(x) = \frac{u(x)}{v(x)}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} = \lim_{h \rightarrow 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{v(x)[u(x+h) - u(x)] - u(x)[v(x+h) - v(x)]}{hv(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)} = \frac{v(x) \cdot u'(x) - u(x)v'(x)}{[v(x)]^2} \end{aligned}$$

38. Given $f(x) = \frac{u(x)}{v(x)}$, multiply both sides by $v(x)$ to obtain

$$f(x)v(x) = u(x).$$

Using the product rule, we have

$$f(x) \cdot v'(x) + v(x) \cdot f'(x) = u'(x).$$

Substitute $\frac{u(x)}{v(x)}$ for $f(x)$ and solve for $\left[\frac{u(x)}{v(x)}\right]'$.

$$\begin{aligned}\frac{u(x)}{v(x)} \cdot v'(x) + v(x) \left[\frac{u(x)}{v(x)}\right]' &= u'(x) \\ u(x) \cdot v'(x) + [v(x)]^2 \left[\frac{u(x)}{v(x)}\right]' &= v(x) \cdot u'(x) \\ [v(x)]^2 \left[\frac{u(x)}{v(x)}\right]' &= v(x) \cdot u'(x) - u(x) \cdot v'(x) \\ \left[\frac{u(x)}{v(x)}\right]' &= \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}\end{aligned}$$

39. Graph the numerical derivative of $f(x) = (x^2 - 2)(x^2 - \sqrt{2})$ for x ranging from -2 to 2 . The derivative crosses the x -axis at 0 and at approximately -1.307 and 1.307 .

40. Graph the numerical derivative of $f(x) = \frac{x-2}{x^2+4}$ for x ranging from -5 to 10 . The derivative crosses the x -axis at approximately -0.828 and 4.828 .

41. $C(x) = \frac{3x+2}{x+4}$

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{3x+2}{x^2+4x}$$

(a) $\bar{C}(10) = \frac{3(10)+2}{10^2+4(10)} = \frac{32}{140} \approx 0.2286$ hundreds of dollars or \$22.86 per unit

(b) $\bar{C}(20) = \frac{3(20)+2}{(20)^2+4(20)} = \frac{62}{480} \approx 0.1292$ hundreds of dollars or \$12.92 per unit

(c) $\bar{C}(x) = \frac{3x+2}{x^2+4x}$ per unit

(d) $\bar{C}'(x) = \frac{(x^2+4x)(3) - (3x+2)(2x+4)}{(x^2+4x)^2} = \frac{3x^2+12x-6x^2-12x-4x-8}{(x^2+4x)^2} = \frac{-3x^2-4x-8}{(x^2+4x)^2}$

42. $P(x) = \frac{5x - 6}{2x + 3}$

$$\bar{P}(x) = \frac{P(x)}{x} = \frac{5x - 6}{2x^2 + 3x}$$

(a) $\bar{P}(8) = \frac{5(8) - 6}{2(8)^2 + 3(8)} = \frac{34}{152} \approx 0.224$ tens of dollars or \$2.24 per book

(b) $\bar{P}(15) = \frac{5(15) - 6}{2(15)^2 + 3(15)} = \frac{69}{495} \approx 0.139$ tens of dollars or \$1.39 per book

(c) $\bar{P}(x) = \frac{5x - 6}{2x^2 + 3x}$ per book

(d)
$$\begin{aligned}\bar{P}'(x) &= \frac{(2x^2 + 3x)(5) - (5x - 6)(4x + 3)}{(2x^2 + 3x)^2} \\ &= \frac{10x^2 + 15x - 20x^2 - 15x + 24x + 18}{(2x^2 + 3x)^2} \\ &= \frac{-10x^2 + 24x + 18}{(2x^2 + 3x)^2}\end{aligned}$$

43. $M(d) = \frac{100d^2}{3d^2 + 10}$

(a) $M'(d) = \frac{(3d^2 + 10)(200d) - (100d^2)(6d)}{(3d^2 + 10)^2} = \frac{600d^3 + 2000d - 600d^3}{(3d^2 + 10)^2} = \frac{2000d}{(3d^2 + 10)^2}$

(b) $M'(2) = \frac{2000(2)}{[3(2)^2 + 10]^2} = \frac{4000}{484} \approx 8.3$

This means the new employee can assemble about 8.3 additional bicycles per day after 2 days of training.

$$M'(5) = \frac{2000(5)}{[3(5)^2 + 10]^2} = \frac{10,000}{7225} \approx 1.4$$

This means the new employee can assemble about 1.4 additional bicycles per day after 5 days of training.

44. The revenue $R(q)$ is given by $R(q) = pq = D(q)q$.

Using the product rule, the derivative of $R(q)$, which is the marginal revenue, is

$$R'(q) = D(q) \cdot 1 + qD'(q) = D(q) + qD'(q).$$

45. $\bar{C}(x) = \frac{C(x)}{x}$

Let $u(x) = C(x)$, with $u'(x) = C'(x)$

Let $v(x) = x$ with $v'(x) = 1$. Then, by the quotient rule,

$$\bar{C}(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} = \frac{x \cdot C'(x) - C(x) \cdot 1}{x^2} = \frac{xC'(x) - C(x)}{x^2}$$

46. Let $d(t)$ be the demand as a function of time and $p(t)$ be the price as a function of time.

Then $R(t) = d(t)p(t)$ is the revenue as a function of time. Let $t = t_1$ represent the beginning of the year.

$$R'(t) = d(t)p'(t) + d'(t)p(t)$$

$$R'(t_1) = d(t_1)p'(t_1) + d'(t_1)p(t_1) = (500)(0.5) + (30)(15) = 250 + 450 = 700$$

Revenue is increasing \$700 per month.

47. Let $C(t)$ be the cost as a function of time and $q(t)$ be the quantity as a function of time.

Then $\bar{C}(t) = \frac{C(t)}{q(t)}$ is the average cost as a function of time. Let $t = t_1$ represent last month.

$$\bar{C}'(t) = \frac{q(t)C'(t) - C(t)q'(t)}{[q(t)]^2}$$

$$\begin{aligned}\bar{C}'(t_1) &= \frac{q(t_1)C'(t_1) - C(t_1)q'(t_1)}{[q(t_1)]^2} \\ &= \frac{(12,500)(1200) - (27,000)(350)}{(12,500)^2} \\ &= 0.03552\end{aligned}$$

The average cost is increasing at a rate of \$0.03552 per gallon per month.

48. $s(x) = \frac{x}{m + nx}$; m and n constants

$$\begin{aligned}\text{(a)} \quad s'(x) &= \frac{(m + nx)1 - x(n)}{(m + nx)^2} \\ &= \frac{m + nx - nx}{(m + nx)^2} \\ &= \frac{m}{(m + nx)^2}\end{aligned}$$

- (b) $x = 50, m = 10, n = 3$

$$\begin{aligned}s'(50) &= \frac{m}{(m + 50n)^2} \\ &= \frac{10}{[10 + 50(3)]^2} \\ &= \frac{1}{2560} \\ &\approx 0.000391 \text{ mm per ml}\end{aligned}$$

49. $f(x) = \frac{Kx}{A + x}$

$$\begin{aligned}\text{(a)} \quad f'(x) &= \frac{(A + x)K - Kx(1)}{(A + x)^2} \\ f'(x) &= \frac{AK}{(A + x)^2}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad f'(A) &= \frac{AK}{(A + A)^2} \\ &= \frac{AK}{4A^2} = \frac{K}{4A}\end{aligned}$$

50. $N(t) = 3t(t - 10)^2 + 40$

$$\begin{aligned}\text{(a)} \quad N(t) &= 3t(t^2 - 20t + 100) + 40 \\ &= 3t^3 - 60t^2 + 300t + 40 \\ N'(t) &= 9t^2 - 120t + 300\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad N'(8) &= 9(8)^2 - 120(8) + 300 \\ &= 9(64) - 960 + 300 \\ &= -84\end{aligned}$$

The rate of change is -84 million per hour.

$$\begin{aligned}\text{(c)} \quad N'(11) &= 9(11)^2 - 120(11) + 300 \\ &= 9(121) - 1320 + 300 \\ &= 69\end{aligned}$$

The rate of change is 69 million per hour.

- (d) The population first declines, and then increases.

$$51. \quad R(w) = \frac{30(w-4)}{w-1.5}$$

$$(a) \quad R(5) = \frac{30(5-4)}{5-1.5} \\ \approx 8.57 \text{ min}$$

$$(b) \quad R(7) = \frac{30(7-4)}{7-1.5} \\ \approx 16.36 \text{ min}$$

(c)

$$\begin{aligned} R'(w) &= \frac{(w-1.5)(30) - 30(w-4)(1)}{(w-1.5)^2} \\ &= \frac{30w - 45 - 30w + 120}{(w-1.5)^2} \\ &= \frac{75}{(w-1.5)^2} \\ R'(5) &= \frac{75}{(5-1.5)^2} \approx 6.12 \frac{\text{min}^2}{\text{kcal}} \\ R'(7) &= \frac{75}{(7-1.5)^2} \approx 2.48 \frac{\text{min}^2}{\text{kcal}} \end{aligned}$$

$$52. \quad W = \left(1 + \frac{20}{H-0.93}\right)H = H + \frac{20H}{H-0.93}$$

(a)

$$\begin{aligned} \frac{dW}{dH} &= 1 + \frac{(H-0.93)(20) - 20H(1)}{(H-0.93)^2} \\ &= \frac{(H-0.93)^2}{(H-0.93)^2} + \frac{20H - 18.6 - 20H}{(H-0.93)^2} \\ &= \frac{H^2 - 1.86H + 0.8649}{(H-0.93)^2} + \frac{-18.6}{(H-0.93)^2} \\ &= \frac{H^2 - 1.86H - 17.7351}{(H-0.93)^2} \end{aligned}$$

(b)

$$\frac{dW}{dH} = 0 \text{ when } H^2 - 1.86H - 17.7351 = 0$$

By the quadratic formula,

$$\begin{aligned} H &= \frac{1.86 \pm \sqrt{1.86^2 + 4(17.7351)}}{2} \\ &\approx -3.38 \quad \text{or} \quad 5.24 \end{aligned}$$

Discarding the negative solution, W is minimized at $H = 5.24$ m.

(c) Crows apply optimal foraging techniques.

$$\begin{aligned} 53. \quad f(t) &= \frac{90t}{99t-90} \\ f'(t) &= \frac{(99t-90)(90) - (90t)(99)}{(99t-90)^2} \\ &= \frac{-8100}{(99t-90)^2} \end{aligned}$$

$$(a) \quad f'(1) = \frac{-8100}{(99-90)^2} = \frac{-8100}{9^2} \\ = \frac{-8100}{81} = -100 \text{ facts/hr}$$

$$\begin{aligned} (b) \quad f'(10) &= \frac{-8100}{[99(10)-90]^2} = \frac{-8100}{(900)^2} \\ &= \frac{-8100}{810,000} \\ &= -\frac{1}{100} \text{ or } -0.01 \text{ facts/hr} \end{aligned}$$

$$\begin{aligned} 54. \quad f(x) &= \frac{x^2}{2(1-x)} \\ f'(x) &= \frac{2(1-x)(2x) - x^2(-2)}{[2(1-x)]^2} \\ &= \frac{4x - 4x^2 + 2x^2}{4(1-x)^2} \\ &= \frac{4x - 2x^2}{4(1-x)^2} \\ &= \frac{2x(2-x)}{4(1-x)^2} \\ &= \frac{x(2-x)}{2(1-x)^2} \end{aligned}$$

$$(a) \quad f'(0.1) = \frac{0.1(2-0.1)}{2(1-0.1)^2} \approx 0.1173$$

$$(b) \quad f'(0.6) = \frac{0.6(2-0.6)}{2(1-0.6)^2} = 2.625$$

4.3 The Chain Rule

Your Turn 1

Let $f(x) = 2x - 1$ and $g(x) = \sqrt{3x + 5}$.

$$g(0) = \sqrt{3 \cdot 0 + 5} = \sqrt{5}$$

$$f[g(0)] = 2\sqrt{5} - 1$$

$$f(0) = 2 \cdot 0 - 1 = -1$$

$$g[f(0)] = \sqrt{3(-1) + 5} = \sqrt{2}$$

Your Turn 2

Let $f(x) = 2x - 3$ and $g(x) = x^2 + 1$.

$$\begin{aligned} g[f(x)] &= g(2x - 3) \\ &= (2x - 3)^2 + 1 \\ &= 4x^2 - 12x + 9 + 1 \\ &= 4x^2 - 12x + 10 \end{aligned}$$

Your Turn 3

Write $h(x) = (2x - 3)^3$ in the form $h(x) = f[g(x)]$.

One possible answer is $h(x) = f[g(x)]$

where $g(x) = 2x - 3$ and $f(x) = x^3$.

Your Turn 4

$$\begin{aligned} y &= (5x^2 - 6x)^{-2} \\ \frac{dy}{dx} &= -2(5x^2 - 6x)^{-3} \cdot (10x - 6) \\ &= \frac{-2(10x - 6)}{(5x^2 - 6x)^3} = \frac{-20x + 12}{(5x^2 - 6x)^3} \end{aligned}$$

Your Turn 5

$$D_x[(x^2 - 7)^{10}] = 10(x^2 - 7)^9(2x) = 20x(x^2 - 7)^9$$

Your Turn 6

$$\begin{aligned} y &= x^2(5x - 1)^3 \\ \frac{dy}{dx} &= x^2 \cdot 3(5x - 1)^2(5) + (5x - 1)^3 \cdot 2x \\ &= 15x^2(5x - 1)^2 + 2x(5x - 1)^3 \\ &= x(5x - 1)^2[15x + 2(5x - 1)] \\ &= x(5x - 1)^2(25x - 2) \end{aligned}$$

Your Turn 7

$$\begin{aligned} D_x \left[\frac{(4x - 1)^3}{x + 3} \right] &= \frac{(x + 3)[3(4x - 1)^2(4)] - (4x - 1)^3(1)}{(x + 3)^2} \\ &= \frac{12(x + 3)(4x - 1)^2 - (4x - 1)^3}{(x + 3)^2} \\ &= \frac{(4x - 1)^2[12(x + 3) - (4x - 1)]}{(x + 3)^2} \\ &= \frac{(4x - 1)^2(8x + 37)}{(x + 3)^2} \end{aligned}$$

4.3 Exercises

In Exercises 1 through 6, $f(x) = 5x^2 - 2x$ and $g(x) = 8x + 3$.

- $$\begin{aligned} g(2) &= 8(2) + 3 = 19 \\ f[g(2)] &= f[19] \\ &= 5(19)^2 - 2(19) \\ &= 1805 - 38 = 1767 \end{aligned}$$
- $$\begin{aligned} g(-5) &= 8(-5) + 3 = -37 \\ f[g(-37)] &= f[-37] \\ &= 5(-37)^2 - 2(-37) \\ &= 6845 + 74 = 6919 \end{aligned}$$
- $$\begin{aligned} f(2) &= 5(2)^2 - 2(2) \\ &= 20 - 4 = 16 \\ g[f(2)] &= g[16] \\ &= 8(16) + 3 \\ &= 128 + 3 = 131 \end{aligned}$$
- $$\begin{aligned} f(-5) &= 5(-5)^2 - 2(-5) \\ &= 125 + 10 = 135 \\ g[f(-5)] &= f[135] \\ &= 8(135) + 3 \\ &= 1080 + 3 = 1083 \end{aligned}$$
- $$\begin{aligned} g(k) &= 8k + 3 \\ f[g(k)] &= f[8k + 3] \\ &= 5(8k + 3)^2 - 2(8k + 3) \\ &= 5(64k^2 + 48k + 9) - 16k - 6 \\ &= 320k^2 + 224k + 39 \end{aligned}$$

$$\begin{aligned} 6. \quad f(5z) &= 5(5z)^2 - 2(5z) \\ &= 125z^2 - 10z \end{aligned}$$

$$\begin{aligned} g[f(5z)] &= g(125z^2 - 10z) \\ &= 8(125z^2 - 10z) + 3 \\ &= 1000z^2 - 80z + 3 \end{aligned}$$

$$7. \quad f(x) = \frac{x}{8} + 7; g(x) = 6x - 1$$

$$\begin{aligned} f[g(x)] &= \frac{6x - 1}{8} + 7 \\ &= \frac{6x - 1}{8} + \frac{56}{8} \\ &= \frac{6x + 55}{8} \end{aligned}$$

$$\begin{aligned} g[f(x)] &= 6\left[\frac{x}{8} + 7\right] - 1 \\ &= \frac{6x}{8} + 42 - 1 \\ &= \frac{3x}{4} + 41 \\ &= \frac{3x}{4} + \frac{164}{4} \\ &= \frac{3x + 164}{4} \end{aligned}$$

$$8. \quad f(x) = -8x + 9; g(x) = \frac{x}{5} + 4$$

$$\begin{aligned} f[g(x)] &= -8\left[\frac{x}{5} + 4\right] + 9 \\ &= \frac{-8x}{5} - 32 + 9 \\ &= \frac{-8x}{5} - 23 \\ &= \frac{-8x - 115}{5} \end{aligned}$$

$$\begin{aligned} g[f(x)] &= \frac{-8x + 9}{5} + 4 \\ &= \frac{-8x + 9}{5} + \frac{20}{5} \\ &= \frac{-8x + 29}{5} \end{aligned}$$

$$9. \quad f(x) = \frac{1}{x}; g(x) = x^2$$

$$f[g(x)] = \frac{1}{x^2}$$

$$\begin{aligned} g[f(x)] &= \left(\frac{1}{x}\right)^2 \\ &= \frac{1}{x^2} \end{aligned}$$

$$10. \quad f(x) = \frac{2}{x^4}; g(x) = 2 - x$$

$$f[g(x)] = \frac{2}{(2 - x)^4}$$

$$g[f(x)] = 2 - \left(\frac{2}{x^4}\right) = 2 - \frac{2}{x^4}$$

$$11. \quad f(x) = \sqrt{x + 2}; g(x) = 8x^2 - 6$$

$$\begin{aligned} f[g(x)] &= \sqrt{(8x^2 - 6) + 2} \\ &= \sqrt{8x^2 - 4} \end{aligned}$$

$$\begin{aligned} g[f(x)] &= 8(\sqrt{x + 2})^2 - 6 \\ &= 8x + 16 - 6 \\ &= 8x + 10 \end{aligned}$$

$$12. \quad f(x) = 9x^2 - 11x; g(x) = 2\sqrt{x + 2}$$

$$\begin{aligned} f[g(x)] &= 9(2\sqrt{x + 2})^2 - 11(2\sqrt{x + 2}) \\ &= 9[4(x + 2)] - 22\sqrt{x + 2} \\ &= 36(x + 2) - 22\sqrt{x + 2} \\ &= 36x + 72 - 22\sqrt{x + 2} \end{aligned}$$

$$\begin{aligned} g[f(x)] &= 2\sqrt{(9x^2 - 11x) + 2} \\ &= 2\sqrt{9x^2 - 11x + 2} \end{aligned}$$

$$13. \quad f(x) = \sqrt{x + 1}; g(x) = \frac{-1}{x}$$

$$f[g(x)] = \sqrt{\frac{-1}{x} + 1}$$

$$= \sqrt{\frac{x - 1}{x}}$$

$$g[f(x)] = \frac{-1}{\sqrt{x + 1}}$$

14. $f(x) = \frac{8}{x}; g(x) = \sqrt{3-x}$

$$f[g(x)] = \frac{8}{\sqrt{3-x}}$$

$$\begin{aligned} g[f(x)] &= \sqrt{3 - \frac{8}{x}} \\ &= \sqrt{\frac{3x-8}{x}} \end{aligned}$$

15. $y = (5 - x^2)^{3/5}$

If $f(x) = x^{3/5}$ and $g(x) = 5 - x^2$, then

$$y = f[g(x)] = (5 - x^2)^{3/5}.$$

16. $y = (3x^2 - 7)^{2/3}$

If $f(x) = x^{2/3}$ and $g(x) = 3x^2 - 7$, then

$$y = f[g(x)] = (3x^2 - 7)^{2/3}.$$

17. $y = -\sqrt{13 + 7x}$

If $f(x) = -\sqrt{x}$ and

$$g(x) = 13 + 7x,$$

then $y = f[g(x)] = -\sqrt{13 + 7x}$.

18. $y = \sqrt{9 - 4x}$

If $f(x) = \sqrt{x}$ and

$$g(x) = 9 - 4x,$$

then $y = f[g(x)] = \sqrt{9 - 4x}$.

19. $y = (x^2 + 5x)^{1/3} - 2(x^2 + 5x)^{2/3} + 7$

If $f(x) = x^{1/3} - 2x^{2/3} + 7$ and

$$g(x) = x^2 + 5x,$$

then

$$y = f[g(x)] = (x^2 + 5x)^{1/3} - 2(x^2 + 5x)^{2/3} + 7.$$

20. $y = (x^{1/2} - 3)^2 + (x^{1/2} - 3) + 5$

If $f(x) = x^2 + x + 5$ and

$$g(x) = x^{1/2} - 3, \text{ then}$$

$$\begin{aligned} y &= f[g(x)] \\ &= (x^{1/2} - 3)^2 + (x^{1/2} - 3) + 5. \end{aligned}$$

21. $y = (8x^4 - 5x^2 + 1)^4$

Let $f(x) = x^4$ and $g(x) = 8x^4 - 5x^2 + 1$.

Then $(8x^4 - 5x^2 + 1)^4 = f[g(x)]$.

Use the alternate form of the chain rule.

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$

$$f'(x) = 4x^3$$

$$f'[g(x)] = 4[g(x)]^3 = 4(8x^4 - 5x^2 + 1)^3$$

$$g'(x) = 32x^3 - 10x$$

$$\frac{dy}{dx} = 4(8x^4 - 5x^2 + 1)^3(32x^3 - 10x)$$

22. $y = (2x^3 + 9x)^5$

Let $f(x) = x^5$ and $g(x) = 2x^3 + 9x$.

Then $(2x^3 + 9x)^5 = f[g(x)]$.

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x)$$

$$f'(x) = 5x^4$$

$$f'[g(x)] = 5[g(x)]^4 = 5(2x^3 + 9x)^4$$

$$g'(x) = 6x^2 + 9$$

$$\frac{dy}{dx} = 5(2x^3 + 9x)^4(6x^2 + 9)$$

23. $k(x) = -2(12x^2 + 5)^{-6}$

Use the generalized power rule with

$u = 12x^2 + 5$, $n = -6$, and $u' = 24x$.

$$\begin{aligned} k'(x) &= -2[-6(12x^2 + 5)^{-6-1} \cdot 24x] \\ &= -2[-144x(12x^2 + 5)^{-7}] \\ &= 288x(12x^2 + 5)^{-7} \end{aligned}$$

24. $f(x) = -7(3x^4 + 2)^{-4}$

Use the generalized power rule with

$u = 3x^4 + 2$, $n = -4$ and $u' = 12x^3$.

$$\begin{aligned} f'(x) &= -7[-4(3x^4 + 2)^{-4-1} \cdot 12x^3] \\ &= -7[-48x^3(3x^4 + 2)^{-5}] \\ &= 336x^3(3x^4 + 2)^{-5} \end{aligned}$$

25. $s(t) = 45(3t^3 - 8)^{3/2}$

Use the generalized power rule with

$$u = 3t^3 - 8, n = \frac{3}{2}, \text{ and } u' = 9t^2.$$

$$\begin{aligned} s'(t) &= 45 \left[\frac{3}{2} (3t^3 - 8)^{1/2} \cdot 9t^2 \right] \\ &= 45 \left[\frac{27}{2} t^2 (3t^3 - 8)^{1/2} \right] \\ &= \frac{1215}{2} t^2 (3t^3 - 8)^{1/2} \end{aligned}$$

26. $s(t) = 12(2t^4 + 5)^{3/2}$

Use the generalized power rule with

$$u = 2t^4 + 5, n = \frac{3}{2}, \text{ and } u' = 8t^3.$$

$$\begin{aligned} s'(t) &= 12 \left[\frac{3}{2} (2t^4 + 5)^{1/2} \cdot 8t^3 \right] \\ &= 12 [12t^3 (2t^4 + 5)^{1/2}] \\ &= 144t^3 (2t^4 + 5)^{1/2} \end{aligned}$$

27. $g(t) = -3\sqrt{7t^3 - 1}$
 $= -3\sqrt{(7t^3 - 1)^{1/2}}$

Use generalized power rule with

$$u = 7t^3 - 1, n = \frac{1}{2}, \text{ and } u' = 21t^2.$$

$$\begin{aligned} g'(t) &= -3 \left[\frac{1}{2} (7t^3 - 1)^{-1/2} \cdot 21t^2 \right] \\ &= -3 \left[\frac{21}{2} t^2 (7t^3 - 1)^{-1/2} \right] \\ &= \frac{-63}{2} t^2 \cdot \frac{1}{(7t^3 - 1)^{1/2}} \\ &= \frac{-63t^2}{2\sqrt{7t^3 - 1}} \end{aligned}$$

28. $f(t) = 8\sqrt{4t^2 + 7}$
 $= 8(4t^2 + 7)^{1/2}$

Use generalized power rule with

$$u = 4t^2 + 7, n = \frac{1}{2}, \text{ and } u' = 8t.$$

$$\begin{aligned} f'(t) &= 8 \left[\frac{1}{2} (4t^2 + 7)^{-1/2} \cdot 8t \right] \\ &= 8 [4t(4t^2 + 7)^{-1/2}] \\ &= 32t(4t^2 + 7)^{-1/2} \\ &= \frac{32t}{(4t^2 + 7)^{1/2}} \\ &= \frac{32t}{\sqrt{4t^2 + 7}} \end{aligned}$$

29. $m(t) = -6t(5t^4 - 1)^4$

Use the product rule and the power rule.

$$\begin{aligned} m'(t) &= -6t[4(5t^4 - 1)^3 \cdot 20t^3] + (5t^4 - 1)^4(-6) \\ &= -480t^4(5t^4 - 1)^3 - 6(5t^4 - 1)^4 \\ &= -6(5t^4 - 1)^3[80t^4 + (5t^4 - 1)] \\ &= -6(5t^4 - 1)^3(85t^4 - 1) \end{aligned}$$

30. $r(t) = 4t(2t^5 + 3)^4$

Use the product rule and the power rule.

$$\begin{aligned} r'(t) &= 4t[4(2t^5 + 3)^3 \cdot 10t^4] + (2t^5 + 3)^4 \cdot 4 \\ &= 160t^5(2t^5 + 3)^3 + 4(2t^5 + 3)^4 \\ &= 4(2t^5 + 3)^3[40t^5 + (2t^5 + 3)] \\ &= 4(2t^5 + 3)^3(42t^5 + 3) \end{aligned}$$

31. $y = (3x^4 + 1)^4(x^3 + 4)$

Use the product rule and the power rule.

$$\begin{aligned} \frac{dy}{dx} &= (3x^4 + 1)^4(3x^2) + (x^3 + 4)[4(3x^4 + 1)^3 \cdot 12x^3] \\ &= 3x^2(3x^4 + 1)^4 + 48x^3(x^3 + 4)(3x^4 + 1)^3 \\ &= 3x^2(3x^4 + 1)^3[3x^4 + 1 + 16x(x^3 + 4)] \\ &= 3x^2(3x^4 + 1)^3(3x^4 + 1 + 16x^4 + 64x) \\ &= 3x^2(3x^4 + 1)^3(19x^4 + 64x + 1) \end{aligned}$$

32. $y = (x^3 + 2)(x^2 - 1)^4$

Use the product rule and the power rule.

$$\begin{aligned} \frac{dy}{dx} &= (x^3 + 2)[4(x^2 - 1)^3 \cdot 2x] + (x^2 - 1)^4(3x^2) \\ &= 8x(x^3 + 2)(x^2 - 1)^3 + 3x^2(x^2 - 1)^4 \\ &= x(x^2 - 1)^3[8(x^3 + 2) + 3x(x^2 - 1)] \\ &= x(x^2 - 1)^3(8x^3 + 16 + 3x^3 - 3x) \\ &= x(x^2 - 1)^3(11x^3 - 3x + 16) \end{aligned}$$

33. $q(y) = 4y^2(y^2 + 1)^{5/4}$

Use the product rule and the power rule.

$$\begin{aligned} q'(y) &= 4y^2 \cdot \frac{5}{4}(y^2 + 1)^{1/4}(2y) + 8y(y^2 + 1)^{5/4} \\ &= 10y^3(y^2 + 1)^{1/4} + 8y(y^2 + 1)^{5/4} \\ &= 2y(y^2 + 1)^{1/4}[5y^2 + 4(y^2 + 1)^{4/4}] \\ &= 2y(y^2 + 1)^{1/4}(9y^2 + 4) \end{aligned}$$

34. $p(z) = z(6z + 1)^{4/3}$

Use the product rule and the power rule.

$$\begin{aligned} p'(z) &= z \cdot \frac{4}{3}(6z + 1)^{1/3} \cdot 6 + 1 \cdot (6z + 1)^{4/3} \\ &= 8z(6z + 1)^{1/3} + (6z + 1)^{4/3} \\ &= (6z + 1)^{1/3}[8z + (6z + 1)^1] \\ &= (6z + 1)^{1/3}(14z + 1) \end{aligned}$$

35. $y = \frac{-5}{(2x^3 + 1)^2} = -5(2x^3 + 1)^{-2}$

$$\begin{aligned} \frac{dy}{dx} &= -5[-2(2x^3 + 1)^{-3} \cdot 6x^2] \\ &= -5[-12x^2(2x^3 + 1)^{-3}] \\ &= 60x^2(2x^3 + 1)^{-3} \\ &= \frac{60x^2}{(2x^3 + 1)^3} \end{aligned}$$

36. $y = \frac{1}{(3x^2 - 4)^5} = (3x^2 - 4)^{-5}$

$$\begin{aligned} \frac{dy}{dx} &= -5(3x^2 - 4)^{-6} \cdot 6x \\ &= -30x(3x^2 - 4)^{-6} \\ &= \frac{-30x}{(3x^2 - 4)^6} \end{aligned}$$

37. $r(t) = \frac{(5t - 6)^4}{3t^2 + 4}$

$$\begin{aligned} r'(t) &= \frac{(3t^2 + 4)[4(5t - 6)^3 \cdot 5] - (5t - 6)^4(6t)}{(3t^2 + 4)^2} \\ &= \frac{20(3t^2 + 4)(5t - 6)^3 - 6t(5t - 6)^4}{(3t^2 + 4)^2} \\ &= \frac{2(5t - 6)^3[10(3t^2 + 4) - 3t(5t - 6)]}{(3t^2 + 4)^2} \\ &= \frac{2(5t - 6)^3(30t^2 + 40 - 15t^2 + 18t)}{(3t^2 + 4)^2} \\ &= \frac{2(5t - 6)^3(15t^2 + 18t + 40)}{(3t^2 + 4)^2} \end{aligned}$$

38. $p(t) = \frac{(2t + 3)^3}{4t^2 - 1}$

$$\begin{aligned} p'(t) &= \frac{(4t^2 - 1)[3(2t + 3)^2 \cdot 2] - (2t + 3)^3(8t)}{(4t^2 - 1)^2} \\ &= \frac{6(4t^2 - 1)(2t + 3)^2 - 8t(2t + 3)^3}{(4t^2 - 1)^2} \\ &= \frac{(2t + 3)^2[6(4t^2 - 1) - 8t(2t + 3)]}{(4t^2 - 1)^2} \\ &= \frac{(2t + 3)^2[24t^2 - 6 - 16t^2 - 24t]}{(4t^2 - 1)^2} \\ &= \frac{(2t + 3)^2[8t^2 - 24t - 6]}{(4t^2 - 1)^2} \\ &= \frac{2(2t + 3)^2(4t^2 - 12t - 3)}{(4t^2 - 1)^2} \end{aligned}$$

39. $y = \frac{3x^2 - x}{(2x - 1)^5}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x - 1)^5(6x - 1) - (3x^2 - x)[5(2x - 1)^4 \cdot 2]}{[(2x - 1)^5]^2} \\ &= \frac{(2x - 1)^5(6x - 1) - 10(3x^2 - x)(2x - 1)^4}{(2x - 1)^{10}} \\ &= \frac{(2x - 1)^4[(2x - 1)(6x - 1) - 10(3x^2 - x)]}{(2x - 1)^{10}} \\ &= \frac{12x^2 - 2x - 6x + 1 - 30x^2 + 10x}{(2x - 1)^6} \\ &= \frac{-18x^2 + 2x + 1}{(2x - 1)^6} \end{aligned}$$

40.

$$\begin{aligned}
 y &= \frac{x^2 + 4x}{(3x^3 + 2)^4} \\
 \frac{dy}{dx} &= \frac{(3x^3 + 2)^4(2x + 4) - (x^2 + 4x)[4(3x^3 + 2)^3 \cdot 9x^2]}{[(3x^3 + 2)^4]^2} \\
 &= \frac{(3x^3 + 2)^4(2x + 4) - 36x^2(x^2 + 4x)(3x^3 + 2)^3}{(3x^3 + 2)^8} \\
 &= \frac{2(3x^3 + 2)^3[(3x^3 + 2)(x + 2) - 18x^2(x^2 + 4x)]}{(3x^3 + 2)^8} \\
 &= \frac{2(3x^4 + 6x^3 + 2x + 4 - 18x^4 - 72x^3)}{(3x^3 + 2)^5} \\
 &= \frac{-30x^4 - 132x^3 + 4x + 8}{(3x^3 + 2)^5}
 \end{aligned}$$

42. Let $f(x) = x^n$. Then $y = f(g(x)) = [g(x)]^n$.

Using the chain rule,

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

Then, using the power rule, $f'(x) = nx^{n-1}$.

$$\frac{dy}{dx} = n[g(x)]^{n-1} \cdot g'(x).$$

43. (a) $D_x(f[g(x)])$ at $x = 1$

$$= f'[g(1)] \cdot g'(1)$$

$$= f'(2) \cdot \left(\frac{2}{7}\right)$$

$$= -7\left(\frac{2}{7}\right) = -2$$

(b) $D_x(f[g(x)])$ at $x = 2$

$$= f'[g(2)] \cdot g'(2)$$

$$= f'(3) \cdot \left(\frac{3}{7}\right)$$

$$= -8\left(\frac{3}{7}\right) = -\frac{24}{7}$$

44. (a) $D_x(g[f(x)])$ at $x = 1$

$$= g'[f(1)] \cdot f'(1)$$

$$= g'(2) \cdot (-6)$$

$$= \frac{3}{7}(-6) = -\frac{18}{7}$$

(b) $D_x(g[f(x)])$ at $x = 2$

$$= g'[f(2)] \cdot f'(2)$$

$$= g'(4) \cdot (-7)$$

$$= \frac{5}{7}(-7) = -5$$

45. $f(x) = \sqrt{x^2 + 16}$; $x = 3$

$$f(x) = (x^2 + 16)^{1/2}$$

$$f'(x) = \frac{1}{2}(x^2 + 16)^{-1/2}(2x)$$

$$f'(x) = \frac{x}{\sqrt{x^2 + 16}}$$

$$f'(3) = \frac{3}{\sqrt{3^2 + 16}} = \frac{3}{5}$$

$$f(3) = \sqrt{3^2 + 16} = 5$$

We use $m = \frac{3}{5}$ and the point $P(3, 5)$ in the point-slope form

$$y - 5 = \frac{3}{5}(x - 3)$$

$$y - 5 = \frac{3}{5}x - \frac{9}{5}$$

$$y = \frac{3}{5}x + \frac{16}{5}$$

46. $f(x) = (x^3 + 7)^{2/3}$; $x = 1$

$$f'(x) = \frac{2}{3}(x^3 + 7)^{-1/3}(3x^2)$$

$$f'(x) = \frac{2x^2}{(x^3 + 7)^{1/3}}$$

$$f'(1) = \frac{2}{2} = 1$$

$$f(1) = 8^{2/3} = 4$$

We use $m = 1$ and the point $P(1, 4)$

$$y - 4 = 1(x - 1)$$

$$y = x + 3$$

$$\begin{aligned}
 47. \quad f(x) &= x(x^2 - 4x + 5)^4; \quad x = 2 \\
 f'(x) &= x \cdot 4(x^2 - 4x + 5)^3 \cdot (2x - 4) \\
 &\quad + 1 \cdot (x^2 - 4x + 5)^4 \\
 &= (x^2 - 4x + 5)^3 \\
 &\quad \cdot [4x(2x - 4) + (x^2 - 4x + 5)] \\
 &= (x^2 - 4x + 5)^3(9x^2 - 20x + 5) \\
 f'(2) &= (1)^3(1) = 1 \\
 f(2) &= 2(1)^4 = 2
 \end{aligned}$$

We use $m = 1$ and the point $P(2, 2)$.

$$\begin{aligned}
 y - 2 &= 1(x - 2) \\
 y - 2 &= x - 2 \\
 y &= x
 \end{aligned}$$

48.

$$\begin{aligned}
 f(x) &= x^2\sqrt{x^4 - 12}; \quad x = 2 \\
 f(x) &= x^2(x^4 - 12)^{1/2} \\
 f'(x) &= x^2 \cdot \frac{1}{2}(x^4 - 12)^{-1/2}(4x^3) + 2x(x^4 - 12)^{1/2} \\
 f'(x) &= \frac{2x^5}{(x^4 - 12)^{1/2}} + 2x(x^4 - 12)^{1/2} \\
 f'(2) &= \frac{64}{4^{1/2}} + 4(4)^{1/2} \\
 f'(2) &= 32 + 8 = 40 \\
 f(2) &= 4\sqrt{4} = 8
 \end{aligned}$$

We use $m = 40$ and the point $P(2, 8)$.

$$\begin{aligned}
 y - 8 &= 40(x - 2) \\
 y &= 40x - 72
 \end{aligned}$$

$$\begin{aligned}
 49. \quad f(x) &= \sqrt{x^3 - 6x^2 + 9x + 1} \\
 f(x) &= (x^3 - 6x^2 + 9x + 1)^{1/2} \\
 f'(x) &= \frac{1}{2}(x^3 - 6x^2 + 9x + 1)^{-1/2} \\
 &\quad \cdot (3x^2 - 12x + 9) \\
 f'(x) &= \frac{3(x^2 - 4x + 3)}{2\sqrt{x^3 - 6x^2 + 9x + 1}}
 \end{aligned}$$

If the tangent line is horizontal, its slope is zero and $f'(x) = 0$.

$$\begin{aligned}
 \frac{3(x^2 - 4x + 3)}{2\sqrt{x^3 - 6x^2 + 9x + 1}} &= 0 \\
 3(x^2 - 4x + 3) &= 0 \\
 3(x - 1)(x - 3) &= 0 \\
 x &= 1 \text{ or } x = 3
 \end{aligned}$$

The tangent line is horizontal $x = 1$ and $x = 3$.

$$\begin{aligned}
 50. \quad f(x) &= \frac{x}{(x^2 + 4)^4} \\
 f'(x) &= \frac{(x^2 + 4)^4 \cdot 1 - x \cdot 4(x^2 + 4)^3(2x)}{(x^2 + 4)^8} \\
 &= \frac{(x^2 + 4)^3[(x^2 + 4) - 8x^2]}{(x^2 + 4)^8} \\
 &= \frac{4 - 7x^2}{(x^2 + 4)^5}
 \end{aligned}$$

If the tangent line is horizontal, its slope is zero and $f'(x) = 0$.

$$\begin{aligned}
 \frac{4 - 7x^2}{(x^2 + 4)^5} &= 0 \\
 4 - 7x^2 &= 0 \\
 x^2 &= \frac{4}{7} \\
 x &= \pm \frac{2}{\sqrt{7}}
 \end{aligned}$$

The tangent line is horizontal at $x = \pm \frac{2}{\sqrt{7}}$.

$$53. \quad D(p) = \frac{-p^2}{100} + 500; \quad p(c) = 2c - 10$$

The demand in terms of the cost is

$$\begin{aligned}
 D(c) &= D[p(c)] \\
 &= \frac{-(2c - 10)^2}{100} + 500 \\
 &= \frac{-4(c - 5)^2}{100} + 500 \\
 &= \frac{-c^2 + 10c - 25}{25} + 500 \\
 &= \frac{-c^2 + 10c - 25 + 12,500}{25} \\
 &= \frac{-c^2 + 10c + 12,475}{25}
 \end{aligned}$$

54. $R(x) = 24(x^2 + x)^{2/3}$

$$R'(x) = 24 \left[\frac{2}{3} (x^2 + x)^{-1/3} \cdot (2x + 1) \right]$$

$$= 16(x^2 + x)^{-1/3} (2x + 1)$$

(a) $R'(100) = 16(100^2 + 100)^{-1/3} (2 \cdot 100 + 1)$
 ≈ 148.78

The marginal revenue is \$148.78.

(b) $R'(200) = 16(200^2 + 200)^{-1/3} (2 \cdot 200 + 1)$
 ≈ 187.29

The marginal revenue is \$187.29.

(c) $R'(300) = 16(300^2 + 300)^{-1/3} (2 \cdot 300 + 1)$
 ≈ 214.34

The marginal revenue is \$214.34.

(d) $\bar{R}(x) = \frac{R(x)}{x} = \frac{24(x^2 + x)^{2/3}}{x}$

(e)

$$\bar{R}'(x) = \frac{x \cdot 16(x^2 + x)^{-1/3} (2x + 1) - 24(x^2 + x)^{2/3} \cdot 1}{x^2}$$

$$= \frac{16x(x^2 + x)^{-1/3} (2x + 1) - 24(x^2 + x)^{2/3}}{x^2}$$

$$= \frac{16x(2x + 1) - 24(x^2 + x)}{x^2(x^2 + x)^{1/3}}$$

$$= \frac{32x^2 + 16x - 24x^2 - 24x}{x^2(x^2 + x)^{1/3}}$$

$$= \frac{8x^2 - 8x}{x^2(x^2 + x)^{1/3}}$$

$$= \frac{8(x - 1)}{x(x^2 + x)^{1/3}}$$

55. $A = 1500 \left(1 + \frac{r}{36,500} \right)^{1825}$

$\frac{dA}{dr}$ is the rate of change of A with respect to r .

$$\frac{dA}{dr} = 1500(1825) \left(1 + \frac{r}{36,500} \right)^{1824} \left(\frac{1}{36,500} \right)$$

$$= 75 \left(1 + \frac{r}{36,500} \right)^{1824}$$

(a) For $r = 6\%$,

$$\frac{dA}{dr} = 75 \left(1 + \frac{6}{36,500} \right)^{1824} = \$101.22.$$

(b) For $r = 8\%$,

$$\frac{dA}{dr} = 75 \left(1 + \frac{8}{36,500} \right)^{1824} = \$111.86.$$

(c) For $r = 9\%$,

$$\frac{dA}{dr} = 75 \left(1 + \frac{9}{36,500} \right)^{1824} = \$117.59.$$

56. $q = D(p) = 30 \left(5 - \frac{p}{\sqrt{p^2 + 1}} \right)$

$$= 150 - \frac{30p}{\sqrt{p^2 + 1}} = 150 - \frac{30p}{(p^2 + 1)^{1/2}}$$

$$\frac{dq}{dp} = 0 - \left[\frac{(p^2 + 1)^{1/2} D_p(30p) - (30p)(D_p(p^2 + 1)^{1/2})}{[(p^2 + 1)^{1/2}]^2} \right]$$

$$= - \left[\frac{(p^2 + 1)^{1/2} (30) - (30p) \left(\frac{1}{2} \right) (p^2 + 1)^{-1/2} (2p)}{(p^2 + 1)} \right]$$

$$= - \left[\frac{(p^2 + 1)^{1/2} (30) - (30p)(p)(p^2 + 1)^{-1/2}}{(p^2 + 1)} \right]$$

$$= - \left[\frac{-30(p^2 + 1)^{-1/2} [(p^2 + 1) - p^2]}{(p^2 + 1)} \right]$$

$$= - \frac{-30(p^2 + 1)^{-1/2} (1)}{(p^2 + 1)} = - \frac{30}{(p^2 + 1)^{3/2}}$$

57. $V = \frac{60,000}{1 + 0.3t + 0.1t^2}$

The rate of change of the value is

$$V'(t) = \frac{(1 + 0.3t + 0.1t^2)(0) - 60,000(0.3 + 0.2t)}{(1 + 0.3t + 0.1t^2)^2}$$

$$= \frac{-60,000(0.3 + 0.2t)}{(1 + 0.3t + 0.1t^2)^2}.$$

(a) 2 years after purchase, the rate of change in the value is

$$V'(2) = \frac{-60,000[0.3 + 0.2(2)]}{[1 + 0.3(2) + 0.1(2)^2]^2}$$

$$= \frac{-60,000(0.3 + 0.4)}{(1 + 0.6 + 0.4)^2}$$

$$= \frac{-42,000}{4}$$

$$= -\$10,500.$$

- (b) 4 years after purchase, the rate of change in the value is

$$\begin{aligned} V'(4) &= \frac{-60,000[0.3 + 0.2(4)]}{[1 + 0.3(4) + 0.1(4)^2]^2} \\ &= \frac{-66,000}{14.44} \\ &= -\$4570.64. \end{aligned}$$

58. $C = 2000q + 3500$

$$q = \sqrt{15,000 - 1.5p}$$

Solve for p .

$$q = \sqrt{15,000 - 1.5p}$$

$$q^2 = 15,000 - 1.5p$$

$$\frac{q^2 - 15,000}{-1.5} = p$$

$$\frac{q^2}{-1.5} + \frac{15,000}{1.5} = p$$

$$\frac{-2q^2}{3} + 10,000 = p$$

$$\begin{aligned} \text{(a)} \quad R(q) &= qp = q \left(\frac{-2q^2}{3} + 10,000 \right) \\ &= \frac{-2q^3}{3} + 10,000q \\ &= \frac{-2q^3 + 30,000q}{3} \\ &= \frac{30,000q - 2q^3}{3} \end{aligned}$$

(b)

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= \frac{30,000q - 2q^3}{3} - (2000q + 3500) \\ &= \left(\frac{-2q^3}{3} + 10,000q \right) - (2000q + 3500) \\ &= \frac{-2q^3}{3} + 8000q - 3500 \\ &= 8000q - \frac{2q^3}{3} - 3500 \end{aligned}$$

$P(q)$ is the profit function.

$$\begin{aligned} \text{(c)} \quad \frac{dP}{dq} &= D_q(8000q - \frac{2q^3}{3} - 3500) \\ &= 8000 - 2q^2 \end{aligned}$$

$\frac{dp}{dq}$ or $P'(q)$ gives the marginal profit.

- (d) If $p = \$5000$ and

$$\begin{aligned} \text{then } q &= \sqrt{15,000 - 1.5p}, \\ q &= \sqrt{15,000 - 1.5(5000)} \\ &= \sqrt{15,000 - 7500} = \sqrt{7500}. \end{aligned}$$

Thus, $q^2 = 7500$.

$$\begin{aligned} P'(q) &= 8000 - 2q^2 \\ &= 8000 - 2(7500) \\ &= 8000 - 15,000 = -7000 \end{aligned}$$

When the price is \$5000, the marginal profit is $-\$7000$.

59. $P(x) = 2x^2 + 1; x = f(a) = 3a + 2$

$$\begin{aligned} P[f(a)] &= 2(3a + 2)^2 + 1 \\ &= 2(9a^2 + 12a + 4) + 1 \\ &= 18a^2 + 24a + 9 \end{aligned}$$

$$\begin{aligned} \text{60. (a)} \quad A(r) &= \pi r^2 \\ r(t) &= t^2 \\ A[r(t)] &= \pi[t^2]^2 \\ &= \pi(t^4) \\ &= \pi t^4 \end{aligned}$$

This function represents the area of the oil slick as a function of time t after the beginning of the leak.

$$\begin{aligned} \text{(b)} \quad D_t A[r(t)] &= 4\pi t^3 \\ D_t A[r(100)] &= 4\pi(100)^3 \\ &= 4,000,000\pi \end{aligned}$$

At 100 minutes the area of the spill is changing at the rate of $4,000,000\pi \text{ ft}^2/\text{min}$.

$$\begin{aligned} \text{61. (a)} \quad r(t) &= 2t; A(r) = \pi r^2 \\ A[r(t)] &= \pi(2t)^2 \\ &= 4\pi t^2 \end{aligned}$$

$A = 4\pi t^2$ gives the area of the pollution in terms of the time since the pollutants were first emitted.

$$\begin{aligned} \text{(b)} \quad D_t A[r(t)] &= 8\pi t \\ D_t A[r(4)] &= 8\pi(4) = 32\pi \end{aligned}$$

At 12 P.M., the area of pollution is changing at the rate of $32\pi \text{ mi}^2/\text{hr}$.

$$\begin{aligned}
 62. \quad N(t) &= 2t(5t + 9)^{1/2} + 12 \\
 N'(t) &= (2t) \left[\frac{1}{2}(5t + 9)^{-1/2}(5) \right] \\
 &\quad + 2(5t + 9)^{1/2} + 0 \\
 &= 5t(5t + 9)^{-1/2} + 2(5t + 9)^{1/2} \\
 &= (5t + 9)^{-1/2}[5t + 2(5t + 9)] \\
 &= (5t + 9)^{-1/2}(15t + 18) \\
 &= \frac{15t + 18}{(5t + 9)^{1/2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(a)} \quad N'(0) &= \frac{15(0) + 18}{[5(0) + 9]^{1/2}} \\
 &= \frac{18}{9^{1/2}} = 6
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad N'\left(\frac{7}{5}\right) &= \frac{15\left(\frac{7}{5}\right) + 18}{[5\left(\frac{7}{5}\right) + 9]^{1/2}} \\
 &= \frac{21 + 18}{(7 + 9)^{1/2}} \\
 &= \frac{39}{(16)^{1/2}} \\
 &= \frac{39}{4} = 9.75
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad N'(8) &= \frac{15(8) + 18}{[5(8) + 9]^{1/2}} \\
 &= \frac{120 + 18}{(49)^{1/2}} \\
 &= \frac{138}{7} \approx 19.71
 \end{aligned}$$

$$\begin{aligned}
 63. \quad C(t) &= \frac{1}{2}(2t + 1)^{-1/2} \\
 C'(t) &= \frac{1}{2} \left(-\frac{1}{2} \right) (2t + 1)^{-3/2} (2) \\
 &= -\frac{1}{2}(2t + 1)^{-3/2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(a)} \quad C'(0) &= -\frac{1}{2}[2(0) + 1]^{-3/2} \\
 &= -\frac{1}{2} \\
 &= -0.5
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad C'(4) &= -\frac{1}{2}[2(4) + 1]^{-3/2} \\
 &= -\frac{1}{2}(9)^{-3/2} \\
 &= \frac{-1}{2} \cdot \frac{1}{(\sqrt{9})^3} \\
 &= -\frac{1}{54} \\
 &\approx -0.02
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad C'(7.5) &= -\frac{1}{2}[2(7.5) + 1]^{-3/2} \\
 &= -\frac{1}{2}(16)^{-3/2} \\
 &= -\frac{1}{2} \left(\frac{1}{(\sqrt{16})^3} \right) \\
 &= -\frac{1}{128} \\
 &\approx -0.008
 \end{aligned}$$

(d) C is always decreasing because

$$C' = -\frac{1}{2}(2t + 1)^{-3/2}$$

is always negative for $t \geq 0$.

(The amount of calcium in the bloodstream will continue to decrease over time.)

64. (a)

$$\begin{aligned}
 R(Q) &= Q \left(C - \frac{Q}{3} \right)^{1/2} \\
 R'(Q) &= Q \left[\frac{1}{2} \left(C - \frac{Q}{3} \right)^{-1/2} \left(-\frac{1}{3} \right) \right] + \left(C - \frac{Q}{3} \right)^{1/2} \quad (1) \\
 &= -\frac{1}{6} Q \left(C - \frac{Q}{3} \right)^{-1/2} + \left(C - \frac{Q}{3} \right)^{1/2} \\
 &= -\frac{Q}{6 \left(C - \frac{Q}{3} \right)^{1/2}} + \left(C - \frac{Q}{3} \right)^{1/2}
 \end{aligned}$$

$$\text{(b)} \quad R'(Q) = -\frac{Q}{6 \left(C - \frac{Q}{3} \right)^{1/2}} + \left(C - \frac{Q}{3} \right)^{1/2}$$

If $Q = 87$ and $C = 59$, then

$$\begin{aligned}
 R'(Q) &= \left(59 - \frac{87}{3}\right)^{1/2} - \frac{87}{6\left(59 - \frac{87}{3}\right)^{1/2}} \\
 &= (30)^{1/2} - \frac{87}{6(30)^{1/2}} \\
 &= 5.48 - \frac{87}{32.88} \\
 &= 5.48 - 2.65 \\
 &= 2.83.
 \end{aligned}$$

- (c) Because $R'(Q)$ is positive, the patient's sensitivity to the drug is increasing.

65. $V(r) = \frac{4}{3}\pi r^3$, $S(r) = 4\pi r^2$, $r(t) = 6 - \frac{3}{17}t$

(a) $r(t) = 0$ when $6 - \frac{3}{17}t = 0$;

$$t = \frac{17(6)}{3} = 34 \text{ min.}$$

(b) $\frac{dV}{dr} = 4\pi r^2$, $\frac{dS}{dr} = 8\pi r$, $\frac{dr}{dt} = -\frac{3}{17}$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = -\frac{12}{17}\pi r^2$$

$$= -\frac{12}{17}\pi \left(6 - \frac{3}{17}t\right)^2$$

$$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt} = -\frac{24}{17}\pi r$$

$$= -\frac{24}{17}\pi \left(6 - \frac{3}{17}t\right)$$

When $t = 17$,

$$\begin{aligned}
 \frac{dV}{dt} &= -\frac{12}{17}\pi \left[6 - \frac{3}{17}(17)\right]^2 \\
 &= -\frac{108}{17}\pi \text{ mm}^3/\text{min}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dS}{dt} &= -\frac{24}{17}\pi \left[6 - \frac{3}{17}(17)\right] \\
 &= -\frac{72}{17}\pi \text{ mm}^2/\text{min}
 \end{aligned}$$

At $t = 17$ minutes, the volume is decreasing by $\frac{108}{17}\pi \text{ mm}^3$ per minute and the surface area is decreasing by $\frac{72}{17}\pi \text{ mm}^2$ per minute.

66. (a) $y = ((x^2)^2)^2$

$$\begin{aligned}
 \frac{dy}{dx} &= 2((x^2)^2) \cdot \frac{d}{dx}((x^2)^2) \\
 &= 2x^4 \cdot 2(x^2) \cdot \frac{d}{dx}(x^2) \\
 &= 2x^4 \cdot 2x^2 \cdot 2x \\
 &= 8x^7
 \end{aligned}$$

(b) $y = ((x^2)^2)^2 = x^8$

$$\frac{dy}{dx} = 8x^{8-1} = 8x^7$$

67. (a) $y = ((x^3)^2)^2$

$$\begin{aligned}
 \frac{dy}{dx} &= 2((x^3)^2) \cdot \frac{d}{dx}((x^3)^2) \\
 &= 2x^6 \cdot 2(x^3) \cdot \frac{d}{dx}(x^3) \\
 &= 2x^6 \cdot 2x^3 \cdot 3x^2 \\
 &= 12x^{11}
 \end{aligned}$$

(b) $y = ((x^3)^2)^2 = x^{12}$

$$\frac{dy}{dx} = 12x^{12-1} = 12x^{11}$$

4.4 Derivatives of Exponential Functions

Your Turn 1

(a) $y = 4^{3x}$

$$\frac{dy}{dx} = (\ln 4) \cdot 4^{3x} \cdot 3 = 3(\ln 4)4^{3x}$$

(b) $y = e^{7x^3+5}$

$$\frac{dy}{dx} = e^{7x^3+5} \cdot 21x^2 = 21x^2 e^{7x^3+5}$$

Your Turn 2

$$\begin{aligned}
 y &= (x^2 + 1)^2 e^{2x} \\
 \frac{dy}{dx} &= (x^2 + 1) \cdot e^{2x} \cdot 2 + e^{2x} \cdot 2(x^2 + 1) \cdot 2x \\
 &= 2(x^2 + 1)e^{2x}(x^2 + 2x + 1) \\
 &= 2e^{2x}(x^2 + 1)(x + 1)^2
 \end{aligned}$$

Your Turn 3

$$f(x) = \frac{100}{5 + 2e^{-0.01x}}$$

$$f'(x) = \frac{(5 + 2e^{-0.01x}) \cdot 0 - 100 \cdot 2e^{-0.01x} \cdot (-0.01)}{(5 + 2e^{-0.01x})^2}$$

$$= \frac{2e^{-0.01x}}{(5 + 2e^{-0.01x})^2}$$

Your Turn 4

$$Q(t) = 100e^{-0.421t}$$

$$\frac{dQ}{dt} = 100 \cdot e^{-0.421t}(-0.421)$$

$$= -42.1e^{-0.421t}$$

After 2 years ($t = 2$), the rate of change of the quantity present is

$$\frac{dQ}{dt} = -42.1e^{-0.421(2)}$$

$$= -42.1e^{-0.842}$$

$$\approx -18.1 \text{ grams per year}$$

4.4 Exercises

1. $y = e^{4x}$

Let $g(x) = 4x$,

with $g'(x) = 4$.

$$\frac{dy}{dx} = 4e^{4x}$$

2. $y = e^{-2x}$

Let $g(x) = -2x$,

with $g'(x) = -2$,

$$\frac{dy}{dx} = -2e^{-2x}.$$

3. $y = -8e^{3x}$

$$\frac{dy}{dx} = -8(3e^{3x}) = -24e^{3x}$$

4. $y = 1.2e^{5x}$

$$\frac{dy}{dx} = 1.2(5e^{5x}) = 6e^{5x}$$

5. $y = -16e^{2x+1}$

$$g(x) = 2x + 1$$

$$g'(x) = 2$$

$$\frac{dy}{dx} = -16(2e^{2x+1}) = -32e^{2x+1}$$

6. $y = -4e^{-0.3x}$

$$\frac{dy}{dx} = -4(-0.3e^{-0.3x})$$

$$= 1.2e^{-0.3x}$$

7. $y = e^{x^2}$

$$g(x) = x^2$$

$$g'(x) = 2x$$

$$\frac{dy}{dx} = 2xe^{x^2}$$

8. $y = e^{-x^2}$

$$g(x) = -x^2$$

$$g'(x) = -2x$$

$$\frac{dy}{dx} = -2xe^{-x^2}$$

9. $y = 3e^{2x^2}$

$$g(x) = 2x^2$$

$$g'(x) = 4x$$

$$\frac{dy}{dx} = 3(4xe^{2x^2})$$

$$= 12xe^{2x^2}$$

10. $y = -5e^{4x^3}$

$$g(x) = 4x^3$$

$$g'(x) = 12x^2$$

$$\frac{dy}{dx} = (-5)(12x^2)e^{4x^3}$$

$$= -60x^2e^{4x^3}$$

11. $y = 4e^{2x^2-4}$

$$g(x) = 2x^2 - 4$$

$$g'(x) = 4x$$

$$\frac{dy}{dx} = 4[(4x)e^{2x^2-4}]$$

$$= 16xe^{2x^2-4}$$

$$\begin{aligned}
 12. \quad y &= -3e^{3x^2+5} \\
 g(x) &= 3x^2 + 5 \\
 g'(x) &= 6x \\
 y' &= (-3)(6x)e^{3x^2+5} \\
 &= -18xe^{3x^2+5}
 \end{aligned}$$

$$13. \quad y = xe^x$$

Use the product rule.

$$\frac{dy}{dx} = xe^x + e^x \cdot 1 = e^x(x + 1)$$

$$14. \quad y = x^2e^{-2x}$$

Use the product rule.

$$\begin{aligned}
 \frac{dy}{dx} &= x^2(-2e^{-2x}) + 2xe^{-2x} \\
 &= -2x^2e^{-2x} + 2xe^{-2x} \\
 &= 2x(1 - x)e^{-2x}
 \end{aligned}$$

$$15. \quad y = (x + 3)^2e^{4x}$$

Use the product rule.

$$\begin{aligned}
 \frac{dy}{dx} &= (x + 3)^2(4)e^{4x} + e^{4x} \cdot 2(x + 3) \\
 &= 4(x + 3)^2e^{4x} + 2(x + 3)e^{4x} \\
 &= 2(x + 3)e^{4x}[2(x + 3) + 1] \\
 &= 2(x + 3)(2x + 7)e^{4x}
 \end{aligned}$$

$$16. \quad y = (3x^3 - 4x)e^{-5x}$$

Use the product rule.

$$\begin{aligned}
 \frac{dy}{dx} &= (3x^3 - 4x)(-5e^{-5x}) + e^{-5x}(9x^2 - 4) \\
 &= (-15x^3 + 20x)e^{-5x} + (9x^2 - 4)e^{-5x} \\
 &= (-15x^3 + 9x^2 + 20x - 4)e^{-5x}
 \end{aligned}$$

$$17. \quad y = \frac{x^2}{e^x}$$

Use the quotient rule.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{e^x(2x) - x^2e^x}{(e^x)^2} \\
 &= \frac{xe^x(2 - x)}{e^{2x}} \\
 &= \frac{x(2 - x)}{e^x}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad y &= \frac{e^x}{2x + 1} \\
 \frac{dy}{dx} &= \frac{e^x(2x + 1) - (2)(e^x)}{(2x + 1)^2} \\
 &= \frac{e^x(2x + 1 - 2)}{(2x + 1)^2} \\
 &= \frac{e^x(2x - 1)}{(2x + 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad y &= \frac{e^x + e^{-x}}{x} \\
 \frac{dy}{dx} &= \frac{x(e^x - e^{-x}) - (e^x + e^{-x})(1)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad y &= \frac{e^x - e^{-x}}{x} \\
 \frac{dy}{dx} &= \frac{x(e^x - (-1)e^{-x}) - (e^x - e^{-x})(1)}{x^2} \\
 &= \frac{xe^x + xe^{-x} - e^x + e^{-x}}{x^2} \\
 &= \frac{e^x(x - 1) + e^{-x}(x + 1)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad p &= \frac{10,000}{9 + 4e^{-0.2t}} \\
 \frac{dp}{dt} &= \frac{(9 + 4e^{-0.2t}) \cdot 0 - 10,000[0 + 4(-0.2)e^{-0.2t}]}{(9 + 4e^{-0.2t})^2} \\
 &= \frac{8000e^{-0.2t}}{(9 + 4e^{-0.2t})^2}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad p &= \frac{500}{12 + 5e^{-0.5t}} \\
 \frac{dp}{dt} &= \frac{(12 + 5e^{-0.5t}) \cdot 0 - 500[0 + 5(0.5)e^{-0.5t}]}{(12 + 5e^{-0.5t})^2} \\
 &= \frac{1250e^{-0.5t}}{(12 + 5e^{-0.5t})^2}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad f(z) &= \left(2z + e^{-z^2}\right)^2 \\
 f'(z) &= 2\left(2z + e^{-z^2}\right)^1 \left(2 - 2ze^{-z^2}\right) \\
 &= 4\left(2z + e^{-z^2}\right)\left(1 - ze^{-z^2}\right)
 \end{aligned}$$

$$\begin{aligned}
 24. \quad f(t) &= \left(e^{t^2} + 5t\right)^3 \\
 f'(t) &= 3\left(e^{t^2} + 5t\right)^2 \cdot \left(e^{t^2} \cdot 2t + 5\right) \\
 &= 3\left(e^{t^2} + 5t\right)^2 \left(2te^{t^2} + t\right)
 \end{aligned}$$

$$\begin{aligned}
 25. \quad y &= 7^{3x+1} \\
 \text{Let } g(x) &= 3x + 1, \text{ with } g'(x) = 3. \text{ Then} \\
 \frac{dy}{dx} &= (\ln 7)(7^{3x+1}) \cdot 3 = 3(\ln 7)7^{3x+1}
 \end{aligned}$$

$$\begin{aligned}
 26. \quad y &= 4^{-5x+2} \\
 \text{Let } g(x) &= -5x + 2, \text{ with } g'(x) = -5. \text{ Then} \\
 \frac{dy}{dx} &= (\ln 4)(4^{-5x+2}) \cdot (-5) = -5(\ln 4)4^{-5x+2}
 \end{aligned}$$

$$\begin{aligned}
 27. \quad y &= 3 \cdot 4^{x^2+2} \\
 \text{Let } g(x) &= x^2 + 2, \text{ with } g'(x) = 2x. \text{ Then} \\
 \frac{dy}{dx} &= 3(\ln 4)4^{x^2+2} \cdot 2x = 6x(\ln 4)4^{x^2+2}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad y &= -10^{3x^2-4} \\
 \text{Let } g(x) &= 3x^2 - 4, \text{ with } g'(x) = 6x. \\
 \frac{dy}{dx} &= -(\ln 10)10^{3x^2-4} \cdot 6x \\
 &= -6x \left(10^{3x^2-4}\right) \ln 10
 \end{aligned}$$

$$\begin{aligned}
 29. \quad s &= 2 \cdot 3^{\sqrt{t}} \\
 \text{Let } g(t) &= \sqrt{t}, \text{ with } g'(t) = \frac{1}{2\sqrt{t}}. \text{ Then}
 \end{aligned}$$

$$\begin{aligned}
 \frac{ds}{dt} &= 2(\ln 3)3^{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} \\
 &= \frac{(\ln 3)3^{\sqrt{t}}}{\sqrt{t}}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad s &= 5 \cdot 2^{\sqrt{t-2}} \\
 \text{Let } g(t) &= t - 2, \text{ with } g'(t) = \frac{1}{2\sqrt{t-2}}. \text{ Then}
 \end{aligned}$$

$$\begin{aligned}
 \frac{ds}{dt} &= 5(\ln 2) \left(2^{\sqrt{t-2}}\right) \cdot \frac{1}{2\sqrt{t-2}} \\
 &= \frac{(5 \ln 2)2^{\sqrt{t-2}}}{2\sqrt{t-2}}
 \end{aligned}$$

$$31. \quad y = \frac{te^t + 2}{e^{2t} + 1}$$

Use the quotient rule and product rule.

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{(e^{2t} + 1)(te^t + e^t \cdot 1) - (te^t + 2)(2e^{2t})}{(e^{2t} + 1)^2} \\
 &= \frac{(e^{2t} + 1)(te^t + e^t) - (te^t + 2)(2e^{2t})}{(e^{2t} + 1)^2} \\
 &= \frac{te^{3t} + e^{3t} + te^t + e^t - 2te^{3t} - 4e^{2t}}{(e^{2t} + 1)^2} \\
 &= \frac{-te^{3t} + e^{3t} + te^t + e^t - 4e^{2t}}{(e^{2t} + 1)^2} \\
 &= \frac{(1-t)e^{3t} - 4e^{2t} + (1+t)e^t}{(e^{2t} + 1)^2}
 \end{aligned}$$

$$32. \quad y = \frac{t^2 e^{2t}}{t + e^{3t}}$$

Use the quotient rule and product rule.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(t + e^{3t})(2te^{2t} + t^2 \cdot 2e^{2t}) - t^2 e^{2t}(1 + 3e^{3t})}{(t + e^{3t})^2} \\
 &= \frac{(t + e^{3t})(2te^{2t} + 2t^2 e^{2t}) - t^2 e^{2t}(1 + 3e^{3t})}{(t + e^{3t})^2} \\
 &= \frac{(2t^2 e^{2t} + 2t^3 e^{2t} + 2te^{5t} + 2t^2 e^{5t}) - (t^2 e^{2t} + 3t^2 e^{5t})}{(t + e^{3t})^2} \\
 &= \frac{t^2 e^{2t} + 2t^3 e^{2t} + 2te^{5t} - t^2 e^{5t}}{(t + e^{3t})^2} \\
 &= \frac{(2t^3 + t^2)e^{2t} + (2t - t^2)e^{5t}}{(t + e^{3t})^2}
 \end{aligned}$$

33. $f(x) = e^{x\sqrt{3x+2}}$

Let $g(x) = x\sqrt{3x+2}$.

$$\begin{aligned} g'(x) &= 1 \cdot \sqrt{3x+2} + x \left(\frac{3}{2\sqrt{3x+2}} \right) \\ &= \sqrt{3x+2} + \frac{3x}{2\sqrt{3x+2}} \\ &= \frac{2(3x+2)}{2\sqrt{3x+2}} + \frac{3x}{2\sqrt{3x+2}} \\ &= \frac{9x+4}{2\sqrt{3x+2}} \\ f'(x) &= e^{x\sqrt{3x+2}} \cdot \left(\frac{9x+4}{2\sqrt{3x+2}} \right) \end{aligned}$$

34. $f(x) = e^{x^2/(x^3+2)}$

Let $g(x) = \frac{x^2}{x^3+2}$

$$\begin{aligned} g'(x) &= \frac{(x^3+2)(2x) - x^2(3x^2)}{(x^3+2)^2} = \frac{2x^4+4x-3x^4}{(x^3+2)^2} \\ &= \frac{4x-x^4}{(x^3+2)^2} = \frac{x(4-x^3)}{(x^3+2)^2} \\ f'(x) &= e^{x^2/(x^3+2)} \cdot \left[\frac{x(4-x^3)}{(x^3+2)^2} \right] \\ &= \frac{x(4-x^3)e^{x^2/(x^3+2)}}{(x^3+2)^2} \end{aligned}$$

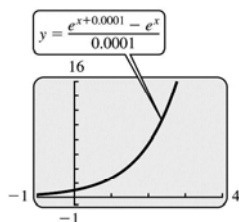
35. $y = y_0 e^{kt}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dt} [y_0 e^{kt}] = y_0 k e^{kt} \\ &= k(y_0 e^{kt}) \\ &= ky \end{aligned}$$

36. Graph

$$y = \frac{e^{x+0.0001} - e^x}{0.0001}$$

on a graphing calculator. A good choice for the viewing window is $[-1, 4]$ by $[-1, 16]$ with Xscl = 1, Yscl = 2.



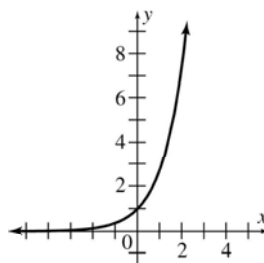
If we graph $y = e^x$ on the same screen, we see that the two graphs coincide. They are close enough to being identical that they are indistinguishable. By the definition of the derivative, if $f(x) = e^x$,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}, \end{aligned}$$

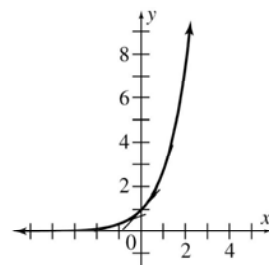
and $h = 0.0001$ is very close to 0.

Comparing the two graphs provides graphical evidence that $f'(x) = e^x$.

37. Graph the function $y = e^x$.



Sketch the lines tangent to the graph at $x = -1, 0, 1, 2$.



Estimate the slopes of the tangent lines at these points.

At $x = -1$ the slope is a little steeper than $\frac{1}{3}$ or approximately $0.\bar{3}$.

At $x = 0$ the slope is 1.

At $x = 1$ the slope is a little steeper than $\frac{5}{2}$ or 2.5.

At $x = 2$ the slope is a little steeper than $7\frac{1}{3}$ or $7.\bar{3}$.

Note that $e^{-1} \approx 0.36787944$, $e^0 = 1$,

$e^1 = e \approx 2.7182812$, and $e^2 \approx 7.3890561$.

The values are close enough to the slopes of the tangent lines to convince us that $\frac{de^x}{dx} = e^x$.

$$\begin{aligned}
 38. \quad S(t) &= 100 - 90e^{-0.3t} \\
 S'(t) &= -90(-0.3)e^{-0.3t} \\
 &= 27e^{-0.3t}
 \end{aligned}$$

$$\begin{aligned}
 (a) \quad S'(1) &= 27e^{-0.3(1)} \\
 &= 27e^{-0.3} \\
 &\approx 20
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad S'(5) &= 27e^{-0.3(5)} \\
 &= 27e^{-1.5} \\
 &\approx 6
 \end{aligned}$$

(c) As time goes on, the rate of change of sales is decreasing.

$$(d) \quad S'(t) = 27e^{-0.3t} \neq 0, \text{ but}$$

$$\lim_{t \rightarrow \infty} S'(t) = \lim_{t \rightarrow \infty} 27e^{-0.3t} = 0.$$

Although the rate of change of sales never equals zero, it gets closer and closer to zero as t increases.

$$\begin{aligned}
 39. \quad C(x) &= \sqrt{900 - 800 \cdot 1.1^{-x}} \\
 C(x) &= [900 - 800(1.1^{-x})]^{1/2} \\
 &= \frac{1}{2}[900 - 800(1.1^{-x})]^{-1/2} \\
 &\quad \cdot [-800(\ln 1.1)(1.1^{-x})(-1)] \\
 C'(x) &= \frac{(400 \ln 1.1)(1.1^{-x})}{\sqrt{900 - 800(1.1^{-x})}}
 \end{aligned}$$

$$(a) \quad C'(0) = \frac{400 \ln 1.1}{\sqrt{100}} \approx 3.81$$

The marginal cost is \$3.81.

$$(b) \quad C'(20) = \frac{(400 \ln 1.1)(1.1^{-20})}{\sqrt{900 - 800(1.1^{-20})}} \approx 0.20$$

The marginal cost is \$.20.

(c) As x becomes larger and larger, $C'(x)$ approaches zero.

$$\begin{aligned}
 40. \quad A(t) &= 10t^2 2^{-t} \\
 A'(t) &= 10t^2 (\ln 2) 2^{-t} (-1) + 20t 2^{-t} \\
 A'(t) &= 10t 2^{-t} (-t \ln 2 + 2)
 \end{aligned}$$

$$\begin{aligned}
 (a) \quad A'(2) &= 10(2)(2^{-2})(-2 \ln 2 + 2) \\
 &\approx 3.07
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad A'(4) &= 10(4)(2^{-4})(-4 \ln 2 + 2) \\
 &\approx -1.93
 \end{aligned}$$

(c) Public awareness increased at first and then decreased.

$$41. \quad y = 100e^{-0.03045t}$$

$$\begin{aligned}
 (a) \quad \text{For } t = 0, \\
 y &= 100e^{-0.03045(0)} \\
 &= 100e^0 \\
 &= 100\%.
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \text{For } t = 2, \\
 y &= 100e^{-0.03045(2)} \\
 &= 100e^{-0.0609} \\
 &\approx 94\%.
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad \text{For } t = 4, \\
 y &= 100e^{-0.03045(4)} \\
 &\approx 89\%.
 \end{aligned}$$

$$\begin{aligned}
 (d) \quad \text{For } t = 6, \\
 y' &= 100e^{-0.03045(6)} \\
 &\approx 83\%.
 \end{aligned}$$

$$\begin{aligned}
 (e) \quad y' &= 100(-0.03045)e^{-0.3045t} \\
 &= -3.045e^{-0.03045t} \\
 \text{For } t = 0, \\
 y' &= -3.045e^{-0.03045(0)} \\
 &= -3.045.
 \end{aligned}$$

$$\begin{aligned}
 (f) \quad \text{For } t = 2, \\
 y' &= -3.045e^{-0.03045(2)} \\
 &\approx -2.865.
 \end{aligned}$$

(g) The percent of these cars on the road is decreasing, but at a slower rate as they age.

$$42. \quad S(t) = 5000e^{0.1(e^{0.25t})}$$

Let $g(t) = 0.1e^{0.25t}$, with

$$g'(t) = 0.1(e^{0.25t})(0.25) = 0.025e^{0.25t}$$

$$\begin{aligned}
 S'(t) &= 5000e^{0.1(e^{0.25t})}(0.025e^{0.25t}) \\
 &= 125e^{0.1(e^{0.25t})}e^{0.25t} \\
 &= 125e^{0.1(e^{0.25t})+0.25t}
 \end{aligned}$$

$$\begin{aligned}
 S'(8) &= 125e^{0.1(e^{0.25 \cdot 8})+0.25(8)} \\
 &= 125e^{0.1(e^2)+2} \approx 1934
 \end{aligned}$$

The answer is b.

43. (a) $G_0 = 2$, $m = 250$, $k = 0.0018$

$$G(t) = \frac{250}{1 + \left(\frac{250}{2} - 1\right)e^{-0.0018(250)t}}$$

$$= \frac{250}{1 + 124e^{-0.45t}}$$

(b)

$$G'(t) = \frac{(1 + 124e^{-0.45t})(0) - 250(124e^{-0.45t})(-0.45)}{(1 + 124e^{-0.45t})^2}$$

$$= \frac{13,950e^{-0.45t}}{(1 + 124e^{-0.45t})^2}$$

1995 when $t = 5$:

$$G(5) = \frac{250}{1 + 124e^{-0.45(5)}} \approx 17.8$$

$$G'(5) = \frac{13,950e^{-0.45(5)}}{(1 + 124e^{-0.45(5)})^2} \approx 7.4$$

The number of Internet users in 1995 is about 17.8 million, and the growth rate is about 7.4 million users per year.

- (c) 2000 when $t = 10$:

$$G(10) = \frac{250}{1 + 124e^{-0.45(10)}} \approx 105.2$$

$$G'(10) = \frac{13,950e^{-0.45(10)}}{(1 + 124e^{-0.45(10)})^2} \approx 27.4$$

The number of Internet users in 2000 is about 105.2 million, and the growth rate is about 27.4 million users per year.

- (d) 2010 when $t = 20$:

$$G(20) = \frac{250}{1 + 124e^{-0.45(20)}} \approx 246.2$$

$$G'(20) = \frac{13,950e^{-0.45(20)}}{(1 + 124e^{-0.45(20)})^2} \approx 1.7$$

The number of Internet users in 2010 is about 246.2 million, and the growth rate is about 1.7 million users per year.

- (e) The rate of growth increases for a while and then gradually decreases to 0.

44. $A(t) = 3100e^{0.0166t}$

$$A'(t) = 3100e^{0.0166t}(0.0166)$$

$$= 51.46e^{0.0166t}$$

- (a) For 2010, $t = 50$:

$$A'(50) = 51.46e^{0.0166(50)} \approx 118$$

The instantaneous rate of growth is 118,000,000 people per year.

- (b) For 2015, $t = 55$:

$$A'(55) = 51.46e^{0.0166(55)} \approx 128$$

The instantaneous rate of growth will be 128,000,000 people per year.

45. $h(t) = 37.79(1.021)^t$

$$h'(t) = 37.79(\ln 1.021)(1.021)^t$$

$$= 0.789(1.021)^t$$

- (a) For 2015, $t = 15$:

$$h(15) = 37.79(1.021)^{15} \approx 51.6$$

The Hispanic population in the United States will be about 51,600,000 in 2015.

(b) $h'(15) = 0.789(\ln 1.021)(1.021)^{15} \approx 1.07$

The Hispanic population in the United States will be increasing at the rate of 1,070,000 people per year at the end of the year 2015.

46. $G(t) = \frac{mG_0}{G_0 + (m - G_0)e^{-kmt}}$;

Where $G_0 = 200$; $m = 10,000$;
and $k = 0.00001$.

(a)

$$G(t) = \frac{10,000(200)}{200 + (10,000 - 200)e^{-(0.00001)(10,000)t}}$$

$$G(t) = \frac{10,000}{1 + 49e^{-0.1t}}$$

(b)

$$G(t) = 10,000(1 + 49e^{-0.1t})^{-1}$$

$$G'(t) = -10,000(1 + 49e^{-0.1t})^{-2}(-4.9e^{-0.1t})$$

$$= \frac{49,000e^{-0.1t}}{(1 + 49e^{-0.1t})^2}$$

$$G(6) = \frac{10,000}{1 + 49e^{-0.6}} \approx 359$$

$$G'(6) = \frac{49,000e^{-0.6}}{(1 + 49e^{-0.6})^2} \approx 34.6$$

- (c) After 3 years,
- $t = 36$
- .

$$G(36) = \frac{10,000}{1 + 49e^{-3.6}} \approx 4276$$

$$G'(36) = \frac{49,000e^{-3.6}}{(1 + 49e^{-3.6})^2} \approx 245$$

- (d) After 7 years,
- $t = 84$
- .

$$G(84) = \frac{10,000}{1 + 49e^{-8.4}} \approx 9891$$

$$G'(84) = \frac{49,000e^{-8.4}}{(1 + 49e^{-8.4})^2} \approx 10.8$$

- (e) It increases for a while and then gradually decreases to 0.

47. $G(t) = \frac{mG_0}{G_0 + (m - G_0)e^{kmt}}$, where $G_0 = 400$;
 $m = 5200$; and $k = 0.0001$.

$$\begin{aligned} \text{(a)} \quad G(t) &= \frac{(5200)(400)}{400 + (5200 - 400)e^{(-0.0001)(5200)t}} \\ &= \frac{(400)(5200)}{400 + 4800e^{-0.52t}} \\ &= \frac{5200}{1 + 12e^{-0.52t}} \end{aligned}$$

- (b)

$$\begin{aligned} G(t) &= 5200(1 + 12e^{-0.52t})^{-1} \\ G'(t) &= -5200(1 + 12e^{-0.52t})^{-2}(-6.24e^{-0.52t}) \\ &= \frac{32,448e^{-0.52t}}{(1 + 12e^{-0.52t})^2} \\ G(1) &= \frac{5200}{1 + 12e^{-0.52}} \approx 639 \\ G'(1) &= \frac{32,448e^{-0.52}}{(1 + 12e^{-0.52})^2} \approx 292 \end{aligned}$$

$$\text{(c)} \quad G(4) = \frac{5200}{1 + 12e^{-2.08}} \approx 2081$$

$$G'(4) = \frac{32,448e^{-2.08}}{(1 + 12e^{-2.08})^2} \approx 649$$

$$\text{(d)} \quad G(10) = \frac{5200}{1 + 12e^{-5.2}} \approx 4877$$

$$G'(10) = \frac{32,448e^{-5.2}}{(1 + 12e^{-5.2})^2} \approx 167$$

- (e) It increases for a while and then gradually decreases to 0.

48. $P(x) = 0.04e^{-4x}$

$$\begin{aligned} \text{(a)} \quad P(0.5) &= 0.04e^{-4(0.5)} \\ &= 0.04e^{-2} \\ &\approx 0.005 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(1) &= 0.04e^{-4(1)} \\ &= 0.04e^{-4} \\ &\approx 0.0007 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(2) &= 0.04e^{-4(2)} \\ &= 0.04e^{-8} \\ &\approx 0.000013 \\ P'(x) &= 0.04(-4)e^{-4x} \\ &= -0.16e^{-4x} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad P'(0.5) &= -0.16e^{-4(0.5)} \\ &= -0.16e^{-2} \\ &\approx -0.022 \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad P'(1) &= -0.16e^{-4(1)} \\ &= -0.16e^{-4} \\ &\approx -0.0029 \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad P'(2) &= -0.16e^{-4(2)} \\ &= -0.16e^{-8} \\ &\approx -0.000054 \end{aligned}$$

49. $V(t) = 1100[1023e^{-0.02415t} + 1]^{-4}$

$$\begin{aligned} \text{(a)} \quad V(240) &= 1100[1023e^{-0.02415(240)} + 1]^{-4} \\ &\approx 3.857 \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad V &= \frac{4}{3}\pi r^3, \text{ so } r(V) = \sqrt[3]{\frac{3V}{4\pi}} \\ r(3.857) &= \sqrt[3]{\frac{3(3.857)}{4\pi}} \approx 0.973 \text{ cm} \end{aligned}$$

- (c)

$$\begin{aligned} V(t) &= 1100[1023e^{-0.02415t} + 1]^{-4} = 0.5 \\ [1023e^{-0.02415t} + 1]^{-4} &= \frac{1}{2200} \\ (1023e^{-0.02415t} + 1)^4 &= 2200 \\ 1023e^{-0.02415t} + 1 &= 2200^{1/4} \\ 1023e^{-0.02415t} &= 2200^{1/4} - 1 \\ e^{-0.02415t} &= \frac{2200^{1/4} - 1}{1023} \end{aligned}$$

$$-0.02415t = \ln\left(\frac{2200^{1/4} - 1}{1023}\right)$$

$$t = \frac{1}{-0.02415} \ln\left(\frac{2200^{1/4} - 1}{1023}\right) \approx 214 \text{ months}$$

The tumor has been growing for almost 18 years.

- (d) As t goes to infinity, $e^{-0.02415t}$ goes to zero, and $V(t) = 1100[1023e^{-0.02415t} + 1]^{-4}$ goes to 1100 cm^3 , which corresponds to a sphere with a radius of $\sqrt[3]{\frac{3(1100)}{4\pi}} \approx 6.4 \text{ cm}$.

It makes sense that a tumor growing in a person's body reaches a maximum volume of this size.

- (e) By the chain rule,

$$\frac{dV}{dt} = 1100(-4)[1023e^{-0.02415t} + 1]^{-5}$$

$$\cdot (1023)(e^{-0.02415t})(-0.02415)$$

$$= 108,703.98[1023e^{-0.02415t} + 1]^{-5}e^{-0.02415t}$$

$$\text{At } t = 240, \frac{dV}{dt} \approx 0.282.$$

At 240 months old, the tumor is increasing in volume at the instantaneous rate of $0.282 \text{ cm}^3/\text{month}$.

50. $P(t) = 0.00239e^{0.0957t}$

(a) $P(25) = 0.00239e^{0.0957(25)}$

$$\approx 0.026\%$$

$$P(50) = 0.00239e^{0.0957(50)}$$

$$\approx 0.286\%$$

$$P(75) = 0.00239e^{0.0957(75)}$$

$$\approx 3.130\%$$

(b) $P'(t) = 0.00239e^{0.0957t}(0.0957)$

$$= 0.000228723e^{0.0957t}$$

$$P'(25) = 0.000228723e^{0.0957(25)}$$

$$\approx 0.0025\%/\text{year}$$

$$P'(50) = 0.000228723e^{0.0957(50)}$$

$$\approx 0.0274\%/\text{year}$$

$$P'(75) = 0.000228723e^{0.0957(75)}$$

$$\approx 0.300\%/\text{year}$$

- (c) The percentage of people in each of the age groups that die in a given year is increasing as indicated by the answers in parts (a) and

(b). A person who is 75 years old has a 3% chance of dying during year and the rate is increasing by almost 0.3%. The formula implies that everyone will be dead by age 112.

51. $p(x) = 0.001131e^{0.1268x}$

(a) $p(40) = 0.001131e^{0.1268(40)} \approx 0.180$

- (b) When $p(x) = 1$,

$$0.001131e^{0.1268x} = 1$$

$$e^{0.1268x} = \frac{1}{0.001131}$$

$$0.1268x = \ln \frac{1}{0.001131}$$

$$x = \frac{1}{0.1268} \ln \frac{1}{0.001131}$$

$$\approx 54$$

This represents the year 2024.

(c) $p'(x) = 0.001131e^{0.1268x}(0.1268)$

$$= 0.0001434108e^{0.1268x}$$

$$p'(40) = 0.0001434108e^{0.1268(40)}$$

$$\approx 0.023$$

The marginal increase in the proportion per year in 2010 is approximately 0.023.

52. $M(t) = 3102e^{-0.022(t-56)}$

(a) $M(200) = 3102e^{-0.022(200-56)}$

$$\approx 2974.15 \text{ grams,}$$

or about 3 kilograms.

- (b) As t gets very large, $-e^{-0.022(t-56)}$ goes to

zero, $e^{-e^{-0.022(t-56)}}$ goes to 1, and $M(t)$ approaches 3102 grams or about 3.1 kilograms.

- (c) 80% of 3102 is 2481.6.

$$2481.6 = 3102e^{-0.022(t-56)}$$

$$-\ln \frac{2481.6}{3102} = e^{-0.022(t-56)}$$

$$\ln \left(\ln \frac{3102}{2481.6} \right) = -0.022(t-56)$$

$$t = -\frac{1}{0.022} \ln \left(\ln \frac{3102}{2481.6} \right) + 56$$

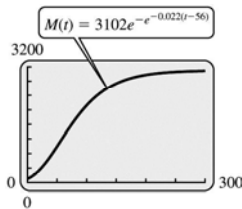
$$\approx 124 \text{ days}$$

(d)

$$\begin{aligned}
 D_t M(t) &= 3102e^{-e^{-0.022(t-56)}} D_t \left(-e^{-0.022(t-56)} \right) \\
 &= 3102e^{-e^{-0.022(t-56)}} \left(-e^{-0.022(t-56)} \right) (-0.022) \\
 &= 68.244e^{-e^{-0.022(t-56)}} e^{-0.022(t-56)}
 \end{aligned}$$

When $t = 200$, $D_t M(t) \approx 2.75$ g/day.

(e)



Growth is initially rapid, then tapers off.

(f)

Day	Weight	Rate
50	991	24.88
100	2122	17.73
150	2734	7.60
200	2974	2.75
250	3059	0.94
300	3088	0.32

53. $W_1(t) = 509.7(1 - 0.941e^{-0.00181t})$
 $W_2(t) = 498.4(1 - 0.889e^{-0.00219t})^{1.25}$

(a) Both W_1 and W_2 are strictly increasing functions, so they approach their maximum values as t approaches ∞ .

$$\begin{aligned}
 \lim_{t \rightarrow \infty} W_1(t) &= \lim_{t \rightarrow \infty} 509.7(1 - 0.941e^{-0.00181t}) \\
 &= 509.7(1 - 0) = 509.7
 \end{aligned}$$

$$\begin{aligned}
 \lim_{t \rightarrow \infty} W_2(t) &= \lim_{t \rightarrow \infty} 498.4(1 - 0.889e^{-0.00219t})^{1.25} \\
 &= 498.4(1 - 0)^{1.25} = 498.4
 \end{aligned}$$

So, the maximum values of W_1 and W_2 are 509.7 kg and 498.4 kg respectively.

(b) $0.9(509.7) = 509.7(1 - 0.941e^{-0.00181t})$

$$0.9 = 1 - 0.941e^{-0.00181t}$$

$$\frac{0.1}{0.941} \approx e^{-0.00181t}$$

$$1239 \approx t$$

$$0.9(498.4) = 498.4(1 - 0.889e^{-0.00219t})^{1.25}$$

$$0.9 = (1 - 0.889e^{-0.00219t})^{1.25}$$

$$\frac{1 - 0.9^{0.8}}{0.889} = e^{-0.00219t}$$

$$1095 \approx t$$

Respectively, it will take the average beef cow about 1239 days or 1095 days to reach 90% of its maximum.

(c)

$$\begin{aligned}
 W_1'(t) &= (509.7)(-0.941)(-0.00181)e^{-0.00181t} \\
 &\approx 0.868126e^{-0.00181t}
 \end{aligned}$$

$$\begin{aligned}
 W_1'(750) &\approx 0.868126e^{-0.00181(750)} \\
 &\approx 0.22 \text{ kg/day}
 \end{aligned}$$

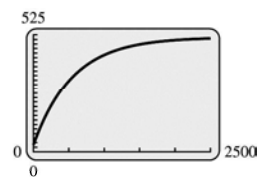
$$\begin{aligned}
 W_2'(t) &= (498.4)(1.25)(1 - 0.889e^{-0.00219t})^{0.25} \\
 &\quad \cdot (-0.889)(-0.00219)e^{-0.00219t} \\
 &\approx 1.21292e^{-0.00219t}(1 - 0.889e^{-0.00219t})^{0.25}
 \end{aligned}$$

$$\begin{aligned}
 W_2'(750) &\approx 1.12192e^{-0.00219(750)} \\
 &\quad \cdot (1 - 0.889e^{-0.00219(750)})^{0.25}
 \end{aligned}$$

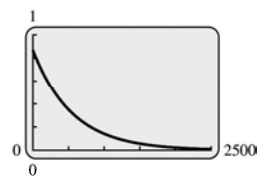
$$\approx 0.22 \text{ kg/day}$$

Both functions yield a rate of change of about 0.22 kg per day.

(d) Looking at the graph, the growth patterns of the two functions are very similar.



(e) The graphs of the rate of change of the two functions are also very similar.



54. $R(c) = 3.19(1.006^c)$

$$\frac{dR}{dt} = 3.19(\ln 1.006)(1.006^c) \frac{dc}{dt}$$

If $c = 180$ and $\frac{dc}{dt} = 15$, then

$$\frac{dR}{dt} = 3.19(\ln 1.006)(1.006^{180})(15) \approx 0.840$$

55. (a) $G_0 = 0.00369, m = 1, k = 3.5$

$$G(t) = \frac{1}{1 + \left(\frac{1}{0.00369} - 1 \right) e^{-3.5(1)t}}$$

$$= \frac{1}{1 + 270e^{-3.5t}}$$

(b)

$$G'(t) = -(1 + 270e^{-3.5t})^{-2} \cdot 270e^{-3.5t}(-3.5)$$

$$= \frac{945e^{-3.5t}}{(1 + 270e^{-3.5t})^2}$$

$$G(1) = \frac{1}{1 + 270e^{-3.5(1)}} \approx 0.109$$

$$G'(1) = \frac{945e^{-3.5(1)}}{[1 + 270e^{-3.5(1)}]^2} \approx 0.341$$

The proportion is 0.109 and the rate of growth is 0.341 per century.

(c) $G(2) = \frac{1}{1 + 270e^{-3.5(2)}} \approx 0.802$

$$G'(2) = \frac{945e^{-3.5(2)}}{[1 + 270e^{-3.5(2)}]^2} \approx 0.555$$

57. $G_0 = 1.603, m = 6.8, k = 0.0440$

(a) $G(t) = \frac{6.8}{1 + \left(\frac{6.8}{1.603} - 1 \right) e^{-0.0440(6.8)t}}$

$$= \frac{6.8}{1 + 3.242e^{-0.2992t}}$$

(b) $G'(t) = \frac{(1 + 3.242e^{-0.2992t})(0) - 6.8(3.242e^{-0.2992t})(-0.2992)}{(1 + 3.242e^{-0.2992t})^2}$

$$= \frac{6.59604e^{-0.2992t}}{(1 + 3.242e^{-0.2992t})^2}$$

The proportion is 0.802 and the rate of growth is 0.555 per century.

(d) $G(3) = \frac{1}{1 + 270e^{-3.5(3)}} \approx 0.993$

$$G'(3) = \frac{945e^{-3.5(3)}}{[1 + 270e^{-3.5(3)}]^2} \approx 0.0256$$

The proportion is 0.993 and the rate of growth is 0.0256 per century.

(e) The rate of growth increases for a while and then gradually decreases to 0.

56. $H(N) = 1000(1 - e^{-kN}), k = 0.1$

$$H' = -1000e^{-0.1N}(-0.1)$$

$$= 100e^{-0.1N}$$

(a) $H'(10) = 100e^{-0.1(10)} \approx 36.8$

(b) $H'(100) = 100e^{-0.1(100)} \approx .00454$

(c) $H'(1000) = 100e^{-0.1(1000)} \approx 0$

(d) $100e^{-0.1N}$ is always positive since powers of e are never negative. This means that repetition always makes a habit stronger.

For 2004, $t = 2$:

$$G(2) = \frac{6.8}{1 + 3.242e^{-0.2992(2)}} \approx 2.444$$

$$G'(2) = \frac{6.59604e^{-0.2992(2)}}{(1 + 3.242e^{-0.2992(2)})^2} \approx 0.468$$

In 2004, about 2.444 million students enrolled in at least one online course and the growth rate is about 0.468 million students per year.

(c) For 2006, $t = 4$:

$$G(4) = \frac{6.8}{1 + 3.242e^{-0.2992(4)}} \approx 3.435$$

$$G'(4) = \frac{6.59604e^{-0.2992(4)}}{(1 + 3.242e^{-0.2992(4)})^2} \approx 0.509$$

In 2006, about 3.435 million students enrolled in at least one online course and the growth rate is about 0.509 million students per year.

(d) For 2010, $t = 8$:

$$G(8) = \frac{6.8}{1 + 3.242e^{-0.2992(8)}} \approx 5.247$$

$$G'(8) = \frac{6.59604e^{-0.2992(8)}}{(1 + 3.242e^{-0.2992(8)})^2} \approx 0.359$$

In 2010, about 5.247 million students enrolled in at least one online course and the growth rate is about 0.359 million students per year.

(e) The growth rate increases at first and then decreases.

58. $A(t) = 500e^{-0.31t}$
 $A'(t) = 500(-0.31)e^{-0.31t} = -155e^{-0.31t}$

(a) $A'(4) = -155e^{-0.31(4)}$
 $= -155e^{-1.24}$
 ≈ -44.9 grams per year

(b) $A'(6) = -155e^{-0.31(6)}$
 $= -155e^{-1.86}$
 ≈ -24.1 grams per year

(c) $A'(10) = -155e^{-0.31(10)}$
 $= -155e^{-3.1}$
 ≈ -6.98 grams per year

(d) $\lim_{t \rightarrow \infty} A'(t) = \lim_{t \rightarrow \infty} -155e^{-0.31t} = 0$

As the number of years increases, the rate of change approaches zero.

(e) $A'(t)$ will never equal zero, although as t increases, it approaches zero. Powers of e are always positive.

59. $Q(t) = CV(1 - e^{-t/RC})$

(a) $I_c = \frac{dQ}{dt} = CV \left[0 - e^{-t/RC} \left(-\frac{1}{RC} \right) \right]$
 $= CV \left(\frac{1}{RC} \right) e^{-t/RC}$
 $= \frac{V}{R} e^{-t/RC}$

(b) When $C = 10^{-5}$ farads, $R = 10^7$ ohms, and $V = 10$ volts, after 200 seconds

$$I_c = \frac{10}{10^7} e^{-200/(10^7 \cdot 10^{-5})}$$

$$\approx 1.35 \times 10^{-7} \text{ amps}$$

$$\begin{aligned}
 60. \quad (a) \quad H(T) &= T - 0.9971e^{0.02086T} \left[1 - e^{0.445(85-57.2)} \right] \\
 &= T + 2.4385e^{0.02086T}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad H(80) &= 80 + 2.4385e^{0.0286(80)} \\
 &\approx 92.94
 \end{aligned}$$

The heat index when the temperature is 80°F is 92.94°F.

$$\begin{aligned}
 (c) \quad H'(T) &= 1 + 2.4385e^{0.02086T} (0.02086) \\
 H'(80) &= 1 + 2.4385e^{0.0286(80)} (0.02086) \\
 &= 1.2699
 \end{aligned}$$

The rate of change of the heat index when the temperature is 80°F is 1.2699°F per degree increase in the temperature.

$$61. \quad t(r) = 218 + 31(0.933)^r$$

$$\begin{aligned}
 (a) \quad t(60) &= 218 + 31(0.933)^{60} \\
 &\approx 218.5 \text{ sec}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad t'(n) &= (31 \ln 0.933)(0.933)^n \\
 t'(60) &= (31 \ln 0.933)(0.933)^{60} \\
 &\approx -0.034
 \end{aligned}$$

$$\begin{aligned}
 63. \quad (a) \quad s(0) &= 693.9 - 34.38 \left(e^{0.01003(0)} + e^{-0.01003(0)} \right) \\
 &= 693.9 - 34.38(2) \\
 &= 625.14
 \end{aligned}$$

The height of the Gateway Arch at its center is 625.14 ft.

$$\begin{aligned}
 (b) \quad m &= s'(t) = -34.38(e^{0.01003x}(0.01003) + e^{-0.01003x}(-0.01003)) \\
 &= -34.38(0.1003e^{0.01003x} - 0.01003e^{-0.01003x}) \\
 m &= s'(0) = -34.38(0.01003(1) - 0.01003(1)) = 0
 \end{aligned}$$

The slope of the line tangent to the curve when $x = 0$ is 0, which makes sense because the tangent line has to be a horizontal line at the center of the arch (the highest point on the curve)

$$\begin{aligned}
 (c) \quad s'(150) &= -34.38 \left(0.01003e^{0.01003(150)} - 0.01003e^{-0.01003(150)} \right) \\
 &= -1.476
 \end{aligned}$$

The rate of change of the height of the arch $x = 150$ ft is -1.476 ft per foot from the center.

The record is decreasing by 0.034 seconds per year at the end of 2010.

- (c) As $n \rightarrow \infty$, $(0.933)^n \rightarrow 0$ and $t(n) \rightarrow 218$. If the estimate is correct, then this is the least amount of time that it will ever take a human to run a mile.

$$\begin{aligned}
 62. \quad T(h) &= 80e^{-0.000065h} \\
 \frac{dT}{dh} &= 80e^{-0.000065h} \left(-0.000065 \frac{dh}{dh} \right) \\
 &= -0.0052e^{-0.000065h} \frac{dh}{dh}
 \end{aligned}$$

If $h = 1000$ and $\frac{dh}{dt} = 800$, then

$$\begin{aligned}
 \frac{dT}{dt} &= -0.0052e^{-0.000065(1000)}(800) \\
 &\approx -3.90
 \end{aligned}$$

The temperature is decreasing at 3.90 degrees/hr.

4.5 Derivatives of Logarithmic Functions

Your Turn 1

$$f(x) = \log_3 x$$

$$f'(x) = \frac{1}{(\ln 3)x}$$

Your Turn 2

$$(a) \quad y = \ln(2x^3 - 3)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2x^3 - 3} \cdot \frac{d}{dx}(2x^3 - 3) \\ &= \frac{1}{2x^3 - 3}(6x^2) \\ &= \frac{6x^2}{2x^3 - 3} \end{aligned}$$

$$(b) \quad f(x) = \log_4(5x + 3x^3)$$

$$\begin{aligned} f'(x) &= \frac{1}{(\ln 4)(5x + 3x^3)} \cdot \frac{d}{dx}(5x + 3x^3) \\ &= \frac{5 + 9x^2}{(\ln 4)(5x + 3x^3)} \end{aligned}$$

Your Turn 3

$$(a) \quad y = \ln|2x + 6|$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2x + 6} \cdot \frac{d}{dx}(2x + 6) \\ &= \frac{2}{2x + 6} = \frac{2}{2(x + 3)} = \frac{1}{x + 3} \end{aligned}$$

$$(b) \quad f(x) = x^2 \ln 3x$$

$$\begin{aligned} f'(x) &= x^2 \cdot \frac{1}{3x}(3) + \ln 3x(2x) \\ &= x + 2x \ln 3x \end{aligned}$$

$$(c) \quad s(t) = \frac{\ln(t^2 - 1)}{t + 1}$$

$$\begin{aligned} s'(t) &= \frac{(t + 1)\left(\frac{1}{t^2 - 1}\right)(2t) - \ln(t^2 - 1)(1)}{(t + 1)^2} \\ &= \frac{\frac{2t}{t - 1} - \ln(t^2 - 1)}{(t + 1)^2} \\ &= \frac{2t - (t - 1)\ln(t^2 - 1)}{(t - 1)(t + 1)^2} \end{aligned}$$

4.5 Exercises

$$1. \quad y = \ln(8x)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\ln 8x) \\ &= \frac{d}{dx}(\ln 8 + \ln x) \\ &= \frac{d}{dx}(\ln 8) + \frac{d}{dx}(\ln x) \\ &= 0 + \frac{1}{x} = \frac{1}{x} \end{aligned}$$

$$2. \quad y = \ln(-4x)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}[\ln(-4x)] \\ &= \frac{d}{dx}[\ln 4 + \ln(-x)] \\ &= \frac{d}{dx} \ln 4 + \frac{d}{dx} \ln(-x) \\ &= 0 + \frac{-1}{-x} = \frac{1}{x} \end{aligned}$$

$$3. \quad y = \ln(8 - 3x)$$

$$g(x) = 8 - 3x$$

$$g'(x) = -3$$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{-3}{8 - 3x} \text{ or } \frac{3}{3x - 8}$$

$$4. \quad y = \ln(1 + x^3)$$

$$g(x) = 1 + x^3$$

$$g'(x) = 3x^2$$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{3x^2}{1 + x^3}$$

$$5. \quad y = \ln|4x^2 - 9x|$$

$$g(x) = 4x^2 - 9x$$

$$g'(x) = 8x - 9$$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{8x - 9}{4x^2 - 9x}$$

$$6. \quad y = \ln|-8x^3 + 2x|$$

$$g(x) = -8x^3 + 2x$$

$$g'(x) = -24x^2 + 2$$

$$\frac{dy}{dx} = \frac{-24x^2 + 2}{-8x^3 + 2x}$$

7. $y = \ln \sqrt{x+5}$

$$g(x) = \sqrt{x+5} = (x+5)^{1/2}$$

$$g'(x) = \frac{1}{2}(x+5)^{-1/2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(x+5)^{-1/2}}{(x+5)^{1/2}} = \frac{1}{2(x+5)}$$

8. $y = \ln \sqrt{2x+1} = \ln(2x+1)^{1/2}$

$$g(x) = (2x+1)^{1/2}$$

$$g'(x) = \frac{1}{2}(2x+1)^{-1/2}(2) = (2x+1)^{-1/2}$$

$$\frac{dy}{dx} = \frac{(2x+1)^{-1/2}}{(2x+1)^{1/2}} = \frac{1}{2x+1}$$

9. $y = \ln(x^4 + 5x^2)^{3/2}$

$$= \frac{3}{2} \ln(x^4 + 5x^2)$$

$$\frac{dy}{dx} = \frac{3}{2} D_x [\ln(x^4 + 5x^2)]$$

$$g(x) = x^4 + 5x^2$$

$$g'(x) = 4x^3 + 10x$$

$$\frac{dy}{dx} = \frac{3}{2} \left(\frac{4x^3 + 10x}{x^4 + 5x^2} \right)$$

$$= \frac{3}{2} \left[\frac{2x(2x^2 + 5)}{x^2(x^2 + 5)} \right]$$

$$= \frac{3(2x^2 + 5)}{x(x^2 + 5)}$$

10. $y = \ln(5x^3 - 2x)^{3/2}$

$$= \frac{3}{2} \ln(5x^3 - 2x)$$

$$\frac{dy}{dx} = \frac{3}{2} D_x \ln(5x^3 - 2x)$$

$$g(x) = 5x^3 - 2x$$

$$g'(x) = 15x^2 - 2$$

$$\frac{dy}{dx} = \frac{3}{2} \left(\frac{15x^2 - 2}{5x^3 - 2x} \right)$$

$$= \frac{3(15x^2 - 2)}{2(5x^3 - 2x)}$$

11. $y = -5x \ln(3x+2)$

Use the product rule.

$$\frac{dy}{dx} = -5x \left[\frac{d}{dx} \ln(3x+2) \right] + \ln(3x+2) \left[\frac{d}{dx} (-5x) \right]$$

$$= -5x \left(\frac{3}{3x+2} \right) + [\ln(3x+2)](-5)$$

$$= -\frac{15x}{3x+2} - 5 \ln(3x+2)$$

12. $y = (3x+7) \ln(2x-1)$

Use the product rule.

$$\frac{dy}{dx} = (3x+7) \left(\frac{2}{2x-1} \right) + [\ln(2x-1)](3)$$

$$= \frac{2(3x+7)}{2x-1} + 3 \ln(2x-1)$$

13. $s = t^2 \ln|t|$

$$\frac{ds}{dt} = t^2 \cdot \frac{1}{t} + 2t \ln|t|$$

$$= t + 2t \ln|t|$$

$$= t(1 + 2 \ln|t|)$$

14. $y = x \ln|2 - x^2|$

Use the product rule.

$$\frac{dy}{dx} = x \left(\frac{1}{2 - x^2} \right) (-2x) + \ln|2 - x^2|$$

$$= -\frac{2x^2}{2 - x^2} + \ln|2 - x^2|$$

15. $y = \frac{2 \ln(x+3)}{x^2}$

Use the quotient rule.

$$\frac{dy}{dx} = \frac{x^2 \left(\frac{2}{x+3} \right) - 2 \ln(x+3) \cdot 2x}{(x^2)^2}$$

$$= \frac{\frac{2x^2}{x+3} - 4x \ln(x+3)}{x^4}$$

$$= \frac{2x^2 - 4x(x+3) \ln(x+3)}{x^4(x+3)}$$

$$= \frac{x[2x - 4(x+3) \ln(x+3)]}{x^4(x+3)}$$

$$= \frac{2x - 4(x+3) \ln(x+3)}{x^3(x+3)}$$

16. $v = \frac{\ln u}{u^3}$

Use the quotient rule.

$$\begin{aligned}\frac{dv}{du} &= \frac{u^3\left(\frac{1}{u}\right) - (\ln u)(3u^2)}{(u^3)^2} \\ &= \frac{u^2 - 3u^2 \ln u}{u^6} \\ &= \frac{u^2(1 - 3 \ln u)}{u^6} \\ &= \frac{1 - 3 \ln u}{u^4}\end{aligned}$$

17. $y = \frac{\ln x}{4x + 7}$

Use the quotient rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(4x + 7)\left(\frac{1}{x}\right) - (\ln x)(4)}{(4x + 7)^2} \\ &= \frac{\frac{4x+7}{x} - 4 \ln x}{(4x + 7)^2} \\ &= \frac{4x + 7 - 4x \ln x}{x(4x + 7)^2}\end{aligned}$$

18. $y = \frac{-2 \ln x}{3x - 1}$

Use the quotient rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x - 1)(-2)\left(\frac{1}{x}\right) - (-2 \ln x)(3)}{(3x - 1)^2} \\ &= \frac{\frac{-2(3x-1)}{x} + 6 \ln x}{(3x - 1)^2} \\ &= \frac{-2(3x - 1) + 6x \ln x}{x(3x - 1)^2} \\ &= \frac{-2(3x - 1 - 3x \ln x)}{x(3x - 1)^2}\end{aligned}$$

19. $y = \frac{3x^2}{\ln x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\ln x)(6x) - 3x^2\left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{6x \ln x - 3x}{(\ln x)^2}\end{aligned}$$

20. $y = \frac{x^3 - 1}{2 \ln x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2 \ln x)(3x^2) - (x^3 - 1)(2)\left(\frac{1}{x}\right)}{(2 \ln x)^2} \\ &= \frac{6x^2 \ln x - \frac{2}{x}(x^3 - 1)}{(2 \ln x)^2} \cdot \frac{x}{x} \\ &= \frac{6x^3 \ln x - 2(x^3 - 1)}{4x(\ln x)^2} \\ &= \frac{3x^3 \ln x - (x^3 - 1)}{2x(\ln x)^2}\end{aligned}$$

21. $y = (\ln|x + 1|)^4$

$$\begin{aligned}\frac{dy}{dx} &= 4(\ln|x + 1|)^3 \left(\frac{1}{x + 1} \right) \\ &= \frac{4(\ln|x + 1|)^3}{x + 1}\end{aligned}$$

22. $y = \sqrt{\ln|x - 3|} = (\ln|x - 3|)^{1/2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(\ln|x - 3|)^{1/2} \cdot \frac{d}{dx}(\ln|x - 3|) \\ &= \frac{1}{2(\ln|x - 3|)^{1/2}} \left(\frac{1}{x - 3} \right) \\ &= \frac{1}{2(x - 3)(\ln|x - 3|)^{1/2}} \\ &= \frac{1}{2(x - 3)\sqrt{\ln|x - 3|}}\end{aligned}$$

23. $y = \ln|\ln x|$

$$g(x) = \ln x$$

$$g'(x) = \frac{1}{x}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{g'(x)}{g(x)} \\ &= \frac{\frac{1}{x}}{\ln x} \\ &= \frac{1}{x \ln x}\end{aligned}$$

24. $y = (\ln 4)(\ln|3x|)$

$$\begin{aligned}\frac{dy}{dx} &= (\ln 4) \left(\frac{1}{3x} \right) (3) \\ &= \frac{3 \ln 4}{3x} = \frac{\ln 4}{x}\end{aligned}$$

(Recall that $\ln 4$ is a constant.)

$$\begin{aligned}
 25. \quad y &= e^{x^2} \ln x, x > 0 \\
 \frac{dy}{dx} &= e^{x^2} \left(\frac{1}{x} \right) + (\ln x)(2x)e^{x^2} \\
 &= \frac{e^{x^2}}{x} + 2xe^{x^2} \ln x
 \end{aligned}$$

$$\begin{aligned}
 26. \quad y &= e^{2x-1} \ln(2x-1) \\
 \frac{dy}{dx} &= (2)e^{2x-1} \ln(2x-1) \\
 &\quad + (e^{2x-1}) \left(\frac{1}{2x-1} \right) (2) \\
 &= 2e^{2x-1} \ln(2x-1) + \frac{2e^{2x-1}}{2x-1}
 \end{aligned}$$

$$27. \quad y = \frac{e^x}{\ln x}, x > 0$$

Use the quotient rule.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(\ln x)e^x - e^x \left(\frac{1}{x} \right)}{(\ln x)^2} \cdot \frac{x}{x} \\
 &= \frac{xe^x \ln x - e^x}{x(\ln x)^2}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad p(y) &= \frac{\ln y}{e^y} \\
 p'(y) &= \frac{e^y \cdot \frac{1}{y} - (\ln y)e^y}{(e^y)^2} \\
 &= \frac{e^y - y(\ln y)e^y}{ye^{2y}} \\
 &= \frac{e^y(1 - y \ln y)}{ye^{2y}} \\
 &= \frac{1 - y \ln y}{ye^y}
 \end{aligned}$$

$$\begin{aligned}
 29. \quad g(z) &= (e^{2z} + \ln z)^3 \\
 g'(z) &= 3(e^{2z} + \ln z)^2 \left(e^{2z} \cdot 2 + \frac{1}{z} \right) \\
 &= 3(e^{2z} + \ln z)^2 \left(\frac{2ze^{2z} + 1}{z} \right)
 \end{aligned}$$

$$\begin{aligned}
 30. \quad s(t) &= \sqrt{e^{-t} + \ln 2t} = (e^{-t} + \ln 2t)^{1/2} \\
 s'(t) &= \frac{1}{2}(e^{-t} + \ln 2t)^{-1/2} \left(e^{-t}(-1) + \frac{1}{2t}(2) \right) \\
 &= \frac{1}{2}(e^{-t} + \ln 2t)^{-1/2} \left(-e^{-t} + \frac{1}{t} \right) \\
 &= \frac{1}{2}(e^{-t} + \ln 2t)^{-1/2} \left(\frac{-te^{-t} + 1}{t} \right) \\
 &= \frac{1 - te^{-t}}{2t\sqrt{e^{-t} + \ln 2t}}
 \end{aligned}$$

$$\begin{aligned}
 31. \quad y &= \log(6x) \\
 g(x) &= 6x \text{ and } g'(x) = 6. \\
 \frac{dy}{dx} &= \frac{1}{\ln 10} \left(\frac{6}{6x} \right) \\
 &= \frac{1}{x \ln 10}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad y &= \log(4x-3) \\
 g(x) &= 4x-3 \\
 g'(x) &= 4 \\
 \frac{dy}{dx} &= \frac{1}{\ln 10} \cdot \frac{4}{4x-3} \\
 &= \frac{4}{(\ln 10)(4x-3)}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad y &= \log|1-x| \\
 g(x) &= 1-x \text{ and } g'(x) = -1. \\
 \frac{dy}{dx} &= \frac{1}{\ln 10} \cdot \frac{-1}{1-x} \\
 &= -\frac{1}{(\ln 10)(1-x)} \\
 &\text{or } \frac{1}{(\ln 10)(x-1)}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad y &= \log|3x| \\
 g(x) &= 3x \text{ and } g'(x) = 3. \\
 \frac{dy}{dx} &= \frac{1}{\ln 10} \cdot \frac{3}{3x} = \frac{1}{x \ln 10}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad y &= \log_5 \sqrt{5x+2} \\
 g(x) &= \sqrt{5x+2} \text{ and } g'(x) = \frac{5}{2\sqrt{5x+2}} \\
 \frac{dy}{dx} &= \frac{1}{\ln 5} \cdot \frac{\frac{5}{2\sqrt{5x+2}}}{\sqrt{5x+2}} \\
 &= \frac{5}{2\ln 5(5x+2)}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad y &= \log_7 \sqrt{4x-3} \\
 g(x) &= \sqrt{4x-3} \\
 g'(x) &= \frac{4}{2\sqrt{4x-3}} = \frac{2}{\sqrt{4x-3}} \\
 \frac{dy}{dx} &= \frac{1}{\ln 7} \cdot \frac{\frac{2}{\sqrt{4x-3}}}{\sqrt{4x-3}} \\
 &= \frac{2}{(\ln 7)(4x-3)}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad y &= \log_3 (x^2 + 2x)^{3/2} \\
 g(x) &= (x^2 + 2x)^{3/2} \text{ and } \\
 g'(x) &= \frac{3}{2}(x^2 + 2x)^{1/2} \cdot (2x + 2) \\
 &= 3(x+1)(x^2 + 2x)^{1/2} \\
 \frac{dy}{dx} &= \frac{1}{\ln 3} \cdot \frac{3(x+1)(x^2 + 2x)^{1/2}}{(x^2 + 2x)^{3/2}} \\
 &= \frac{3(x+1)}{(\ln 3)(x^2 + 2x)}
 \end{aligned}$$

$$\begin{aligned}
 38. \quad y &= \log_2 (2x^2 - x)^{5/2} \\
 g(x) &= (2x^2 - x)^{5/2} \text{ and } \\
 g'(x) &= \frac{5}{2}(2x^2 - x)^{3/2} \cdot (4x - 1) \\
 \frac{dy}{dx} &= \frac{1}{\ln 2} \cdot \frac{\frac{5}{2}(2x^2 - x)^{3/2} \cdot (4x - 1)}{(2x^2 - x)^{5/2}} \\
 &= \frac{5(4x - 1)}{(2\ln 2)(2x^2 - x)}
 \end{aligned}$$

$$\begin{aligned}
 39. \quad w &= \log_8 (2^p - 1) \\
 g(p) &= 2^p - 1 \\
 g'(p) &= (\ln 2)2^p \\
 \frac{dw}{dp} &= \frac{1}{\ln 8} \cdot \frac{(\ln 2)2^p}{(2^p - 1)} \\
 &= \frac{(\ln 2)2^p}{(\ln 8)(2^p - 1)}
 \end{aligned}$$

$$\begin{aligned}
 40. \quad z &= 10^y \log y \\
 g(y) &= 10^y \text{ and } g'(y) = (\ln 10)10^y \\
 \frac{dz}{dy} &= 10^y \cdot \frac{1}{(\ln 10)y} + \log y \cdot (\ln 10)10^y \\
 &= \frac{10^y}{(\ln 10)y} + (\log y)(\ln 10)10^y
 \end{aligned}$$

$$\begin{aligned}
 41. \quad f(x) &= e^{\sqrt{x}} \ln(\sqrt{x} + 5) \\
 &\text{Use the product rule.} \\
 f'(x) &= e^{\sqrt{x}} \left(\frac{\frac{1}{2\sqrt{x}}}{\sqrt{x} + 5} \right) + [\ln(\sqrt{x} + 5)]e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) \\
 &= \frac{e^{\sqrt{x}}}{2} \left[\frac{1}{\sqrt{x}(\sqrt{x} + 5)} + \frac{\ln(\sqrt{x} + 5)}{\sqrt{x}} \right]
 \end{aligned}$$

$$\begin{aligned}
 42. \quad f(x) &= \ln(xe^{\sqrt{x}} + 2) \\
 g(x) &= xe^{\sqrt{x}} + 2 \\
 g'(x) &= x \left[e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) \right] + e^{\sqrt{x}}(1) \\
 &= \frac{e^{\sqrt{x}}\sqrt{x}}{2} + e^{\sqrt{x}} \\
 &= \frac{e^{\sqrt{x}}}{2}(\sqrt{x} + 2) \\
 f'(x) &= \frac{g'(x)}{g(x)} \\
 &= \frac{\frac{e^{\sqrt{x}}}{2}(\sqrt{x} + 2)}{xe^{\sqrt{x}} + 2} \\
 &= \frac{e^{\sqrt{x}}(\sqrt{x} + 2)}{2(xe^{\sqrt{x}} + 2)}
 \end{aligned}$$

43. $f(t) = \frac{\ln(t^2 + 1) + t}{\ln(t^2 + 1) + 1}$

Use the quotient rule.

$$\begin{aligned}
 u(t) &= \ln(t^2 + 1) + t, u'(t) = \frac{2t}{t^2 + 1} + 1 \\
 v(t) &= \ln(t^2 + 1) + 1, v'(t) = \frac{2t}{t^2 + 1} \\
 f'(t) &= \frac{[\ln(t^2 + 1) + 1]\left(\frac{2t}{t^2 + 1} + 1\right) - [\ln(t^2 + 1) + t]\left(\frac{2t}{t^2 + 1}\right)}{[\ln(t^2 + 1) + 1]^2} \\
 &= \frac{\frac{2t \ln(t^2 + 1)}{t^2 + 1} + \frac{2t}{t^2 + 1} + \ln(t^2 + 1) + 1 - \frac{2t \ln(t^2 + 1)}{t^2 + 1} - \frac{2t^2}{t^2 + 1}}{[\ln(t^2 + 1) + 1]^2} \\
 &= \frac{\frac{2t - 2t^2}{t^2 + 1} + \ln(t^2 + 1) + 1}{[\ln(t^2 + 1) + 1]^2} \\
 &= \frac{2t - 2t^2 + (t^2 + 1)\ln(t^2 + 1) + t^2 + 1}{(t^2 + 1)[\ln(t^2 + 1) + 1]^2} \\
 &= \frac{-t^2 + 2t + 1 + (t^2 + 1)\ln(t^2 + 1)}{(t^2 + 1)[\ln(t^2 + 1) + 1]^2}
 \end{aligned}$$

44. $f(t) = \frac{2t^{3/2}}{\ln(2t^{3/2} + 1)}$

Use the quotient rule.

$$\begin{aligned}
 u(t) &= 2t^{3/2}, u'(t) = 3t^{1/2} \\
 v(t) &= \ln(2t^{3/2} + 1), v'(t) = \frac{3t^{1/2}}{2t^{3/2} + 1} \\
 f'(t) &= \frac{\ln(2t^{3/2} + 1)(3t^{1/2}) - 2t^{3/2}\left[\frac{3t^{1/2}}{2t^{3/2} + 1}\right]}{[\ln(2t^{3/2} + 1)]^2} \\
 &= \frac{(3t^{1/2})\ln(2t^{3/2} + 1) - \frac{6t^2}{2t^{3/2} + 1}}{[\ln(2t^{3/2} + 1)]^2} \\
 &= \frac{(6t^2 + 3t^{1/2})\ln(2t^{3/2} + 1) - 6t^2}{(2t^{3/2} + 1)[\ln(2t^{3/2} + 1)]^2}
 \end{aligned}$$

46. Note that a is a constant.

$$\begin{aligned}\frac{d}{dx} \ln|ax| &= \frac{d}{dx} (\ln|a| + \ln|x|) \\ &= \frac{d}{dx} \ln|a| + \frac{d}{dx} \ln|x| \\ &= 0 + \frac{d}{dx} \ln|x| \\ &= \frac{d}{dx} \ln|x|\end{aligned}$$

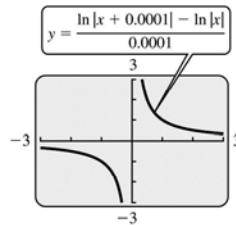
Therefore,

$$\frac{d}{dx} \ln|ax| = \frac{d}{dx} \ln|x|.$$

48. Graph

$$y = \frac{\ln|x + 0.0001| - \ln|x|}{0.0001}$$

on a graphing calculator. A good choice for the viewing window is $[-3, 3]$.



If we graph $y = \frac{1}{x}$ on the same screen, we see that the two graphs coincide.

By the definition of the derivative, if $f(x) = \ln|x|$,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln|x+h| - \ln|x|}{h},$$

and $h = 0.0001$ is very close to 0.

Comparing the two graphs provides graphical evidence that

$$f'(x) = \frac{1}{x}.$$

49. Use the derivative of $\ln x$.

$$\begin{aligned}\frac{d \ln[u(x)v(x)]}{dx} &= \frac{1}{u(x)v(x)} \cdot \frac{d[u(x)v(x)]}{dx} \\ \frac{d \ln u(x)}{dx} &= \frac{1}{u(x)} \cdot \frac{d[u(x)]}{dx} \\ \frac{d \ln v(x)}{dx} &= \frac{1}{v(x)} \cdot \frac{d[v(x)]}{dx}\end{aligned}$$

Then since $\ln[u(x)v(x)] = \ln u(x) + \ln v(x)$,

$$\frac{1}{u(x)v(x)} \cdot \frac{d[u(x)v(x)]}{dx} = \frac{1}{u(x)} \cdot \frac{d[u(x)]}{dx} + \frac{1}{v(x)} \cdot \frac{d[v(x)]}{dx}.$$

Multiply both sides of this equation by $u(x)v(x)$. Then $\frac{d[u(x)v(x)]}{dx} = v(x) \frac{d[u(x)]}{dx} + u(x) \frac{d[v(x)]}{dx}$.

This is the product rule.

50. Use the derivative of $\ln x$.

$$\begin{aligned}\frac{d \ln \frac{u(x)}{v(x)}}{dx} &= \frac{1}{\frac{u(x)}{v(x)}} \cdot \frac{d \left[\frac{u(x)}{v(x)} \right]}{dx} = \frac{v(x)}{u(x)} \cdot \frac{d \left[\frac{u(x)}{v(x)} \right]}{dx} \\ d \ln u(x) &= \frac{1}{u(x)} \cdot \frac{d[u(x)]}{dx} \\ d \ln v(x) &= \frac{1}{v(x)} \cdot \frac{d[v(x)]}{dx}\end{aligned}$$

Then, since $\ln \frac{u(x)}{v(x)} = \ln u(x) - \ln v(x)$,

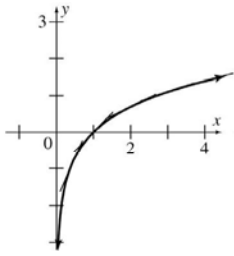
$$\frac{v(x)}{u(x)} \cdot \frac{d \left[\frac{u(x)}{v(x)} \right]}{dx} = \frac{1}{u(x)} \cdot \frac{d[u(x)]}{dx} - \frac{1}{v(x)} \cdot \frac{d[v(x)]}{dx}.$$

Multiply both sides of this equation by $\frac{u(x)}{v(x)}$. Then

$$\begin{aligned}\frac{d \left[\frac{u(x)}{v(x)} \right]}{dx} &= \frac{1}{v(x)} \cdot \frac{d[u(x)]}{dx} - \frac{u(x)}{[v(x)]^2} \cdot \frac{d[v(x)]}{dx} \\ &= \frac{v(x)}{[v(x)]^2} \cdot \frac{d[u(x)]}{dx} - \frac{u(x)}{[v(x)]^2} \cdot \frac{d[v(x)]}{dx} \\ &= \frac{v(x) \cdot \frac{d[u(x)]}{dx} - u(x) \cdot \frac{d[v(x)]}{dx}}{[v(x)]^2}\end{aligned}$$

This is the quotient rule.

51. Graph the function $y = \ln x$. Sketch lines tangent to the graph at $x = \frac{1}{2}, 1, 2, 3, 4$. Estimate the slopes of the tangent lines at these points.



x	slope of tangent
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
4	$\frac{1}{4}$

The values of the slopes at x are $\frac{1}{x}$.

Thus we see that $\frac{d \ln x}{dx} = \frac{1}{x}$.

52. The change-of-base theorem for logarithms states $\log_a x = \frac{\ln x}{\ln a}$. Find the derivative of each side.

$$\begin{aligned} \frac{d \log_a x}{dx} &= \frac{\ln a \cdot \frac{d \ln x}{dx} - \ln x \cdot \frac{d \ln a}{dx}}{(\ln a)^2} \\ &= \frac{\ln a \cdot \frac{1}{x} - \ln x \cdot 0}{(\ln a)^2} \\ &= \frac{\frac{1}{x}}{\ln a} = \frac{1}{x \ln a} \end{aligned}$$

53. (a) $h(x) = u(x)^{v(x)}$
- $$\begin{aligned} \frac{d}{dx} \ln h(x) &= \frac{d}{dx} \ln[u(x)^{v(x)}] \\ &= \frac{d}{dx} [v(x) \ln u(x)] \\ &= v(x) \frac{d}{dx} \ln u(x) + (\ln u(x)) v'(x) \\ &= v(x) \frac{u'(x)}{u(x)} + (\ln u(x)) v'(x) \\ &= \frac{v(x) u'(x)}{u(x)} + (\ln u(x)) v'(x) \end{aligned}$$

- (b) Since $\frac{d}{dx} \ln h(x) = \frac{h'(x)}{h(x)}$,

$$\begin{aligned} h'(x) &= h(x) \frac{d}{dx} \ln h(x) \\ &= h(x) \left[\frac{v(x) u'(x)}{u(x)} + (\ln u(x)) v'(x) \right] \\ &= u(x)^{v(x)} \left[\frac{v(x) u'(x)}{u(x)} + (\ln u(x)) v'(x) \right] \end{aligned}$$

54. $h(x) = x^x$
 $u(x) = x, u'(x) = 1$
 $v(x) = x, v'(x) = 1$
 $h'(x) = x^x \left[\frac{x(1)}{x} + (\ln x)(1) \right]$
 $= x^x (1 + \ln x)$

55. $h(x) = (x^2 + 1)^{5x}$
 $u(x) = x^2 + 1, u'(x) = 2x$
 $v(x) = 5x, v'(x) = 5$
 $h'(x) = (x^2 + 1)^{5x} \left[\frac{5x(2x)}{x^2 + 1} + \ln(x^2 + 1) \cdot (5) \right]$
 $= (x^2 + 1)^{5x} \left[\frac{10x^2}{x^2 + 1} + 5 \ln(x^2 + 1) \right]$

56. $R(x) = 30 \ln(2x + 1)$
 $C(x) = \frac{x}{2}$

- (a) The marginal revenue is

$$R'(x) = 30 \left(\frac{1}{\ln(2x + 1)} \right) (2) = \frac{60}{2x + 1}$$

- (b) The profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 30 \ln(2x + 1) - \frac{x}{2} \end{aligned}$$

- (c) The marginal profit when $x = 60$ is

$$\begin{aligned} P'(x) &= R'(x) - C'(x) \\ &= \frac{60}{2x + 1} - \frac{1}{2} \\ P'(60) &= \frac{60}{2(60) + 1} - \frac{1}{2} \\ &= \frac{60}{121} - \frac{1}{2} \approx 0 \end{aligned}$$

- (d) When 60 items are manufactured, there is not profit from selling additional items.

57. $p = 100 + \frac{50}{\ln q}, q > 1$

(a) $R = pq$

$$R = 100q + \frac{50q}{\ln q}$$

The marginal revenue is

$$\begin{aligned}\frac{dR}{dq} &= 100 + \frac{(\ln q)(50) - 50q\left(\frac{1}{q}\right)}{(\ln q)^2} \\ &= 100 + \frac{50(\ln q - 1)}{(\ln q)^2}.\end{aligned}$$

(b) The revenue from one more unit is $\frac{dR}{dq}$ for $q = 8$.

$$100 + \frac{50(\ln 8 - 1)}{(\ln 8)^2} = \$112.48$$

(c) The manager can use the information from (b) to decide if it is reasonable to sell additional items.

58. $C(q) = 100q + 100$

(a) The marginal cost is given by $C'(q)$.

$$C'(q) = 100$$

(b) $P(q) = R(q) - C(q)$

$$\begin{aligned}&= q\left(100 + \frac{50}{\ln q}\right) - (100q + 100) \\ &= 100q + \frac{50q}{\ln q} - 100q - 100 \\ &= \frac{50q}{\ln q} - 100\end{aligned}$$

(c) The profit from one more unit is $\frac{dP}{dq}$ for $q = 8$.

$$\begin{aligned}\frac{dP}{dq} &= \frac{(\ln q)(50) - 50q\left(\frac{1}{q}\right)}{(\ln q)^2} \\ &= \frac{50 \ln q - 50}{(\ln q)^2} \\ &= \frac{50(\ln q - 1)}{(\ln q)^2}\end{aligned}$$

When $q = 8$, the profit from one more unit is

$$\frac{50(\ln 8 - 1)}{(\ln 8)^2} = \$12.48.$$

(d) The manager can use the information from part (c) to decide whether it is profitable to make and sell additional items.

59. $C(x) = 5 \log_2 x + 10$

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{5 \log_2 x + 10}{x}$$

$$\begin{aligned}\bar{C}'(x) &= \frac{x \cdot 5 \cdot \frac{1}{\ln 2} \cdot \frac{1}{x} - (5 \log_2 x + 10) \cdot 1}{x^2} \\ &= \frac{5 - (\ln 2)(5 \log_2 x + 10)}{x^2 \ln 2}\end{aligned}$$

(a) $\bar{C}'(10) = \frac{5 - (\ln 2)(5 \log_2 10 + 10)}{(10^2) \ln 2}$

$$\approx -0.19396$$

(b) $\bar{C}'(20) = \frac{5 - (\ln 2)(5 \log_2 20 + 10)}{(20^2) \ln 2}$

$$\approx -0.06099$$

60. $A(w) = 4.688w^{0.8168-0.0154\log_{10} w}$

(a)

$$\begin{aligned}A(4000) &= 4.688(4000)^{0.8168-0.0154\log_{10} 4000} \\ &\approx 2590 \text{ cm}^2\end{aligned}$$

(b) $\frac{A(w)}{4.688} = w^{0.8166-0.0154\log_{10} w}$

$$\ln A(w) - \ln 4.688 = (\ln w) \left(0.8168 - 0.0154 \frac{\ln w}{\ln 10} \right)$$

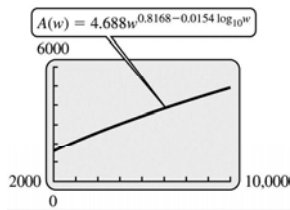
$$\begin{aligned}\frac{A'(w)}{A(w)} &= \frac{1}{w} \left(0.8168 - 0.0154 \frac{\ln w}{\ln 10} \right) \\ &\quad + \ln w \left(\frac{-0.0154}{\ln 10} \right) \frac{1}{w} \\ &= \frac{0.8168}{w} - \frac{0.0308}{\ln 10} \frac{\ln w}{w}\end{aligned}$$

$$\begin{aligned}A'(w) &= \frac{1}{w} \left(0.8168 - \frac{0.0308}{\ln 10} \ln w \right) \\ &\quad \cdot \left(4.688w^{0.8168-0.0154\log_{10} w} \right)\end{aligned}$$

$$A'(4000) \approx 0.4571 \approx 0.46 \text{ g/cm}^2$$

When the infant has a mass of 4000 g, it is gaining 0.46 square centimeters per gram of mass increase.

(c)



$$61. \quad \ln \left(\frac{N(t)}{N_0} \right) = 9.8901e^{-e^{2.54197-0.2167t}}$$

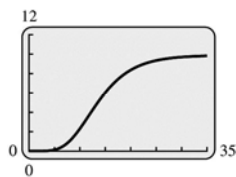
$$(a) \quad \frac{N(t)}{1000} = e^{9.8901e^{-e^{2.54197-0.2167t}}}$$

$$N(t) = 1000e^{9.8901e^{-e^{2.54197-0.2167t}}}$$

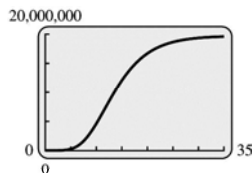
$$(b) \quad N'(20) \approx 1,307,416 \text{ bacteria/hour}$$

Twenty hours into the experiment, the number of bacteria is increasing at a rate of 1,307,416 per hour

$$(c) \quad S(t) = \ln \left(\frac{N(t)}{N_0} \right)$$



(d)



The two graphs have the same general shape, but $N(t)$ is scaled much larger.

$$(e) \quad \lim_{t \rightarrow \infty} S(t) = \lim_{t \rightarrow \infty} 9.8901e^{-e^{2.54197-0.2167t}} = 9.8901$$

$$S(t) = \ln \left(\frac{N(t)}{N_0} \right)$$

$$N(t) = N_0 e^{S(t)}$$

$$\lim_{t \rightarrow \infty} N(t) = N_0 e^{\lim_{t \rightarrow \infty} S(t)} = 1000e^{9.8901} \approx 19,734,033 \text{ bacteria}$$

$$62. \quad F(x) = 0.774 + 0.727 \log(x)$$

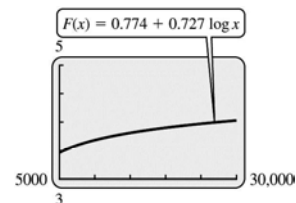
$$(a) \quad F(25,000) = 0.774 + 0.727 \log(25,000) = 3.9713... \approx 4 \text{ kJ/day}$$

$$(b) \quad F'(x) = 0.727 \frac{1}{x \ln 10} = \frac{0.727}{\ln 10} x^{-1}$$

$$F'(25,000) = \frac{0.727}{\ln 10} 25,000^{-1} \approx 0.000012629... \approx 1.3 \times 10^{-5}$$

When a fawn is 25 kg in size the rate of change of the energy expenditure of the fawn is about 1.3×10^{-5} kJ/day per gram.

(c)



$$63. \quad \log y = 1.54 - 0.008x - 0.658 \log x$$

$$(a) \quad y(x) = 10^{(1.54-0.008x-0.658 \log x)} = 10^{1.54} (10^{-0.008x}) (10^{-0.658 \log x}) = 10^{1.54} (10^{0.008})^{-x} (10^{\log x})^{-0.658} \approx 34.7(1.0186)^{-x} x^{-0.658}$$

$$(b) \quad (i) \quad y(20) = 34.7(1.0186)^{-20} 20^{-0.658} \approx 3.343$$

$$(ii) \quad y(40) = 34.7(1.0186)^{-40} 40^{-0.658} \approx 1.466$$

(c)

$$\frac{dy}{dx} = -34.7(1.0186)^{-x} x^{-0.658} \left(\ln(1.0186) + \frac{0.658}{x} \right)$$

$$(i) \quad \frac{dy}{dx}(20) = -0.17160... \approx -0.172$$

$$(ii) \quad \frac{dy}{dx}(40) = -0.051120... \approx -0.0511$$

64. $M(t) = (0.1t + 1) \ln \sqrt{t}$

(a) $M(15) = [0.1(15) + 1] \ln \sqrt{15}$
 ≈ 3.385

When the temperature is 15°C , the number of matings is about 3.

(b) $M(25) = [0.1(25) + 1] \ln \sqrt{25}$
 ≈ 5.663

When the temperature is 25°C , the number of matings is about 6.

(c) $M(t) = (0.1t + 1) \ln \sqrt{t}$
 $= (0.1t + 1) \ln t^{1/2}$
 $M'(t) = (0.1t + 1) \left(\frac{1}{2} \cdot \frac{1}{t} \right) + (\ln t^{1/2})(0.1)$
 $= 0.1 \ln \sqrt{t} + \frac{1}{2t} (0.1t + 1)$
 $M'(15) = 0.1 \ln \sqrt{15} + \frac{1}{2 \cdot 15} [(0.1)(15) + 1]$
 ≈ 0.22

When the temperature is 15°C , the rate of change of the number of matings is about 0.22.

65. $P(t) = (t + 100) \ln(t + 2)$
 $P'(t) = (t + 100) \left(\frac{1}{t + 2} \right) + \ln(t + 2)$
 $P'(2) = (102) \left(\frac{1}{4} \right) + \ln(4) \approx 26.9$
 $P'(8) = (108) \left(\frac{1}{10} \right) + \ln(10) \approx 13.1$

66. $P(t) = 30.60 - 5.79 \ln t$
 $P'(t) = -5.79 \left(\frac{1}{t} \right) = -\frac{5.79}{t}$

(a) For 1970, $t = 5$:

$$P(5) = 30.60 - 5.79 \ln(5) \approx 21.28$$

$$P'(5) = -\frac{5.79}{5} \approx -1.158$$

In 1970, the percent of persons 65 years and over with family income below the poverty level was about 21.28% and the rate of change was about -1.158% per year.

(b) For 1990, $t = 25$:

$$P(25) = 30.60 - 5.79 \ln(25) \approx 11.96$$

$$P'(25) = -\frac{5.79}{25} \approx -0.2316$$

In 1990, the percent of persons 65 years and over with family income below the poverty level was about 11.96% and the rate of change was about -0.2316% per year.

(c) For 2010 $t = 45$:

$$P(45) = 30.60 - 5.79 \ln(45) \approx 8.56$$

$$P'(45) = -\frac{5.79}{45} \approx -0.1287$$

In 2010, the percent of persons 65 years and over with family income below the poverty level was about 8.56% and the rate of change was about -0.1287% per year.

(d) The rate of change is approaching 0.

67. $M = \frac{2}{3} \log \frac{E}{0.007}$

(a) $8.9 = \frac{2}{3} \log \frac{E}{0.007}$

$$13.35 = \log \frac{E}{0.007}$$

$$10^{13.35} = \frac{E}{0.007}$$

$$E = 0.007(10^{13.35})$$

$$\approx 1.567 \times 10^{11} \text{ kWh}$$

(b) $10,000,000 \times 247 \text{ kWh/month}$

$$= 2,470,000,000 \text{ kWh/month}$$

$$\frac{1.567 \times 10^{11} \text{ kWh}}{2,470,000,000 \text{ kWh/month}} \approx 63.4 \text{ months}$$

(c) $M = \frac{2}{3} \log E - \frac{2}{3} \log 0.007$

$$\frac{dM}{dE} = \frac{2}{3} \left(\frac{1}{(\ln 10)E} \right) = \frac{2}{(3 \ln 10)E}$$

When $E = 70,000$,

$$\frac{dM}{dE} = \frac{2}{(3 \ln 10)70,000}$$

$$\approx 4.14 \times 10^{-6}$$

(d) $\frac{dM}{dE}$ varies inversely with E , so as E increases, $\frac{dM}{dE}$ decreases and approaches zero.

$$\begin{aligned}
 68. \quad f(x) &= \frac{29,000(2.322 - \log x)}{x} \\
 f'(x) &= \frac{x \left[29,000 \left(-\frac{1}{(\ln 10)x} \right) \right] - 29,000(2.322 - \log x)(1)}{x^2} \\
 &= 29,000 \cdot \frac{-\frac{1}{\ln 10} - (2.322 - \log x)}{x^2} \\
 &= -29,000 \cdot \frac{1 + (\ln 10)(2.322 - \log x)}{x^2 \ln 10}
 \end{aligned}$$

$$\begin{aligned}
 (a) \quad f(30) &= \frac{29,000(2.322 - \log 30)}{30} \\
 &\approx 817 \\
 f'(30) &= -29,000 \frac{1 + (\ln 10)(2.322 - \log 30)}{(30)^2 \ln 10} \\
 &\approx -41.2
 \end{aligned}$$

For a street 30 ft wide, the maximum traffic flow is 817 vehicles/hr, and the rate of change is -41.2 vehicles/hr per foot.

$$\begin{aligned}
 (b) \quad f(40) &= \frac{29,000(2.322 - \log 40)}{40} \\
 &\approx 522 \\
 f'(40) &= -29,000 \cdot \frac{1 + (\ln 10)(2.322 - \log 40)}{(40)^2 \ln 10} \\
 &\approx -20.9
 \end{aligned}$$

For a street 40 ft wide, the maximum traffic flow is 522 vehicles/hr, and the rate of change is -20.9 vehicles/hr per foot.

Chapter 4 Review Exercises

- False; the derivative of π^3 is 0 because π^3 is a constant.
- True
- False; the derivative of a product $u(x) \cdot v(x)$ is $u(x) \cdot v'(x) + v(x) \cdot u'(x)$.
- True
- False; the chain rule is used to take the derivative of a composition of functions.
- False; the derivative ce^x is ce^x for any constant c .
- False; the derivative of 10^x is $(\ln 10)10^x$.
- True

9. True

10. False; the derivative of $\log x$ is $\frac{1}{(\ln 10)x}$, whereas the derivative of $\ln x$ is $\frac{1}{x}$.

$$\begin{aligned}
 11. \quad y &= 5x^3 - 7x^2 - 9x + \sqrt{5} \\
 \frac{dy}{dx} &= 5(3x^2) - 7(2x) - 9 + 0 \\
 &= 15x^2 - 14x - 9
 \end{aligned}$$

$$\begin{aligned}
 12. \quad y &= 7x^3 - 4x^2 - 5x + \sqrt{2} \\
 \frac{dy}{dx} &= 7(3x^2) - 4(2x) - 5 + 0 \\
 &= 21x^2 - 8x - 5
 \end{aligned}$$

$$\begin{aligned}
 13. \quad y &= 9x^{8/3} \\
 \frac{dy}{dx} &= 9\left(\frac{8}{3}x^{5/3}\right) \\
 &= 24x^{5/3}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad y &= -4x^{-3} \\
 \frac{dy}{dx} &= -4(-3x^{-4}) \\
 &= 12x^{-4} \text{ or } \frac{12}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 15. \quad f(x) &= 3x^{-4} + 6\sqrt{x} \\
 &= 3x^{-4} + 6x^{1/2} \\
 f'(x) &= 3(-4x^{-5}) + 6\left(\frac{1}{2}x^{-1/2}\right) \\
 &= -12x^{-5} + 3x^{-1/2} \text{ or } -\frac{12}{x^5} + \frac{3}{x^{1/2}}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad f(x) &= 19x^{-1} - 8\sqrt{x} \\
 &= 19x^{-1} - 8x^{1/2} \\
 f'(x) &= 19(-x^{-2}) - 8\left(\frac{1}{2}x^{-1/2}\right) \\
 &= -19x^{-2} - 4x^{-1/2} \text{ or } -\frac{19}{x^2} - \frac{4}{x^{1/2}}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad k(x) &= \frac{3x}{4x+7} \\
 k'(x) &= \frac{(4x+7)(3) - (3x)(4)}{(4x+7)^2} \\
 &= \frac{12x+21-12x}{(4x+7)^2} \\
 &= \frac{21}{(4x+7)^2}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad r(x) &= \frac{-8x}{2x+1} \\
 r'(x) &= \frac{(2x+1)(-8) - (-8x)(2)}{(2x+1)^2} \\
 &= \frac{-16x-8+16x}{(2x+1)^2} \\
 &= \frac{-8}{(2x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 19. \quad y &= \frac{x^2 - x + 1}{x - 1} \\
 \frac{dy}{dx} &= \frac{(x-1)(2x-1) - (x^2 - x + 1)(1)}{(x-1)^2} \\
 &= \frac{2x^2 - 3x + 1 - x^2 + x - 1}{(x-1)^2} \\
 &= \frac{x^2 - 2x}{(x-1)^2}
 \end{aligned}$$

$$\begin{aligned}
 20. \quad y &= \frac{2x^3 - 5x^2}{x + 2} \\
 \frac{dy}{dx} &= \frac{(x+2)(6x^2 - 10x) - (2x^3 - 5x^2)(1)}{(x+2)^2} \\
 &= \frac{6x^3 + 12x^2 - 10x^2 - 20x - 2x^3 + 5x^2}{(x+2)^2} \\
 &= \frac{4x^3 + 7x^2 - 20x}{(x+2)^2}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad f(x) &= (3x^2 - 2)^4 \\
 f'(x) &= 4(3x^2 - 2)^3[3(2x)] \\
 &= 24x(3x^2 - 2)^3
 \end{aligned}$$

$$\begin{aligned}
 22. \quad k(x) &= (5x^3 - 1)^6 \\
 k'(x) &= 6(5x^3 - 1)^5[5(3x^2)] \\
 &= 90x^2(5x^3 - 1)^5
 \end{aligned}$$

$$\begin{aligned}
 23. \quad y &= \sqrt{2t^7 - 5} = (2t^7 - 5)^{1/2} \\
 \frac{dy}{dt} &= \frac{1}{2}(2t^7 - 5)^{-1/2}[2(7t^6)] \\
 &= 7t^6(2t^7 - 5)^{-1/2} \text{ or } \frac{7t^6}{(2t^7 - 5)^{1/2}}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad y &= -3\sqrt{8t^4 - 1} = -3(8t^4 - 1)^{1/2} \\
 \frac{dy}{dx} &= -3\left(\frac{1}{2}\right)(8t^4 - 1)^{-1/2}[8(4t^3)] \\
 &= -48t^3(8t^4 - 1)^{-1/2} \text{ or } \frac{-48t^3}{(8t^4 - 1)^{1/2}}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad y &= 3x(2x + 1)^3 \\
 \frac{dy}{dx} &= 3x(3)(2x + 1)^2(2) + (2x + 1)^3(3) \\
 &= (18x)(2x + 1)^2 + 3(2x + 1)^3 \\
 &= 3(2x + 1)^2[6x + (2x + 1)] \\
 &= 3(2x + 1)^2(8x + 1)
 \end{aligned}$$

$$\begin{aligned}
 26. \quad y &= 4x^2(3x - 2)^5 \\
 \frac{dy}{dx} &= (4x^2)[5(3x - 2)^4(3)] + (3x - 2)^5(8x) \\
 &= 60x^2(3x - 2)^4 + 8x(3x - 2)^5 \\
 &= 4x(3x - 2)^4[15x + 2](3x - 2) \\
 &= 4x(3x - 2)^4(15x + 6x - 4) \\
 &= 4x(3x - 2)^4(21x - 4)
 \end{aligned}$$

$$\begin{aligned}
 27. \quad r(t) &= \frac{5t^2 - 7t}{(3t + 1)^3} \\
 r'(t) &= \frac{(3t + 1)^3(10t - 7) - (5t^2 - 7t)(3)(3t + 1)^2(3)}{[(3t + 1)^3]^2} \\
 &= \frac{(3t + 1)^3(10t - 7) - 9(5t^2 - 7t)(3t + 1)}{(3t + 1)^6} \\
 &= \frac{(3t + 1)(10t - 7) - 9(5t^2 - 7t)}{(3t + 1)^4} \\
 &= \frac{30t^2 - 11t - 7 - 45t^2 + 63t}{(3t + 1)^4} \\
 &= \frac{-15t^2 + 52t - 7}{(3t + 1)^4}
 \end{aligned}$$

$$\begin{aligned}
 28. \quad s(t) &= \frac{t^3 - 2t}{(4t - 3)^4} \\
 s'(t) &= \frac{(4t - 3)^4(3t^2 - 2) - (t^3 - 2t)(4)(4t - 3)^3(4)}{[(4t - 3)^4]^2} \\
 &= \frac{(4t - 3)^4(3t^2 - 2) - 16(t^3 - 2t)(4t - 3)^3}{(4t - 3)^8} \\
 &= \frac{(4t - 3)^3[(4t - 3)(3t^2 - 2) - 16(t^3 - 2t)]}{(4t - 3)^8} \\
 &= \frac{(4t - 3)^3(12t^3 - 9t^2 - 8t + 6 - 16t^3 + 32t)}{(4t - 3)^8} \\
 &= \frac{-4t^3 - 9t^2 + 24t + 6}{(4t - 3)^5}
 \end{aligned}$$

$$\begin{aligned}
 29. \quad p(t) &= t^2(t^2 + 1)^{5/2} \\
 p'(t) &= t^2 \cdot \frac{5}{2}(t^2 + 1)^{3/2} \cdot 2t + 2t(t^2 + 1)^{5/2} \\
 &= 5t^3(t^2 + 1)^{3/2} + 2t(t^2 + 1)^{5/2} \\
 &= t(t^2 + 1)^{3/2}[5t^2 + 2(t^2 + 1)] \\
 &= t(t^2 + 1)^{3/2}(7t^2 + 2)
 \end{aligned}$$

$$\begin{aligned}
 30. \quad g(t) &= t^3(t^4 + 5)^{7/2} \\
 g'(t) &= t^3 \cdot \frac{7}{2}(t^4 + 5)^{5/2}(4t^3) + 3t^2 \cdot (t^4 + 5)^{7/2} \\
 &= 14t^6(t^4 + 5)^{5/2} + 3t^2(t^4 + 5)^{7/2} \\
 &= t^2(t^4 + 5)^{5/2}[14t^4 + 3(t^4 + 5)] \\
 &= t^2(t^4 + 5)^{5/2}(17t^4 + 15)
 \end{aligned}$$

$$\begin{aligned}
 31. \quad y &= -6e^{2x} \\
 \frac{dy}{dx} &= -6(2e^{2x}) = -12e^{2x}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad y &= 8e^{0.5x} \\
 \frac{dy}{dx} &= 8(0.5e^{0.5x}) = 4e^{0.5x}
 \end{aligned}$$

$$\begin{aligned}
 33. \quad y &= e^{-2x^3} \\
 g(x) &= -2x^3 \\
 g'(x) &= -6x^2 \\
 y' &= -6x^2e^{-2x^3}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad y &= -4e^{x^2} \\
 g(x) &= x^2 \\
 g'(x) &= 2x \\
 \frac{dy}{dx} &= (2x)(-4e^{x^2}) \\
 &= -8xe^{x^2}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad y &= 5x \cdot e^{2x} \\
 \text{Use the product rule.} \\
 \frac{dy}{dx} &= 5x(2e^{2x}) + e^{2x(5)} \\
 &= 10xe^{2x} + 5e^{2x} \\
 &= 5e^{2x}(2x + 1)
 \end{aligned}$$

36. $y = -7x^2e^{-3x}$

Use the product rule.

$$\begin{aligned}\frac{dy}{dx} &= (-7x^2)(-3e^{-3x}) + e^{-3x}(-14x) \\ &= 21x^2e^{-3x} - 14xe^{-3x} \\ &\text{or } 7xe^{-3x}(3x - 2)\end{aligned}$$

37. $y = \ln(2 + x^2)$

$$g(x) = 2x + x^2$$

$$g'(x) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2 + x^2}$$

38. $y = \ln(5x + 3)$

$$g(x) = 5x + 3$$

$$g'(x) = 5$$

$$\frac{dy}{dx} = \frac{5}{5x + 3}$$

39. $y = \frac{\ln|3x|}{x - 3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x - 3)\left(\frac{1}{3x}\right)(3) - (\ln|3x|)(1)}{(x - 3)^2} \\ &= \frac{x - 3 - x\ln|3x|}{x(x - 3)^2}\end{aligned}$$

40.

$$\begin{aligned}y &= \frac{\ln|2x - 1|}{x + 3} \\ \frac{dy}{dx} &= \frac{(x + 3)\left(\frac{2}{2x - 1}\right) - (\ln|2x - 1|)(1)}{(x + 3)^2} \cdot \frac{2x - 1}{2x - 1} \\ &= \frac{2(x + 3) - (2x - 1)\ln|2x - 1|}{(2x - 1)(x + 3)^2}\end{aligned}$$

41. $y = \frac{xe^x}{\ln(x^2 - 1)}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\ln(x^2 - 1)[xe^x + e^x] - xe^x\left(\frac{1}{x^2 - 1}\right)(2x)}{[\ln(x^2 - 1)]^2} \\ &= \frac{e^x(x + 1)\ln(x^2 - 1) - \frac{2x^2e^x}{x^2 - 1}}{[\ln(x^2 - 1)]^2} \cdot \frac{x^2 - 1}{x^2 - 1} \\ &= \frac{e^x(x + 1)(x^2 - 1)\ln(x^2 - 1) - 2x^2e^x}{(x^2 - 1)[\ln(x^2 - 1)]^2}\end{aligned}$$

42.

$$\begin{aligned}y &= \frac{(x^2 + 1)e^{2x}}{\ln x} \\ \frac{dy}{dx} &= \frac{\left[\ln x[(x^2 + 1)(2e^{2x}) + (e^{2x})(2x)] - (x^2 + 1)e^{2x}\left(\frac{1}{x}\right) \right]}{(\ln x)^2} \cdot \frac{x}{x} \\ &= \frac{x \ln x[2e^{2x}(x^2 + 1) + 2xe^{2x}] - (x^2 + 1)e^{2x}}{x(\ln x)^2} \\ &= \frac{e^{2x}[2x(\ln x)(x^2 + 1 + x) - (x^2 + 1)]}{x(\ln x)^2}\end{aligned}$$

43. $s = (t^2 + e^t)^2$

$$s' = 2(t^2 + e^t)(2t + e^t)$$

44. $q = (e^{2p+1} - 2)^4$

Use the chain rule.

$$\begin{aligned}\frac{dq}{dp} &= 4(e^{2p+1} - 2)^3[2e^{2p+1}] \\ &= 8e^{2p+1}(e^{2p+1} - 2)^3\end{aligned}$$

45. $y = 3 \cdot 10^{-x^2}$

$$\begin{aligned}\frac{dy}{dx} &= 3 \cdot (\ln 10)10^{-x^2}(-2x) \\ &= -6x(\ln 10) \cdot 10^{-x^2}\end{aligned}$$

46. $y = 10 \cdot 2^{\sqrt{x}}$

$$\begin{aligned}\frac{dy}{dx} &= 10 \cdot (\ln 2) \cdot 2^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2} \\ &= \frac{5(\ln 2)2^{\sqrt{x}}}{x^{1/2}}\end{aligned}$$

47. $g(z) = \log_2(z^3 + z + 1)$

$$\begin{aligned}g'(z) &= \frac{1}{\ln 2} \cdot \frac{3z^2 + 1}{z^3 + z + 1} \\ &= \frac{3z^2 + 1}{(\ln 2)(z^3 + z + 1)}\end{aligned}$$

48. $h(z) = \log(1 + e^z)$

$$\begin{aligned}h'(z) &= \frac{1}{\ln 10} \cdot \frac{e^z}{1 + e^z} \\ &= \frac{e^z}{(\ln 10)(1 + e^z)}\end{aligned}$$

49. $f(x) = e^{2x} \ln(xe^x + 1)$

Use the product rule.

$$\begin{aligned} f'(x) &= e^{2x} \left(\frac{e^x + xe^x}{xe^x + 1} \right) + [\ln(xe^x + 1)](2e^{2x}) \\ &= \frac{(1+x)e^{3x}}{xe^x + 1} + 2e^{2x} \ln(xe^x + 1) \end{aligned}$$

50. $f(x) = \frac{e^{\sqrt{x}}}{\ln(\sqrt{x} + 1)}$

Use the quotient rule.

$$\begin{aligned} u(x) &= e^{\sqrt{x}}, u'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \\ v(x) &= \ln(\sqrt{x} + 1), v'(x) = \frac{1}{2\sqrt{x}(\sqrt{x} + 1)} \\ f'(x) &= \frac{[\ln(\sqrt{x} + 1)] \left(\frac{e^{\sqrt{x}}}{2\sqrt{x}} \right) - e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}(\sqrt{x} + 1)} \right)}{[\ln(\sqrt{x} + 1)]^2} \\ &= \frac{e^{\sqrt{x}}[(\sqrt{x} + 1)\ln(\sqrt{x} + 1) - 1]}{2\sqrt{x}(\sqrt{x} + 1)[\ln(\sqrt{x} + 1)]^2} \end{aligned}$$

51. (a) $D_x(f[g(x)])$ at $x = 2$

$$\begin{aligned} &= f'[g(2)]g'(2) \\ &= f'(1) \left(\frac{3}{10} \right) \\ &= -5 \left(\frac{3}{10} \right) \\ &= -\frac{3}{2} \end{aligned}$$

(b) $D_x(f[g(x)])$ at $x = 3$

$$\begin{aligned} &= f'[g(3)]g'(3) \\ &= f'(2) \left(\frac{4}{11} \right) \\ &= -6 \left(\frac{4}{11} \right) \\ &= -\frac{24}{11} \end{aligned}$$

52. (a) $D_x(g[f(x)])$ at $x = 2$

$$\begin{aligned} &= g'[f(2)]f'(2) \\ &= g'(4)(-6) \\ &= \frac{6}{13}(-6) \\ &= -\frac{36}{13} \end{aligned}$$

(b) $D_x(g[f(x)])$ at $x = 3$

$$\begin{aligned} &= g'[f(3)]f'(3) \\ &= g'(2)(-7) \\ &= \frac{3}{10}(-7) \\ &= -\frac{21}{10} \end{aligned}$$

53. $y = x^2 - 6x$; tangent at $x = 2$

$$\frac{dy}{dx} = 2x - 6$$

Slope = $y'(2) = 2(2) - 6 = -2$

Use $(2, -8)$ and point-slope form.

$$\begin{aligned} y - (-8) &= -2(x - 2) \\ y + 8 &= -2x + 4 \\ y + 2x &= -4 \\ y &= -2x - 4 \end{aligned}$$

54. $y = 8 - x^2$; $x = 1$

$$\begin{aligned} y &= 8 - x^2 \\ \frac{dy}{dx} &= -2x \end{aligned}$$

slope = $y'(1) = -2(1) = -2$

Use $(1, 7)$ and $m = -2$ in the point-slope form.

$$\begin{aligned} y - 7 &= -2(x - 1) \\ y - 7 &= -2x + 2 \\ 2x + y &= 9 \\ y &= -2x + 9 \end{aligned}$$

55. $y = \frac{3}{x-1}$; tangent at $x = -1$

$$\begin{aligned} y &= \frac{3}{x-1} = 3(x-1)^{-1} \\ \frac{dy}{dx} &= 3(-1)(x-1)^{-2}(1) \\ &= -3(x-1)^{-2} \end{aligned}$$

Slope = $y'(-1) = -3(-1-1)^{-2} = -\frac{3}{4}$

Use $(-1, -\frac{3}{2})$ and point-slope form.

$$y - \left(-\frac{3}{2}\right) = -\frac{3}{4}[x - (-1)]$$

$$y + \frac{3}{2} = -\frac{3}{4}(x + 1)$$

$$y + \frac{6}{4} = -\frac{3}{4}x - \frac{3}{4}$$

$$y = -\frac{3}{4}x - \frac{9}{4}$$

56. $y = \frac{x}{x^2 - 1}; x = 2$

$$\frac{dy}{dx} = \frac{(x^2 - 1) \cdot 1 - x(2x)}{(x^2 - 1)^2}$$

$$= \frac{-x^2 - 1}{(x^2 - 1)^2}$$

The value of $\frac{dy}{dx}$ when $x = 2$ is the slope.

$$m = \frac{-(2^2) - 1}{(2^2 - 1)^2} = \frac{-5}{9} = -\frac{5}{9}$$

When $x = 2$,

$$y = \frac{2}{4 - 1} = \frac{2}{3}.$$

Use $m = -\frac{5}{9}$ with $P\left(2, \frac{2}{3}\right)$.

$$y - \frac{2}{3} = -\frac{5}{9}(x - 2)$$

$$y - \frac{6}{9} = -\frac{5}{9}x + \frac{10}{9}$$

$$y = -\frac{5}{9}x + \frac{16}{9}$$

57. $y = \sqrt{6x - 2}$; tangent at $x = 3$

$$y = \sqrt{6x - 2} = (6x - 2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(6x - 2)^{-1/2}(6)$$

$$= 3(6x - 2)^{-1/2}$$

$$\text{slope} = y'(3) = 3(6 \cdot 3 - 2)^{-1/2}$$

$$= 3(16)^{-1/2}$$

$$= \frac{3}{16^{1/2}} = \frac{3}{4}$$

Use (3, 4) and point-slope form.

$$y - 4 = \frac{3}{4}(x - 3)$$

$$y - \frac{16}{4} = \frac{3}{4}x - \frac{9}{4}$$

$$y = \frac{3}{4}x + \frac{7}{4}$$

58. $y = -\sqrt{8x + 1}; x = 3$

$$y = -(8x + 1)^{1/2}$$

$$\frac{dy}{dx} = -\frac{1}{2}(8x + 1)^{-1/2}(8)$$

$$\frac{dy}{dx} = -\frac{4}{(8x + 1)^{1/2}}$$

The value of $\frac{dy}{dx}$ when $x = 3$ is the slope.

$$m = -\frac{4}{(24 + 1)^{1/2}} = -\frac{4}{5}$$

When $x = 3$,

$$y = -\sqrt{24 + 1} = -5.$$

Use $m = -\frac{4}{5}$ with $P(3, -5)$.

$$y + 5 = -\frac{4}{5}(x - 3)$$

$$y + \frac{25}{5} = -\frac{4}{5}x + \frac{12}{5}$$

$$y = -\frac{4}{5}x - \frac{13}{5}$$

59. $y = e^x; x = 0$

$$\frac{dy}{dx} = e^x$$

The value of $\frac{dy}{dx}$ when $x = 0$ is the slope

$$m = e^0 = 1.$$

When $x = 0$, $y = e^0 = 1$. Use $m = 1$ with $P(0, 1)$.

$$y - 1 = 1(x - 0)$$

$$y = x + 1$$

60. $y = xe^x; x = 1$

$$\frac{dy}{dx} = xe^x + 1 \cdot e^x$$

$$= e^x(x + 1)$$

The value of $\frac{dy}{dx}$ when $x = 1$ is the slope.

$$m = e^1(1 + 1) = 2e$$

When $x = 1$, $y = 1e^1 = e$.

Use $m = 2e$ with $P(1, e)$.

$$\begin{aligned}y - e &= 2e(x - 1) \\y &= 2ex - e\end{aligned}$$

61. $y = \ln x$; $x = 1$

$$\frac{dy}{dx} = \frac{1}{x}$$

The value of $\frac{dy}{dx}$ when $x = 1$ is the

$$\text{slope } m = \frac{1}{1} = 1.$$

When $x = 1$, $y = \ln 1 = 0$.

Use $m = 1$ with $P(1, 0)$.

$$\begin{aligned}y - 0 &= 1(x - 1) \\y &= x - 1\end{aligned}$$

62. $y = x \ln x$; $x = e$

$$\begin{aligned}\frac{dy}{dx} &= x \cdot \frac{1}{x} + 1 \cdot \ln x \\&= 1 + \ln x\end{aligned}$$

The value of $\frac{dy}{dx}$ when $x = e$ is the slope.

$$m = 1 + \ln e = 1 + 1 = 2$$

When $x = e$, $y = e \ln e = e \cdot 1 = e$.

Use $m = 2$ with $P(e, e)$.

$$\begin{aligned}y - e &= 2(x - e) \\y &= 2x - e\end{aligned}$$

63. The slope of the graph of $y = x + k$ is 1. First, we find the point on the graph of $f(x) = \sqrt{2x - 1}$ at which the slope is also 1.

$$\begin{aligned}f(x) &= (2x - 1)^{1/2} \\f'(x) &= \frac{1}{2}(2x - 1)^{-1/2}(2) \\f'(x) &= \frac{1}{\sqrt{2x - 1}}\end{aligned}$$

The slope is 1 when

$$\begin{aligned}\frac{1}{\sqrt{2x - 1}} &= 1 \\1 &= \sqrt{2x - 1} \\1 &= 2x - 1 \\2x &= 2 \\x &= 1,\end{aligned}$$

and

$$f(1) = 1.$$

Therefore, at $P(1, 1)$ on the graph of

$$f(x) = \sqrt{2x - 1}, \text{ the slope is } 1.$$

An equation of the tangent line is

$$\begin{aligned}y - 1 &= 1(x - 1) \\y - 1 &= x - 1 \\y &= x + 0.\end{aligned}$$

Any tangent line intersects the curve in exactly one point.

From this we see that if $k = 0$, there is one point of intersection.

The graph of f is below the line $y = x + 0$.

Therefore, if $k > 0$, the graph of $y = x + k$ will not intersect the graph.

Consider the point $Q\left(\frac{1}{2}, 0\right)$ on the graph. We find an equation of the line through Q with slope 1.

$$\begin{aligned}y - 0 &= 1\left(x - \frac{1}{2}\right) \\y &= x - \frac{1}{2}\end{aligned}$$

The line with a slope of 1 through $Q\left(\frac{1}{2}, 0\right)$ will intersect the graph in two points. One is Q and the other is some point on the graph to the right of Q .

The graph of $y = x + 0$ intersects the graph in one point, while the graph of $y = x - \frac{1}{2}$ intersects it in two points. If we use a value of k in $y = x + k$ with $-\frac{1}{2} < k < 0$, we will have a line with a y-intercept between $-\frac{1}{2}$ and 0 and a slope of 1 which will intersect the graph in two points.

If k , the y-intercept, is less than $-\frac{1}{2}$, the graph of $y = x + k$ will be below point Q and will intersect the graph of f in exactly one point.

To summarize, the graph of $y = x + k$ will intersect the graph of $f(x) = \sqrt{2x - 1}$ in

- (1) no points if $k > 0$;
- (2) exactly one point if $k = 0$ or if $k < -\frac{1}{2}$;
- (3) exactly two points if $-\frac{1}{2} \leq k < 0$

64. (a) Use the chain rule.

Let $g(x) = \ln x$. Then $g'(x) = \frac{1}{x}$.

Let $y = g[f(x)]$.

Then $\frac{dy}{dx} = g'[f(x)] \cdot f'(x)$.

so

$$\frac{d \ln f(x)}{dx} = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}.$$

$$\hat{f} = \frac{f'(x)}{f(x)}, \hat{g} = f \frac{g'(x)}{g(x)}$$

$$\begin{aligned} \widehat{fg} &= \frac{(fg)'(x)}{(fg)(x)} \\ &= \frac{f(x) \cdot g'(x) + g(x) \cdot f'(x)}{f(x)g(x)} \\ &= \frac{f(x) \cdot g'(x)}{f(x) \cdot g(x)} + \frac{g(x) \cdot f'(x)}{f(x) \cdot g(x)} \\ &= \frac{g'(x)}{g(x)} + \frac{f'(x)}{f(x)} \\ \text{(b)} \quad &= \hat{g} + \hat{f} \text{ or } \hat{f} + \hat{g} \end{aligned}$$

65. Using the result $\widehat{fg} = \hat{f} + \hat{g}$, the total amount of tuition collected goes up by approximately $2\% + 3\% = 5\%$.

Let T = tuition per person before the increase and S the number of students before the increase. Then the new tuition is $1.03T$ and the new numbers of students is $1.02S$, so the total amount of tuition collected is $(1.03T)(1.02S) = 1.0506TS$, which is an increase of 5.06%.

67. $C(x) = \sqrt{x+1}$
 $\bar{C}(x) = \frac{C(x)}{x} = \frac{\sqrt{x+1}}{x}$
 $= \frac{(x+1)^{1/2}}{x}$
 $\bar{C}'(x) = \frac{x[\frac{1}{2}(x+1)^{-1/2}] - (x+1)^{1/2}(1)}{x^2}$
 $= \frac{\frac{1}{2}x(x+1)^{-1/2} - (x+1)^{1/2}}{x^2}$
 $= \frac{x(x+1)^{-1/2} - 2(x+1)^{1/2}}{2x^2}$
 $= \frac{(x+1)^{-1/2}[x - 2(x+1)]}{2x^2}$
 $= \frac{(x+1)^{-1/2}(-x-2)}{2x^2}$
 $= \frac{-x-2}{2x^2(x+1)^{1/2}}$
68. $C(x) = \sqrt{3x+2}$
 $\bar{C}(x) = \frac{C(x)}{x} = \frac{\sqrt{3x+2}}{x} = \frac{(3x+2)^{1/2}}{x}$
 $\bar{C}'(x) = \frac{x[\frac{1}{2}(3x+2)^{-1/2}(3)] - (3x+2)^{1/2}(1)}{x^2}$
 $= \frac{\frac{3}{2}x(3x+2)^{-1/2} - (3x+2)^{1/2}}{x^2}$
 $= \frac{3x(3x+2)^{-1/2} - 2(3x+2)^{1/2}}{2x^2}$
 $= \frac{(3x+2)^{-1/2}[3x - 2(3x+2)]}{2x^2}$
 $= \frac{3x - 6x - 4}{2x^2(3x+2)^{1/2}} = \frac{-3x-4}{2x^2(3x+2)^{1/2}}$
69. $C(x) = (x^2+3)^3$
 $\bar{C}(x) = \frac{C(x)}{x} = \frac{(x^2+3)^3}{x}$
 $\bar{C}'(x) = \frac{x[3(x^2+3)^2(2x)] - (x^2+3)^3(1)}{x^2}$
 $= \frac{6x^2(x^2+3)^2 - (x^2+3)^3}{x^2}$
 $= \frac{(x^2+3)^2[6x^2 - (x^2+3)]}{x^2}$
 $= \frac{(x^2+3)^2(5x^2-3)}{x^2}$

$$\begin{aligned}
 70. \quad C(x) &= (4x + 3)^4 \\
 \bar{C}(x) &= \frac{C(x)}{x} = \frac{(4x + 3)^4}{x} \\
 \bar{C}'(x) &= \frac{x[4(4x + 3)^3(4)] - (4x + 3)^4(1)}{x^2} \\
 &= \frac{16x(4x + 3)^3 - (4x + 3)^4}{x^2} \\
 &= \frac{(4x + 3)^3[16x - (4x + 3)]}{x^2} \\
 &= \frac{(4x + 3)^3(12x - 3)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 71. \quad C(x) &= 10 - e^{-x} \\
 \bar{C}(x) &= \frac{C(x)}{x} \\
 \bar{C}(x) &= \frac{10 - e^{-x}}{x} \\
 \bar{C}'(x) &= \frac{x(e^{-x}) - (10 - e^{-x}) \cdot 1}{x^2} \\
 &= \frac{e^{-x}(x + 1) - 10}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 72. \quad C(x) &= \ln(x + 5) \\
 \bar{C}(x) &= \frac{\ln(x + 5)}{x} \\
 \bar{C}'(x) &= \frac{x \cdot \frac{1}{x+5} - \ln(x + 5) \cdot 1}{x^2} \\
 &= \frac{x - (x + 5)\ln(x + 5)}{x^2(x + 5)}
 \end{aligned}$$

$$\begin{aligned}
 73. \quad S(x) &= 1000 + 60\sqrt{x} + 12x \\
 &= 1000 + 60x^{1/2} + 12x \\
 \frac{dS}{dx} &= 60\left(\frac{1}{2}x^{-1/2}\right) + 12 \\
 &= 30x^{-1/2} + 12 = \frac{30}{\sqrt{x}} + 12 \\
 (a) \quad \frac{dS}{dx}(9) &= \frac{30}{\sqrt{9}} + 12 = \frac{30}{3} + 12 = 22 \\
 &\text{Sales will increase by \$22 million when} \\
 &\text{\$1000 more is spent on research.} \\
 (b) \quad \frac{dS}{dx}(16) &= \frac{30}{\sqrt{16}} + 12 = \frac{30}{4} + 12 = 19.5 \\
 &\text{Sales will increase by \$19.5 million when} \\
 &\text{\$1000 more is spent on research.}
 \end{aligned}$$

$$(c) \quad \frac{dS}{dx}(25) = \frac{30}{\sqrt{25}} + 12 = \frac{30}{5} + 12 = 18$$

Sales will increase by \$18 million when \$1000 more is spent on research.

- (d) As more money is spent on research, the increase in sales is decreasing.

$$\begin{aligned}
 74. \quad P(x) &= \frac{x^2}{2x + 1} \\
 P'(x) &= \frac{(2x + 1)(2x) - x^2(2)}{(2x + 1)^2} \\
 &= \frac{4x^2 + 2x - 2x^2}{(2x + 1)^2} \\
 &= \frac{2x^2 + 2x}{(2x + 1)^2}
 \end{aligned}$$

$$(a) \quad P'(4) = \frac{2(4)^2 + 2(4)}{[2(4) + 1]^2} = \frac{40}{81} \approx 0.4938$$

In dollars, this is $\frac{40}{81}(100) \approx \49.38 , which is the approximate increase in profit from selling the fifth unit.

$$\begin{aligned}
 (b) \quad P'(12) &= \frac{2(12)^2 + 2(12)}{[2(12) + 1]^2} \\
 &= \frac{312}{625} = 0.4992
 \end{aligned}$$

In dollars, this is $\frac{312}{625}(100) \approx \49.92 , which is the approximate increase in profit from selling the thirteenth unit.

$$\begin{aligned}
 (c) \quad P'(20) &= \frac{2(20)^2 + 2(20)}{[2(20) + 1]^2} \\
 &= \frac{840}{1681} \approx 0.4997
 \end{aligned}$$

In dollars, this is $\frac{840}{1681}(100) \approx \49.97 , which is the approximate increase in profit from selling the twenty-first unit.

- (d) As the number sold increases, the marginal profit increases.

- (e) The average profit is defined by

$$\bar{P}(x) = \frac{P(x)}{x} = \frac{\frac{x^2}{2x+1}}{x} = \frac{x}{2x+1}$$

The marginal average profit is given by

$$\begin{aligned}\frac{d}{dx}(\bar{P}(x)) &= \frac{(2x+1)(1) - (x)(2)}{(2x+1)^2} \\ &= \frac{2x+1-2x}{(2x+1)^2} \\ &= \frac{1}{(2x+1)^2}\end{aligned}$$

The marginal average profit when 4 units are sold is

$$\bar{P}'(4) = \frac{1}{[2(4)+1]^2} = \frac{1}{81} \approx 0.0123$$

The average profit is going up at a rate of $100\left(\frac{1}{81}\right) \approx \1.23 per unit when 4 units are sold.

$$\begin{aligned}75. \quad T(x) &= \frac{1000 + 60x}{4x + 5} \\ T'(x) &= \frac{(4x+5)(60) - (1000+60x)(4)}{(4x+5)^2} \\ &= \frac{240x + 300 - 4000 - 240x}{(4x+5)^2} \\ &= \frac{-3700}{(4x+5)^2}\end{aligned}$$

$$(a) \quad T'(9) = \frac{-3700}{[4(9)+5]^2} = \frac{-3700}{1681} \approx -2.201$$

Costs will decrease \$2201 for the next \$100 spent on training.

$$\begin{aligned}(b) \quad T'(19) &= \frac{-3700}{[4(19)+5]^2} \\ &= \frac{-3700}{6561} \approx -0.564\end{aligned}$$

Costs will decrease \$564 for the next \$100 spent on training.

(c) Costs will always decrease because

$$T'(x) = \frac{-3700}{(4x+5)^2}$$

will always be negative.

$$\begin{aligned}76. \quad A(r) &= 1000\left(1 + \frac{r}{400}\right)^{48} \\ A'(r) &= 1000 \cdot 48\left(1 + \frac{r}{400}\right)^{47} \cdot \frac{1}{400} \\ &= 120\left(1 + \frac{r}{400}\right)^{47} \\ A'(5) &= 120\left(1 + \frac{5}{400}\right)^{47} \approx 215.15\end{aligned}$$

The balance increases by approximately \$215.15 for every 1% increase in the interest rate when the rate is 5%.

$$\begin{aligned}77. \quad A(r) &= 1000e^{12r/100} \\ A'(r) &= 1000e^{12r/100} \cdot \frac{12}{100} \\ &= 120e^{12r/100} \\ A'(5) &= 120e^{0.6} \approx 218.65\end{aligned}$$

The balance increases by approximately \$218.65 for every 1% increase in the interest rate when the rate is 5%.

$$\begin{aligned}78. \quad T(r) &= \frac{\ln 2}{\ln\left(1 + \frac{r}{100}\right)} \\ T(r) &= \ln 2 \left[\ln\left(1 + \frac{r}{100}\right) \right]^{-1} \\ T'(r) &= (\ln 2)(-1) \left[\ln\left(1 + \frac{r}{100}\right) \right]^{-2} \cdot \frac{\frac{1}{100}}{1 + \frac{r}{100}} \\ T'(r) &= \frac{-\ln 2}{(100+r) \left[\ln\left(1 + \frac{r}{100}\right) \right]^2} \\ T'(5) &= -\frac{\ln 2}{105 (\ln 1.05)^2} \approx -2.77\end{aligned}$$

The doubling time decreases by approximately 2.77 years for every 1% increase in the interest rate when the interest rate is 5%.

$$\begin{aligned}79. \quad p(t) &= -0.00132t^4 + 0.0665t^3 \\ &\quad - 1.127t^2 + 11.581t + 105.655 \\ p'(t) &= -0.00528t^3 + 0.1995t^2 \\ &\quad - 2.254t + 11.581\end{aligned}$$

(a) For 1995, $t = 15$:

$$\begin{aligned}p'(15) &= -0.00528(15)^3 + 0.1995(15)^2 \\ &\quad - 2.254(15) + 11.581 \\ &\approx 4.839\end{aligned}$$

In 1995 the volume of mail was increasing by about 4,839,000,000 pieces per year.

- (b) For 2005, $t = 25$:

$$\begin{aligned} p'(25) &= -0.00528(25)^3 + 0.1995(25)^2 \\ &\quad - 2.254(25) + 11.581 \\ &\approx -2.582 \end{aligned}$$

In 1995 the volume of mail was decreasing by about 2,582,000,000 pieces per year.

80. (a) Using the regression feature on a graphing calculator, a cubic function that models the data is

$$\begin{aligned} y &= (1.108 \times 10^{-4})t^3 - 0.0164t^2 \\ &\quad + 0.0913t + 61.99. \end{aligned}$$

Using the regression feature on a graphing calculator, a quartic function that models the data is

$$\begin{aligned} y &= (1.749 \times 10^{-6})t^4 - 2.712 \times 10^{-4}t^3 \\ &\quad + 0.00961t^2 - 0.464t + 63.11 \end{aligned}$$

- (b) Using the cubic function, $\frac{dy}{dx}$ at $x = 105$ is about 0.31 percent per year. Using the quartic function, $\frac{dy}{dx}$ at $x = 105$ is about 0.68 percent per year.

81. (a) Using the regression feature on a graphing calculator, a cubic function that models the data is

$$\begin{aligned} y &= (1.799 \times 10^{-5})t^3 - (3.177 \times 10^{-4})t^2 \\ &\quad - 0.06866t + 2.504. \end{aligned}$$

Using the regression feature on a graphing calculator, a quartic function that models the data is

$$\begin{aligned} y &= (-1.112 \times 10^{-6})t^4 + (2.920 \times 10^{-4})t^3 \\ &\quad - 0.02238t^2 + 0.6483t - 4.278. \end{aligned}$$

- (b) Using the cubic function, $\frac{dy}{dx}$ at $x = 105$ is about 0.59 dollar per year. Using the quartic function, $\frac{dy}{dx}$ at $x = 105$ is about 0.46 dollar per year.

82. $P(t) = ae^{0.05t}$

$$P'(t) = ae^{0.05t}(0.05)$$

$$P'(t) = P(t)(0.05)$$

If $P(t) = 1,000,000$, then

Chapter 4 CALCULATING THE DERIVATIVE

$$P'(t) = 1,000,000(0.05) = 50,000.$$

The population is growing at a rate of 50,000 per year.

83. $G(t) = \frac{mG_0}{G_0 + (m - G_0)e^{-kmt}}$, where
 $m = 30,000$, $G_0 = 2000$, and $k = 5.10^{-6}$.

(a)

$$\begin{aligned} G(t) &= \frac{(30,000)(2000)}{2000 + (30,000 - 2000)e^{-5.10^{-6}(30,000)t}} \\ &= \frac{30,000}{1 + 14e^{-0.15t}} \end{aligned}$$

(b)

$$\begin{aligned} G(t) &= 30,000(1 + 14e^{-0.15t})^{-1} \\ G(6) &= 30,000(1 + 14e^{-0.90})^{-1} \approx 4483 \\ G'(t) &= -30,000(1 + 14e^{-0.15t})^{-2}(-2.1e^{-0.15t}) \\ &= \frac{63,000e^{-0.15t}}{(1 + 14e^{-0.15t})^2} \\ G'(6) &= \frac{63,000e^{-0.90}}{(1 + 14e^{-0.90})^2} \approx 572 \end{aligned}$$

The population is 4483, and the rate of growth is 572.

84. $L(t) = 71.5(1 - e^{-0.1t})$ and
 $W(L) = 0.01289 \cdot L^{2.9}$

(a) $L(5) = 71.5(1 - e^{-0.5}) \approx 28.1$

The approximate length of a 5-year-old monkey-face is 28.1 cm.

(b) $L'(t) = 71.5(0.1e^{-0.1t})$
 $L'(5) = 71.5(0.1e^{-0.5})$
 ≈ 4.34

The length is growing by about 4.34 cm/year.

(c) $W[L(5)] \approx 0.01289(28.1)^{2.9} \approx 205$

The approximate weight is 205 grams.

(d) $W'(L) = 0.01289(2.9)L^{1.9}$
 $= 0.037381L^{1.9}$

$$\begin{aligned} W'[L(5)] &\approx 0.037381(28.1)^{1.9} \\ &\approx 21.2 \end{aligned}$$

The rate of change of the weight with respect to length is 21.2 grams/cm.

$$\begin{aligned} \text{(e)} \quad \frac{dW}{dt} &= \frac{dW}{dL} \cdot \frac{dL}{dt} \\ &\approx (21.2)(4.34) \\ &\approx 92.0 \end{aligned}$$

The weight is growing at about 92.0 grams/year.

$$85. \quad M(t) = 3583e^{-0.020(t-66)}$$

$$\begin{aligned} \text{(a)} \quad M(250) &= 3583e^{-0.020(250-66)} \\ &\approx 3493.76 \text{ grams,} \end{aligned}$$

or about 3.5 kilograms

$$\begin{aligned} \text{(b)} \quad \text{As } t \rightarrow \infty, -e^{-0.020(t-66)} &\rightarrow 0, \\ e^{-e^{-0.020(t-66)}} &\rightarrow 1, \text{ and } M(t) \rightarrow 3583 \\ &\text{grams or about 3.6 kilograms.} \end{aligned}$$

$$\text{(c)} \quad 50\% \text{ of } 3583 \text{ is } 1791.5.$$

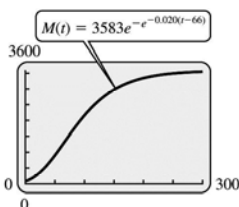
$$\begin{aligned} 1791.5 &= 3583e^{-0.020(t-66)} \\ \ln\left(\frac{1791.5}{3583}\right) &= -e^{-0.020(t-66)} \\ \ln\left(\ln\frac{3583}{1791.5}\right) &= -0.020(t-66) \\ t &= -\frac{1}{0.020} \ln\left(\ln\frac{3583}{1791.5}\right) + 66 \\ &\approx 84 \text{ days} \end{aligned}$$

(d)

$$\begin{aligned} D_t M(t) &= 3583e^{-0.020(t-66)} D_t \left(-e^{-0.020(t-66)} \right) \\ &= 3583e^{-0.020(t-66)} \left(-e^{-0.020(t-66)} \right) (-0.020) \\ &= 71.66e^{-0.020(t-66)} \left(e^{-0.020(t-66)} \right) \end{aligned}$$

When $t = 250$, $D_t M(t) \approx 1.76$ g/day.

(e)



Growth is initially rapid, then tapers off.

(f)

Day	Weight	Rate
50	904	24.90
100	2159	21.87
150	2974	11.08
200	3346	4.59
250	3494	1.76
300	3550	0.66

$$86. \quad a(t) = 11.14(1.023)^t$$

$$a'(t) = 11.14(\ln 1.023)(1.023)^t$$

(a) For 2005, $t = 5$:

$$\begin{aligned} a'(5) &= 11.14(\ln 1.023)(1.023)^5 \\ &\approx 0.284 \end{aligned}$$

In 2005 the instantaneous rate of change is 284,000 per year.

(b) For 2025, $t = 25$:

$$\begin{aligned} a'(25) &= 11.14(\ln 1.023)(1.023)^{25} \\ &\approx 0.447 \end{aligned}$$

In 2025 the instantaneous rate of change will be 447,000 per year.

$$87. \quad C(t) = 19,231(1.2647)^t$$

$$C'(t) = 19,231(\ln 1.2647)(1.2647)^t$$

(a) For 2005, $t = 5$:

$$\begin{aligned} C'(5) &= 19,231(\ln 1.2647)(1.2647)^5 \\ &\approx 14,612 \end{aligned}$$

In 2005 the rate of change in the energy capacity is 14,612 megawatts per year.

(b) For 2010, $t = 10$:

$$\begin{aligned} C'(10) &= 19,231(\ln 1.2647)(1.2647)^{10} \\ &\approx 47,276 \end{aligned}$$

In 2010 the rate of change in the energy capacity is 47,276 megawatts per year.

(c) For 2015, $t = 15$:

$$\begin{aligned} C'(15) &= 19,231(\ln 1.2647)(1.2647)^{15} \\ &\approx 152,960 \end{aligned}$$

In 2015 the rate of change in the energy capacity is 152,960 megawatts per year.

$$88. \quad f(t) = \frac{8}{t+1} + \frac{20}{t^2+1}$$

- (a) The average velocity from $t = 1$ to $t = 3$ is given by

$$\begin{aligned} \text{average velocity} &= \frac{f(3) - f(1)}{3 - 1} \\ &= \frac{\left(\frac{8}{4} + \frac{20}{10}\right) - \left(\frac{8}{2} + \frac{20}{2}\right)}{2} \\ &= \frac{4 - 14}{2} \\ &= -5 \end{aligned}$$

Belmar's average velocity between 1 sec and 3 sec is -5 ft/sec.

$$\begin{aligned} (b) \quad f(t) &= 8(t+1)^{-1} + 20(t^2+1)^{-1} \\ f'(t) &= -8(t+1)^{-2} \cdot 1 - 20(t^2+1)^{-2} \cdot 2t \\ &= -\frac{8}{(t+1)^2} - \frac{40t}{(t^2+1)^2} \\ f'(3) &= -\frac{8}{16} - \frac{120}{100} \\ &= -0.5 - 1.2 \\ &= -1.7 \end{aligned}$$

Belmar's instantaneous velocity at 3 sec is -1.7 ft/sec.

$$\begin{aligned} 89. \quad p(x) &= 1.757(1.0249)^{x-1930} \\ p'(x) &= 1.757(\ln 1.0249)(1.0249)^{x-1930} \\ p'(2000) &= 1.757(\ln 1.0249)(1.0249)^{2000-1930} \\ &\approx 0.242 \end{aligned}$$

The production of corn is increasing at a rate of 0.242 billion bushels per year in 2000.

90. (a) $N(t) = N_0 e^{-0.217t}$, where $t = 1$ and $N_0 = 210$

$$\begin{aligned} N(1) &= 210e^{-0.217(1)} \\ &\approx 169 \end{aligned}$$

The number of words predicted to be in use in 1950 is 169, and the actual number in use was 167.

$$(b) \quad N(1.1) = 210e^{-0.217(1.1)} \approx 165$$

In 2050 there will be about 165 words still being used.

$$\begin{aligned} (c) \quad N(t) &= 210e^{-0.217t} \\ N'(t) &= 210e^{-0.217t} \cdot (-0.217) \\ &= -45.57e^{-0.217t} \\ N'(1.1) &= -45.57e^{-0.217(1.1)} \\ &\approx -36 \end{aligned}$$

In the year 2050 the number of words in use will be decreasing by 36 words per millennium.

$$\begin{aligned} 91. \quad f(x) &= k(x-49)^6 + .8 \\ f'(x) &= k \cdot 6(x-49)^5 \\ &= (3.8 \times 10^{-9})(6)(x-49)^5 \\ &= (2.28 \times 10^{-8})(x-49)^5 \end{aligned}$$

$$(a) \quad f'(20) = (2.28 \times 10^{-8})(20-49)^5 \approx -0.4677$$

fatalities per 1000 licensed drivers per 100 million miles per year.

At the age of 20, each extra year results in a decrease of 0.4677 fatalities per 1000 licensed drivers per 100 million miles.

$$(b) \quad f'(60) = (2.28 \times 10^{-8})(60-49)^5 \approx 0.003672$$

fatalities per 1000 licensed drivers per 100 million miles per year.

At the age of 60, each extra year results in an increase of 0.003672 fatalities per 1000 licensed drivers per 100 million miles.

Extended Application: Electric Potential and Electric Field

$$\begin{aligned} 1. \quad E &= -\frac{dV}{dz} \\ V &= k_1[(z^2 + R^2)^{1/2} - z] \\ \frac{dV}{dz} &= k_1 \cdot \frac{d}{dz}[(z^2 + R^2)^{1/2} - z] \\ &= k_1 \left[\frac{1}{2}(z^2 + R^2)^{-1/2}(2z + 0) - 1 \right] \\ &= k_1 \left[\frac{z}{\sqrt{z^2 + R^2}} - 1 \right] \\ E &= -\frac{dV}{dz} = -k_1 \left[\frac{z}{\sqrt{z^2 + R^2}} - 1 \right] \end{aligned}$$

$$2. \quad E = -\frac{dV}{dz}$$

$$V = \frac{k_2}{z} = k_2 z^{-1}$$

$$\frac{dV}{dz} = k_2(-1)z^{-2} = -\frac{k_2}{z^2}$$

$$E = -\frac{dV}{dz} = \frac{k_2}{z^2}$$

$$3. \quad E = -\frac{dV}{dz}$$

$$V = k_1 \left(R - z + \frac{z^2}{2R} \right)$$

$$\frac{dV}{dz} = k_1 \left(0 - 1 + \frac{1}{2R} \cdot 2z \right)$$

$$= k_1 \left(\frac{z}{R} - 1 \right)$$

$$E = -\frac{dV}{dz} = -k_1 \left(\frac{z}{R} - 1 \right)$$

$$= k_1 \left(1 - \frac{z}{R} \right)$$

4. If z is very close to the disk, then $\frac{z^2}{2R}$ approaches

zero, and $V = k_1 \left(R - z + \frac{z^2}{2R} \right)$

becomes $V = k_1(R - z)$.

$$V = k_1(R - z)$$

$$\frac{dV}{dz} = k_1(0 - 1) = -k_1$$

Therefore, $E = -\frac{dV}{dz} = -(-k_1) = k_1$,

where k_1 is a constant.

Chapter 5

GRAPHS AND THE DERIVATIVE

5.1 Increasing and Decreasing Functions

Your Turn 1

Scanning across the x -values from left to right we see that the function is increasing on $(-1, 2)$ and $(4, \infty)$ and decreasing on $(-\infty, -1)$ and $(2, 4)$.

Your Turn 2

$$f(x) = -x^3 - 2x^2 + 15x + 10$$

$$f'(x) = -3x^2 - 4x + 15$$

Set $f'(x)$ equal to 0 and solve for x .

$$-3x^2 - 4x + 15 = 0$$

$$3x^2 + 4x - 15 = 0$$

$$(3x - 5)(x + 3) = 0$$

$$x = \frac{5}{3} \text{ or } x = -3$$

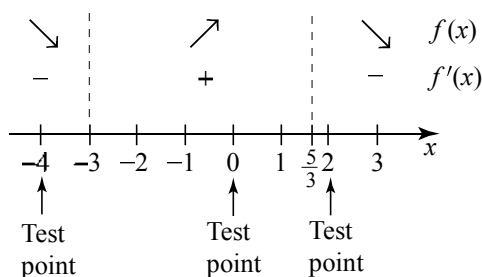
The only critical numbers are -3 and $5/3$. These points determine three intervals: $(-\infty, -3)$, $(-3, 5/3)$, and $(5/3, \infty)$. Determine the sign of f' in each interval by picking a test point and evaluating $f'(x)$.

$$f'(-4) = -(3(-4) - 5)((-4) + 3) = -(-17)(-1) < 0$$

$$f'(0) = -(3(0) - 5)((0) + 3) = -(-5)(3) > 0$$

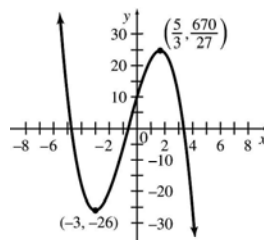
$$f'(2) = -(3(2) - 5)((2) + 3) = -(1)(5) < 0$$

The arrows in each interval in the figure below indicate where f is increasing or decreasing.



The function $f(x) = -x^3 - 2x^2 + 15x + 10$ is increasing on $(-3, 5/3)$ and decreasing on $(-\infty, -3)$ and $(5/3, \infty)$.

Since $f(-3) = -26$ and $f(\frac{5}{3}) = \frac{670}{27} \approx 24.8$, the graph goes through $(-3, -26)$ and $(\frac{5}{3}, \frac{670}{27})$.



Your Turn 3

$$f(x) = (2x + 4)^{2/5}$$

$$f'(x) = \frac{2}{5}(2x + 4)^{-3/5}(2)$$

$$= \frac{4}{5(2x + 4)^{3/5}}$$

Find the critical numbers. $f'(x)$ is never 0 so we find any values where $f'(x)$ fails to be defined by setting the denominator equal to 0.

$$5(2x + 4)^{3/5} = 0$$

$$(2x + 4)^{3/5} = 0$$

$$2x + 4 = 0^{5/3} = 0$$

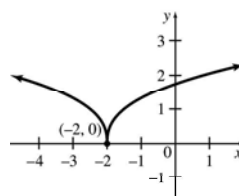
$$2x = -4$$

$$x = -2$$

Since $f'(-2)$ does not exist but $f(-2)$ is defined, $x = -2$ is a critical number.

Choosing -3 and -1 as test points, we find that $f'(-3) < 0$ and $f'(-1) > 0$. Thus the function f is decreasing on $(-\infty, -2)$ and increasing on $(-2, \infty)$.

Using the fact that $f(-2) = 0$ and $f(0) = 4^{2/5} \approx 1.74$ we can sketch the following graph. Note that the graph is not smooth at $x = -2$, where the derivative is not defined.



Your Turn 4

$$f(x) = \frac{-2x}{x+2}; \text{ the domain of } f \text{ excludes } -2$$

$$\begin{aligned} f'(x) &= \frac{(x+2)(-2) - (-2x)(1)}{(x+2)^2} \\ &= \frac{-2x - 4 + 2x}{(x+2)^2} \\ &= -\frac{4}{(x+2)^2} \end{aligned}$$

$f'(x)$ is never 0 and fails to exist at $x = -2$; however, f is undefined at $x = -2$ so there are no critical numbers. Apply the first derivative test for numbers on either side of $x = -2$.

$$f'(-3) = -\frac{4}{(-3+2)^2} < 0$$

$$f'(-1) = -\frac{4}{(-1+2)^2} < 0$$

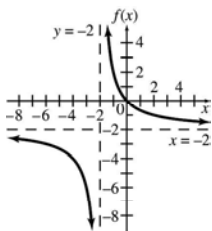
Since $f'(x)$ is negative wherever it is defined, the function is never increasing and is decreasing on $(-\infty, -2)$ and $(-2, \infty)$.

Next find any asymptotes. Since $x = -2$ makes the denominator of f equal to 0, the line $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow \infty} \frac{-2x}{x+2} = \lim_{x \rightarrow \infty} \frac{-2}{1 + \frac{2}{x}} = -2$$

The line $y = -2$ is a horizontal asymptote.

The x -intercept is $(0, 0)$. The graph is shown below.

**5.1 Exercises**

- By reading the graph, f is
 - increasing on $(1, \infty)$ and
 - decreasing on $(-\infty, 1)$.
- By reading the graph, f is
 - increasing on $(-\infty, 4)$ and
 - decreasing on $(4, \infty)$.
- By reading the graph, g is
 - increasing on $(-\infty, -2)$ and
 - decreasing on $(-2, \infty)$.
- By reading the graph, g is
 - increasing on $(3, \infty)$ and
 - decreasing on $(-\infty, 3)$.
- By reading the graph, h is
 - increasing on $(-\infty, -4)$ and $(-2, \infty)$ and
 - decreasing on $(-4, -2)$.
- By reading the graph, h is
 - increasing on $(1, 5)$ and
 - decreasing on $(-\infty, 1)$ and $(5, \infty)$.
- By reading the graph, f is
 - increasing on $(-7, -4)$ and $(-2, \infty)$ and
 - decreasing on $(-\infty, -7)$ and $(-4, -2)$.
- By reading the graph, f is
 - increasing on $(-3, 0)$ and $(3, \infty)$ and
 - decreasing on $(-\infty, -3)$ and $(0, 3)$.
- Since the graph of $f'(x)$ is positive for $x < -1$ and $x > 3$, the intervals where $f(x)$ is increasing are $(-\infty, -1)$ and $(3, \infty)$.
 - Since the graph of $f'(x)$ is negative for $-1 < x < 3$, the interval where $f(x)$ is decreasing is $(-1, 3)$.
- Since the graph of the $f'(x)$ is positive for $3 < x < 5$, the intervals where $f(x)$ is increasing is $(3, 5)$.
 - Since the graph of $f'(x)$ is negative for $x < 3$ and $x > 5$, the intervals where $f(x)$ is decreasing are $(-\infty, 3)$ and $(5, \infty)$.

11. (a) Since the graph of $f'(x)$ is positive for $x < -8$, $-6 < x < -2.5$ and $x > -1.5$, the intervals where $f(x)$ is increasing are $(-\infty, -8)$, $(-6, -2.5)$, and $(-1.5, \infty)$.
- (b) Since the graph of $f'(x)$ is negative for $-8 < x < -6$ and $-2.5 < x < -1.5$, the intervals where $f(x)$ is decreasing are $(-8, -6)$ and $(-2.5, -1.5)$.

12. (a) Since the graph of $f'(x)$ is positive for $x < -3$, $-3 < x < 3$, and $x > 3$, the intervals where $f(x)$ is increasing are $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$.
- (b) Since the graph of $f'(x)$ is never negative, there are no intervals where $f(x)$ is decreasing.

13. $y = 2.3 + 3.4x - 1.2x^2$

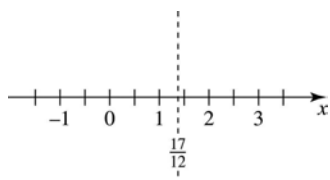
(a) $y' = 3.4 - 2.4x$

y' is zero when

$$3.4 - 2.4x = 0$$

$$x = \frac{3.4}{2.4} = \frac{17}{12}$$

and there are no values of x where y' does not exist, so the only critical number is $x = \frac{17}{12}$.



Test a point in each interval.

When

$$x = 0, y' = 3.4 - 2.4(0) = 3.4 > 0.$$

When

$$x = 2, y' = 3.4 - 2.4(2) = -1.4 < 0.$$

- (b) The function is increasing on $(-\infty, \frac{17}{12})$.
- (c) The function is decreasing on $(\frac{17}{12}, \infty)$.

14. $y = 1.1 - 0.3x - 0.3x^2$

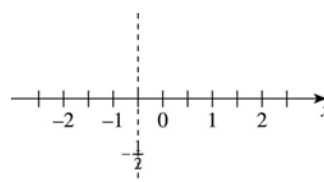
(a) $y' = -0.3 - 0.6x$

y' is zero when

$$-0.3 - 0.6x = 0$$

$$x = -\frac{0.3}{0.6} = -\frac{1}{2}$$

and there are no values of x where y' does not exist, so the only critical number is $x = -\frac{1}{2}$.



Test a point in each interval.

When

$$x = -1, y' = -0.3 - 0.6(-1) = 0.3 > 0.$$

When

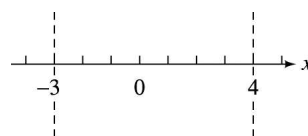
$$x = 0, y' = -0.3 - 0.6(0) = -0.3 < 0.$$

- (b) The function is increasing on $(-\infty, -\frac{1}{2})$.
- (c) The function is decreasing on $(-\frac{1}{2}, \infty)$.

15. $f(x) = \frac{2}{3}x^3 - x^2 - 24x - 4$

(a) $f'(x) = 2x^2 - 2x - 24$
 $= 2(x^2 - x - 12)$
 $= 2(x + 3)(x - 4)$

$f'(x)$ is zero when $x = -3$ or $x = 4$, so the critical numbers are -3 and 4 .



Test a point in each interval.

$$f'(-4) = 16 > 0$$

$$f'(0) = -24 < 0$$

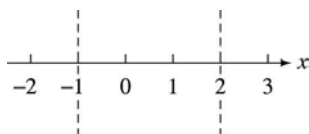
$$f'(5) = 16 > 0$$

- (b) f is increasing on $(-\infty, -3)$ and $(4, \infty)$.
- (c) f is decreasing on $(-3, 4)$.

16. $f(x) = \frac{2}{3}x^3 - x^2 - 4x + 2$

(a) $f'(x) = 2x^2 - 2x - 4$
 $= 2(x^2 - x - 2)$
 $= 2(x + 1)(x - 2)$

$f'(x)$ is zero when $x = -1$ or $x = 2$, so the critical numbers are -1 and 2 .



Test a point in each interval.

$f'(-2) = 8 > 0$
 $f'(0) = -4 < 0$
 $f'(3) = 8 > 0$

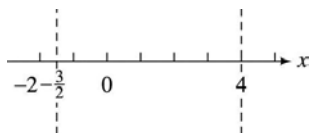
(b) f is increasing on $(-\infty, -1)$ and $(2, \infty)$.

(c) f is decreasing on $(-1, 2)$.

17. $f(x) = 4x^3 - 15x^2 - 72x + 5$

(a) $f'(x) = 12x^2 - 30x - 72$
 $= 6(2x^2 - 5x - 12)$
 $= 6(2x + 3)(x - 4)$

$f'(x)$ is zero when $x = -\frac{3}{2}$ or $x = 4$, so the critical numbers are $-\frac{3}{2}$ and 4 .



$f'(-2) = 36 > 0$
 $f'(0) = -72 < 0$
 $f'(5) = 78 > 0$

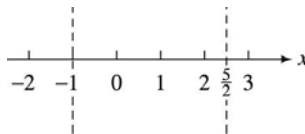
(b) f is increasing on $(-\infty, -\frac{3}{2})$ and $(4, \infty)$.

(c) f is decreasing on $(-\frac{3}{2}, 4)$.

18. $f(x) = 4x^3 - 9x^2 - 30x + 6$

(a) $f'(x) = 12x^2 - 18x - 30$
 $= 6(2x^2 - 3x - 5)$
 $= 6(2x - 5)(x + 1)$

$f'(x)$ is zero when $x = \frac{5}{2}$ or $x = -1$, so the critical numbers are $\frac{5}{2}$ and -1 .



Test a point in each interval.

$f(-2) = 54 > 0$
 $f(0) = -30 < 0$
 $f(3) = 24 > 0$

(b) f is increasing on $(-\infty, -1)$ and $(\frac{5}{2}, \infty)$.

(c) f is decreasing on $(-1, \frac{5}{2})$.

19. $f(x) = x^4 + 4x^3 + 4x^2 + 1$

(a) $f'(x) = 4x^3 + 12x^2 + 8x$
 $= 4x(x^2 + 3x + 2)$
 $= 4x(x + 2)(x + 1)$

$f'(x)$ is zero when $x = 0$, $x = -2$, or $x = -1$, so the critical numbers are 0 , -2 , and -1 .



Test a point in each interval.

$f'(-3) = -12(-1)(-2) = -24 < 0$
 $f'(-1.5) = -6(.5)(-.5) = 1.5 > 0$
 $f'(-.5) = -2(1.5)(.5) = -1.5 < 0$
 $f'(1) = 4(3)(2) = 24 > 0$

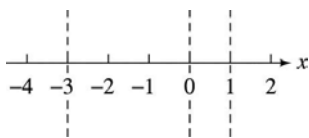
(b) f is increasing on $(-2, -1)$ and $(0, \infty)$.

(c) f is decreasing on $(-\infty, -2)$ and $(-1, 0)$.

20. $f(x) = 3x^4 + 8x^3 - 18x^2 + 5$

(a) $f'(x) = 12x^3 + 24x^2 - 36x$
 $= 12x(x^2 + 2x - 3)$
 $= 12x(x + 3)(x - 1)$

$f'(x)$ is zero when $x = 0$, $x = -3$ or $x = 1$, so the critical numbers are 0 , -3 , and 1 .



Test a point each interval.

$$f'(-4) = -240 < 0$$

$$f'(-1) = 48 > 0$$

$$f'\left(\frac{1}{2}\right) = -\frac{21}{2} < 0$$

$$f'(2) = 120 > 0$$

(b) f is increasing on $(-3, 0)$ and $(1, \infty)$.

(c) f is decreasing on $(-\infty, -3)$ and $(0, 1)$.

21. $y = -3x + 6$

(a) $y' = -3 < 0$

There are no critical numbers since y' is never 0 and always exists.

(b) Since y' is always negative, the function is increasing on no interval.

(c) y' is always negative, so the function is decreasing everywhere, or on the interval $(-\infty, \infty)$.

22. $y = 6x - 9$

(a) $y' = 6 > 0$

There are no critical numbers since y' can never be 0 and always exists.

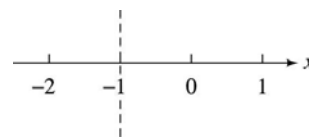
(b) Since y' is always positive, the function is increasing everywhere, or on the interval $(-\infty, \infty)$.

(c) y' is never negative, so the function is decreasing on no interval.

23. $f(x) = \frac{x+2}{x+1}$

$$\begin{aligned} \text{(a)} \quad f'(x) &= \frac{(x+1)(1) - (x+2)(1)}{(x+1)^2} \\ &= \frac{-1}{(x+1)^2} \end{aligned}$$

The derivative is never 0, but it fails to exist at $x = -1$. Since -1 is not in the domain of f , however, -1 is not a critical number.



$$f'(-2) = -1 < 0$$

$$f'(0) = -1 < 0$$

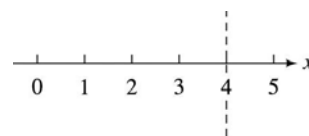
(b) f is increasing on no interval.

(c) f is decreasing everywhere that it is defined, on $(-\infty, -1)$ and on $(-1, \infty)$.

24. $f(x) = \frac{x+3}{x-4}$

$$\begin{aligned} \text{(a)} \quad f'(x) &= \frac{(1)(x-4) - (1)(x+3)}{(x-4)^2} \\ &= \frac{x-4-x-3}{(x-4)^2} = \frac{-7}{(x-4)^2} \end{aligned}$$

f' is never 0, but it fails to exist when $x = 4$. Since 4 is not in the domain of f , 4 is not a critical number. Thus, there are no critical numbers. However, the line $x = 4$ is an asymptote of the graph, so the function might change direction from one side of the asymptote to the other.



$$f'(0) = -\frac{7}{16} < 0$$

$$f'(5) = -7 < 0$$

(b) $f'(x)$ is always negative, so $f(x)$ is increasing on no interval.

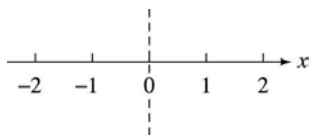
(c) $f'(x)$ is always negative, so $f(x)$ is decreasing everywhere that is defined. Since $f(x)$ is not defined at $x = 4$, these intervals are $(-\infty, 4)$ and $(4, \infty)$.

25. $y = \sqrt{x^2 + 1}$
 $= (x^2 + 1)^{1/2}$

$$\begin{aligned} \text{(a)} \quad y' &= \frac{1}{2}(x^2 + 1)^{-1/2}(2x) \\ &= x(x^2 + 1)^{-1/2} \\ &= \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

$$y' = 0 \text{ when } x = 0.$$

Since y does not fail to exist for any x , and since $y' = 0$ when $x = 0$, 0 is the only critical number.



$$y'(1) = \frac{1}{\sqrt{2}} > 0$$

$$y'(-1) = \frac{-1}{\sqrt{2}} < 0$$

(b) y is increasing on $(0, \infty)$.

(c) y is decreasing on $(-\infty, 0)$.

26. $y = x\sqrt{9 - x^2} = x(9 - x^2)^{1/2}$

(a) Use the product rule.

$$\begin{aligned} y' &= (1)(9 - x^2)^{1/2} \\ &\quad + \frac{1}{2}(9 - x^2)^{-1/2}(-2x)(x) \\ &= (9 - x^2)^{1/2} - x^2(9 - x^2)^{-1/2} \\ &= (9 - x^2)^{-1/2}(9 - x^2 - x^2) \\ &= (9 - x^2)^{-1/2}(9 - 2x^2) \\ &= \frac{9 - 2x^2}{\sqrt{9 - x^2}} \end{aligned}$$

Critical numbers occur when $y' = 0$ or when y' fails to exist.

$y' = 0$ when

$$9 - 2x^2 = 0$$

$$x = \frac{\pm 3}{\sqrt{2}} = \frac{\pm 3\sqrt{2}}{2}.$$

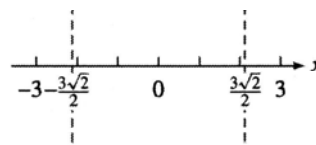
y' fails to exist when

$$9 - x^2 = 0$$

$$x = \pm 3.$$

Thus, the critical numbers are $\frac{\pm 3\sqrt{2}}{2}$ and ± 3 .

These four values determine three intervals since $f(x)$ is defined only on $[-3, 3]$. Note that $\pm \frac{3\sqrt{2}}{2} \approx \pm 2.12$.



$$f'(-2.5) = -2.11 < 0$$

$$f'(0) = 3 > 0$$

$$f'(2.5) = -2.11 < 0$$

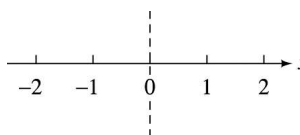
(b) f is increasing on $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.

(c) f is decreasing on $\left(-3, -\frac{3\sqrt{2}}{2}\right)$ and $\left(\frac{3\sqrt{2}}{2}, 3\right)$.

27. $f(x) = x^{2/3}$

(a) $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

$f'(x)$ is never zero, but fails to exist when $x = 0$, so 0 is the only critical number.



$$f'(-1) = -\frac{2}{3} < 0$$

$$f'(1) = \frac{2}{3} > 0$$

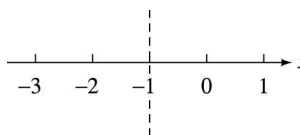
(b) f is increasing on $(0, \infty)$.

(c) f is decreasing on $(-\infty, 0)$.

28. $f(x) = (x + 1)^{4/5}$

(a) $f'(x) = \frac{4}{5}(x + 1)^{-1/5} = \frac{4}{5(x + 1)^{1/5}}$

$f'(x)$ is never zero, but fails to exist when $x = -1$, so the critical number is -1 .



$$f'(-2) = -\frac{4}{5} < 0$$

$$f'(0) = \frac{4}{5} > 0$$

(b) f is increasing on $(-1, \infty)$.

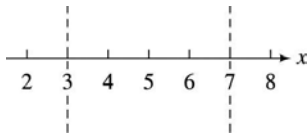
(c) f is decreasing on $(-\infty, -1)$.

29. $y = x - 4 \ln(3x - 9)$

$$\begin{aligned} \text{(a)} \quad y' &= 1 - \frac{12}{3x - 9} = 1 - \frac{4}{x - 3} \\ &= \frac{x - 7}{x - 3} \end{aligned}$$

y' is zero when $x = 7$. The derivative does not exist at $x = 3$, but note that the domain of f is $(3, \infty)$.

Thus, the only critical number is 7.



Choose a value in the intervals $(3, 7)$ and $(7, \infty)$.

$$f'(4) = -3 < 0$$

$$f'(8) = \frac{1}{5} > 0$$

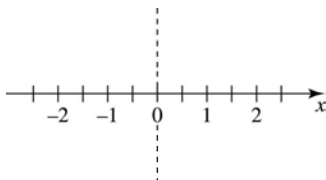
(b) The function is increasing on $(7, \infty)$.

(c) The function is decreasing on $(3, 7)$.

30. $f(x) = \ln \frac{5x^2 + 4}{x^2 + 1}$
 $= \ln(5x^2 + 4) - \ln(x^2 + 1)$

$$\begin{aligned} \text{(a)} \quad f'(x) &= \frac{10x}{5x^2 + 4} - \frac{2x}{x^2 + 1} \\ &= \frac{10x(x^2 + 1)}{(5x^2 + 4)(x^2 + 1)} - \frac{2x(5x^2 + 4)}{(5x^2 + 4)(x^2 + 1)} \\ &= \frac{10x^3 + 10x - 10x^3 - 8x}{(5x^2 + 4)(x^2 + 1)} \\ &= \frac{2x}{(5x^2 + 4)(x^2 + 1)} \end{aligned}$$

$f'(x)$ is zero when $x = 0$ and there are no values of x where $f'(x)$ does not exist, so the critical number is 0.



Test a point in each interval.

$$f'(-1) = \frac{2(-1)}{(5(-1)^2 + 4)((-1)^2 + 1)} = -\frac{1}{9} < 0$$

$$f'(1) = \frac{2(1)}{(5(1)^2 + 4)((1)^2 + 1)} = \frac{1}{9} > 0$$

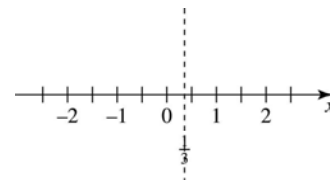
(b) The function is increasing on $(0, \infty)$.

(c) The function is decreasing on $(-\infty, 0)$.

31. $f(x) = xe^{-3x}$

$$\begin{aligned} \text{(a)} \quad f'(x) &= e^{-3x} + x(-3e^{-3x}) \\ &= (1 - 3x)e^{-3x} \\ &= \frac{1 - 3x}{e^{3x}} \end{aligned}$$

$f'(x)$ is zero when $x = \frac{1}{3}$ and there are no values of x where $f'(x)$ does not exist, so the critical number is $\frac{1}{3}$.



Test a point in each interval.

$$f'(0) = \frac{1 - 3(0)}{e^{3(0)}} = 1 > 0$$

$$f'(1) = \frac{1 - 3(1)}{e^{3(1)}} = -\frac{2}{e^3} < 0$$

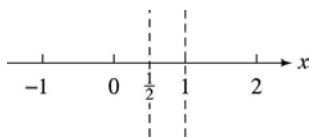
(b) The function is increasing on $(-\infty, \frac{1}{3})$.

(c) The function is decreasing on $(\frac{1}{3}, \infty)$.

32. $y = xe^{x^2 - 3x}$

$$\begin{aligned} \text{(a)} \quad y' &= xe^{x^2 - 3x}(2x - 3) + e^{x^2 - 3x}(1) \\ &= e^{x^2 - 3x}[x(2x - 3) + 1] \\ &= e^{x^2 - 3x}(2x^2 - 3x + 1) \\ &= e^{x^2 - 3x}(2x - 1)(x - 1) \end{aligned}$$

y' is zero when $x = \frac{1}{2}$ or $x = 1$, so the critical numbers are $\frac{1}{2}$ and 1.



Test a point in each interval.

$$f'(0) = 1(-1)(-1) = 1 > 0$$

$$f'(0.6) = e^{-1.44}(0.2)(-0.4) = \frac{-0.08}{e^{1.44}} < 0$$

$$f'(2) = e^{-2}(3)(1) = \frac{3}{e^2} > 0$$

(b) The function is increasing on $(-\infty, \frac{1}{\ln 2})$ and $(1, \infty)$.

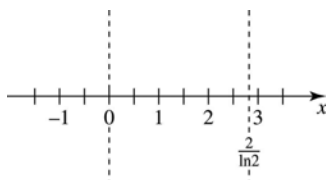
(c) The function is decreasing on $(\frac{1}{\ln 2}, 1)$.

33. $f(x) = x^2 2^{-x}$

$$\begin{aligned} \text{(a)} \quad f'(x) &= x^2[\ln 2(2^{-x})(-1)] + (2^{-x})2x \\ &= 2^{-x}(-x^2 \ln 2 + 2x) \\ &= \frac{x(2 - x \ln 2)}{2^x} \end{aligned}$$

$$f'(x) \text{ is zero when } x = 0 \text{ or } x = \frac{2}{\ln 2}$$

and there are no values of x where $f'(x)$ does not exist. The critical numbers are 0 and $\frac{2}{\ln 2}$.



Test a point in each interval.

$$f'(-1) = \frac{(-1)(2 - (-1)\ln 2)}{2^{-1}} = -2(2 + \ln 2) < 0$$

$$f'(1) = \frac{(1)(2 - (1)\ln 2)}{2^1} = \frac{2 - \ln 2}{2} > 0$$

$$f'(3) = \frac{(3)(2 - (3)\ln 2)}{2^3} = \frac{3(2 - 3\ln 2)}{8} < 0$$

(b) The function is increasing on $(0, \frac{2}{\ln 2})$.

(c) The function is decreasing on $(-\infty, 0)$ and $(\frac{2}{\ln 2}, \infty)$.

34. $f(x) = x2^{-x^2}$

$$\begin{aligned} \text{(a)} \quad f'(x) &= x\left[\ln 2\left(2^{-x^2}(-2x)\right)\right] + \left(2^{-x^2}\right) \\ &= 2^{-x^2}(-2x^2 \ln 2 + 1) \\ &= \frac{1 - 2x^2 \ln 2}{2^x} \end{aligned}$$

$$f'(x) = 0 \text{ when}$$

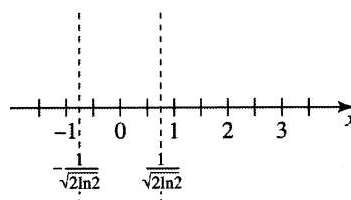
$$1 - 2x^2 \ln 2 = 0$$

$$x^2 = \frac{1}{2 \ln 2}$$

$$x = \pm \frac{1}{\sqrt{2 \ln 2}}$$

and there are no values of x where $f'(x)$ does not exist. The critical numbers are

$$\pm \frac{1}{\sqrt{2 \ln 2}}.$$



Test a point in each interval.

$$f'(-1) = \frac{1 - 2(-1)^2 \ln 2}{2^{-1}} = 2(1 - 2 \ln 2) < 0$$

$$f'(0) = \frac{1 - 2(0)^2 \ln 2}{2^0} = 1 > 0$$

$$f'(1) = \frac{1 - 2(1)^2 \ln 2}{2^1} = \frac{1 - 2 \ln 2}{2} < 0$$

(b) The function is increasing on

$$\left(-\frac{1}{\sqrt{2 \ln 2}}, \frac{1}{\sqrt{2 \ln 2}}\right).$$

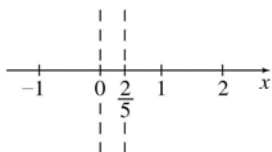
(c) The function is decreasing on

$$\left(-\infty, -\frac{1}{\sqrt{2 \ln 2}}\right) \text{ and } \left(\frac{1}{\sqrt{2 \ln 2}}, \infty\right).$$

35. $y = x^{2/3} - x^{5/3}$

$$\text{(a)} \quad y' = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{2 - 5x}{3x^{1/3}}$$

$y' = 0$ when $x = \frac{2}{5}$. The derivative does not exist at $x = 0$. So the critical numbers are 0 and $\frac{2}{5}$.



Test a point in each interval.

$$y'(-1) = \frac{7}{-3} < 0$$

$$y'\left(\frac{1}{5}\right) = \frac{1}{3\left(\frac{1}{5}\right)^{1/3}} = \frac{5^{1/3}}{2} > 0$$

$$y'(1) = \frac{-3}{3} = -1 < 0$$

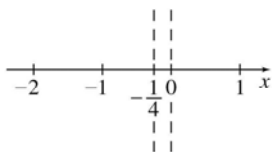
(b) y is increasing on $\left(0, \frac{2}{5}\right)$.

(c) y is decreasing on $(-\infty, 0)$ and $\left(\frac{2}{5}, \infty\right)$.

36. $y = x^{1/3} + x^{4/3}$

(a) $y' = \frac{1}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = \frac{1+4x}{3x^{2/3}}$

$y' = 0$ when $x = -\frac{1}{4}$. The derivative does not exist at $x = 0$. So the critical numbers are $-\frac{1}{4}$ and 0.



Test a number in each interval.

$$y'(-1) = -1 < 0$$

$$y'\left(-\frac{1}{8}\right) = \frac{2}{3} > 0$$

$$y'(1) = \frac{5}{3} > 0$$

(b) The derivative is positive on $\left(-\frac{1}{4}, 0\right)$ and $(0, \infty)$ and the function is defined at $x = 0$. Thus, y is increasing on $\left(-\frac{1}{4}, \infty\right)$.

(c) y is decreasing on $(-\infty, -\frac{1}{4})$.

38. $f(x) = ax^2 + bx + c, a > 0$

$$f'(x) = 2ax + b$$

Let $f'(x) = 0$ to find the critical number.

$$2ax + b = 0$$

$$2ax = -b$$

$$x = \frac{-b}{2a}$$

Choose a value in the interval $(-\infty, \frac{-b}{2a})$.

Since $a > 0$,

$$\frac{-b}{2a} - \frac{1}{2a} = \frac{-b-1}{2a} < \frac{-b}{2a}.$$

$$f'\left(\frac{-b-1}{2a}\right) = 2a\left(\frac{-b-1}{2a}\right) + b = -1 < 0$$

Choose a value in the interval $(\frac{-b}{2a}, \infty)$.

Since $a > 0$,

$$\frac{-b}{2a} + \frac{1}{2a} = \frac{-b+1}{2a} > \frac{-b}{2a}.$$

$$f'\left(\frac{-b+1}{2a}\right) = 1 > 0$$

$f(x)$ is increasing on $(\frac{-b}{2a}, \infty)$ and decreasing on $(-\infty, \frac{-b}{2a})$.

This tells us that the curve opens upward and $x = \frac{-b}{2a}$ is the x -coordinate of the vertex.

$$\begin{aligned} f\left(\frac{-b}{2a}\right) &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c \\ &= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \\ &= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} \\ &= \frac{4ac - b^2}{4a} \end{aligned}$$

The vertex is $\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$ or $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

39. $f(x) = ax^2 + bx + c, a < 0$
 $f'(x) = 2ax + b$

Let $f'(x) = 0$ to find the critical number.

$$\begin{aligned} 2ax + b &= 0 \\ 2ax &= -b \\ x &= \frac{-b}{2a} \end{aligned}$$

Choose a value in the interval $(-\infty, \frac{-b}{2a})$.

Since $a < 0$,

$$\begin{aligned} \frac{-b}{2a} - \frac{-1}{2a} &= \frac{-b+1}{2a} < \frac{-b}{2a} \\ f'\left(\frac{-b+1}{2a}\right) &= 2a\left(\frac{-b+1}{2a}\right) + b \\ &= 1 < 0 \end{aligned}$$

Choose a value in the interval $(\frac{-b}{2a}, \infty)$.

Since $a < 0$,

$$\begin{aligned} \frac{-b}{2a} - \frac{-1}{2a} &= \frac{-b-1}{2a} < \frac{-b}{2a} \\ f'\left(\frac{-b-1}{2a}\right) &= 2a\left(\frac{-b-1}{2a}\right) + b \\ &= -1 > 0 \end{aligned}$$

f is increasing on $(-\infty, \frac{-b}{2a})$ and decreasing on $(\frac{-b}{2a}, \infty)$.

This tells us that the curve opens downward and $x = \frac{-b}{2a}$ is the x -coordinate of the vertex.

$$\begin{aligned} f\left(\frac{-b}{2a}\right) &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c \\ &= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \\ &= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4bc}{4a} \\ &= \frac{4ac - b^2}{4a} \end{aligned}$$

The vertex is $\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$ or $\left(-\frac{b^2}{2a}, \frac{4ac-b^2}{4a}\right)$.

40. $f(x) = e^x$
 $f'(x) = e^x > 0$

$f(x) = e^x$ is increasing on $(-\infty, \infty)$.

$f(x) = e^x$ is decreasing nowhere.

Since $f'(x)$ is always positive, it never equals zero. Therefore, the tangent line is horizontal nowhere.

41. $f(x) = \ln x$
 $f'(x) = \frac{1}{x}$

$f'(x)$ is undefined at $x = 0$. $f'(x)$ never equals zero. Note that $f(x)$ has a domain of $(0, \infty)$. Pick a value in the interval $(0, \infty)$.

$$f'(2) = \frac{1}{2} > 0$$

$f(x)$ is increasing on $(0, \infty)$.

$f(x)$ is never decreasing.

Since $f(x)$ never equals zero, the tangent line is horizontal nowhere.

42. (a) The critical number are -3 and 4 .
The average is

$$\frac{-3+4}{2} = 0.5.$$

- (b) Use a graphing calculator to find the zeros of the function

$$f(x) = \frac{2}{3}x^3 - x^2 - 24x - 4.$$

The roots are -5.2005 , -0.1680 , and 6.8685 . The average is

$$\frac{-5.2005 + (-0.1680) + 6.8685}{3} = 0.5.$$

- (c) The answers are the same.

- (d) The critical numbers are $-\frac{3}{2}$ and 4 .
The average is

$$\frac{-\frac{3}{2} + 4}{2} = 1.25.$$

- (e) Use a graphing calculator to find the zeros of the function $f(x) = 4x^3 - 15x^2 - 72x + 5$. The roots are -2.8112 , 0.0685 , and 6.4927 . The average is

$$\frac{-2.8112 + 0.0685 + 6.4927}{3} = 1.25.$$

- (f) The answers are the same.

43. $f(x) = e^{0.001x} - \ln x$
 $f'(x) = 0.001e^{0.001x} - \frac{1}{x}$

Note that $f(x)$ is only defined for $x > 0$. Use a graphing calculator to plot $f'(x)$ for $x > 0$.

- (a) $f'(x) > 0$ on about $(567, \infty)$, so $f(x)$ is increasing on about $(567, \infty)$.
 (b) $f'(x) < 0$ on about $(0, 567)$, so $f(x)$ is decreasing on about $(0, 567)$.

44. $f(x) = \ln(x^2 + 1) - x^{0.3}$
 $f'(x) = \frac{2x}{x^2 + 1} - 0.3x^{-0.7}$

Note that $f(x)$ is only defined for $x > 0$. Use a graphing calculator to plot $f'(x)$ for $x > 0$.

- (a) $f'(x) > 0$ on about $(0, 558)$, so $f(x)$ is increasing on about $(558, \infty)$.
 (b) $f'(x) < 0$ on $(558, \infty)$, so $f(x)$ is decreasing on $(558, \infty)$.

45. $H(r) = \frac{300}{1 + 0.03r^2} = 300(1 + 0.03r^2)^{-1}$
 $H'(r) = 300[-1(1 + 0.03r^2)^{-2}(0.06r)]$
 $= \frac{-18r}{(1 + 0.03r^2)^2}$

Since r is a mortgage rate (in percent), it is always positive. Thus, $H'(r)$ is always negative.

- (a) H is increasing on nowhere.
 (b) H is decreasing on $(0, \infty)$.

46. $C(x) = x^3 - 2x^2 + 8x + 50$
 $C'(x) = 3x^2 - 4x + 8$

Since $C(x)$ is a polynomial function, the only critical numbers are found by solving $C'(x) = 0$.

$$3x^2 - 4x + 8 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 96}}{6}$$

$$= \frac{4 \pm \sqrt{-80}}{6}$$

Since $\sqrt{-80}$ is not a real number, there are no critical points. The function either always increases or always decreases.

Test any point, say $x = 0$.

$$C'(0) = 8 > 0$$

- (a) $C(x)$ is decreasing nowhere.
 (b) $C(x)$ is increasing on $(-\infty, \infty)$.

47. $C(x) = 0.32x^2 - 0.00004x^3$
 $R(x) = 0.848x^2 - 0.0002x^3$
 $P(x) = R(x) - C(x)$
 $= (0.848x^2 - 0.0002x^3) - (0.32x^2 - 0.00004x^3)$
 $= 0.528x^2 - 0.00016x^3$
 $P'(x) = 1.056x - 0.00048x^2$

$$1.056x - 0.00048x^2 = 0$$

$$x(1.056 - 0.00048x) = 0$$

$$x = 0 \text{ or } x = 2200$$

Choose $x = 1000$ and $x = 3000$ as test points.

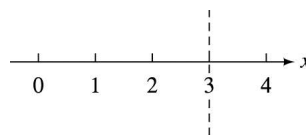
$$P'(1000) = 1.056(1000) - 0.00048(1000)^2 = 576$$

$$P'(3000) = 1.056(3000) - 0.00048(3000)^2 = -1152$$

The function is increasing on $(0, 2200)$.

48. $P(x) = -(x - 4)e^x - 4(0 < x \leq 3.9)$
 $P'(x) = -(x - 4)e^x + e^x(-1)$
 $= -e^x[(x - 4) + 1]$
 $= -e^x(x - 3)$

$$P'(x) = 0 \text{ when } x = 3.$$



$$P'(1) = -e^1(-2) = 2e > 0$$

$$P'(3.5) = -e^{3.5}(0.5) = \frac{-0.5}{e^{3/5}} < 0$$

- (a) The profit is increasing on $(0, 3)$, which represents production between 0 and 300 players.

- (b) The profit is decreasing on $(3, 3.9)$, which represents production between 300 and 390 players.

$$49. \quad A(t) = 0.0000329t^3 - 0.00450t^2 + 0.0613t + 2.34$$

$$A'(t) = 0.0000987t^2 - 0.009t + 0.0613$$

Set $A'(t)$ equal to 0 and solve for t .

$$0.0000987t^2 - 0.009t + 0.0613 = 0$$

$$t = \frac{0.009 \pm \sqrt{(0.009)^2 - 4(0.0000987)(0.0613)}}{(2)(0.0000987)}$$

$$t \approx 83.8 \text{ or } t = 7.4$$

Since $0 \leq t \leq 50$, $t \approx 7.4$.

Check the sign of A' on either side of this critical number.

$$A'(7) \approx 0.00314 > 0$$

$$A'(8) \approx -0.00438 < 0$$

- (a) The projected year-end assets function is increasing on the interval $(0, 7.4)$, or from 2000 to about the middle of 2007.
- (b) The projected year-end assets function is decreasing on the interval $(7.4, 50)$, or from about the middle of 2007 to 2050.

50. Following the graph from left to right, we find that the function is increasing on the intervals $(1990, 1992)$, $(2000, 2003)$, and $(2007, 2009)$. The function is decreasing on the intervals $(1992, 2000)$ and $(2003, 2006)$. The function is constant on the interval $(2006, 2007)$.

51. (a) These curves are graphs of functions since they all pass the vertical line test.
- (b) The graph for particulates increases from April to July; it decreases from July to November; it is constant from January to April and November to December.
- (c) All graphs are constant from January to April and November to December. When the temperature is low, as it is during these months, air pollution is greatly reduced.

$$52. \quad P(t) = \frac{10 \ln(0.19t + 1)}{0.19t + 1}$$

$$\begin{aligned} P'(t) &= 10 \left(\frac{(0.19t + 1) \cdot \frac{1}{0.19t + 1} \cdot 0.19 - 0.19 \ln(0.19t + 1)}{(0.19t + 1)^2} \right) \\ &= 10 \left(\frac{0.19 - 0.19 \ln(0.19t + 1)}{(0.19t + 1)^2} \right) \\ &= 1.9 \left(\frac{1 - \ln(0.19t + 1)}{(0.19t + 1)^2} \right) \end{aligned}$$

Note that $P(t)$ is defined when $0.19t + 1 > 0$, which implies that $t > \frac{-1}{0.19} \approx -5.26$, so the domain of P is $\{t | t > \frac{-100}{0.19} \approx -5.26\}$.

Find critical numbers. Set numerator equal to 0.

$$1 - \ln(0.19t + 1) = 0$$

$$\ln(0.19t + 1) = 1$$

$$0.19t + 1 = e$$

$$t = \frac{e - 1}{0.19} \approx 9.04$$

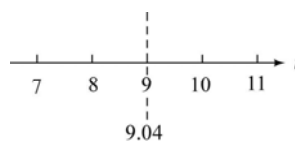
So, $P'(t) = 0$ when $t = \frac{100}{19}(e - 1) \approx 9.04$. Set denominator equal to 0.

$$(0.19t + 1)^2 = 0$$

$$0.19t + 1 = 0$$

$$t = \frac{-1}{0.19} \approx -5.26$$

Since $\frac{-100}{19} \approx -5.26$ is not in the domain of P , $\frac{100}{19}(e - 1)$ is the only critical number.



Test a number in each interval.

$$P'(0) = 1.9 \left(\frac{1 - \ln(1)}{1^2} \right) = 1.9 > 0$$

$$P'(10) = 1.9 \left(\frac{1 - \ln(2.9)}{2.9^2} \right) < 0$$

since $\ln(2.9) > \ln(e) = 1$.

Therefore, P increases on $\left(\frac{-100}{19}, \frac{100}{19}(e - 1)\right)$ and decreases on $\left(\frac{100}{19}(e - 1), \infty\right)$, so the number of people infected will start to decline after day $\frac{100}{19}(e - 1) \approx 9.04$, or day 9.

$$53. \quad A(x) = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$$

$$A'(x) = 0.010893x^2 - 0.07492x + 0.1012$$

Solve for $A'(x) = 0$.

$$x \approx 1.85 \text{ or } x \approx 5.03$$

Choose $x = 1$ and $x = 4$ as test points.

$$A'(1) = 0.010893(1)^2 - 0.07492(1) + 0.1012 = 0.037173$$

$$A'(4) = 0.010893(4)^2 - 0.07492(4) + 0.1012 = -0.024192$$

(a) The function is increasing on $(0, 1.85)$.

(b) The function is decreasing on $(1.85, 5)$.

$$54. \quad K(x) = \frac{4x}{3x^2 + 27}$$

$$K'(x) = \frac{(3x^2 + 27)4 - 4x(6x)}{(3x^2 + 27)^2}$$

$$= \frac{108 - 12x^2}{(3x^2 + 27)^2}$$

$K'(x)$ is zero when

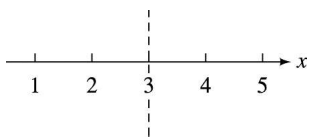
$$108 - 12x^2 = 0$$

$$12(9 - x^2) = 0$$

$$(3 - x)(3 + x) = 0$$

$$x = 3 \text{ or } x = -3.$$

However, in this problem x represents time, which cannot be negative, so 3 is the only critical number.



$$K'(0) = \frac{108}{27^2} > 0$$

$$K'(4) = \frac{-84}{(48 + 27)^2} < 0$$

(a) $K(x)$ is increasing on $(0, 3)$.

(Note: x must be at least 0.)

(c) $K(x)$ is decreasing on $(3, \infty)$.

$$55. \quad K(t) = \frac{5t}{t^2 + 1}$$

$$K'(t) = \frac{5(t^2 + 1) - 2t(5t)}{(t^2 + 1)^2}$$

$$= \frac{5t^2 + 5 - 10t^2}{(t^2 + 1)^2}$$

$$= \frac{5 - 5t^2}{(t^2 + 1)^2}$$

$K'(t) = 0$ when

$$\frac{5 - 5t^2}{(t^2 + 1)^2} = 0$$

$$5 - 5t^2 = 0$$

$$5t^2 = 5$$

$$t = \pm 1.$$

Since t is the time after a drug is administered, the function applies only for $[0, \infty)$, so we discard $t = -1$. Then 1 divides the domain into two intervals.



$$K'(0.5) = 2.4 > 0$$

$$K'(2) = -0.6 < 0$$

(a) K is increasing on $(0, 1)$.

(b) K is decreasing on $(1, \infty)$.

$$56. \quad D(p) = 0.000002p^3 - 0.0008p^2 + 0.1141p + 16.683$$

$$D'(p) = 0.000006p^2 - 0.0016p + 0.1141$$

By the quadratic formula, there are no real number solutions for p when $D'(p) = 0$.

Choose a value in the interval $(55, 130)$.

$$D'(60) = 0.0397 > 0$$

The function is increasing on the entire interval $(55, 130)$, so it is decreasing nowhere.

57. (a) $F(t) = -10.28 + 175.9te^{-t/1.3}$
 $F'(t) = (175.9)(e^{-t/1.3})$
 $+ (175.9 \cdot 9t) \left(-\frac{1}{1.3} e^{-t/1.3} \right)$
 $= (175.9)(e^{-t/1.3}) \left(1 - \frac{t}{1.3} \right)$
 $\approx 175.9e^{-t/1.3}(1 - 0.769t)$
- (b) $F'(t)$ is equal to 0 at $t = 1.3$. Therefore, 1.3 is a critical number. Since the domain is $(0, \infty)$, test values in the intervals from $(0, 1.3)$ and $(1.3, \infty)$.

$$F'(1) \approx 18.83 > 0 \text{ and } F'(2) \approx -20.32 < 0$$

$F'(t)$ is increasing on $(0, 1.3)$ and decreasing on $(1.3, \infty)$.

58. $W_1(t) = 619(1 - 0.905e^{-0.002t})^{1.2386}$
 $W'_1(t) = 1.388e^{-0.002t}(1 - 0.905e^{-0.002t})$
 $0.2386 > 0$
 Since $W'_1(t) > 0$ and t is a positive number, this function is increasing on $(0, \infty)$.

59. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
 $f'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} (-x)$
 $= \frac{-x}{\sqrt{2\pi}} e^{-x^2/2}$
 $f'(x) = 0$ when $x = 0$.

Choose a value from each of the intervals $(-\infty, 0)$ and $(0, \infty)$.

$$f'(-1) = \frac{1}{\sqrt{2\pi}} e^{-1/2} > 0$$

$$f'(1) = \frac{-1}{\sqrt{2\pi}} e^{-1/2} < 0$$

The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

60. (a) The total inventories of both countries were increasing on the following approximate intervals: (1945, 1965) and (1970, 1973).
- (b) The total inventories of both countries were decreasing on the following approximate interval: (1987, 2010).

61. (a) (2500, 5750)
 (b) (5750, 6000)
 (c) (2800, 4800)
 (d) (2500, 2800) and (4800, 6000)
62. (a) As x increases, $f(x)$ decreases, so f is decreasing for these x -values. Therefore, $f'(x)$ is negative.
- (b) $\frac{\text{mpg}}{\text{lb}}$, or miles per gallon per pound.

5.2 Relative Extrema

Your Turn 1

There is an open interval containing x_1 for which $f(x) \geq f(x_1)$.

There is an open interval containing x_2 for which $f(x) \leq f(x_2)$.

There is an open interval containing x_3 for which $f(x) \geq f(x_3)$.

There are relative minima of $f(x_1)$ at $x = x_1$ and $f(x_3)$ at $x = x_3$ and a relative maximum of $f(x_2)$ at $x = x_2$.

Your Turn 2

$$f(x) = -x^3 - 2x^2 + 15x + 10$$

$$f'(x) = -3x^2 - 4x + 15$$

Set $f'(x)$ equal to 0 and solve for x to find the critical numbers.

$$-3x^2 - 4x + 15 = 0$$

$$-(3x^2 + 4x - 15) = 0$$

$$-(3x - 5)(x + 3) = 0$$

$$3x - 5 = 0 \text{ or } x + 3 = 0$$

$$x = 5/3 \text{ or } x = -3$$

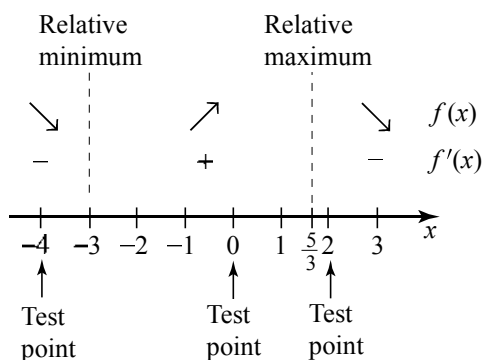
The critical numbers are -3 and $5/3$. They divide the domain of f into the three intervals $(-\infty, -3)$, $(-3, 5/3)$, and $(5/3, \infty)$. Pick test points in each interval to determine the sign of f' on that interval. For example, choose $x = -4$, $x = 0$, and $x = 2$.

$$f'(-4) = -17$$

$$f'(0) = 15$$

$$f'(2) = -5$$

Thus the derivative is negative on $(-\infty, -3)$, positive on $(-3, 5/3)$, and negative on $(5/3, \infty)$. The graph below shows this information and indicates where f is increasing or decreasing.



By the first derivative test, we know that the function has a relative minimum of $f(-3) = -26$ at $x = -3$ and a relative maximum of $f(5/3) = 670/27 \approx 24.8$ at $x = 5/3$.

Your Turn 3

$$\begin{aligned} f(x) &= x^{2/3} - x^{5/3} \\ f'(x) &= \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} \\ &= \frac{1}{3} \left(\frac{2 - 5x}{x^{1/3}} \right) \end{aligned}$$

$x = 0$ is a critical number because $f(x)$ is defined at $x = 0$ but $f'(x)$ is not. To find any other critical numbers, assume $x \neq 0$ and set $f'(x) = 0$ and solve for x .

$$\begin{aligned} \frac{1}{3} \left(\frac{2 - 5x}{x^{1/3}} \right) &= 0 \\ 2 - 5x &= 0 \\ x &= \frac{2}{5} \end{aligned}$$

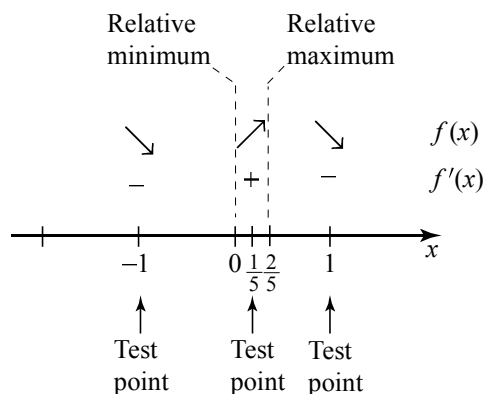
Use the critical numbers to divide the line into three intervals, $(-\infty, 0)$, $(0, 2/5)$, and $(2/5, \infty)$. Choose test points in each interval; for example, choose -1 , $-1/5$, and 1 .

$$f'(-1) = -2.33$$

$$f'(1/5) = 0.57$$

$$f'(1) = -1$$

Thus the derivative is negative on $(-\infty, -1)$, positive on $(-1, 2/5)$, and negative on $(2/5, \infty)$. The graph below shows this information and indicates where f is increasing or decreasing.



The first derivative test identifies $f(2/5)$ as a relative maximum, and there is a relative minimum at $x = 0$. Thus the function has a relative maximum of

$$f\left(\frac{2}{5}\right) = \frac{3}{5} \left(\frac{2}{5}\right)^{2/3} \approx 0.3257 \text{ at } x = \frac{2}{5}$$

and a relative minimum of $f(0) = 0$ at $x = 0$.

Your Turn 4

$$f(x) = x^2 e^x$$

Using the product rule:

$$\begin{aligned} f'(x) &= x^2 e^x + 2x e^x \\ &= e^x (x^2 + 2x) \end{aligned}$$

Find the critical values:

$$\begin{aligned} e^x (x^2 + 2x) &= 0 \\ x^2 + 2x &= 0 \\ x(x + 2) &= 0 \\ x + 2 = 0 \text{ or } x &= 0 \\ x = -2 \text{ or } x &= 0 \end{aligned}$$

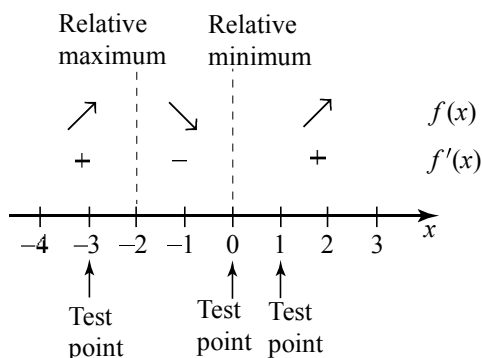
There are two critical numbers, dividing the line into the intervals $(-\infty, -2)$, $(-2, 0)$ and $(0, \infty)$. Choose a test point in each interval, for example -3 , -1 , and 1 .

$$f'(-3) = 0.149$$

$$f'(-1) = -0.368$$

$$f'(1) = 8.155$$

Thus the derivative is positive on $(-\infty, -2)$, negative on $(-2, 0)$, and positive on $(0, \infty)$. The graph below shows this information and indicates where f is increasing or decreasing.



The function has a relative maximum of $4e^{-2} \approx 0.5413$ at $x = -2$ and a relative minimum of $f(0) = 0$ at $x = 0$.

Your Turn 5

$$\begin{aligned}
 C(q) &= 100 + 10q \\
 p &= D(q) = 50 - 2q \\
 P(q) &= R(q) - C(q) \\
 &= qD(q) - C(q) \\
 &= q(50 - 2q) - (100 + 10q) \\
 &= 50q - 2q^2 - 100 + 10q \\
 &= -2q^2 + 40q - 100 \\
 P'(q) &= -4q + 40 \\
 &= -4(q - 10)
 \end{aligned}$$

P' is 0 only when $q = 10$, and this is the only critical number. $P'(5) = 20$ and $P(20) = -40$ so P is increasing as q approaches 10 from the left and decreasing to the right of $q = 10$. The value of P at $q = 10$ is $P(10) = -2(100) + 40(10) - 100 = 100$.

Thus the maximum weekly profit is \$100 when the demand is 10 items per week. The company should charge $D(10) = 50 - 2(10) = 30$, or \$30 per item.

5.2 Exercises

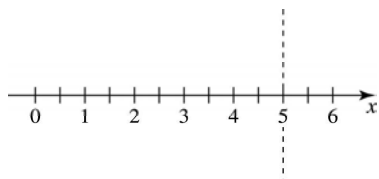
- As shown on the graph, the relative minimum of -4 occurs when $x = 1$.
- As shown on the graph, the relative maximum of 1 occurs when $x = 4$.
- As shown on the graph, the relative maximum of 3 occurs when $x = -2$.

- As shown on the graph, the relative minimum of -4 occurs when $x = 3$.
- As shown on the graph, the relative maximum of 3 occurs when $x = -4$ and the relative minimum of 1 occurs when $x = -2$.
- As shown on the graph, the relative minimum of -6 occurs when $x = 1$ and the relative maximum of 2 occurs when $x = 5$.
- As shown on the graph, the relative maximum of 3 occurs when $x = -4$; the relative minimum of -2 occurs when $x = -7$ and $x = -2$.
- As shown on the graph, the relative maximum of 4 occurs when $x = 0$; the relative minimum of 0 occurs when $x = -3$ and $x = 3$.
- Since the graph of the function is zero at $x = -1$ and $x = 3$, the critical numbers are -1 and 3 . Since the graph of the derivative is positive on $(-\infty, -1)$ and negative on $(-1, 3)$, there is a relative maximum at -1 . Since the graph of the function is negative on $(-1, 3)$ and positive on $(3, \infty)$, there is a relative minimum at 3 .
- Since the graph of the derivative is zero at $x = 3$ and $x = 5$, the critical numbers are 3 and 5 . Since the graph of the derivative is negative on $(-\infty, 3)$ and positive on $(3, 5)$, there is a relative minimum at 3 . Since the graph of the derivative is positive on $(3, 5)$ and negative on $(5, \infty)$, there is a relative maximum at 5 .
- Since the graph of the derivative is zero at $x = -8$, $x = -6$, $x = -2.5$ and $x = -1.5$, the critical numbers are -8 , -6 , -2.5 , and -1.5 . Since the graph of the derivative is positive on $(-\infty, -8)$ and negative on $(-8, -6)$, there is a relative maximum at -8 . Since the graph of the derivative is negative on $(-8, -6)$ and positive on $(-6, -2.5)$, there is a relative minimum at -6 . Since the graph of the derivative is positive on $(-6, -2.5)$ and negative on $(-2.5, -1.5)$, there is a relative maximum at -2.5 . Since the graph of the derivative is negative on $(-2.5, -1.5)$ and positive on $(-1.5, \infty)$, there is a relative minimum at -1.5 .

12. Since the graph of the derivative is zero at $x = -3$ and $x = 3$, the critical numbers are -3 and 3 .

Since the graph of the derivative is positive on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$, there are no relative extrema.

13. $f(x) = x^2 - 10x + 33$
 $f'(x) = 2x - 10$
 $f'(x)$ is zero when $x = 5$.



$$f'(0) = -10 < 0$$

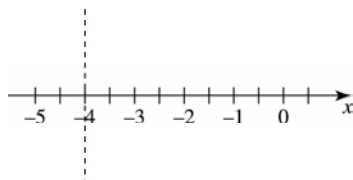
$$f'(6) = 2 > 0$$

f is decreasing on $(-\infty, 5)$ and increasing on $(5, \infty)$. Thus, a relative minimum occurs at $x = 5$.

$$f(5) = 8$$

Relative minimum of 8 at 5

14. $f(x) = x^2 + 8x + 5$
 $f'(x) = 2x + 8$
 $f'(x)$ is zero when $x = -4$.



$$f'(-5) = -2 < 0$$

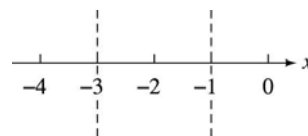
$$f'(0) = 8 > 0$$

f is decreasing on $(-\infty, -4)$ and increasing on $(-4, \infty)$. Thus, a relative minimum occurs at $x = -4$.

$$f(-4) = -11$$

Relative minimum of -11 at -4

15. $f(x) = x^3 + 6x^2 + 9x - 8$
 $f'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3)$
 $= 3(x + 3)(x + 1)$
 $f'(x)$ is zero when $x = -1$ or $x = -3$.



$$f'(-4) = 9 > 0$$

$$f'(-2) = -3 < 0$$

$$f'(0) = 9 > 0$$

Thus, f is increasing on $(-\infty, -3)$, decreasing on $(-3, -1)$, and increasing on $(-1, \infty)$.

f has a relative maximum at -3 and a relative minimum at -1 .

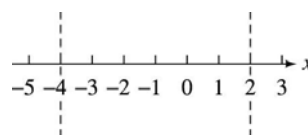
$$f(-3) = -8$$

$$f(-1) = -12$$

Relative maximum of -8 at -3 ; relative minimum of -12 at -1

16. $f(x) = x^3 + 3x^2 - 24x + 2$
 $f'(x) = 3x^2 + 6x - 24$
 $= 3(x^2 + 2x - 8)$
 $= 3(x + 4)(x - 2)$

$f'(x)$ is zero when $x = -4$ and $x = 2$.



$$f'(-5) = 21 > 0$$

$$f'(0) = -24 < 0$$

$$f'(3) = 21 > 0$$

f is increasing on $(-\infty, -4)$ and decreasing on $(-4, 2)$. Thus, a relative maximum occurs at $x = -4$. f is decreasing on $(-4, 2)$ and increasing on $(2, \infty)$. Thus, a relative minimum occurs at $x = 2$.

$$f(-4) = (-4)^3 + 3(-4)^2 - 24(-4) + 2$$

$$= 82$$

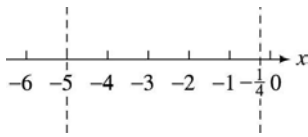
$$f(2) = (2)^3 + 3(2)^2 - 24(2) + 2$$

$$= -26$$

Relative maximum of 82 at -4 ; relative minimum of -26 at 2.

17. $f(x) = -\frac{4}{3}x^3 - \frac{21}{2}x^2 - 5x + 8$
 $f'(x) = -4x^2 - 21x - 5$
 $= (-4x - 1)(x + 5)$

$f'(x)$ is zero when $x = -5$, or $x = -\frac{1}{4}$.



$$\begin{aligned}f'(-6) &= -23 < 0 \\f'(-4) &= 15 > 0 \\f'(0) &= -5 < 0\end{aligned}$$

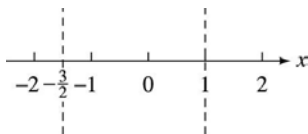
f is decreasing on $(-\infty, -5)$, increasing on $(-5, -\frac{1}{4})$, and decreasing on $(-\frac{1}{4}, \infty)$. f has a relative minimum at -5 and a relative maximum at $-\frac{1}{4}$.

$$\begin{aligned}f(-5) &= -\frac{377}{6} \\f\left(-\frac{1}{4}\right) &= \frac{827}{96}\end{aligned}$$

Relative maximum of $\frac{827}{96}$ at $-\frac{1}{4}$; relative minimum of $-\frac{377}{6}$ at -5 .

$$\begin{aligned}18. \quad f(x) &= -\frac{2}{3}x^3 - \frac{1}{2}x^2 + 3x - 4 \\f'(x) &= -2x^2 - x + 3 \\&= -(2x^2 + x - 3) \\&= -(2x + 3)(x - 1)\end{aligned}$$

$f'(x)$ is zero when $x = -\frac{3}{2}$ or $x = 1$.



$$\begin{aligned}f'(-2) &= -3 < 0 \\f'(0) &= 3 > 0 \\f'(2) &= -7 < 0\end{aligned}$$

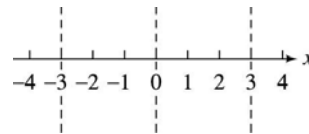
f is decreasing on $(-\infty, -\frac{3}{2})$ and increasing on $(-\frac{3}{2}, 1)$. Thus, a relative minimum occurs at $x = -\frac{3}{2}$. $f(x)$ is increasing on $(-\frac{3}{2}, 1)$ and decreasing on $(1, \infty)$. Thus, a relative maximum occurs at $x = 1$.

$$\begin{aligned}f\left(-\frac{3}{2}\right) &= -\frac{59}{8} \\f(1) &= -\frac{13}{6}\end{aligned}$$

Relative maximum of $-\frac{13}{6}$ at 1; relative minimum of $-\frac{59}{8}$ at $-\frac{3}{2}$.

$$\begin{aligned}19. \quad f(x) &= x^4 - 18x^2 - 4 \\f'(x) &= 4x^3 - 36x \\&= 4x(x^2 - 9) \\&= 4x(x + 3)(x - 3)\end{aligned}$$

$f'(x)$ is zero when $x = 0$ or $x = -3$ or $x = 3$.



$$\begin{aligned}f'(-4) &= 4(-4)^3 - 36(-4) = -112 < 0 \\f'(-1) &= -4 + 36 = 32 > 0 \\f'(1) &= 4 - 36 = -32 < 0 \\f'(4) &= 4(4)^3 - 36(4) = 112 > 0\end{aligned}$$

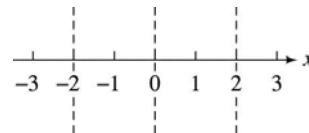
f is decreasing on $(-\infty, -3)$ and $(0, 3)$; f is increasing on $(-3, 0)$ and $(3, \infty)$.

$$\begin{aligned}f(-3) &= -85 \\f(0) &= -4 \\f(3) &= -85\end{aligned}$$

Relative maximum of -4 at 0; relative minimum of -85 at 3 and -3 .

$$\begin{aligned}20. \quad f(x) &= x^4 - 8x^2 - 9 \\f'(x) &= 4x^3 - 16x \\&= 4x(x^2 - 4) \\&= 4x(x + 2)(x - 2)\end{aligned}$$

$f'(x)$ is zero when $x = 0$ or $x = -2$ or $x = 2$.



$$\begin{aligned}f'(-3) &= -60 < 0 \\f'(-1) &= 12 > 0 \\f'(1) &= -12 < 0 \\f'(3) &= 60 > 0\end{aligned}$$

f is increasing on $(-2, 0)$ and $(2, \infty)$; f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

$$f(-2) = -7$$

$$f(0) = 9$$

$$f(2) = -7$$

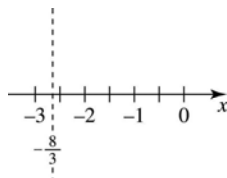
Relative maximum of 9 at 0; relative minimum of -7 at -2 and 2

$$\begin{aligned} 21. \quad f(x) &= 3 - (8 + 3x)^{2/3} \\ f'(x) &= -\frac{2}{3}(8 + 3x)^{-1/3}(3) \\ &= -\frac{2}{(8 + 3x)^{1/3}} \end{aligned}$$

Critical number:

$$8 + 3x = 0$$

$$x = -\frac{8}{3}$$



$$f'(-3) = 2 > 0$$

$$f'(0) = -1 < 0$$

f is increasing on $(-\infty, -\frac{8}{3})$ and decreasing on $(-\frac{8}{3}, \infty)$.

$$f\left(-\frac{8}{3}\right) = 3$$

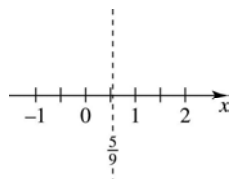
Relative maximum of 3 at $-\frac{8}{3}$.

$$\begin{aligned} 22. \quad f(x) &= \frac{(5 - 9x)^{2/3}}{7} + 1 \\ f'(x) &= \left(\frac{2}{3}\right) \frac{(5 - 9x)^{-1/3}}{7} (-9) \\ &= -\frac{6}{7(5 - 9x)^{1/3}} \end{aligned}$$

Critical number:

$$5 - 9x = 0$$

$$x = \frac{5}{9}$$



$$f'(0) \approx -0.5013 < 0$$

$$f'(1) \approx 0.5400 > 0$$

f is decreasing on $(-\infty, \frac{5}{9})$ and increasing on $(\frac{5}{9}, \infty)$.

$$f\left(\frac{5}{9}\right) = 1$$

Relative minimum of 1 at $\frac{5}{9}$.

$$\begin{aligned} 23. \quad f(x) &= 2x + 3x^{2/3} \\ f'(x) &= 2 + 2x^{-1/3} \\ &= 2 + \frac{2}{\sqrt[3]{x}} \end{aligned}$$

Find the critical numbers.

$$f'(x) = 0 \text{ when}$$

$$2 + \frac{2}{\sqrt[3]{x}} = 0$$

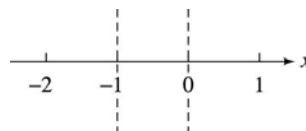
$$\frac{2}{\sqrt[3]{x}} = -2$$

$$x = (-1)^3$$

$$x = -1.$$

$f'(x)$ does not exist when

$$\sqrt[3]{x} = 0, \text{ that is, when } x = 0.$$



$$f'(-2) = 2 + \frac{2}{\sqrt[3]{-2}} \approx 0.41 > 0$$

$$\begin{aligned} f'\left(-\frac{1}{2}\right) &= 2 + \frac{2}{\sqrt[3]{-\frac{1}{2}}} \\ &= 2 + \frac{2\sqrt[3]{2}}{-1} \approx -0.52 < 0 \end{aligned}$$

$$f'(1) = 2 + \frac{2}{\sqrt[3]{1}} = 4 > 0$$

f is increasing on $(-\infty, -1)$ and $(0, \infty)$.

f is decreasing on $(-1, 0)$.

$$f(-1) = 2(-1) + 3(-1)^{2/3} = 1$$

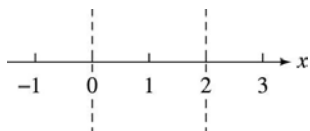
$$f(0) = 0$$

Relative maximum of 1 at -1; relative minimum of 0 at 0.

$$\begin{aligned}
 24. \quad f(x) &= 3x^{5/3} - 15x^{2/3} \\
 f'(x) &= 5x^{2/3} - 10x^{-1/3} \\
 &= 5x^{-1/3}(x - 2) \\
 &= \frac{5(x - 2)}{x^{1/3}}
 \end{aligned}$$

Find the critical numbers:

$$\begin{aligned}
 5(x - 2) &= 0 & x^{1/3} &= 0 \\
 x &= 2 & x &= 0
 \end{aligned}$$



$$\begin{aligned}
 f'(-1) &= 15 > 0 \\
 f'(1) &= -5 < 0 \\
 f'(3) &= 3.47 > 0
 \end{aligned}$$

f is increasing on $(-\infty, 0)$ and $(2, \infty)$.

f is decreasing on $(0, 2)$.

$$\begin{aligned}
 f(x) &= 3x^{2/3}(x - 5) \\
 f(0) &= 3 \cdot 0(0 - 5) = 0 \\
 f(2) &= 3 \cdot 2^{2/3}(2 - 5) \\
 &= -9 \cdot 2^{2/3} \approx -14.287
 \end{aligned}$$

Relative maximum of 0 at 0; relative minimum of $-9 \cdot 2^{2/3} \approx -14.287$ at 2

$$\begin{aligned}
 25. \quad f(x) &= x - \frac{1}{x} \\
 f'(x) &= 1 + \frac{1}{x^2} \text{ is never zero, but fails to exist at } x = 0
 \end{aligned}$$

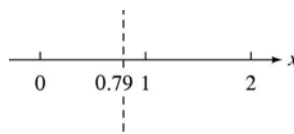
Since $f(x)$ also fails to exist at $x = 0$, there are no critical numbers and no relative extrema.

$$\begin{aligned}
 26. \quad f(x) &= x^2 + \frac{1}{x} \\
 f'(x) &= 2x - \frac{1}{x^2} \\
 &= \frac{2x^3 - 1}{x^2}
 \end{aligned}$$

Find the critical number:

$$\begin{aligned}
 2x^3 - 1 &= 0 \\
 x &= \frac{1}{\sqrt[3]{2}} = \frac{\sqrt[3]{4}}{2} \approx 0.79
 \end{aligned}$$

Note that both $f(x)$ and $f'(x)$ do not exist at $x = 0$, so 0 is not a critical number.



$$\begin{aligned}
 f'(-1) &= -3 < 0 \\
 f'(1) &= 1 > 0
 \end{aligned}$$

$f(x)$ is decreasing on $(-\infty, \frac{\sqrt[3]{4}}{2})$ and increasing on $(\frac{\sqrt[3]{4}}{2}, \infty)$.

$$\begin{aligned}
 f\left(\frac{\sqrt[3]{4}}{2}\right) &= \left(\frac{\sqrt[3]{4}}{2}\right)^2 + \left(\frac{\sqrt[3]{4}}{2}\right)^{-1} \\
 &= \left(\frac{\sqrt[3]{4}}{2}\right)^{-1} \left[\left(\frac{\sqrt[3]{4}}{2}\right)^3 + 1 \right] \\
 &= \frac{2}{\sqrt[3]{4}} \left(\frac{4}{8} + 1 \right) = \frac{2}{\sqrt[3]{4}} \left(\frac{3}{2} \right) \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} \\
 &= \frac{3\sqrt[3]{2}}{2} \approx 1.890
 \end{aligned}$$

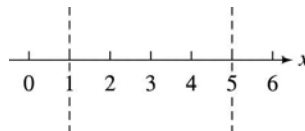
Relative minimum of $\frac{3\sqrt[3]{2}}{2} \approx 1.890$ at $\frac{\sqrt[3]{4}}{2}$.

$$\begin{aligned}
 27. \quad f(x) &= \frac{x^2 - 2x + 1}{x - 3} \\
 f'(x) &= \frac{(x - 3)(2x - 2) - (x^2 - 2x + 1)(1)}{(x - 3)^2} \\
 &= \frac{x^2 - 6x + 5}{(x - 3)^2}
 \end{aligned}$$

Find the critical numbers:

$$\begin{aligned}
 x^2 - 6x + 5 &= 0 \\
 (x - 5)(x - 1) &= 0 \\
 x &= 5 \text{ or } x = 1
 \end{aligned}$$

Note that $f(x)$ and $f'(x)$ do not exist at $x = 3$, so the only critical numbers are 1 and 5.



$$f'(0) = \frac{5}{9} > 0$$

$$f'(2) = -3 < 0$$

$$f'(6) = \frac{5}{9} > 0$$

$f(x)$ is increasing on $(-\infty, 1)$ and $(5, \infty)$.

$f(x)$ is decreasing on $(1, 5)$.

$$f(1) = 0$$

$$f(5) = 8$$

Relative maximum of 0 at 1; relative minimum of 8 at 5

$$28. \quad f(x) = \frac{x^2 - 6x + 9}{x + 2}$$

$$f'(x) = \frac{(2x - 6)(x + 2) - (1)(x^2 - 6x + 9)}{(x + 2)^2}$$

$$= \frac{x^2 + 4x - 21}{(x + 2)^2}$$

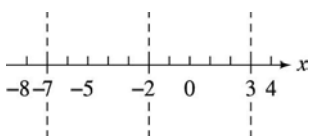
$$= \frac{(x + 7)(x - 3)}{(x + 2)^2}$$

Find the critical numbers:

$$(x + 7)(x - 3) = 0$$

$$x = -7 \text{ or } x = 3$$

Note that $f(x)$ and $f'(x)$ do not exist at $x = -2$, so the only critical numbers are -7 and 3 .



$$f'(-8) = \frac{11}{36} > 0$$

$$f'(-3) = -24 < 0$$

$$f'(0) = -\frac{21}{4} < 0$$

$$f'(4) = \frac{11}{36} > 0$$

f is increasing on $(-\infty, -7)$ and $(3, \infty)$.

f is decreasing on $(-7, -2)$ and $(-2, 3)$.

$$f(-7) = -20$$

$$f(3) = 0$$

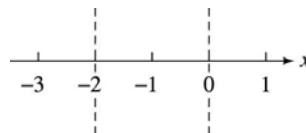
Relative maximum of -20 at -7 ; relative minimum of 0 at 3 .

$$29. \quad f(x) = x^2 e^x - 3$$

$$f'(x) = x^2 e^x + 2x e^x$$

$$= x e^x (x + 2)$$

$f'(x)$ is zero at $x = 0$ and $x = -2$.



$$f'(-3) = 3e^{-3} = \frac{3}{e^3} > 0$$

$$f'(-1) = -e^{-1} = -\frac{1}{e} < 0$$

$$f'(1) = 3e^1 > 0$$

f is increasing on $(-\infty, -2)$ and $(0, \infty)$.

f is decreasing on $(-2, 0)$.

$$f(0) = 0 \cdot e^0 - 3 = -3$$

$$f(-2) = (-2)^2 e^{-2} - 3$$

$$= \frac{4}{e^2} - 3$$

$$\approx -2.46$$

Relative minimum of -3 at 0 ; relative maximum of -2.46 at -2 .

$$30. \quad f(x) = 3x e^x + 2$$

$$f'(x) = 3x e^x + 3e^x$$

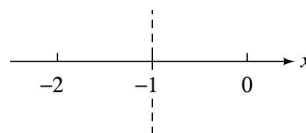
$$= 3e^x (x + 1)$$

Find the critical numbers:

$$3e^x = 0 \text{ or } x + 1 = 0$$

$$e^x = 0 \text{ or } x = -1$$

e^x is always positive, so the only critical number is -1 .



$$f'(-2) = 3e^{-2}(-2 + 1)$$

$$= -3e^{-2} < 0$$

$$f'(0) = 3e^0(0 + 1)$$

$$= 3 > 0$$

f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

$$\begin{aligned} f(-1) &= 3(-1)e^{-1} + 2 \\ &= \frac{-3}{e} + 2 \approx 0.9 \end{aligned}$$

Relative minimum of 0.90 at -1

31. $f(x) = 2x + \ln x$

$$f'(x) = 2 + \frac{1}{x} = \frac{2x+1}{x}$$

$f'(x)$ is zero at $x = -\frac{1}{2}$. The domain of $f(x)$ is $(0, \infty)$. Therefore $f'(x)$ is never zero in the domain of $f(x)$.

$f'(1) = 3 > 0$. Since $f(x)$ is always increasing, f has no relative extrema.

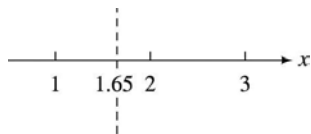
32. $f(x) = \frac{x^2}{\ln x}$

$$\begin{aligned} f'(x) &= \frac{(\ln x)2x - x^2\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{2x \ln x - x}{(\ln x)^2} \\ &= \frac{x(2 \ln x - 1)}{(\ln x)^2} \end{aligned}$$

Find the critical numbers:

$$\begin{aligned} x = 0 \quad \text{or} \quad 2 \ln x - 1 = 0 \quad \text{or} \quad \ln x = 0 \\ 2 \ln x = 1 \quad \quad \quad x = 1 \\ \ln x = \frac{1}{2} \\ x = e^{1/2} \\ = \sqrt{e} \\ \approx 1.65 \end{aligned}$$

Since the domain of $y = \ln x$ is $(0, \infty)$, 0 is not a critical number. Since $\ln 1 = 0$, we see that 1 is also not in the domain of f . Thus $\sqrt{e} \approx 1.65$ is the only critical number.



$$f'(1.5) = \frac{1.5(2 \ln 1.5 - 1)}{(\ln 1.5)^2} < 0$$

$$f'(2) = \frac{2(2 \ln 2 - 1)}{(\ln 2)^2} > 0$$

$f(x)$ is increasing on $(1.65, \infty)$ and decreasing on $(1, 1.65)$.

$$f(1.65) = \frac{(1.65)^2}{\ln(1.65)} \approx 5.44$$

Relative minimum of 5.44 at 1.65

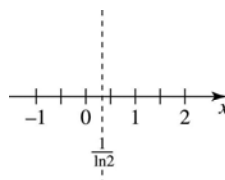
33. $f(x) = \frac{2^x}{x}$

$$\begin{aligned} f'(x) &= \frac{(x) \ln 2(2^x) - 2^x(1)}{x^2} \\ &= \frac{2^x(x \ln 2 - 1)}{x^2} \end{aligned}$$

Find the critical numbers:

$$\begin{aligned} x \ln 2 - 1 = 0 \quad \text{or} \quad x^2 = 0 \\ x = \frac{1}{\ln 2} \quad \quad \quad x = 0 \end{aligned}$$

Since f is defined for $x = 0$, 0 is not a critical number. $x = \frac{1}{\ln 2} \approx 1.44$ is the only critical number.



$$f'(1) \approx -0.6137 < 0$$

$$f'(2) \approx 0.3863 > 0$$

f is decreasing on $(0, \frac{1}{\ln 2})$ and increasing on $(\frac{1}{\ln 2}, \infty)$.

$$\begin{aligned} f\left(\frac{1}{\ln 2}\right) &= \frac{2^{1/\ln 2}}{\frac{1}{\ln 2}} \\ &= \ln 2 \left(e^{\ln 2}\right)^{1/\ln 2} = e \ln 2 \end{aligned}$$

Relative minimum of $e \ln 2$ at $\frac{1}{\ln 2}$.

34. $f(x) = x + 8^{-x}$

$$\begin{aligned} f'(x) &= 1 + (\ln 8)8^{-x}(-1) \\ &= 1 - \frac{\ln 8}{8^x} \end{aligned}$$

Find the critical numbers:

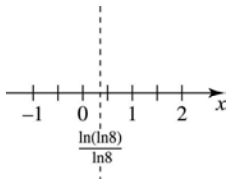
$$1 - \frac{\ln 8}{8^x} = 0$$

$$1 = \frac{\ln 8}{8^x}$$

$$8^x = \ln 8$$

$$x \ln 8 = \ln(\ln 8)$$

$$x = \frac{\ln(\ln 8)}{\ln 8} \approx 0.35$$



$$f'(0) \approx -1.0794 < 0$$

$$f'(1) \approx 0.7401 > 0$$

f is decreasing on $\left(-\infty, \frac{\ln(\ln 8)}{\ln 8}\right)$ and increasing on $\left(\frac{\ln(\ln 8)}{\ln 8}, \infty\right)$.

Note that

$$\begin{aligned} 8^{-\ln(\ln 8)/\ln 8} &= e^{(\ln 8)(-\ln(\ln 8)/\ln 8)} \\ &= e^{-\ln(\ln 8)} \\ &= \frac{1}{\ln 8}. \end{aligned}$$

$$\begin{aligned} f\left(\frac{\ln(\ln 8)}{\ln 8}\right) &= \frac{\ln(\ln 8)}{\ln 8} + 8^{-\ln(\ln 8)/\ln 8} \\ &= \frac{\ln(\ln 8)}{\ln 8} + \frac{1}{\ln 8} \\ &= \frac{\ln(\ln 8) + 1}{\ln 8} \end{aligned}$$

Relative minimum of $\frac{\ln(\ln 8) + 1}{\ln 8}$ at $\frac{\ln(\ln 8)}{\ln 8}$.

$$\begin{aligned} 35. \quad y &= -2x^2 + 12x - 5 \\ y' &= -4x + 12 \\ &= -4(x - 3) \end{aligned}$$

The vertex occurs when $y' = 0$ or when

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

When $x = 3$,

$$y = -2(3)^2 + 12(3) - 5 = 13$$

The vertex is $(3, 13)$.

$$\begin{aligned} 36. \quad y &= ax^2 + bx + c \\ y' &= 2ax + b \end{aligned}$$

The vertex occurs when $y' = 0$.

$$2ax + b = 0$$

$$x = \frac{-b}{2a}$$

$$\begin{aligned} a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c &= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \\ &= \frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} \\ &= \frac{4ac - b^2}{4a}. \end{aligned}$$

The vertex is at $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$.

$$\begin{aligned} 37. \quad f(x) &= x^5 - x^4 + 4x^3 - 30x^2 + 5x + 6 \\ f'(x) &= 5x^4 - 4x^3 + 12x^2 - 60x + 5 \end{aligned}$$

Graph f' on a graphing calculator. A suitable choice for the viewing window is $[-4, 4]$ by $[-50, 50]$, $\text{Yscl} = 10$.

Use the calculator to estimate the x -intercepts of this graph. These numbers are the solutions of the equation $f'(x) = 0$ and thus the critical numbers for f . Rounded to three decimal places, these x -values are 0.085 and 2.161.

Examine the graph of f' near $x = 0.085$ and $x = 2.161$. Observe that $f'(x) > 0$ to the left of $x = 0.085$ and $f'(x) < 0$ to the right of $x = 0.085$. Also observe that $f'(x) < 0$ to the left of $x = 2.161$ and $f'(x) > 0$ to the right of $x = 2.161$. The first derivative test allows us to conclude that f has a relative maximum at $x = 0.085$ and a relative minimum at $x = 2.161$.

$$f(0.085) \approx 6.211$$

$$f(2.161) \approx -57.607$$

Relative maximum of 6.211 at 0.085; relative minimum of -57.607 at 2.161.

$$\begin{aligned} 38. \quad f(x) &= -x^5 - x^4 + 2x^3 - 25x^2 + 9x + 12 \\ f'(x) &= -5x^4 - 4x^3 + 6x^2 - 50x + 9 \end{aligned}$$

Graph f' on a graphing calculator. A suitable choice for the viewing window is $[-4, 4]$ by $[-100, 100]$, $\text{Yscl} = 20$.

Use the calculator to estimate the x -intercepts of this graph. These numbers are the solutions of the equation $f'(x) = 0$ and thus the critical numbers for f . Rounded to three decimal places, these x -values are 0.183 and -2.703 .

Examine the graph of f' near $x = -2.703$ and $x = 0.183$. Observe that $f'(x) < 0$ to the left of $x = -2.703$ and $f'(x) > 0$ to the right of $x = -2.703$. Also observe that $f'(x) > 0$ to the left of $x = 0.183$ and $f'(x) < 0$ to the right of $x = 0.183$. The first derivative test allows us to conclude that f has a relative minimum at $x = -2.703$ and a relative maximum at $x = 0.183$.

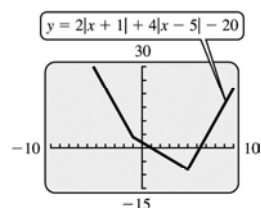
$$f(0.183) \approx 12.821$$

$$f(-2.703) \approx -143.572$$

Relative maximum of 12.821 at 0.183; relative minimum of -143.572 at -2.703

39. $f(x) = 2|x + 1| + 4|x - 5| - 20$

Graph this function in the window $[-10, 10]$ by $[-15, 30]$, Yscl = 5.



The graph shows that f has no relative maxima, but there is a relative minimum at $x = 5$.

(Note that the graph has a sharp point at $(5, -8)$, indicating that $f'(-5)$ does not exist.)

40. (a) When graphing $g(x)$ in the standard window, no graph seems to appear.

(b) $g(x) = \frac{1}{x^{12}} - 2\left(\frac{1000}{x}\right)^6$

$$g'(x) = \frac{-12}{x^{13}} - 12\left(\frac{1000}{x}\right)^5\left(\frac{-1000}{x^2}\right)$$

$$= \frac{-12}{x^{13}} + \frac{1.2 \times 10^{19}}{x^7}$$

$$= \frac{-12}{x^{13}} + \frac{(1.2 \times 10^{19})x^6}{x^{13}}$$

$$= \frac{-12 + (1.2 \times 10^{19})x^6}{x^{13}}$$

Find the critical numbers:

$$-12 + (1.2 \times 10^{19})x^6 = 0 \quad \text{or} \quad x^{13} = 0$$

$$(1.2 \times 10^{19})x^6 = 12 \quad x = 0$$

$$x^6 = \frac{12}{1.2 \times 10^{19}}$$

$$x^6 = \frac{10}{10^{19}} = 10^{-18}$$

$$x = \sqrt[6]{10^{-18}}$$

$$= \pm 0.001 \quad \text{or} \quad x = 0$$

$$g'(-1) = \frac{-12 + (1.2 \times 10^{19})(-1)^6}{(-1)^{13}} < 0$$

$$g'(-0.0001) = \frac{-12 + (1.2 \times 10^{19})(-0.0001)^6}{(-0.0001)^{13}} > 0$$

$$g'(0.0001) = \frac{-12 + (1.2 \times 10^{19})(0.0001)^6}{(0.0001)^{13}} < 0$$

$$g'(1) = \frac{-12 + (1.2 \times 10^{19})1^6}{1^{13}} > 0$$

$$g(0.001) = -10^{-36}; g(-0.001) = -10^{36}$$

Minimum of -10^{36} at $x = \pm 0.001$

41. $C(q) = 80 + 18q; p = 70 - 2q$

$$P(q) = R(q) - C(q) = pq - C(q)$$

$$= (70 - 2q)q - (80 + 18q)$$

$$= -2q^2 + 52q - 80$$

- (a) Since the graph of P is a parabola that opens downward, we know that its vertex is a maximum point. To find the q -value of this point, we find the critical number.

$$P'(q) = -4q + 52$$

$$P'(q) = 0 \quad \text{when}$$

$$-4q + 52 = 0$$

$$4q = 52$$

$$q = 13$$

The number of units that produce maximum profit is 13.

- (b) If $q = 13$,

$$p = 70 - 2(13)$$

$$= 44$$

The price that produces maximum profit is \$44.

(c) $P(13) = -2(13)^2 + 52(13) - 80 = 258$

The maximum profit is \$258.

$$\begin{aligned}
 42. \quad C(q) &= 25q + 5000; p = 90 - 0.02q \\
 P(q) &= R(q) - C(q) = pq - C(q) \\
 &= (90 - 0.02q)q - (25q + 5000) \\
 &= -0.02q^2 + 65q - 5000
 \end{aligned}$$

- (a) Since the graph of P is a parabola that opens downward, we know that its vertex is a maximum point. To find the q -value of this point, we find the critical number.

$$\begin{aligned}
 P'(q) &= -0.04q + 65 \\
 P'(q) &= 0 \text{ when} \\
 -0.04q + 65 &= 0 \\
 0.04q &= 65 \\
 q &= 1625
 \end{aligned}$$

The number of units that produces maximum profit is 1625.

- (b) If $q = 1625$,

$$\begin{aligned}
 p &= 90 - 0.02(1625) \\
 &= 57.5
 \end{aligned}$$

The price per unit that produces maximum profit is \$57.50.

$$\begin{aligned}
 (c) \quad P(1625) &= -0.02(1625)^2 + 65(1625) - 5000 \\
 &= 47,812.5
 \end{aligned}$$

The maximum profit is \$47,812.50.

$$\begin{aligned}
 43. \quad C(q) &= 100 + 20qe^{-0.01q}; p = 40e^{-0.01q} \\
 P(q) &= R(q) - C(q) = pq - C(q) \\
 &= (40e^{-0.01q})q - (100 + 20qe^{-0.01q}) \\
 &= 20qe^{-0.01q} - 100
 \end{aligned}$$

$$\begin{aligned}
 (a) \quad P'(q) &= 20e^{-0.01q} + 20qe^{-0.01q}(-0.01) \\
 &= (20 - 0.2q)e^{-0.01q}
 \end{aligned}$$

Solve $P'(q) = 0$.

$$\begin{aligned}
 (20 - 0.2q)e^{-0.01q} &= 0 \\
 20 - 0.2q &= 0 \\
 q &= 100
 \end{aligned}$$

Since $e^{-0.01q} > 0$ for all values of q , the sign of $P'(q)$ is the same as the sign of $20 - 0.2q$. For $q < 100$, $P'(q) > 0$; for $q > 100$, $P'(q) < 0$. Therefore, the number of units that produces maximum profit is 100.

- (b) If $q = 100$,

$$\begin{aligned}
 p &= 40e^{-0.01(100)} \\
 &= 40e^{-1} \\
 &\approx 14.72
 \end{aligned}$$

The price per unit that produces maximum profit is \$14.72.

$$\begin{aligned}
 (c) \quad P(100) &= 20(100)e^{-0.01(100)} - 100 \\
 &= 2000e^{-1} - 100 \\
 &\approx 635.76
 \end{aligned}$$

The maximum profit is \$635.76.

$$\begin{aligned}
 44. \quad C(q) &= 21.047q + 3; p = 50 - 5 \ln(q + 10) \\
 P(q) &= R(q) - C(q) = pq - C(q) \\
 &= [50 - 5 \ln(q + 10)]q - (21.047q + 3) \\
 &= 28.953q - 3 - 5q \ln(q + 10)
 \end{aligned}$$

$$(a) \quad P'(q) = 28.953 - 5 \ln(q + 10) - \frac{5q}{q+10}$$

Use a graphing calculator to find the critical numbers for $q > 0$. The critical number is 120. For $q < 120$, $P'(q) > 0$; for $q > 120$, $P'(q) < 0$. Therefore, the number of units that produces maximum profit is 120.

- (b) If $q = 120$,

$$\begin{aligned}
 p &= 50 - 5 \ln(120 + 10) \\
 &\approx 25.66
 \end{aligned}$$

The price per unit that produces maximum profit is \$25.66.

$$\begin{aligned}
 (c) \quad P(120) &= 28.953(120) - 3 \\
 &\quad - 5(120) \ln(102 + 10) \\
 &\approx 550.84
 \end{aligned}$$

The maximum profit is \$550.84.

$$\begin{aligned}
 45. \quad P(t) &= -0.005846t^3 + 0.1614t^2 \\
 &\quad - 0.4910t + 20.47 \\
 P'(t) &= -0.017538t^2 + 0.3228t - 0.4190
 \end{aligned}$$

Use a graphing calculator to find the critical numbers for $0 \leq t \leq 24$.

The critical numbers are $t \approx 1.673$ and $t \approx 16.733$. These divide the domain of P into three intervals, and we pick a test point in each: Pick $t = 1$, $t = 10$, and $t = 20$.

$$\begin{aligned}
 P'(1) &\approx -0.186 \\
 P'(10) &\approx 0.983 \\
 P'(20) &\approx -1.050
 \end{aligned}$$

P is decreasing on $(0, 1.673)$ and $(16.733, 24)$ and increasing on $(1.673, 16.733)$. There will be relative maxima at the left endpoint ($t = 0$) and at $t = 16.733$; there will be relative minima at $t = 1.673$ and at the right endpoint ($t = 24$).

$$\begin{aligned}P(0) &= 20.47 \\P(1.673) &\approx 20.07 \\P(16.733) &\approx 30.06 \\P(24) &\approx 20.84\end{aligned}$$

We convert the time values to hours and minutes by multiplying the decimal parts by 60:
 $(0.673)(60) \approx 40$ and $(0.733)(60) \approx 44$.

Recalling that the function P gives the power in thousands of megawatts, we have the final result:

Relative maximum of 20,470 megawatts at midnight ($t = 0$)

Relative minimum of 20,070 megawatts at 1:40 AM.

Relative maximum of 30,060 megawatts at 4:44 PM.

Relative minimum of 20,840 megawatts at midnight ($t = 24$).

46. $P(x) = \ln(-x^3 + 3x^2 + 72x + 1)$, for x in $[0, 10]$.

$$\begin{aligned}\text{(a)} \quad P'(x) &= \frac{-3x^2 + 6x + 72}{-x^3 + 3x^2 + 72x + 1} \\&= \frac{-3(x - 6)(x + 4)}{-x^3 + 3x^2 + 72x + 1}\end{aligned}$$

$P'(x) = 0$ when $x = 6$ or $x = -4$, but 4 is not in the domain of $[0, 10]$. $P'(x)$ fails to exist for $x \approx -0.14$ and $x \approx 10.123$, neither of which are in the domain of $[0, 10]$. Thus, the critical number is 6.

Use the first derivative test to verify that $x = 6$ gives a maximum profit.

$$\begin{aligned}P'(5) &= \frac{27}{311} > 0 \\P'(7) &= -\frac{11}{103} < 0\end{aligned}$$

The maximum profit results when 6 units are sold.

- (b) $P(6) \approx 5.784$

The maximum profit is about \$5784.

47. $p = D(q) = 200e^{-0.1q}$

$$\begin{aligned}R(q) &= pq \\&= 200qe^{-0.1q} \\R'(q) &= 200qe^{-0.1q}(-0.1) + 200e^{-0.1q} \\&= 20e^{-0.1q}(10 - q)\end{aligned}$$

$R'(q) = 0$ when $q = 10$, the only critical number. Use the first derivative test to verify that $q = 10$ gives the maximum revenue.

$$\begin{aligned}R'(9) &= 20e^{-0.9} > 0 \\R'(11) &= -20e^{-1.1} < 0\end{aligned}$$

The maximum revenue results when $q = 10$

$p = D(10) = \frac{200}{e} \approx 73.58$, or when telephones are sold at \$73.58.

48. $p = D(q) = 500qe^{-0.0016q^2}$
- $$\begin{aligned}R(q) &= pq = \left(500qe^{-0.0016q^2}\right)q \\&= 500q^2e^{-0.0016q^2} \\R'(q) &= 500e^{-0.0016q^2}(2q) \\&\quad + 500q^2e^{-0.0016q^2}(-0.0016 \cdot 2q) \\&= (1000q - 1.6q^3)e^{-0.0016q^2}\end{aligned}$$

Since $e^{-0.0016q^2} > 0$ for all values of q ,

$$\begin{aligned}R'(q) &= 0 \\(1000q - 1.6q^3)e^{-0.0016q^2} &= 0 \\1.6q^3 &= 1000q \\q^2 &= 625 \\q &= 25\end{aligned}$$

If $q < 25$, $R'(q) > 0$; if $q > 25$, $R'(q) < 0$. So p has a maximum value at $q = 25$.

$$p = D(25) = 500(25)e^{-0.0016(25)^2} \approx 4600$$

Maximum revenue occurs when the price is about \$4600 and 25 computer systems are sold.

49. $C(x) = 0.002x^3 = 9x + 6912$
- $$\begin{aligned}\bar{C}(x) &= \frac{C(x)}{x} = 0.002x^2 + 9 + \frac{6912}{x} \\ \bar{C}'(x) &= 0.004x - \frac{6912}{x^2} \\ \bar{C}'(x) &= 0 \text{ when}\end{aligned}$$

$$\begin{aligned}
 0.004x - \frac{6912}{x^2} &= 0 \\
 0.004x^3 &= 6912 \\
 x^3 &= 1,728,000 \\
 x &= 120
 \end{aligned}$$

A product level of 120 units will produce the minimum average cost per unit.

50. Reading the graph we can identify the low and high points; we estimate the corresponding unemployment rates from the vertical scale. There is a relative maximum of 7.5% in 1992, or 6% in 2003, and of 9.3% in 2009; there is a relative minimum of 5.6% in 1990, of 4% in 2000, and of 4.6% in 2006 and 2007.

51. $a(t) = 0.008t^3 - 0.288t^2 + 2.304t + 7$
 $a'(t) = 0.024t^2 - 0.576t + 2.304$

Set $a' = 0$ and solve for t .

$$\begin{aligned}
 0.024t^2 - 0.576t + 2.304 &= 0 \\
 0.024(t^2 - 24t + 96) &= 0 \\
 t^2 - 24t + 96 &= 0 \\
 t &\approx 5.07 \text{ or } t \approx 18.93
 \end{aligned}$$

$t = 5.07 = 5 \text{ hours} + 0.07 \cdot 60 \text{ minutes}$
 corresponds to 5:04 P.M.

$t = 18.93 = 18 \text{ hours} + 0.93 \cdot 60 \text{ minutes}$
 corresponds to 6:56 A.M.

52. (a) $M(t) = 6.281t^{0.242}e^{-0.025t}$
 $M'(t) = (1.520002t^{-0.758})(e^{-0.025t})$
 $+ (6.281t^{0.242})(-0.025e^{-0.025t})$
 $= e^{-0.025t}(1.520002t^{-0.758}$
 $- 0.157025t^{0.242})$

$M'(t) = 0$ when

$$\begin{aligned}
 1.520002t^{-0.758} - 0.157025t^{0.242} &= 0 \\
 t &= 9.68
 \end{aligned}$$

Let $t = 9.68$ in $M(t)$.

$$\begin{aligned}
 M(9.68) &= 6.281(9.68)^{0.242}e^{-0.025(9.68)} \\
 &\approx 8.54 \text{ kg}
 \end{aligned}$$

The maximum daily consumption is 8.54 kg and it occurs at 9.68 weeks.

(b) $M(t) = at^be^{-ct}$
 $M'(t) = (bat^{b-1})(e^{-ct}) + (at^b)(-ce^{-ct})$
 $= ae^{-ct}(bt^{b-1} - ct^b)$

$M'(t) = 0$ when

$$\begin{aligned}
 bt^{b-1} - ct^b &= 0 \\
 bt^{b-1} &= ct^b \\
 \frac{b}{c} &= t
 \end{aligned}$$

Let $t = \frac{b}{c}$ in $M(t)$.

$$\begin{aligned}
 M\left(\frac{b}{c}\right) &= a\left(\frac{b}{c}\right)^b e^{-c\left(\frac{b}{c}\right)} \\
 &= a\left(\frac{b}{c}\right)^b e^{-b}
 \end{aligned}$$

The maximum daily consumption is

$a(b/c)^b e^{-b}$ kg and it occurs at b/c weeks.

53. $M(t) = 369(0.93)^t(t)^{0.36}$
 $M'(t) = (369)(0.93)^t \ln(0.93)(t)^{0.36}$
 $+ 369(0.93)^t(0.36)(t)^{-0.64}$
 $= (369t^{0.36})(0.93)^t \ln(0.93)$
 $+ \frac{132.84(0.93)^t}{t^{0.64}}$

$M'(t) = 0$ when $t \approx 4.96$.

Verify that $t \approx 4.96$ gives a maximum.

$$M'(4) > 0$$

$$M'(5) < 0$$

Find $M(4.96)$

$$M(4.96) = 369(0.93)^{4.96}(4.96)^{0.36} \approx 485.22$$

The female moose reaches a maximum weight of about 458.22 kilograms at about 4.96 years.

$$\begin{aligned}
 54. \quad F(t) &= -10.28 + 175.9te^{-t/1.3} \\
 F'(t) &= (175.9t)\left(-\frac{1}{1.3}\right)e^{-t/1.3} \\
 &\quad + (175.9)(e^{-t/1.3}) \\
 &= e^{-t/1.3}\left(-\frac{175.9t}{1.3} + 175.9\right)
 \end{aligned}$$

$F'(t) = 0$ when

$$\begin{aligned}
 -\frac{175.9t}{1.3} + 175.9 &= 0 \\
 t &= 1.3
 \end{aligned}$$

At 1.3 hours, the thermal effect of the food is maximized.

$$\begin{aligned}
 55. \quad D(x) &= -x^4 + 8x^3 + 80x^2 \\
 D'(x) &= -4x^3 + 24x^2 + 160x \\
 &= -4x(x^2 - 6x - 40) \\
 &= -4x(x + 4)(x - 10)
 \end{aligned}$$

$D'(x) = 0$ when $x = 0$, $x = -4$, or $x = 10$.

Disregard the nonpositive values.

Verify that $x = 10$ gives a maximum.

$$\begin{aligned}
 D'(9) &= 468 > 0 \\
 D'(11) &= -660 < 0
 \end{aligned}$$

The speaker should aim for a degree of discrepancy of 10.

$$\begin{aligned}
 56. \quad R(t) &= \frac{20t}{t^2 + 100} \\
 R'(t) &= \frac{20(t^2 + 100) - 20t(2t)}{(t^2 + 100)^2} \\
 &= \frac{2000 - 20t^2}{(t^2 + 100)^2}
 \end{aligned}$$

$R'(t) = 0$ when

$$\begin{aligned}
 2000 - 20t^2 &= 0 \\
 -20t^2 &= -2000 \\
 t^2 &= 100 \\
 t &= \pm 10.
 \end{aligned}$$

Disregard the negative value.

Use the first derivative test to verify that $t = 10$ gives a maximum rating.

$$\begin{aligned}
 R'(9) &= 0.0116 > 0 \\
 R'(11) &= -0.0086 < 0
 \end{aligned}$$

The film should be 10 minutes long.

$$\begin{aligned}
 57. \quad s(t) &= -16t^2 + 40t + 3 \\
 s'(t) &= -32t + 40
 \end{aligned}$$

$$\begin{aligned}
 \text{(a) when } s'(t) &= 0, \\
 -32t + 40 &= 0 \\
 32t &= 40 \\
 t &= \frac{40}{32} = \frac{5}{4}
 \end{aligned}$$

Verify that $t = 5/4$ gives a maximum.

$$\begin{aligned}
 s'(1) &= 8 \\
 s'(2) &= -24
 \end{aligned}$$

Now find the height when $t = 5/4$.

$$\begin{aligned}
 s\left(\frac{5}{4}\right) &= -16\left(\frac{5}{4}\right)^2 + 40\left(\frac{5}{4}\right) + 3 \\
 &= -25 + 50 + 3 \\
 &= 28
 \end{aligned}$$

The maximum height of the cork is 28 feet.

(b) The cork remains in the air as long as $s(t) > 0$. Use the quadratic formula to solve $s(t) = 0$.

$$\begin{aligned}
 -16t^2 + 40t + 3 &= 0 \\
 t &= \frac{-40 \pm \sqrt{40^2 - 4(-16)(3)}}{2(-16)} \\
 &= \frac{-5 \pm \sqrt{28}}{-4} \\
 &= \frac{5 \pm 2\sqrt{7}}{4} \\
 &\approx -0.073, 2.573
 \end{aligned}$$

Only the positive solution is relevant, so the cork stays in the air for about 2.57 seconds.

5.3 Higher Derivatives, Concavity, and the Second Derivative Test

Your Turn 1

$$\begin{aligned}
 f(x) &= 5x^4 - 4x^3 + 3x \\
 f'(x) &= 20x^3 - 12x^2 + 3 \\
 f''(x) &= 60x^2 - 24x \\
 f''(1) &= 60(1^2) - 24(1) \\
 &= 36
 \end{aligned}$$

Your Turn 2**(a)** Use the chain rule.

$$\begin{aligned}
 f(x) &= (x^3 + 1)^2 \\
 f'(x) &= (2)(x^3 + 1)(3x^2) \\
 &= 6x^2(x^3 + 1) \\
 &= 6x^5 + 6x^2 \\
 f''(x) &= 30x^4 + 12x
 \end{aligned}$$

(b) Use the product rule.

$$\begin{aligned}
 f(x) &= xe^x \\
 f'(x) &= xe^x + (1)e^x \\
 &= xe^x + e^x
 \end{aligned}$$

Now note that we have already found the derivative of the first term of $f'(x)$, which is the original function $f(x)$.

Thus

$$\begin{aligned}
 f''(x) &= (xe^x + e^x) + e^x \\
 &= 2e^x + xe^x
 \end{aligned}$$

(c) Use the quotient rule.

$$\begin{aligned}
 h(x) &= \frac{\ln x}{x} \\
 h'(x) &= \frac{(x)\left(\frac{1}{x}\right) - (\ln x)(1)}{x^2} \\
 &= \frac{1 - \ln x}{x^2} \\
 h''(x) &= \frac{(x^2)\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} \\
 &= \frac{-x - 2x + 2x \ln x}{x^4} \\
 &= \frac{-3 + 2 \ln x}{x^3}
 \end{aligned}$$

Your Turn 3

$$\begin{aligned}
 s(t) &= t^3 - 3t^2 - 24t + 10 \\
 v(t) &= s'(t) = 3t^2 - 6t - 24 \\
 a(t) &= v'(t) = s''(t) = 6t - 6
 \end{aligned}$$

First find when v changes sign.

$$\begin{aligned}
 v(t) &= 3t^2 - 6t - 24 = 0 \\
 3(t^2 - 2t - 8) &= 0 \\
 (t - 4)(t + 2) &= 0 \\
 t &= 4 \text{ or } t = -2
 \end{aligned}$$

Only the positive value of t is relevant. Check the velocity at times before and after 4, say at $t = 2$ and $t = 6$.

$$\begin{aligned}
 v(2) &= -24 < 0 \\
 v(6) &= 48 > 0
 \end{aligned}$$

Thus the car backs up for the first four seconds and then goes forward.

Now find where a changes sign.

$$\begin{aligned}
 a(t) &= 6t - 6 = 0 \\
 6t - 6 &= 0 \\
 t &= 1
 \end{aligned}$$

Check the acceleration at times before and after 1, say at $t = 1/2$ and $t = 2$.

$$\begin{aligned}
 a(1/2) &= -3 < 0 \\
 a(2) &= 6 > 0
 \end{aligned}$$

Now we construct the following graph showing the signs of v and a .

$v(t)$	—	—	+
$a(t)$	—	+	+
net result	+	—	+
	0	1	4

The car speeds up (in the backward direction) for $0 < t < 1$; it slows down (still in the backward direction) for $1 < t < 4$; it speeds up (in the forward direction) for $t > 4$. (All times are in seconds.)

Your Turn 4

$$\begin{aligned}
 f(x) &= x^5 - 30x^3 \\
 f'(x) &= 5x^4 - 90x^2 \\
 f''(x) &= 20x^3 - 180x
 \end{aligned}$$

Factor $f''(x)$ and create a number line.

$$\begin{aligned}
 f''(x) &= 20x^3 - 180x \\
 &= 20(x^3 - 9x) \\
 &= 20(x)(x - 3)(x + 3)
 \end{aligned}$$

The zeros of $f''(x)$ are $-3, 0$, and 3 , and they divide the domain of f into four intervals:

$$(-\infty, -3), (-3, 0), (0, 3), (3, \infty)$$

Choose a test point in each interval and find the sign of $f''(x)$ at the test points. For example, choose $-4, -1, 1$, and 4 .

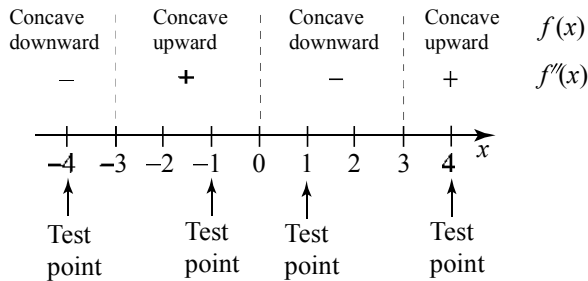
$$f''(-4) = -560$$

$$f''(-1) = 160$$

$$f''(1) = -160$$

$$f''(4) = 560$$

Here is the corresponding number line.



The figure shows that f is concave downward on $(-\infty, -3)$ and $(0, 3)$ and concave upward on $(-3, 0)$ and $(3, \infty)$. Inflection points occur where $f''(x)$ changes sign, so we evaluate f at these points:

$$f(-3) = 567$$

$$f(0) = 0$$

$$f(3) = -567$$

Thus the inflection points are $(-3, 567)$, $(0, 0)$ and $(3, -567)$.

Your Turn 5

$$f(x) = -2x^3 + 3x^2 + 72x$$

$$f'(x) = -6x^2 + 6x + 72$$

Solve the equation $f'(x) = 0$.

$$-6x^2 + 6x + 72 = 0$$

$$x^2 - x - 12 = 0$$

$$(x + 3)(x - 4) = 0$$

$$x = -3 \text{ or } x = 4$$

Now use the second derivative test.

$$f''(x) = -12x + 6$$

$$f''(-3) = 42$$

$$f''(4) = -42$$

Since $f''(-3) > 0$, $x = -3$ leads to a relative minimum; since $f''(4) < 0$, $x = 4$ leads to a relative maximum. Thus we have a relative minimum of $f(-3) = -135$ at $x = -3$ and a relative maximum of $f(4) = 208$ at $x = 4$.

5.3 Exercises

1. $f(x) = 5x^3 - 7x^2 + 4x + 3$

$$f'(x) = 15x^2 - 14x + 4$$

$$f''(x) = 30x - 14$$

$$f''(0) = 30(0) - 14 = -14$$

$$f''(2) = 30(2) - 14 = 46$$

2. $f(x) = 4x^3 + 5x^2 + 6x - 7$

$$f'(x) = 12x^2 + 10x + 6$$

$$f''(x) = 24x + 10$$

$$f''(0) = 24(0) + 10 = 10$$

$$f''(2) = 24(2) + 10 = 58$$

3. $f(x) = 4x^4 - 3x^3 - 2x^2 + 6$

$$f'(x) = 16x^3 - 9x^2 - 4x$$

$$f''(x) = 48x^2 - 18x - 4$$

$$f''(0) = 48(0)^2 - 18(0) - 4 = -4$$

$$f''(2) = 48(2)^2 - 18(2) - 4 = 152$$

4. $f(x) = -x^4 + 7x^3 - \frac{x^2}{2}$

$$f'(x) = -4x^3 + 21x^2 - x$$

$$f''(x) = -12x^2 + 42x - 1$$

$$f''(0) = -12(0)^2 + 42(0) - 1 = -1$$

$$f''(2) = -12(2)^2 + 42(2) - 2 = 35$$

5. $f(x) = 3x^2 - 4x + 8$

$$f'(x) = 6x - 4$$

$$f''(x) = 6$$

$$f''(0) = 6$$

$$f''(2) = 6$$

6. $f(x) = 8x^2 + 6x + 5$

$$f'(x) = 16x + 6$$

$$f''(x) = 16$$

$$f''(0) = 16$$

$$f''(2) = 16$$

$$\begin{aligned}
7. \quad f(x) &= \frac{x^2}{1+x} \\
f'(x) &= \frac{(1+x)(2x) - x^2(1)}{(1+x)^2} \\
&= \frac{2x + x^2}{(1+x)^2} \\
f''(x) &= \frac{(1+x)^2(2+2x) - (2x+x^2)(2)(1+x)}{(1+x)^4} \\
&= \frac{(1+x)(2+2x) - (2x+x^2)(2)}{(1+x)^3} \\
&= \frac{2}{(1+x)^3} \\
f''(0) &= 2 \\
f''(2) &= \frac{2}{27}
\end{aligned}$$

$$\begin{aligned}
8. \quad f(x) &= \frac{-x}{1-x^2} \\
f'(x) &= \frac{(1-x^2)(-1) - (-x)(-2x)}{(1-x^2)^2} \\
&= \frac{-1+x^2-2x^2}{(1-x^2)^2} \\
&= \frac{-1-x^2}{(1-x^2)^2} \\
f''(x) &= \frac{\left[(1-x^2)^2(-2x) - (-1-x^2)(2)(1-x^2)(-2x) \right]}{(1-x^2)^4} \\
&= \frac{(1-x^2)[-2x(1-x^2) + 4x(-1-x^2)]}{(1-x^2)^4} \\
&= \frac{-2x(x^2+3)}{(1-x^2)^3} \\
f''(0) &= \frac{-2(0)(0^2+3)}{(1-0^2)^3} = 0 \\
f''(2) &= \frac{-2(2)(4+3)}{(1-2^2)^3} \\
&= \frac{-28}{-27} \\
&= \frac{28}{27}
\end{aligned}$$

$$\begin{aligned}
9. \quad f(x) &= \sqrt{x^2+4} = (x^2+4)^{1/2} \\
f'(x) &= \frac{1}{2}(x^2+4)^{-1/2} \cdot 2x \\
&= \frac{x}{(x^2+4)^{1/2}} \\
f''(x) &= \frac{(x^2+4)^{1/2}(1) - x\left[\frac{1}{2}(x^2+4)^{-1/2}\right]2x}{x^2+4} \\
&= \frac{(x^2+4)^{1/2} - \frac{x^2}{(x^2+4)^{1/2}}}{x^2+4} \\
&= \frac{(x^2+4) - x^2}{(x^2+4)^{3/2}} \\
&= \frac{4}{(x^2+4)^{3/2}} \\
f''(0) &= \frac{4}{(0^2+4)^{3/2}} \\
&= \frac{4}{4^{3/2}} = \frac{4}{8} = \frac{1}{2} \\
f''(2) &= \frac{4}{(2^2+4)^{3/2}} \\
&= \frac{4}{8^{3/2}} = \frac{4}{16\sqrt{2}} = \frac{1}{4\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
10. \quad f(x) &= \sqrt{2x^2+9} \\
&= (2x^2+9)^{1/2} \\
f'(x) &= \frac{1}{2}(2x^2+9)^{-1/2} \cdot 4x \\
&= \frac{2x}{(2x^2+9)^{1/2}} \\
f''(x) &= \frac{(2x^2+9)^{1/2}(2) - 2x\left[\frac{1}{2}(2x^2+9)^{-1/2}\right]4x}{2x^2+9} \\
&= \frac{2(2x^2+9)^{1/2} - \frac{4x^2}{(2x^2+9)^{1/2}}}{2x^2+9} \\
&= \frac{2(2x^2+9) - 4x^2}{(2x^2+9)^{3/2}} \\
&= \frac{18}{(2x^2+9)^{3/2}} \\
f''(0) &= \frac{18}{[2(0)^2+9]^{3/2}}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{18}{9^{3/2}} = \frac{18}{27} = \frac{2}{3} \\
 f''(2) &= \frac{18}{[2(2)^2 + 9]^{3/2}} \\
 &= \frac{18}{17^{3/2}} = \frac{18}{17\sqrt{17}}
 \end{aligned}$$

11. $f(x) = 32x^{3/4}$
 $f'(x) = 24x^{-1/4}$
 $f''(x) = -6x^{-5/4} = -\frac{6}{x^{5/4}}$
 $f''(0)$ does not exist.

$$\begin{aligned}
 f''(2) &= -\frac{6}{2^{5/4}} \\
 &= -\frac{3}{2^{1/4}}
 \end{aligned}$$

12. $f(x) = -6x^{1/3}$
 $f'(x) = -2x^{-2/3}$
 $f''(x) = \frac{4}{3}x^{-5/3} = \frac{4}{3x^{5/3}}$
 $f''(0)$ does not exist.

$$f''(2) = \frac{4}{3(2^{5/3})} = \frac{2^{1/3}}{3}$$

13. $f(x) = 5e^{-x^2}$
 $f'(x) = 5e^{-x^2}(-2x) = -10xe^{-x^2}$
 $f''(x) = -10xe^{-x^2}(-2x) + e^{-x^2}(-10)$
 $= 20x^2e^{-x^2} - 10e^{-x^2}$
 $f''(0) = 20(0^2)e^{-0^2} - 10e$
 $= 0 - 10 = -10$
 $f''(2) = 20(2^2)e^{-(2^2)} - 10e^{-(2^2)}$
 $= 80e^{-4} - 10e^{-4} = 70e^{-4}$
 ≈ 1.282

14. $f(x) = 0.5e^{x^2}$
 $f'(x) = (0.5)(2x)e^{x^2} = xe^{x^2}$
 $f''(x) = xe^{x^2}(2x) + e^{x^2}(1) = 2x^2e^{x^2} + e^{x^2}$
or $e^{x^2}(2x^2 + 1)$

$$f''(0) = e^{0^2}[2(0)^2 + 1] = 1(1) = 1$$

$$\begin{aligned}
 f''(2) &= e^{2^2}[2(2)^2 + 1] = e^4(9) = 9e^4 \\
 &\approx 491.4
 \end{aligned}$$

15. $f(x) = \frac{\ln x}{4x}$
 $f'(x) = \frac{4x\left(\frac{1}{x}\right) - (\ln x)(4)}{(4x)^2}$
 $= \frac{4 - 4 \ln x}{16x^2} = \frac{1 - \ln x}{4x^2}$
 $f''(x) = \frac{4x^2\left(-\frac{1}{x}\right) - (1 - \ln x)8x}{16x^4}$
 $= \frac{-4x - 8x + 8x \ln x}{16x^4}$
 $= \frac{-12x + 8x \ln x}{16x^4}$
 $= \frac{4x(-3 + 2 \ln x)}{16x^4}$
 $= \frac{-3 + 2 \ln x}{4x^3}$

$f''(0)$ does not exist because $\ln 0$ is undefined.

$$f''(2) = \frac{-3 + 2 \ln 2}{4(2)^3} = \frac{-3 + 2 \ln 2}{32} \approx 0.050$$

16. $f(x) = \ln x + \frac{1}{x}$
 $f''(x) = \frac{1}{x} - \frac{1}{x^2}$
 $f''(x) = \frac{-1}{x^2} + \frac{2}{x^3}$
 $= \frac{-x}{x^3} + \frac{2}{x^3} = \frac{2-x}{x^3}$

$f''(0)$ does not exist since the denominator is 0.

$$f''(2) = \frac{2-2}{2^3} = \frac{0}{8} = 0$$

17. $f(x) = 7x^4 + 6x^3 + 5x^2 + 4x + 3$
 $f'(x) = 28x^3 + 18x^2 + 10x + 4$
 $f''(x) = 84x^2 + 36x + 10$
 $f'''(x) = 168x + 36$
 $f^{(4)}(x) = 168$

$$\begin{aligned}
 18. \quad f(x) &= -2x^4 + 7x^3 + 4x^2 + x \\
 f'(x) &= -8x^3 + 21x^2 + 8x + 1 \\
 f''(x) &= -24x^2 + 42x + 8 \\
 f'''(x) &= -48x + 42 \\
 f^{(4)}(x) &= -48
 \end{aligned}$$

$$\begin{aligned}
 19. \quad f(x) &= 5x^5 - 3x^4 + 2x^3 + 7x^2 + 4 \\
 f'(x) &= 25x^4 - 12x^3 + 6x^2 + 14x \\
 f''(x) &= 100x^3 - 36x^2 + 12x + 14 \\
 f'''(x) &= 300x^2 - 72x + 12 \\
 f^{(4)}(x) &= 600x - 72
 \end{aligned}$$

$$\begin{aligned}
 20. \quad f(x) &= 2x^5 + 3x^4 - 5x^3 + 9x - 2 \\
 f'(x) &= 10x^4 + 12x^3 - 15x^2 + 9 \\
 f''(x) &= 40x^3 + 36x^2 - 30x \\
 f'''(x) &= 120x^2 + 72x - 30 \\
 f^{(4)}(x) &= 240x + 72
 \end{aligned}$$

$$\begin{aligned}
 21. \quad f(x) &= \frac{x-1}{x+2} \\
 f'(x) &= \frac{(x+2) - (x-1)}{(x+2)^2} = \frac{3}{(x+2)^2} \\
 f''(x) &= \frac{-3(2)(x+2)}{(x+2)^4} = \frac{-6}{(x+2)^3} \\
 f'''(x) &= \frac{(-6)(-3)(x+2)^2}{(x+2)^6} \\
 &= 18(x+2)^{-4} \quad \text{or} \quad \frac{18}{(x+2)^4} \\
 f^{(4)}(x) &= \frac{-18(4)(x+2)^3}{(x+2)^8} \\
 &= -72(x+2)^{-5} \quad \text{or} \quad \frac{-72}{(x+2)^5}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad f(x) &= \frac{x+1}{x} \\
 f'(x) &= \frac{(1)(x) - 1(x+1)}{x^2} \\
 &= -\frac{1}{x^2} = -x^{-2} \\
 f''(x) &= 2x^{-3} \quad \text{or} \quad \frac{2}{x^3} \\
 f'''(x) &= -6x^{-4} \quad \text{or} \quad -\frac{6}{x^4} \\
 f^{(4)}(x) &= 24x^{-5} \quad \text{or} \quad \frac{24}{x^5}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad f(x) &= \frac{3x}{x-2} \\
 f'(x) &= \frac{(x-2)(3) - 3x(1)}{(x-2)^2} = \frac{-6}{(x-2)^2} \\
 f''(x) &= \frac{-6(-2)(x-2)}{(x-2)^4} = \frac{12}{(x-2)^3} \\
 f'''(x) &= \frac{-12(3)(x-2)^2}{(x-2)^6} = -36(x-2)^{-4} \\
 &\quad \text{or} \quad \frac{-36}{(x-2)^4} \\
 f^{(4)}(x) &= \frac{-36(-4)(x-2)^3}{(x-2)^8} = 144(x-2)^{-5} \\
 &\quad \text{or} \quad \frac{144}{(x-2)^5}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad f(x) &= \frac{x}{2x+1} \\
 f'(x) &= \frac{(1)(2x+1) - (2)(x)}{(2x+1)^2} \\
 &= \frac{1}{(2x+1)^2} = (2x+1)^{-2} \\
 f''(x) &= -2(2x+1)^{-3}(2) \\
 &= -4(2x+1)^{-3} \\
 f'''(x) &= 12(2x+1)^{-4}(2) \\
 &= 24(2x+1)^{-4} \quad \text{or} \quad \frac{24}{(2x+1)^4} \\
 f^{(4)}(x) &= -96(2x+1)^{-5}(2) \\
 &= -192(2x+1)^{-5} \\
 &\quad \text{or} \quad \frac{-192}{(2x+1)^5}
 \end{aligned}$$

25. $f(x) = \ln x$

(a) $f'(x) = \frac{1}{x} = x^{-1}$

$$f''(x) = -x^{-2} = \frac{-1}{x^2}$$

$$f'''(x) = 2x^{-3} = \frac{2}{x^3}$$

$$f^{(4)}(x) = -6x^{-4} = \frac{-6}{x^4}$$

$$f^{(5)}(x) = 24x^{-5} = \frac{24}{x^5}$$

(b) $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$

26. $f(x) = e^x$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f'''(x) = e^x$$

$$f^{(n)}(x) = e^x$$

27. Concave upward on $(2, \infty)$

Concave downward on $(-\infty, 2)$

Inflection point at $(2, 3)$

28. Concave upward on $(-\infty, 3)$

Concave downward on $(3, \infty)$

Inflection point at $(3, 7)$

29. Concave upward on $(-\infty, -1)$ and $(8, \infty)$

Concave downward on $(-1, 8)$

Inflection points at $(-1, 7)$ and $(8, 6)$

30. Concave upward on $(-2, 6)$

Concave downward on $(-\infty, -2)$ and $(6, \infty)$

Inflection point at $(-2, -4)$ and $(6, -1)$

31. Concave upward on $(2, \infty)$

Concave downward on $(-\infty, 2)$

No points inflection

32. Concave upward on $(-\infty, 0)$

Concave downward on $(0, \infty)$

No inflection points

33. $f(x) = x^2 + 10x - 9$

$$f'(x) = 2x + 10$$

$$f''(x) = 2 > 0 \text{ for all } x.$$

Always concave upward

No inflection points

34. $f(x) = 8 - 6x - x^2$

$$f'(x) = -6 - 2x$$

$$f''(x) = -2 < 0 \text{ for all } x.$$

Always concave downward

No inflection points

35. $f(x) = -2x^3 + 9x^2 + 168x - 3$

$$f'(x) = -6x^2 + 18x + 168$$

$$f''(x) = -12x + 18$$

$$f''(x) = -12x + 18 > 0 \text{ when}$$

$$-6(2x - 3) > 0$$

$$2x - 3 < 0$$

$$x < \frac{3}{2}.$$

Concave upward on $(-\infty, \frac{3}{2})$

$$f''(x) = -12x + 18 < 0 \text{ when}$$

$$-6(2x - 3) < 0$$

$$2x - 3 > 0$$

$$x > \frac{3}{2}.$$

Concave downward on $(\frac{3}{2}, \infty)$

$$f''(x) = -12x + 18 = 0 \text{ when}$$

$$-6(2x + 3) = 0$$

$$2x + 3 = 0$$

$$x = \frac{3}{2}.$$

$$f\left(\frac{3}{2}\right) = \frac{525}{2}$$

Inflection point at $(\frac{3}{2}, \frac{525}{2})$

36. $f(x) = -x^3 - 12x^2 - 45x + 2$

$$f'(x) = -3x^2 - 24x - 45$$

$$f''(x) = -6x - 24$$

$$\begin{aligned}
 f''(x) &= -6x - 24 > 0 \text{ when} \\
 -6(x + 4) &> 0 \\
 x + 4 &< 0 \\
 x &< -4.
 \end{aligned}$$

Concave upward on $(-\infty, -4)$

$$\begin{aligned}
 f''(x) &= -6x - 24 < 0 \text{ when} \\
 -6(x + 4) &< 0 \\
 x + 4 &> 0 \\
 x &> -4.
 \end{aligned}$$

Concave downward on $(-4, \infty)$

$$\begin{aligned}
 f''(x) &= -6x - 24 = 0 \text{ when} \\
 -6(x + 4) &= 0 \\
 x &= -4. \\
 f(-4) &= 54
 \end{aligned}$$

Inflection point at $(-4, 54)$

$$\begin{aligned}
 37. \quad f(x) &= \frac{3}{x-5} \\
 f'(x) &= \frac{-3}{(x-5)^2} \\
 f''(x) &= \frac{-3(-2)(x-5)}{(x-5)^4} = \frac{6}{(x-5)^3} \\
 f''(x) &= \frac{6}{(x-5)^3} > 0 \text{ when} \\
 (x-5)^3 &> 0 \\
 x-5 &> 0 \\
 x &> 5.
 \end{aligned}$$

Concave upward on $(5, \infty)$

$$\begin{aligned}
 f''(x) &= \frac{6}{(x-5)^3} < 0 \text{ when} \\
 (x-5)^3 &< 0 \\
 x-5 &< 0 \\
 x &< 5.
 \end{aligned}$$

Concave downward on $(-\infty, 5)$

$f''(x) \neq 0$ for any value for x ; it does not exist when $x = 5$. There is a change of concavity there, but no inflection point since $f(5)$ does not exist.

$$\begin{aligned}
 38. \quad f(x) &= \frac{-2}{x+1} \\
 &= -2(x+1)^{-1} \\
 f'(x) &= 2(x+1)^{-2} \\
 f''(x) &= -4(x+1)^{-3} = \frac{-4}{(x+1)^3} \\
 f''(x) &= \frac{-4}{(x+1)^3} > 0 \text{ when} \\
 x+1 &< 0 \\
 x &< -1.
 \end{aligned}$$

Concave upward on $(-\infty, -1)$

$$\begin{aligned}
 f''(x) &= \frac{-4}{(x+1)^3} < 0 \text{ when} \\
 x+1 &> 0 \\
 x &> -1.
 \end{aligned}$$

Concave downward on $(-1, \infty)$

$f''(x) \neq 0$ for any value for x ; it does not exist when $x = -1$. There is a change of concavity there, but no inflection point since $f(-1)$ does not exist.

$$\begin{aligned}
 39. \quad f(x) &= x(x+5)^2 \\
 f'(x) &= x(2)(x+5) + (x+5)^2 \\
 &= (x+5)(2x+x+5) \\
 &= (x+5)(3x+5) \\
 f''(x) &= (x+5)(3) + (3x+5) \\
 &= 3x+15+3x+5 = 6x+20 \\
 f''(x) &= 6x+20 > 0 \text{ when} \\
 2(3x+10) &> 0 \\
 3x &> -10 \\
 x &> -\frac{10}{3}.
 \end{aligned}$$

Concave upward on $\left(-\frac{10}{3}, \infty\right)$

$$\begin{aligned}
 f''(x) &= 6x+20 < 0 \text{ when} \\
 2(3x+10) &< 0 \\
 3x &< -10 \\
 x &< -\frac{10}{3}.
 \end{aligned}$$

Concave downward on $\left(-\infty, -\frac{10}{3}\right)$

$$\begin{aligned}
 f\left(-\frac{10}{3}\right) &= -\frac{10}{3}\left(-\frac{10}{3} + 5\right)^2 \\
 &= \frac{-10}{3}\left(\frac{-10 + 15}{3}\right)^2 \\
 &= -\frac{10}{3} \cdot \frac{25}{9} = -\frac{250}{27}
 \end{aligned}$$

Inflection point at $\left(-\frac{10}{3}, -\frac{250}{27}\right)$

40. $f(x) = -x(x-3)^2$

$$\begin{aligned}
 f'(x) &= -1(x-3)^2 + 2(x-3)(-x) \\
 &= -(x-3)^2 - 2x^2 + 6x \\
 f''(x) &= -2(x-3) - 4x + 6 \\
 &= -2x + 6 - 4x + 6 \\
 &= -6x + 12 \\
 f''(x) &= -6x + 12 > 0 \text{ when} \\
 &\quad -6(x-2) > 0 \\
 &\quad x - 2 > 0 \\
 &\quad x < 2.
 \end{aligned}$$

Concave upward on $(-\infty, 2)$

$$\begin{aligned}
 f''(x) &= -6x + 12 < 0 \text{ when} \\
 &\quad -6(x-2) < 0 \\
 &\quad x - 2 < 0 \\
 &\quad x > 2.
 \end{aligned}$$

Concave downward on $(2, \infty)$

$$\begin{aligned}
 f''(x) &= -6x + 12 = 0 \text{ when } x = 2. \\
 f(2) &= -2
 \end{aligned}$$

Inflection point at $(2, -2)$

41. $f(x) = 18x - 18e^{-x}$

$$\begin{aligned}
 f'(x) &= 18 - 18e^{-x}(-1) = 18 + 18e^{-x} \\
 f''(x) &= 18e^{-x}(-1) = -18e^{-x} \\
 f''(x) &= -18e^{-x} < 0 \text{ for all } x \\
 f(x) &\text{ is never concave upward and always} \\
 &\text{concave downward. There are no points of} \\
 &\text{inflection since } -18e^{-x} \text{ is never equal to } 0.
 \end{aligned}$$

42. $f(x) = 2e^{-x^2}$

$$\begin{aligned}
 f'(x) &= 2e^{-x^2}(-2x) = -4xe^{-x^2} \\
 f''(x) &= -4xe^{-x^2}(-2x) + e^{-x^2}(-4) \\
 &= -4e^{-x^2}(-2x^2 + 1)
 \end{aligned}$$

$$\begin{aligned}
 f''(x) &= 0 \text{ when } -2x^2 + 1 = 0 \\
 1 &= 2x^2 \\
 \frac{1}{2} &= x^2 \\
 \pm \frac{\sqrt{2}}{2} &= x
 \end{aligned}$$

Check the sign of $f''(x)$ in each of the intervals determined by $x = \frac{\sqrt{2}}{2}$ and $x = -\frac{\sqrt{2}}{2}$ using test points.

$$\begin{aligned}
 f''(-1) &= -4e^{-(-1)^2}[-2(-1)^2 + 1] \\
 &= \frac{-4}{e}(-1) = \frac{4}{e} > 0 \\
 f''(0) &= -4e^{-0^2}[-2(0)^2 + 1] \\
 &= -4(1) = -4 < 0 \\
 f''(1) &= -4e^{-1^2}[-2(1)^2 + 1] \\
 &= \frac{-4}{e}(-1) = \frac{4}{e} > 0
 \end{aligned}$$

Concave upward on $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, \infty\right)$;

concave downward on $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$$f\left(\pm\frac{\sqrt{2}}{2}\right) = 2e^{-1/2} = \frac{2}{\sqrt{e}}$$

Inflection points at $\left(-\frac{\sqrt{2}}{2}, \frac{2}{\sqrt{e}}\right)$ and $\left(\frac{\sqrt{2}}{2}, \frac{2}{\sqrt{e}}\right)$

43. $f(x) = x^{8/3} - 4x^{5/3}$

$$\begin{aligned}
 f'(x) &= \frac{8}{3}x^{5/3} - \frac{20}{3}x^{2/3} \\
 f''(x) &= \frac{40}{9}x^{2/3} - \frac{40}{9}x^{-1/3} = \frac{40(x-1)}{9x^{1/3}} \\
 f''(x) &= 0 \text{ when } x = 1 \\
 f''(x) &\text{ fails to exist when } x = 0
 \end{aligned}$$

Note that both $f(x)$ and $f'(x)$ exist at $x = 0$.

Check the sign of $f''(x)$ in the three intervals determined by $x = 0$ and $x = 1$ using test points.

$$\begin{aligned}
 f''(-1) &= \frac{40(-2)}{9(-1)} = \frac{80}{9} > 0 \\
 f''\left(\frac{1}{8}\right) &= \frac{40\left(-\frac{7}{8}\right)}{9\left(\frac{1}{2}\right)} = -\frac{70}{9} < 0 \\
 f''(8) &= \frac{40(7)}{9(2)} = \frac{140}{9} > 0
 \end{aligned}$$

Concave upward on $(-\infty, 0)$ and $(1, \infty)$;
concave downward on $(0, 1)$

$$f(0) = (0)^{8/3} - 4(0)^{5/3} = 0$$

$$f(1) = (1)^{8/3} - 4(1)^{5/3} = -3$$

Inflection points at $(0, 0)$ and $(1, -3)$

44. $f(x) = x^{7/3} + 56x^{4/3}$

$$f'(x) = \frac{7}{3}x^{4/3} + \frac{224}{3}x^{1/3}$$

$$f''(x) = \frac{28}{9}x^{1/3} + \frac{224}{9}x^{-2/3}$$

$$= \frac{28(x+8)}{9x^{2/3}}$$

$$f''(x) = 0 \text{ when } x = -8$$

$$f''(x) \text{ fails to exist when } x = 0$$

Note that both $f(x)$ and $f'(x)$ exist at $x = 0$.

Check the sign of $f''(x)$ in three intervals

determined by $x = -8$ and $x = 0$ using test points.

$$f''(-27) = \frac{28(-19)}{9(9)} = -\frac{532}{81} < 0$$

$$f''(-1) = \frac{28(7)}{9(1)} = \frac{196}{9} > 0$$

$$f''(1) = \frac{28(9)}{9(1)} = 28 > 0$$

Concave upward on $(-8, \infty)$; concave
downward on $(-\infty, -8)$

$$f(-8) = (-8)^{7/3} + 56(-8)^{4/3} = -128 + 896$$

$$= 768$$

Inflection point at $(-8, 768)$

45. $f(x) = \ln(x^2 + 1)$

$$f'(x) = \frac{2x}{x^2 + 1}$$

$$f''(x) = \frac{(x^2 + 1)(2) - (2x)(2x)}{(x^2 + 1)^2}$$

$$= \frac{-2x^2 + 2}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2x^2 + 2}{(x^2 + 1)^2} > 0 \text{ when}$$

$$-2x^2 + 2 > 0$$

$$-2x^2 > -2$$

$$x^2 < 1$$

$$-1 < x < 1$$

Concave upward on $(-1, 1)$

$$f''(x) = \frac{-2x^2 + 2}{(x^2 + 1)^2} < 0 \text{ when}$$

$$-2x^2 + 2 < 0$$

$$-2x^2 < -2$$

$$x^2 > 1$$

$$x > 1 \text{ or } x < -1$$

Concave downward on $(-\infty, -1)$ and $(1, \infty)$

$$f(1) = \ln[(1)^2 + 1] = \ln 2$$

$$f(-1) = \ln[(-1)^2 + 1] = \ln 2$$

Inflection points at $(-1, \ln 2)$ and $(1, \ln 2)$

46. $f(x) = x^2 + 8 \ln|x + 1|$

$$f'(x) = 2x + \frac{8}{x + 1}$$

$$f''(x) = 2 - \frac{8}{(x + 1)^2}$$

$$f''(x) = 2 - \frac{8}{(x + 1)^2} > 0 \text{ when}$$

$$2 - \frac{8}{(x + 1)^2} > 0$$

$$2 > \frac{8}{(x + 1)^2}$$

$$(x + 1)^2 > 4$$

$$x + 1 > 2 \text{ or } x + 1 < -2$$

$$x > 1 \text{ or } x < -3$$

Concave upward on $(-\infty, -3)$ and $(1, \infty)$

$$f''(x) = 2 - \frac{8}{(x + 1)^2} < 0 \text{ when}$$

$$2 - \frac{8}{(x+1)^2} < 0$$

$$2 < \frac{8}{(x+1)^2}$$

$$(x+1)^2 < 4$$

$$-2 < x+1 < 2$$

$$-3 < x < 1$$

Note that $f(x)$ is not defined at $x = -1$.

Concave downward on $(-3, -1)$ and $(-1, 1)$

$$f(-3) = (-3)^2 + 8 \ln |-3+1| = 9 + 8 \ln 2$$

$$f(1) = (1)^2 + 8 \ln |1+1| = 1 + 8 \ln 2$$

Inflection points at $(-3, 9 + 8 \ln 2)$ and

$(1, 1 + 8 \ln 2)$

47. $f(x) = x^2 \log |x|$

$$f'(x) = 2x \log |x| + x^2 \left(\frac{1}{x \ln 10} \right)$$

$$= 2x \log |x| + \frac{x}{\ln 10}$$

$$f''(x) = 2 \log |x| + 2x \left(\frac{1}{x \ln 10} \right) + \frac{1}{\ln 10}$$

$$= 2 \log |x| + \frac{3}{\ln 10}$$

$$f''(x) > 0 \text{ when}$$

$$2 \log |x| + \frac{3}{\ln 10} > 0$$

$$2 \log |x| > -\frac{3}{\ln 10}$$

$$\log |x| > -\frac{3}{2 \ln 10}$$

$$\frac{\ln |x|}{\ln 10} > -\frac{3}{2 \ln 10}$$

$$\ln |x| > -\frac{3}{2}$$

$$|x| > e^{-3/2}$$

$$x > e^{-3/2} \text{ or } x < -e^{-3/2}$$

Concave upward on $(-\infty, -e^{-3/2})$

and $(e^{-3/2}, \infty)$

$$f''(x) < 0 \text{ when}$$

$$2 \log |x| + \frac{3}{\ln 10} < 0$$

$$2 \log |x| < -\frac{3}{\ln 10}$$

$$\log |x| < -\frac{3}{2 \ln 10}$$

$$\frac{\ln |x|}{\ln 10} < -\frac{3}{2 \ln 10}$$

$$\ln |x| < -\frac{3}{2}$$

$$|x| < e^{-3/2}$$

$$-e^{-3/2} < x < e^{-3/2}$$

Note that $f(x)$ is not defined at $x = 0$.

Concave downward on $(-e^{-3/2}, 0)$

and $(0, e^{-3/2})$.

$$\begin{aligned} f(-e^{-3/2}) &= (-e^{-3/2})^2 \log |-e^{-3/2}| \\ &= e^{-3} \log e^{-3/2} = -\frac{3e^{-3}}{2 \ln 10} \end{aligned}$$

$$\begin{aligned} f(e^{-3/2}) &= (e^{-3/2})^2 \log |e^{-3/2}| \\ &= e^{-3} \log e^{-3/2} = -\frac{3e^{-3}}{2 \ln 10} \end{aligned}$$

Inflection points at $(-e^{-3/2}, -\frac{3e^{-3}}{2 \ln 10})$

and $(e^{-3/2}, -\frac{3e^{-3}}{2 \ln 10})$

48. $f(x) = 5^{-x^2}$

$$f'(x) = (\ln 5) 5^{-x^2} \cdot (-2x)$$

$$= -(2x \ln 5) 5^{-x^2}$$

$$\begin{aligned} f''(x) &= -2 \ln 5 \cdot 5^{-x^2} + (-2 \ln 5)x \cdot (\ln 5)(5^{-x^2})(-2x) \\ &= (2 \ln 5) 5^{-x^2} [2 \ln 5(x^2) - 1] \end{aligned}$$

$$f''(x) = (2 \ln 5) 5^{-x^2} [2 \ln 5(x^2) - 1] > 0 \text{ when}$$

$$2 \ln 5(x^2) - 1 > 0$$

$$2 \ln 5(x^2) > 1$$

$$x^2 > \frac{1}{2 \ln 5}$$

$$x > \frac{1}{\sqrt{2 \ln 5}} \text{ or } x < -\frac{1}{\sqrt{2 \ln 5}}$$

Concave upward on $\left(-\infty, -\frac{1}{\sqrt{2\ln 5}}\right)$ and $\left(\frac{1}{\sqrt{2\ln 5}}, \infty\right)$

$$f''(x) = (2\ln 5)5^{-x^2} [2\ln 5(x^2) - 1] < 0 \text{ when}$$

$$2\ln 5(x^2) - 1 < 0$$

$$2\ln 5(x^2) < 1$$

$$x^2 < \frac{1}{2\ln 5}$$

$$-\frac{1}{\sqrt{2\ln 5}} < x < \frac{1}{\sqrt{2\ln 5}}$$

Concave downward on $\left(-\frac{1}{\sqrt{2\ln 5}}, \frac{1}{\sqrt{2\ln 5}}\right)$

$$f\left(-\frac{1}{\sqrt{2\ln 5}}\right) = 5^{-(1/\sqrt{2\ln 5})^2} = 5^{-1/(2\ln 5)} = e^{-1/2}$$

$$f\left(\frac{1}{\sqrt{2\ln 5}}\right) = 5^{-(1/\sqrt{2\ln 5})^2} = 5^{-1/(2\ln 5)} = e^{-1/2}$$

Inflection points at $\left(-\frac{1}{\sqrt{2\ln 5}}, e^{-1/2}\right)$ and

$$\left(\frac{1}{\sqrt{2\ln 5}}, e^{-1/2}\right)$$

49. Since the graph of $f'(x)$ is increasing on $(-\infty, 0)$ and $(4, \infty)$, the function is concave upward on $(-\infty, 0)$ and $(4, \infty)$. Since the graph of $f'(x)$ is decreasing on $(0, 4)$, the function is concave downward on $(0, 4)$. The inflection points are at 0 and 4.

50. Since the graph of $f'(x)$ is increasing on $(0, 5)$, the function is concave upward on $(0, 5)$. Since the graph of $f'(x)$ is decreasing on $(-\infty, 0)$ and $(5, \infty)$, the function is concave downward on $(-\infty, 0)$ and $(5, \infty)$. The inflection points are at 0 and 5.

51. Since the graph of $f'(x)$ is increasing on $(-7, 3)$ and $(12, \infty)$, the function is concave upward on $(-7, 3)$ and $(12, \infty)$. Since the graph of $f'(x)$ is decreasing on $(-\infty, -7)$ and $(3, 12)$, the function is concave downward on $(-\infty, -7)$ and $(3, 12)$. The inflection points are at $-7, 3$, and 12 .

52. Since the graph of $f'(x)$ is increasing on $(-\infty, -5)$ and $(1, 9)$, the function is concave upward on $(-\infty, -5)$ and $(1, 9)$. Since the graph of $f'(x)$ is decreasing on $(-5, 1)$ and $(9, \infty)$, the function is concave downward on $(-5, 1)$ and $(9, \infty)$. The inflection points are at $-5, 1$, and 9 .

53. Choose $f(x) = x^k$, where $1 < k < 2$.

If $k = \frac{4}{3}$, then

$$f'(x) = \frac{4}{3}x^{1/3} \quad f''(x) = \frac{4}{9}x^{-2/3} = \frac{4}{9x^{2/3}}$$

Critical number: 0

Since $f'(x)$ is negative when $x < 0$ and positive when $x > 0$, $f(x) = x^{4/3}$ has a relative minimum at $x = 0$.

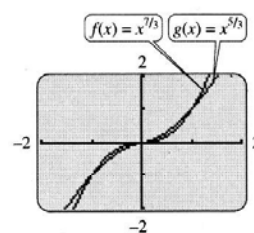
If $k = \frac{5}{3}$, then

$$f(x) = \frac{5}{3}x^{2/3} \quad f''(x) = \frac{10}{9}x^{-1/3} = \frac{10}{9x^{1/3}}$$

$f''(x)$ is never 0, and does not exist when $x = 0$; so, the only candidate for an inflection point is at $x = 0$.

Since $f''(x)$ is negative when $x < 0$ and positive when $x > 0$, $f(x) = x^{5/3}$ has an inflection point at $x = 0$.

54. (a)



- (b) Both $f(x)$ and $g(x)$ are concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Thus, they both have a point of inflection at $(0, 0)$.

- (c) $f(x) = x^{7/3}$ $g(x) = x^{5/3}$
 $f'(x) = \frac{7}{3}x^{4/3}$ $g'(x) = \frac{5}{3}x^{2/3}$
 $f''(x) = \frac{28}{9}x^{1/3}$ $g''(x) = \frac{10}{9}x^{-1/3}$
 $f''(0) = 0$ $g''(0)$ is undefined

- (d) No.

55. (a) The slope of the tangent line to $f(x) = e^x$ as $x \rightarrow -\infty$ is close to 0 since the tangent line is almost horizontal, and a horizontal line has a slope of 0.

- (b) The slope of the tangent line to $f(x) = e^x$ as $x \rightarrow 0$ is close to 1 since the first derivative represents the slope of the tangent line, $f'(x) = e^x$, and $e^0 = 1$.

56. The slope of the tangent line to $f(x) = \ln x$ as $x \rightarrow \infty$ approaches 0. The slope of the tangent line to the graph of $f(x) = \ln x$ as $x \rightarrow 0$ approaches ∞ . For example, the slope of the tangent line at $x = 0.00001$ is 100,000.

57. $f(x) = -x^2 - 10x - 25$
 $f'(x) = -2x - 10$
 $= -2(x + 5) = 0$

Critical number: -5

$$f''(x) = -2 < 0 \text{ for all } x.$$

The curve is concave downward, which means a relative maximum occurs at $x = -5$.

58. $f(x) = x^2 - 12x + 36$
 $f'(x) = 2x - 12$
 $f'(x) = 0$ when

$$\begin{aligned} 2x - 12 &= 0 \\ x &= 6. \end{aligned}$$

Critical number: 6

$$f''(x) = 2 > 0 \text{ for all } x.$$

The curve is concave upward, which means a relative minimum occurs at $x = 6$.

59. $f(x) = 3x^3 - 3x^2 + 1$
 $f'(x) = 9x^2 - 6x$
 $= 3x(3x - 2) = 0$

Critical numbers: 0 and $\frac{2}{3}$

$$f''(x) = 18x - 6$$

$f''(0) = -6 < 0$, which means that a relative maximum occurs at $x = 0$.

$f''\left(\frac{2}{3}\right) = 6 > 0$, which means that a relative minimum occurs at $x = \frac{2}{3}$.

60. $f(x) = 2x^3 - 4x^2 + 2$
 $f'(x) = 6x^2 - 8x$
 $f'(x) = 0$ when

$$\begin{aligned} 6x^2 - 8x &= 0 \\ 2x(3x - 4) &= 0 \\ x = 0 \quad \text{or} \quad x &= \frac{4}{3}. \end{aligned}$$

Critical numbers: 0, $\frac{4}{3}$

$$f''(x) = 12x - 8$$

$f''(0) = -8 < 0$, which means that a relative maximum occurs at $x = 0$.

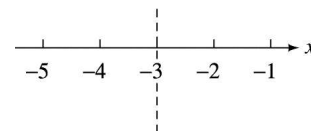
$f''\left(\frac{4}{3}\right) = 8 > 0$, which means that a relative minimum occurs at $x = \frac{4}{3}$.

61. $f(x) = (x + 3)^4$
 $f'(x) = 4(x + 3)^3 = 0$
Critical number: $x = -3$

$$\begin{aligned} f''(x) &= 12(x + 3)^2 \\ f''(-3) &= 12(-3 + 3)^2 = 0 \end{aligned}$$

The second derivative test fails.

Use the first derivative test.



$$\begin{aligned} f'(-4) &= 4(-4 + 3)^2 \\ &= 4(-1)^3 = -4 < 0 \end{aligned}$$

This indicates that f is decreasing on $(-\infty, -3)$.

$$\begin{aligned} f'(0) &= 4(0 + 3)^3 \\ &= 4(3)^3 = 108 > 0 \end{aligned}$$

This indicates that f is increasing on $(-3, \infty)$. A relative minimum occurs at -3 .

62. $f(x) = x^3$
 $f'(x) = 3x^2$
 $f'(x) = 0$ when

$$3x^2 = 0$$

$$x = 0.$$

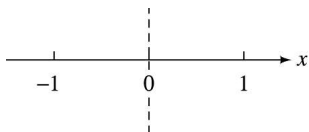
Critical number: 0

$$f''(x) = 6x$$

$$f''(0) = 0$$

The second derivative test fails.

Use the first derivative test.



$$f'(-1) = 3(-1)^2 = 3 > 0$$

This indicates that f is increasing on $(-\infty, 0)$.

$$f'(1) = 3(1)^2 = 3 > 0$$

This indicates that f is increasing on $(0, \infty)$.

Neither a relative maximum nor relative minimum occurs at $x = 0$.

63. $f(x) = x^{7/3} + x^{4/3}$
 $f'(x) = \frac{7}{3}x^{4/3} + \frac{4}{3}x^{1/3}$
 $f'(x) = 0$ when

$$\frac{7}{3}x^{4/3} + \frac{4}{3}x^{1/3} = 0$$

$$\frac{x^{1/3}}{3}(7x + 4) = 0$$

$$x = 0 \text{ or } x = -\frac{4}{7}.$$

Critical numbers: $-\frac{4}{7}, 0$

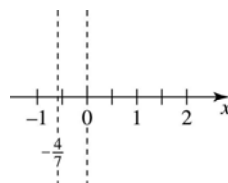
$$f''(x) = \frac{28}{9}x^{1/3} + \frac{4}{9}x^{-2/3}$$

$$f''\left(-\frac{4}{7}\right) = \frac{28}{9}\left(-\frac{4}{7}\right)^{-1/3} + \frac{4}{9}\left(-\frac{4}{7}\right)^{-2/3} \approx -1.9363$$

Relative maximum occurs at $-\frac{4}{7}$.

$f''(0)$ does not exist, so the second derivative test fails.

Use the first derivative test.



$$f'\left(-\frac{1}{2}\right) = \frac{7}{3}\left(-\frac{1}{2}\right)^{4/3} + \frac{4}{3}\left(-\frac{1}{2}\right)^{1/3} \approx -0.1323$$

This indicates that f is decreasing on $\left(-\frac{4}{7}, 0\right)$.

$$f'(1) = \frac{7}{3}(1)^{4/3} + \frac{4}{3}(1)^{1/3} = \frac{11}{3}$$

This indicates that f is increasing on $(0, \infty)$.

Relative minimum occurs at 0.

64. $f(x) = x^{8/3} + x^{5/3}$
 $f'(x) = \frac{8}{3}x^{5/3} + \frac{5}{3}x^{2/3}$
 $f'(x) = 0$ when

$$\frac{8}{3}x^{5/3} + \frac{5}{3}x^{2/3} = 0$$

$$\frac{x^{2/3}}{3}(8x + 5) = 0$$

$$x = 0 \text{ or } x = -\frac{5}{8}.$$

Critical numbers: $-\frac{5}{8}, 0$

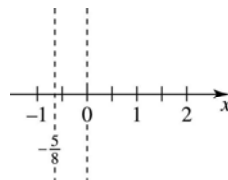
$$f''(x) = \frac{40}{9}x^{2/3} + \frac{10}{9}x^{-1/3}$$

$$f''\left(-\frac{5}{8}\right) = \frac{40}{9}\left(-\frac{5}{8}\right)^{2/3} + \frac{10}{9}\left(-\frac{5}{8}\right)^{-1/3} \approx 1.9493$$

Relative minimum occurs at $-\frac{5}{8}$.

$f''(0)$ does not exist, so the second derivative test fails.

Use the first derivative test.



$$f'\left(-\frac{1}{2}\right) = \frac{8}{3}\left(-\frac{1}{2}\right)^{5/3} + \frac{5}{3}\left(-\frac{1}{2}\right)^{2/3} \approx 0.2100$$

This indicates that f is increasing on $\left(-\frac{5}{8}, 0\right)$.

$$f'(1) = \frac{8}{3}(1)^{5/3} + \frac{5}{3}(1)^{2/3} = \frac{13}{3}$$

This indicates that f is increasing on $(0, \infty)$.

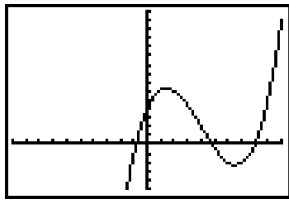
Neither a relative minimum or maximum occurs at 0.

65. $f'(x) = x^3 - 6x^2 + 7x + 4$

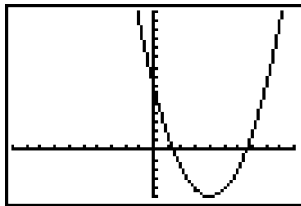
$$f''(x) = 3x^2 - 12x + 7$$

Graph f' and f'' in the window $[-5, 5]$ by $[-5, 15]$, Xscl = 0.5.

Graph of f' :



Graph of f'' :



- (a) f has relative extrema where $f'(x) = 0$.

Use the graph to approximate the x -intercepts of the graph of f' . These numbers are the solutions of the equation $f'(x) = 0$. We find that the critical numbers of f are about -0.4 , 2.4 , and 4.0 .

By either looking at the graph of f' and applying the first derivative test or by looking at the graph of f'' and applying the second derivative test, we see that f has relative minima at about -0.4 and 4.0 and a relative maximum at about 2.4 .

- (b) Examine the graph of f' to determine the intervals where the graph lies above and below the x -axis. We see that $f'(x) > 0$ on about $(-0.4, 2.4)$ and $(4.0, \infty)$, indicating that f is increasing on the same intervals. We also see that $f'(x) < 0$ on about $(-\infty, -0.4)$ and $(2.4, 4.0)$, indicating that f is decreasing on the same intervals.

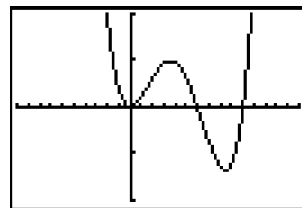
- (c) Examine the graph of f'' . We see that this graph has two x -intercepts, so there are two x -values where $f''(x) = 0$. These x -values are about 0.7 and 3.3 . Because the sign of f'' changes at these two values, we see that the x -values of the inflection points of the graph of f are about 0.7 and 3.3 .

- (d) We observe from the graph of f'' that $f''(x) > 0$ on about $(-\infty, 0.7)$ and $(3.3, \infty)$, so f is concave upward on the same intervals.

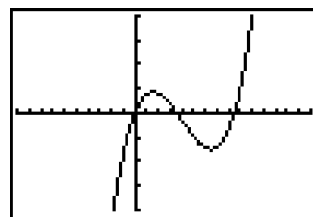
Likewise, we observe that $f''(x) < 0$ on about $(0.7, 3.3)$, so f is concave downward on the same interval.

66.
$$\begin{aligned} f'(x) &= 10x^2(x-1)(5x-3) \\ &= 10x^2(5x^2 - 8x + 3) \\ &= 50x^4 - 80x^3 + 30x^2 \\ f''(x) &= 200x^3 - 240x^2 + 60x \\ &= 20x(10x^2 - 12x + 3) \end{aligned}$$

Graph f' in window $[-1, 1.5]$ by $[-2, 2]$, Xscl = 0.1.



This window does not give a good view of the graph of f'' , so we graph f'' in the window $[-1, 1.5]$ by $[-20, 20]$, Xscl = 0.1. Yscl = 5.



- (a) The critical numbers of f are the x -intercepts of the graph of f' . (Note that there are no values where $f'(x)$ does not exist.) From the graph or by examining the factored expression for f' , we see that the critical numbers of f are 0 , 0.6 , and 1 .

By either looking at the graph of f' and applying the first derivative test or by

looking at the graph of f'' and applying the second derivative test, we see that f has a relative minimum at 1 and a relative maximum at 0.6.

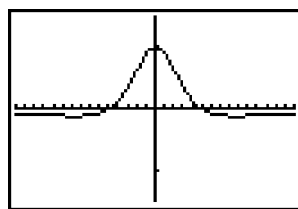
(At $x = 0$, the second derivative test fails since $f''(0) = 0$, and the first derivative does not change sign, so there is no relative extremum at 0.)

- (b) Examine the graph of f' to determine the intervals where the graph lies above and below the x -axis. We see that $f'(x) \geq 0$ on $(-\infty, 0.6)$, $f'(x) < 0$ on $(0.6, 1)$, and $f'(x) > 0$ on $(1, \infty)$. Therefore, f is increasing on $(-\infty, 0.6)$ and $(1, \infty)$ and decreasing on $(0.6, 1)$.
- (c) Examine the graph of f'' . We see that this graph has three x -intercepts, so there are three values where $f''(x) = 0$. These x -values are 0, about 0.36, and about 0.85. Because the sign of f'' and thus the concavity of f changes at these three values, we see that the x -values of the inflection points of the graph of f are 0, about 0.36, and about 0.85.
- (d) We observe from the graph of f'' that $f''(x) > 0$ on $(0, 0.36)$ and $(0.85, \infty)$, so f is concave upward on the same intervals. Likewise, $f''(x) < 0$ on $(-\infty, 0)$ and $(0.36, 0.85)$, so f is concave downward on the same intervals.

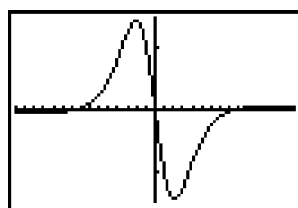
$$\begin{aligned}
 67. \quad f'(x) &= \frac{1 - x^2}{(x^2 + 1)^2} \\
 f''(x) &= \frac{(x^2 + 1)^2(-2x) - (1 - x^2)(2)(x^2 + 1)2x}{(x^2 + 1)^4} \\
 &= \frac{-2x(x^2 + 1)^2[(x^2 + 1) + 2(1 - x^2)]}{(x^2 + 1)^4} \\
 &= \frac{-2x(3 - x^2)}{(x^2 + 1)^3}
 \end{aligned}$$

Graph f' and f'' in the window $[-3, 3]$ by $[-1.5, 1, 5]$, Xscl = 0.2.

Graph of f' :



Graph of f'' :



- (a) The critical numbers of f are the x -intercepts of the graph of f' . (Note that there are no values where f' does not exist.) We see from the graph that these x -values are -1 and 1 .

By either looking at the graph of f' and applying the first derivative test or by looking at the graph of f'' and applying the second derivative test, we see that f has a relative minimum at -1 and a relative maximum at 1 .

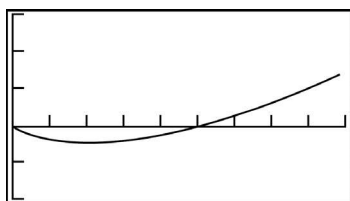
- (b) Examine the graph of f' to determine the intervals where the graph lies above and below the x -axis. We see that $f'(x) > 0$ on $(-1, 1)$, indicating that f is increasing on the same interval. We also see that $f'(x) < 0$ on $(-\infty, -1)$ and $(1, \infty)$, indicating that f is decreasing on the same intervals.
- (c) Examine the graph of f'' . We see that the graph has three x -intercepts, so there are three values where $f''(x) = 0$. These x -values are about -1.7 , 0 , and about 1.7 . Because the sign of f'' and thus the concavity of f changes at these three values, we see that the x -values of the inflection points of the graph of f are about -1.7 , 0 , and about 1.7 .
- (d) We observe from the graph of f'' that $f''(x) > 0$ on about $(-1.7, 0)$ and $(1.7, \infty)$, so f is concave upward on the same intervals.

Likewise, we observe that $f''(x) < 0$ on about $(-\infty, -1.7)$ and $(0, 1.7)$, so f is concave downward on the same intervals.

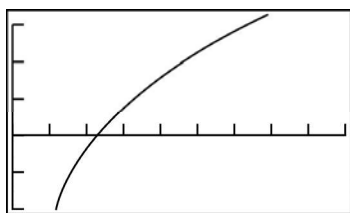
$$\begin{aligned} 68. \quad f'(x) &= x^2 + x \ln x \\ f''(x) &= 2x + (x)\left(\frac{1}{x}\right) + (1)(\ln x) \\ &= 2x + 1 + \ln x \end{aligned}$$

Graph f' and f'' in the window $[0, 1]$ by $[-2, 3]$, Xscl = 0.1, Yscl = 1.

Graph of f' :



Graph of f'' :



- (a) The critical number of f is the x -intercept of the graph of f' . Using the graph, we find a critical number of f is about 0.5671. By looking at the graph of f'' and applying the second derivative test, we see f has a minimum at 0.5671.
- (b) Examine the graph of f' to determine the intervals where the graph lies above and below the x -axis. We see that $f'(x) > 0$ on about $(0.5671, \infty)$, indicating that f is increasing on about $(0.5671, \infty)$. We also see that $f'(x) < 0$ on about $(0, 0.5671)$, indicating that f is decreasing on about $(0, 0.5671)$.
- (c) Examine the graph of f'' . We see that the graph has one x -intercept, so there is one x -value where $f''(x) = 0$. This value is about 0.2315. Because the sign of f'' changes at this value, we see that x -value of the inflection point of the graph of f is about 0.2315.

- (d) We observe from the graph f'' that $f'' > 0$ on about $(0.2315, \infty)$, so f is concave upward on about $(0.2315, \infty)$. Likewise, we observe from the graph f'' that $f'' < 0$ on about $(0, 0.2315)$, so f is concave downward on about $(0, 0.2315)$.

69. There are many examples. The easiest is $f(x) = \sqrt{x}$. This graph is increasing and concave downward.

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$f'(0)$ does not exist, while $f'(x) > 0$ for all $x > 0$. (Note that the domain of f is $[0, \infty)$.)

As x increases, the value of $f'(x)$ decreases, but remains positive. It approaches zero, but never becomes zero or negative.

70. (a) The left side of the graph changes from concave upward to concave downward at the inflection point between voice recognition software and Smart phones. The rate of growth of sales begins to decline at the inflection point.
- (b) Telephone answering machines are closest to the right-hand inflection point. This inflection point indicates that the rate of decline of sales is beginning to slow.
71.
$$\begin{aligned} A(t) &= 0.0000329t^3 - 0.00450t^2 \\ &\quad + 0.0613t + 2.34 \\ A'(t) &= 0.0000987t^2 - 0.009t + 0.0613 \\ A''(t) &= 0.0001974t - 0.009 \end{aligned}$$

Assets will decrease most rapidly when A' has a minimum. A has a single inflection point when $A''(t) = 0$, which occurs when

$$\begin{aligned} 0.0001974t - 0.009 &= 0 \\ 0.0001974t &= 0.009 \\ t &= \frac{0.009}{0.0001974} \approx 45.6. \end{aligned}$$

This t -value corresponds to a minimum of A' and the minimum rate of decrease is $A'(45.6) \approx -0.144$. Since $t = 0$ represents the year 2000, the maximum rate of decrease of Social Security assets will occur in around the middle of 2045.

72. $R(x) = 10,000 - x^3 + 42x^2 + 800x$,
 $0 \leq x \leq 20$

$$R'(x) = -3x^2 + 84x + 800$$

$$R''(x) = -6x + 84$$

A point of diminishing returns occurs at a point of inflection, or where $R''(x) = 0$.

$$-6x + 84 = 0$$

$$x = 14$$

Test $R''(x)$ to determine whether concavity changes at $x = 14$.

$$R''(12) = 12 > 0$$

$$R''(16) = -12 < 0$$

$R(x)$ is concave upward on $(0, 14)$ and concave downward on $(14, 20)$.

$$R(14) = 10,000 - (14)^3 + 42(14)^2 + 800(14)$$

$$= 26,688$$

The point of diminishing returns is $(14, 26,688)$.

73. $R(x) = \frac{4}{27}(-x^3 + 66x^2 + 1050x - 400)$
 $0 \leq x \leq 25$

$$R'(x) = \frac{4}{27}(-3x^2 + 132x + 1050)$$

$$R''(x) = \frac{4}{27}(-6x + 132)$$

A point of diminishing returns occurs at a point of inflection, or where $R''(x) = 0$.

$$\frac{4}{27}(-6x + 132) = 0$$

$$-6x + 132 = 0$$

$$6x = 132$$

$$x = 22$$

Test $R''(x)$ to determine whether concavity changes at $x = 22$.

$$R''(20) = \frac{4}{27}(-6 \cdot 20 + 132) = \frac{16}{9} > 0$$

$$R''(24) = \frac{4}{27}(-6 \cdot 24 + 132) = -\frac{16}{9} < 0$$

$R(x)$ is concave upward on $(0, 22)$ and concave downward on $(22, 25)$.

$$R(22) = \frac{4}{27}[-(22)^3 + 66(22)^2 + 1060(22) - 400]$$

$$\approx 6517.9$$

The point of diminishing returns is $(22, 6517.9)$.

74. $R(x) = -0.3x^3 + x^2 + 11.4x$, $0 \leq x \leq 6$

$$R'(x) = -0.9x^2 + 2x + 11.4$$

$$R''(x) = -1.8x + 2$$

A point of diminishing returns occurs at a point of inflection, or where $R''(x) = 0$.

$$-1.8x + 2 = 0$$

$$2 = 1.8x$$

$$1.11 \approx x$$

Test $R''(x)$ to determine whether concavity changes at $x = 1.11$.

$$R''(2) = -1.8(2) + 2 = -3.6 + 2$$

$$= -1.8 < 0$$

$$R''(1) = -1.8(1) + 2 = 0.2 > 0$$

$R(x)$ is concave upward on $(0, 1.11)$ and concave downward on $(1.11, 6)$.

$$R(1.11) = -0.3(1.11)^3 + (1.11)^2 + 11.4(1.11)$$

$$= 13.476 \approx 13.5$$

The point of diminishing returns is $(1.11, 13.5)$.

75. $R(x) = -0.6x^3 + 3.7x^2 + 5x$, $0 \leq x \leq 6$

$$R'(x) = -1.8x^2 + 7.4x + 5$$

$$R''(x) = -3.6x + 7.4$$

A point of diminishing returns occurs at a point of inflection or where $R''(x) = 0$.

$$-3.6x + 7.4 = 0$$

$$-3.6x = -7.4$$

$$x = \frac{-7.4}{-3.6x} \approx 2.06$$

Test $R''(x)$ to determine whether concavity changes at $x = 2.05$.

$$R''(2) = -3.6(2) + 7.4$$

$$= -7.2 + 7.4 = 0.2 > 0$$

$$R''(3) = -3.6(3) + 7.4$$

$$= -10.8 + 7.4 = -3.4 < 0$$

$R(x)$ is concave upward on $(0, 2.06)$ and concave downward on $(2.06, 6)$.

$$R(2.06) = -0.6(2.06)^3 + 3.7(2.06)^2 + 5(2.06) \\ \approx 20.8$$

The point of diminishing returns is (2.06, 20.8).

76. $I(M) = \frac{-U''(M)}{U'(M)}$

For $U(M) = \sqrt{M} = M^{1/2}$,

$$U'(M) = \frac{1}{2}M^{-1/2}$$

$$U''(M) = -\frac{1}{4}M^{-3/2}.$$

Then

$$I(M) = \frac{-\left(-\frac{1}{4}M^{-3/2}\right)}{\frac{1}{2}M^{-1/2}} \\ = \frac{1}{2}M^{-1} \\ = \frac{1}{2M}.$$

For $U(M) = M^{2/3}$,

$$U'(M) = \frac{2}{3}M^{-1/3}$$

$$U''(M) = -\frac{2}{9}M^{-4/3}.$$

Then

$$I(M) = \frac{-\left(-\frac{2}{9}M^{-4/3}\right)}{\frac{2}{3}M^{-1/3}} \\ = \frac{1}{3}M^{-1} \\ = \frac{1}{3M}.$$

$U(M) = \sqrt{M}$ indicates a greater risk aversion because $\frac{1}{2M} > \frac{1}{3M}$ for $0 < M < 1$.

77. Let $D(q)$ represent the demand function. The revenue function, $R(q)$, is $R(q) = qD(q)$. The marginal revenue is given by

$$R'(q) = qD'(q) + D(q)(1) \\ = qD'(q) + D(q).$$

$$R''(q) = qD''(q) + D'(q)(1) + D'(q) \\ = qD''(q) + 2D'(q)$$

gives the rate of decline of marginal revenue. $D'(q)$ gives the rate of decline of price. If marginal revenue declines more quickly than price,

$$qD''(q) + 2D'(q) - D'(q) < 0 \\ \text{or } qD''(q) + D'(q) < 0.$$

78. (a) f_0 represents initial population ($t = 0$).
 (b) $(a, f(a))$ is the point where the graph changes concavity or the inflection point.
 (c) f_M is the maximum carrying capacity.

79. (a) $R(t) = t^2(t - 18) + 96t + 1000,$
 $0 < t < 8$
 $= t^3 - 18t^2 + 96t + 1000$
 $R'(t) = 3t^2 - 36t + 96$

Set $R'(t) = 0$.

$$3t^2 - 36t + 96 = 0$$

$$t^2 - 12t + 32 = 0$$

$$(t - 8)(t - 4) = 0$$

$$t = 8 \text{ or } t = 4$$

8 is not in the domain of $R(t)$.

$$R''(t) = 6t - 36$$

$R''(4) = -12 < 0$ implies that $R(t)$ is maximized at $t = 4$, so the population is maximized at 4 hours.

(b) $R(4) = 16(-14) + 96(4) + 1000$
 $= -224 + 384 + 1000$
 $= 1160$

The maximum population is 1160 million.

80. The chemicals are accumulating, but at a slower rate. Therefore, $c(t)$ is increasing and concave downward. Thus, $c'(t) > 0$ and $c''(t) < 0$.

81. $K(x) = \frac{3x}{x^2 + 4}$

$$\begin{aligned} \text{(a)} \quad K'(x) &= \frac{3(x^2 + 4) - (2x)(3x)}{(x^2 + 4)^2} \\ &= \frac{-3x^2 + 12}{(x^2 + 4)^2} = 0 \\ -3x^2 + 12 &= 0 \\ x^2 &= 4 \\ x &= 2 \quad \text{or} \quad x = -2 \end{aligned}$$

For this application, the domain of K is $[0, \infty)$, so the only critical number is 2.

$$\begin{aligned} K''(x) &= \frac{(x^2 + 4)^2(-6x) - (-3x^2 + 12)(2)(x^2 + 4)(2x)}{(x^2 + 4)^4} \\ &= \frac{-6x(x^2 + 4) - 4x(-3x^2 + 12)}{(x^2 + 4)^3} \\ &= \frac{6x^3 - 72x}{(x^2 + 4)^3} \end{aligned}$$

$$K''(2) = \frac{-96}{512} = -\frac{3}{16} < 0 \text{ implies that}$$

$$K(x) \text{ is maximized at } x = 2.$$

Thus, the concentration is a maximum after 2 hours.

$$\text{(b)} \quad K(2) = \frac{3(2)}{(2)^2 + 4} = \frac{3}{4}$$

The maximum concentration is $\frac{3}{4}\%$.

82. $K(x) = \frac{4x}{3x^2 + 27}$

$$\begin{aligned} \text{(a)} \quad K'(x) &= \frac{(3x^2 + 27)(4) - 4x(6x)}{(3x^2 + 27)^2} \\ &= \frac{-12x^2 + 108}{(3x^2 + 27)^2} \end{aligned}$$

Set $K'(x) = 0$.

$$\begin{aligned} -12(x^2 - 9) &= 0 \\ x &= -3 \quad \text{or} \quad x = 3 \end{aligned}$$

For this application, the domain of K is $[0, \infty)$, so the only critical number is 3.

To determine whether a relative maximum or minimum occurs at $x = 3$, we find $K''(3)$ and use the second derivative test.

To find $K''(x)$, apply the quotient rule to find the derivative of $K'(x)$. The numerator of $K''(x)$ will be

$$\begin{aligned} (3x^2 + 27)^2(-24x) \\ - (-12x^2 + 108)(2)(3x^2 + 27)(6x) \\ = (3x^2 + 27)^2(-24x) \\ - (12x)(-12x^2 + 108)(3x^2 + 27). \end{aligned}$$

The denominator of K'' is $(3x^2 + 27)^4$, so

$$\begin{aligned} K''(3) &= \frac{(54)^2(-72) - (36)(0)(54)}{(54)^4} \\ &= -\frac{72}{54^2} < 0. \end{aligned}$$

This indicates that the concentration is a maximum after 3 hours.

$$\text{(b)} \quad K(3) = \frac{4(3)}{3(3)^2 + 27} = \frac{2}{9}$$

The maximum concentration is $\frac{2}{9}\%$.

83. $G(t) = \frac{10,000}{1 + 49e^{-0.1t}}$

$$\begin{aligned} G'(t) &= \frac{(1 + 49e^{-0.1t})(0) - (10,000)(-4.9e^{-0.1t})}{(1 + 49e^{-0.1t})^2} \\ &= \frac{49,000e^{-0.1t}}{(1 + 49e^{-0.1t})^2} \end{aligned}$$

To find $G''(t)$, apply the quotient rule to find the derivative of $G'(t)$.

The numerator of $G''(t)$ will be

$$\begin{aligned} (1 + 49e^{-0.1t})^2(-4900e^{-0.1t}) \\ - (49,000e^{-0.1t})(2)(1 + 49e^{-0.1t})(-4.9e^{-0.1t}) \\ = (1 + 49e^{-0.1t})(-4900e^{-0.1t}) \\ \cdot [(1 + 49e^{-0.1t}) - 20(4.9e^{-0.1t})] \\ = (-4900e^{-0.1t})[1 + 49e^{-0.1t} - 98e^{-0.1t}] \\ = (-4900e^{-0.1t})(1 - 49e^{-0.1t}). \end{aligned}$$

Thus,

$$G''(t) = \frac{(-4900e^{-0.1t})(1 - 49e^{-0.1t})}{(1 + 49e^{-0.1t})^4}.$$

$$\begin{aligned} G''(t) = 0 \quad \text{when} \quad -4900e^{-0.1t} = 0 \quad \text{or} \\ 1 - 49e^{-0.1t} = 0. \end{aligned}$$

$-4900e^{-0.1t} < 0$, and thus never equals zero.

$$1 - 49e^{-0.1t} = 0$$

$$1 = 49e^{-0.1t}$$

$$\frac{1}{49} = e^{-0.1t}$$

$$\ln\left(\frac{1}{49}\right) = -0.1t$$

$$\ln 1 - \ln 49 = 0.1t$$

$$t = 10 \ln 49$$

$$t \approx 38.9182$$

The point of inflection is (38.9182, 5000).

$$84. \quad G(t) = \frac{5200}{1 + 12e^{-0.52t}}$$

$$\begin{aligned} G'(t) &= \frac{(1 + 12e^{-0.52t})(0) - 5200(-6.24e^{-0.52t})}{(1 + 12e^{-0.52t})^2} \\ &= \frac{32,448e^{-0.52t}}{(1 + 12e^{-0.52t})^2} \end{aligned}$$

To find $G''(t)$, use the quotient rule to find the derivative of $G'(t)$. The numerator of $G''(t)$ will be

$$\begin{aligned} &(1 + 12e^{-0.52t})^2(-16,873e^{-0.52t}) \\ &- (32,448e^{-0.52t})(2)(1 + 12e^{-0.52t})(-6.24e^{-0.52t}) \\ &= -16,873e^{-0.52t}(1 + 12e^{-0.52t}) \\ &\quad \cdot [(1 + 12e^{-0.52t}) - 24e^{-0.52t}] \\ &= -16,873e^{-0.52t}(1 + 12e^{-0.52t})(1 - 12e^{-0.52t}). \end{aligned}$$

Thus,

$$\begin{aligned} G''(t) &= \frac{-16,873e^{-0.52t}(1 + 12e^{-0.52t})(1 - 12e^{-0.52t})}{(1 + 12e^{-0.52t})^4} \\ &= \frac{-16,873e^{-0.52t}(1 - 12e^{-0.52t})}{(1 + 12e^{-0.52t})^3}. \end{aligned}$$

$$G''(t) = 0 \text{ when } 1 - 12e^{-0.52t} = 0.$$

$$1 = 12e^{-0.52t}$$

$$\ln \frac{1}{12} = -0.52t$$

$$-\ln 12 = -0.52t$$

$$1.9231 \ln 12 = t \approx 4.779$$

The point of inflection is (4.779, 2600).

$$85. \quad L(t) = Be^{-ce^{-kt}}$$

$$L'(t) = Be^{-ce^{-kt}}(-ce^{-kt})'$$

$$= Be^{-ce^{-kt}}[-ce^{-kt}(-kt)']$$

$$= Bcke^{-ce^{-kt}-kt}$$

$$L''(t) = Bcke^{-ce^{-kt}-kt}(-ce^{-kt} - kt)'$$

$$= Bcke^{-ce^{-kt}-kt}[-ce^{-kt}(-kt)' - k]$$

$$= Bcke^{-ce^{-kt}-kt}(cke^{-kt} - k)$$

$$= Bck^2e^{-ce^{-kt}-kt}(ce^{-kt} - 1)$$

$$L''(t) = 0 \text{ when } ce^{-kt} - 1 = 0$$

$$ce^{-kt} - 1 = 0$$

$$\frac{c}{e^{kt}} = 1$$

$$e^{kt} = c$$

$$kt = \ln c$$

$$t = \frac{\ln c}{k}$$

Letting $c = 7.267963$ and $k = 0.670840$

$$t = \frac{\ln 7.267963}{0.670840} \approx 2.96 \text{ years}$$

Verify that there is a point of inflection at

$$t = \frac{\ln c}{k} \approx 2.96. \text{ For}$$

$$L''(t) = Bck^2e^{-ce^{-kt}-kt}(ce^{-kt} - 1),$$

we only need to test the factor $ce^{-kt} - 1$ on the intervals determined by $t \approx 2.96$ since the other factors are always positive.

$L''(1)$ has the same sign as

$$7.267963e^{-0.670840(1)} - 1 \approx 2.72 > 0.$$

$L''(3)$ has the same sign as

$$7.267963e^{-0.670840(3)} - 1 \approx -0.029 < 0.$$

Therefore L is concave up on $\left(0, \frac{\ln c}{k} \approx 2.96\right)$

and concave down on $\left(\frac{\ln c}{k}, \infty\right)$, so there is a

point of inflection at $t = \frac{\ln c}{k} \approx 2.96$ years.

This signifies the time when the rate of growth begins to slow down since L changes from concave up to concave down at this inflection point.

$$86. \quad N(t) = e^{c(1-e^{-kt})}$$

$$\begin{aligned} N'(t) &= e^{c(1-e^{-kt})} [c(1-e^{-kt})]' \\ &= e^{c(1-e^{-kt})} (-ce^{-kt})(-kt)' \\ &= cke^{c-ce^{-kt}-kt} \end{aligned}$$

$$\begin{aligned} N''(t) &= cke^{c-ce^{-kt}-kt} (c - ce^{-kt} - kt)' \\ &= cke^{c-ce^{-kt}-kt} [-ce^{-kt}(-kt)' - k] \\ &= cke^{c-ce^{-kt}-kt} (cke^{-kt} - k) \\ &= ck^2 e^{c-ce^{-kt}-kt} (ce^{-kt} - 1) \end{aligned}$$

$$\begin{aligned} N''(t) &= 0 \text{ when } ce^{-kt} - 1 = 0 \\ ce^{-kt} - 1 &= 0 \end{aligned}$$

$$\frac{c}{e^{kt}} = 1$$

$$e^{kt} = c$$

$$kt = \ln c$$

$$t = \frac{\ln c}{k}$$

Letting $c = 27.3$ and $k = 0.011$

$$t = \frac{\ln 27.3}{0.011} \approx 301$$

Verify that there is a point of inflection at

$$t = \frac{\ln 27.3}{0.011}.$$

For $N''(t) = ck^2 e^{c-ce^{-kt}-kt} (ce^{-kt} - 1)$, we only need to test the factor $ce^{-kt} - 1$ on the intervals determined by $t = \frac{\ln 27.3}{0.011} \approx 301$ since the other factors are always positive.

$N''(200)$ has the same sign as

$$\begin{aligned} 27.3e^{-0.011(200)} - 1 &= 27.3e^{-2.2} - 1 \\ &\approx 2.02 > 0. \end{aligned}$$

$N''(400)$ has the same sign as

$$\begin{aligned} 27.3e^{-0.011(400)} - 1 &= 27.3e^{-4.4} - 1 \\ &\approx -0.66 < 0. \end{aligned}$$

Therefore, N is concave up on $\left(0, \frac{\ln 27.3}{0.011}\right)$ and N

is concave down on $\left(\frac{\ln 27.3}{0.011}, \infty\right)$, so there is a

point of inflection at $t = \frac{\ln 27.3}{0.011} \approx 301$ days.

This signifies the time when the rate of growth begins to slow down since L changes from concave up to concave down at this inflection point.

$$87. \quad v'(x) = -35.98 + 12.09x - 0.4450x^2$$

$$v'(x) = 12.09 - 0.89x$$

$$v''(x) = -0.89$$

Since $-0.89 < 0$, the function is always concave down.

$$88. \quad N(t) = 71.8e^{-8.96e^{-0.0685t}}$$

$$N'(t) = 71.8(-8.96)(-0.0685)e^{-0.0685t}$$

$$\times \left(e^{-8.96e^{-0.0685t}} \right)$$

$$= (44.067968e^{-0.0685t})(e^{-8.96e^{-0.0685t}})$$

$$\begin{aligned} N''(t) &= (44.067968)(0.61376e^{-0.0685t} \\ &\quad - 0.0685)e^{(-8.96e^{-0.0685t}-0.0685t)} \end{aligned}$$

$$N''(t) = 0 \text{ when } 0.61376e^{-0.0685t} - 0.0685 = 0$$

$$0.61376e^{-0.0685t} - 0.0685 = 0$$

$$e^{-0.0685t} \approx 0.11162$$

$$-0.0685t \approx \ln 0.11162$$

$$t \approx 32.01$$

$N(t)$ has an inflection point at $(32.01, N(32.01))$ or $(32.01, 26.41)$.

89. Since the rate of violent crimes is decreasing but at a slower rate than in previous years, we know that $f'(t) < 0$ but $f''(t) > 0$. Note that since $f'(t) < 0$, f is decreasing, and since $f''(t) > 0$, the graph of f is concave upward.

$$90. \quad V(x) = 12x(100 - x)$$

$$= 1200x - 12x^2$$

$$V'(x) = 1200 - 24x$$

$$\text{Set } V'(x) = 0.$$

$$1200 - 24x = 0$$

$$x = 50$$

$$V''(x) = -24$$

$$V''(50) = -24$$

Thus, a value of 50 will produce a maximum rate of reaction.

91. $s(t) = -16t^2$
 $v(t) = s'(t) = -32t$

(a) $v(3) = -32(3) = -96$ ft/sec

(b) $v(5) = -32(5) = -160$ ft/sec

(c) $v(8) = -32(8) = -256$ ft/sec

(d) $a(t) = v'(t) = s''(t)$
 $= -32$ ft/sec²

92. $s(t) = -16t^2 + 140t + 37$
 $s'(t) = -32t + 140$
 $s'(t) = 0$ when $-32t + 140 = 0$
 $140 = 32t$
 $4.375 = t$

$s'(1) = -32(1) + 140 > 0$

$s'(5) = -32(5) + 140 < 0$

$s(4.375) = 343.25$

(a) Maximum height of 343.25 ft is reached 4.375 sec after the ball is thrown.

(b) The ball hits ground when $s(t) = 0$.

$$\frac{-140 \pm \sqrt{140^2 - 4(-16)(37)}}{-32} \approx 9$$

$s'(9) = -32(9) + 140 = -148$

The ball hits ground in about 9 seconds with a speed of about -148 ft/sec.

93. $s(t) = 256t - 16t^2$
 $v(t) = s'(t) = 256 - 32t$
 $a(t) = v'(t) = s''(t) = -32$

To find when the maximum height occurs, set $s'(t) = 0$.

$256 - 32t = 0$

$t = 8$

Find the maximum height.

$s(8) = 256(8) - 16(8^2)$
 $= 1024$

The maximum height of the ball is 1024 ft. The ball hits the ground when $s = 0$.

$256t - 16t^2 = 0$

$16t(16 - t) = 0$

$t = 0$ (initial moment)

$t = 16$ (final moment)

The ball hits the ground 16 seconds after being thrown.

94. $s(t) = 1.5t^2 + 4t$

(a) $s(10) = 1.5(10^2) + 4(10)$
 $= 150 + 40 = 190$

The car will move 190 ft in 10 seconds.

(b) $v(t) = s'(t) = 3t + 4$
 $s'(5) = 3 \cdot 5 + 4 = 19$
 $s'(10) = 3 \cdot 10 + 4 = 34$

The velocity at 5 sec is 19 ft/sec, and the velocity at 10 sec is 34 ft/sec.

(c) The car stops when $v(t) = 0$, but $v(t) > 0$ for all $t \geq 0$.

(d) $a(t) = v'(t) = s''(t) = 3$

The acceleration at 5 sec is 3 ft/sec², and the acceleration at 10 sec is also 3 ft/sec².

(e) As t increases, the velocity increases, but the acceleration is constant.

95. The car was moving most rapidly when $t \approx 6$, because acceleration was positive on $(0, 6)$ and negative after $t = 6$, so velocity was a maximum at $t = 6$.

5.4 Curve Sketching

Your Turn 1

$f(x) = -x^3 + 3x^2 + 9x - 10$

$f'(x) = -3x^2 + 6x + 9$

$f''(x) = -6x + 6$

Use the first derivative to find intervals where the function is increasing or decreasing.

$-3x^2 + 6x + 9 = 0$

$-3(x^2 - 2x - 3) = 0$

$(x + 1)(x - 3) = 0$

$x = -1$ or $x = 3$

These critical numbers divide the line into three intervals, $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$. Evaluate the derivative at a test point in each region.

$f'(-2) = -15$

$f'(0) = 9$

$f'(4) = -4$

This shows that f is decreasing on $(-\infty, -1)$, increasing on $(-1, 3)$, and decreasing on $(3, \infty)$. By the first derivative test, f has a relative minimum of $f(-1) = -15$ at $x = -1$ and a relative maximum of $f(3) = 17$ at $x = 3$.

Use the second derivative to find the intervals where the function is concave upward or downward.

$$-6x + 6 = 0$$

$$6x = x$$

$$x = 1$$

The value where the second derivative is 0 divides the line into two intervals, $(-\infty, 1)$ and $(1, \infty)$. Evaluate

$f''(x)$ at a point in each interval.

$$f''(0) = 6$$

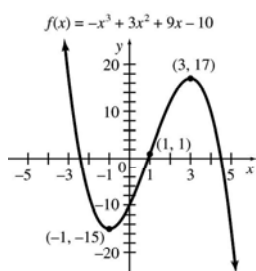
$$f''(2) = -6$$

This shows that f is concave upward on $(-\infty, 1)$ and concave downward on $(1, \infty)$. The graph has an inflection point at $(1, f(1))$, or $(1, 1)$.

This information is summarized in the following table.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, 3)$	$(3, \infty)$
Sign of f'	-	+	+	-
Sign of f''	+	+	-	-
f increasing or decreasing	Decreasing	Increasing	Increasing	Decreasing
Concavity of f	Upward	Upward	Downward	Downward
Shape of graph				

Now sketch the graph using this information.



Your Turn 2

$$f(x) = 4x + \frac{1}{x} = \frac{4x^2 + 1}{x}$$

Because $x = 0$ makes the denominator 0 but not the numerator, the line $x = 0$ (that is, the y -axis) is a vertical asymptote. Neither $\lim_{x \rightarrow -\infty} f(x)$ nor $\lim_{x \rightarrow \infty} f(x)$ exists, so there is no horizontal asymptote. But since the term $1/x$ gets very small as $|x|$ gets large, the graph of f approaches the line $y = 4x$ as $|x|$ becomes larger and larger, so this line is an oblique asymptote.

$f(-x) = -f(x)$, so the graph of the left side can be found by rotating the right side around the origin by 180° .

Find any critical numbers.

$$f'(x) = 4 - \frac{1}{x^2}$$

$$4 - \frac{1}{x^2} = 0$$

$$x^2 = \frac{1}{4}$$

$$x = \frac{1}{2} \text{ or } -\frac{1}{2}$$

The critical numbers are $-1/2$ and $1/2$, which together with the location of the vertical asymptote divide the line into four regions, $(-\infty, -1/2)$, $(-1/2, 0)$, $(0, 1/2)$, and $(1/2, \infty)$. Evaluate the derivative at a test point in each region.

$$f'(-1) = 3$$

$$f'\left(-\frac{1}{4}\right) = -12$$

$$f'\left(\frac{1}{4}\right) = -12$$

$$f'(1) = 3$$

This shows that f has a relative maximum of $f(-1/2) = -4$ at $x = -1/2$ and a relative minimum of $f(1/2) = 4$ at $0 < x = 1/2$.

Now find the intervals where the graph is concave upward or concave downward.

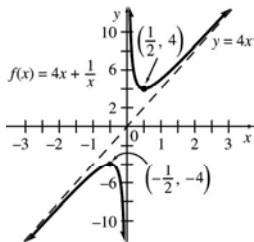
$$f''(x) = \frac{2}{x^3}$$

The second derivative is never 0, but concavity might change where the second derivative is undefined, at $x = 0$. In fact,

$$f''(x) < 0 \text{ for } x < 0,$$

$$f''(x) > 0 \text{ for } x > 0,$$

so the graph is concave downward to the left of the origin and concave upward to the right of the origin. This provides enough information to sketch the graph.



Your Turn 3

$$f(x) = \frac{4x^2}{x^2 + 4}$$

The denominator of f is never 0, so f has no vertical asymptotes. However,

$$\lim_{x \rightarrow -\infty} \frac{4x^2}{x^2 + 4} = 4 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{4x^2}{x^2 + 4} = 4,$$

so the line $y = 4$ is a horizontal asymptote. Note that $f(-x) = f(x)$, so the graph of f is symmetrical around the y -axis,

Find any critical numbers.

$$\begin{aligned} f'(x) &= \frac{(x^2 + 4)(8x) - (4x^2)(2x)}{(x^2 + 4)^2} \\ &= \frac{8x^3 + 32x - 8x^3}{(x^2 + 4)^2} \\ &= \frac{32x}{(x^2 + 4)^2} \end{aligned}$$

The derivative is 0 only when $x = 0$, and the denominator of the derivative is never 0. Therefore the critical number divides the line into just two regions, $(-\infty, 0)$ and $(0, \infty)$. Evaluate the derivative at a test point in each region.

$$\begin{aligned} f'(-1) &= -1.28 \\ f'(1) &= 1.28 \end{aligned}$$

Thus f is decreasing as x approaches 0 from the left, and increasing to the right of $x = 0$, so f has a relative minimum of $f(0) = 0$ at $x = 0$.

Now find the intervals where the graph is concave upward or concave downward.

$$\begin{aligned} f'(x) &= \frac{32x}{(x^2 + 4)^2} \\ f''(x) &= \frac{(x^2 + 4)^2(32) - (32x)(2)(x^2 + 4)(2x)}{(x^2 + 4)^4} \end{aligned}$$

$$\begin{aligned} &= \frac{(x^2 + 4)[(x^2 + 4)(32) - 128x^2]}{(x^2 + 4)^4} \\ &= \frac{128 - 96x^2}{(x^2 + 4)^3} \end{aligned}$$

The denominator is never 0, but the numerator is 0 when

$$128 - 96x^2 = 0$$

$$96x^2 = 128$$

$$3x^2 = 4$$

$$x^2 = \frac{4}{3}$$

$$x = \pm \sqrt{\frac{4}{3}} \approx \pm 1.155$$

The zeros of the second derivative divide the line into three regions,

$$\left(-\infty, -\sqrt{\frac{4}{3}}\right), \left(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}}\right) \text{ and } \left(\sqrt{\frac{4}{3}}, \infty\right).$$

Evaluate $f''(x)$ at a point in each interval.

$$f''(-2) = -0.5$$

$$f''(0) = 2$$

$$f''(2) = -0.5$$

According to this information, the graph is

concave downward on the interval $\left(-\infty, -\sqrt{\frac{4}{3}}\right)$,

concave upward on the interval $\left(-\sqrt{\frac{4}{3}}, \sqrt{\frac{4}{3}}\right)$,

and concave downward on the interval $\left(\sqrt{\frac{4}{3}}, \infty\right)$.

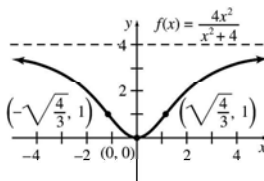
Thus there will be two inflection points. Find the corresponding values of f .

$$f\left(-\sqrt{\frac{4}{3}}\right) = 1$$

$$f\left(\sqrt{\frac{4}{3}}\right) = 1$$

The inflection points are at $\left(-\sqrt{\frac{4}{3}}, 1\right)$ and $\left(\sqrt{\frac{4}{3}}, 1\right)$.

This provides enough information to sketch the graph.



Your Turn 4

$$f(x) = (x + 2)e^{-x}$$

Since $\lim_{x \rightarrow \infty} (x + 2)e^{-x} = 0$, the line $y = 0$ (the x -axis)

is a horizontal asymptote for the graph. Neither $f(-x) = -f(x)$ nor $f(-x) = f(x)$ is true, so the graph has no symmetry.

Find any critical numbers.

$$\begin{aligned} f'(x) &= (x + 2)(-e^{-x}) + (1)(e^{-x}) \\ &= -(1 + x)e^{-x} \end{aligned}$$

The derivative is 0 at only one point, where $1 + x = 0$ or $x = -1$. This critical number divides the line into two regions, $(-\infty, -1)$ and $(-1, \infty)$. Evaluate the derivative at a test point in each region.

$$\begin{aligned} f'(-2) &\approx 7.389 \\ f'(0) &= -1 \end{aligned}$$

Thus f is increasing on the interval $(-\infty, -1)$ and decreasing on the interval $(-1, \infty)$, and has a relative maximum of $f(-1) \approx 2.718$ at $x = -1$.

Now find the intervals where the graph is concave upward or concave downward.

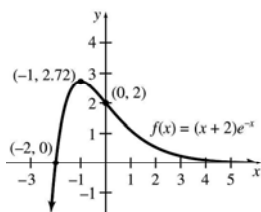
$$\begin{aligned} f'(x) &= -(1 + x)e^{-x} \\ f''(x) &= (1 + x)e^{-x} + (-1)e^{-x} \\ &= xe^{-x} \end{aligned}$$

The second derivative is 0 only at $x = 0$. Evaluate $f''(x)$ at points on either side of $x = 0$.

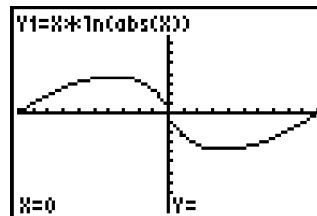
$$\begin{aligned} f''(-1) &\approx -2.718 \\ f''(1) &\approx 0.368 \end{aligned}$$

The graph is concave downward to the left of $x = 0$ and concave upward to the right of $x = 0$. There is an inflection point at $(0, f(0))$, or $(0, 2)$.

This provides enough information to sketch the graph.

**5.4 Exercises**

- Graph $y = x \ln |x|$ on a graphing calculator. A suitable choice for the viewing window is $[-1, 1]$ by $[-1, 1]$, $Xscl = 0.1$, $Yscl = 0.1$.



The calculator shows no y -value when $x = 0$ because 0 is not in the domain of this function. However, we see from the graph that

$$\lim_{x \rightarrow 0^-} x \ln |x| = 0$$

and

$$\lim_{x \rightarrow 0^+} x \ln |x| = 0.$$

Thus,

$$\lim_{x \rightarrow 0} x \ln |x| = 0.$$

- $f(x) = -2x^3 - 9x^2 + 108x - 10$

Domain is $(-\infty, \infty)$.

$$\begin{aligned} f(-x) &= -2(-x)^3 - 9(-x)^2 + 108(-x) - 10 \\ &= 2x^3 - 9x^2 - 108x - 10 \end{aligned}$$

No symmetry

$$\begin{aligned} f'(x) &= -6x^2 - 18x + 108 \\ &= -6(x^2 + 3x - 18) \\ &= -6(x + 6)(x - 3) \end{aligned}$$

$$f'(x) = 0 \text{ when } x = -6 \text{ or } x = 3.$$

Critical numbers: -6 and 3

Critical points: $(-6, -550)$ and $(3, 179)$

$$f''(x) = -12x - 18$$

$$f''(-6) = 54 > 0$$

$$f''(3) = -54 < 0$$

Relative maximum at 3 , relative minimum at -6

Increasing on $(-6, 3)$

Decreasing on $(-\infty, -6)$ and $(3, \infty)$

$$f''(x) = -12x - 18 = 0$$

$$-6(2x + 3) = 0$$

$$x = -\frac{3}{2}$$

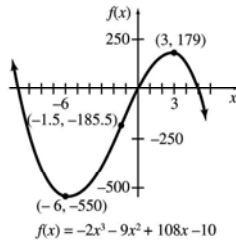
Point of inflection at $(-1.5, -185.5)$

Concave upward on $(-\infty, -1.5)$

Concave downward on $(-1.5, \infty)$

y-intercept:

$$y = -2(0)^3 - 9(0)^2 + 108(0) - 10 = -10$$



4. $f(x) = x^3 - \frac{15}{2}x^2 - 18x - 1$

Domain is $(-\infty, \infty)$.

$$\begin{aligned} f(-x) &= (-x)^3 - \frac{15}{2}(-x)^2 - 18(-x) - 1 \\ &= -x^3 - \frac{15}{2}x^2 + 18x - 1 \end{aligned}$$

No symmetry

$$\begin{aligned} f'(x) &= 3x^2 - 15x - 18 \\ &= 3(x^2 - 5x - 6) \\ &= 3(x - 6)(x + 1) = 0 \end{aligned}$$

Critical numbers: 6 and -1

Critical points: $(6, -163)$ and $(-1, 8.5)$

$$\begin{aligned} f''(x) &= 6x - 15 \\ f''(6) &= 21 > 0 \\ f''(-1) &= -21 < 0 \end{aligned}$$

Relative maximum at $x = -1$, relative minimum at $x = 6$

Increasing on $(-\infty, -1)$ and $(6, \infty)$

Decreasing on $(-1, 6)$

$$f''(x) = 6x - 15 = 0$$

$$x = \frac{5}{2}$$

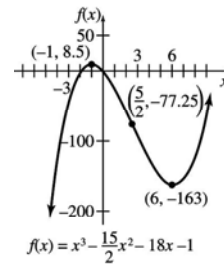
Point of inflection at $(\frac{5}{2}, -77.25)$

Concave upward on $(\frac{5}{2}, \infty)$

Concave downward on $(-\infty, \frac{5}{2})$

y-intercept:

$$y = (0)^3 - \frac{15}{2}(0)^2 - 18(0) - 1 = -1$$



5. $f(x) = -3x^3 + 6x^2 - 4x - 1$

Domain is $(-\infty, \infty)$.

$$\begin{aligned} f(-x) &= -3(-x)^3 + 6(-x)^2 - 4(-x) - 1 \\ &= 3x^3 + 6x^2 + 4x - 1 \end{aligned}$$

No symmetry

$$\begin{aligned} f'(x) &= -9x^2 + 12x - 4 \\ &= -(3x - 2)^2 \end{aligned}$$

$$(3x - 2)^2 = 0$$

$$x = \frac{2}{3}$$

Critical number: $\frac{2}{3}$

$$f\left(\frac{2}{3}\right) = -3\left(\frac{2}{3}\right)^3 + 6\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) - 1 = -\frac{17}{9}$$

Critical point: $(\frac{2}{3}, -\frac{17}{9})$

$$f'(0) = -9(0)^2 + 12(0) - 4 = -4 < 0$$

$$f'(1) = -9(1)^2 + 12(1) - 4 = -1 < 0$$

No relative extremum at $(\frac{2}{3}, -\frac{17}{9})$

Decreasing on $(-\infty, \infty)$

$$\begin{aligned} f''(x) &= -18x + 12 \\ &= -6(3x - 2) \end{aligned}$$

$$3x - 2 = 0$$

$$x = \frac{2}{3}$$

Point of inflection at $(\frac{2}{3}, -\frac{17}{9})$

$$f''(0) = -18(0) + 12 = 12 > 0$$

$$f''(1) = -18(1) + 12 = -6 < 0$$

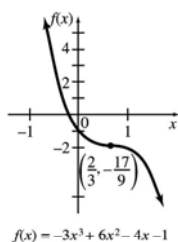
Concave upward on $(-\infty, \frac{2}{3})$

Concave downward on $(\frac{2}{3}, \infty)$

Point of inflection at $(\frac{2}{3}, -\frac{17}{9})$

y-intercept:

$$y = -3(0)^3 + 6(0)^2 - 4(0) - 1 = -1$$



6. $f(x) = x^3 - 6x^2 + 12x - 11$

Domain is $(-\infty, \infty)$.

$$\begin{aligned} f(-x) &= (-x)^3 - 6(-x)^2 + 12(-x) - 11 \\ &= -x^3 - 6x^2 - 12x - 11 \end{aligned}$$

No symmetry

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) \\ &= 3(x - 2)^2 \end{aligned}$$

Critical number: 2

Critical point: $(2, -3)$

$$f''(x) = 6x - 12$$

$$f''(0) = 6(0) - 12 = -12 < 0$$

$$f''(3) = 6(3) - 12 = 6 > 0$$

No relative extrema

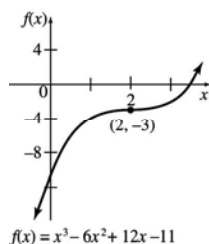
Increasing on $(-\infty, \infty)$

Point of inflection at $(2, -3)$

Concave upward on $(2, \infty)$; concave downward on $(-\infty, 2)$

y-intercept:

$$y = 0^3 - 6(0)^2 + 12(0) - 11 = -11$$



7. $f(x) = x^4 - 24x^2 + 80$

Domain is $(-\infty, \infty)$.

$$\begin{aligned} f(-x) &= (-x)^4 - 24(-x)^2 + 80 \\ &= x^4 - 24x^2 + 80 = f(x) \end{aligned}$$

The graph is symmetric about the y-axis.

$$f'(x) = 4x^3 - 48x$$

$$4x^3 - 48x = 0$$

$$4x(x^2 - 12) = 0$$

$$4x(x - 2\sqrt{3})(x + 2\sqrt{3}) = 0$$

Critical numbers: $-2\sqrt{3}$, 0, and $2\sqrt{3}$

Critical points: $(-2\sqrt{3}, -64)$, $(0, 80)$, and $(2\sqrt{3}, -64)$

$$f''(x) = 12x^2 - 48$$

$$f''(-2\sqrt{3}) = 12(-2\sqrt{3})^2 - 48 = 96 > 0$$

$$f''(0) = 12(0)^2 - 48 = -48 < 0$$

$$f''(2\sqrt{3}) = 12(2\sqrt{3})^2 - 48 = 96 > 0$$

Relative maximum at 0, relative minima at $-2\sqrt{3}$ and $2\sqrt{3}$

Increasing on $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, \infty)$

Decreasing on $(-\infty, -2\sqrt{3})$ and $(0, 2\sqrt{3})$

$$12x^2 - 48 = 0$$

$$12(x^2 - 4) = 0$$

$$x = \pm 2$$

Points of inflection at $(-2, 0)$ and $(2, 0)$

Concave upward on $(-\infty, -2)$ and $(2, \infty)$

Concave downward on $(-2, 2)$

$$x\text{-intercepts: } 0 = x^4 - 24x^2 + 80$$

Let $u = x^2$.

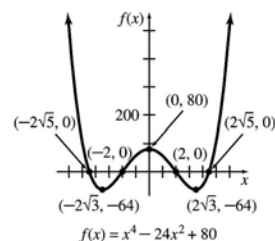
$$u^2 - 24u + 80 = 0$$

$$(u - 4)(u - 20) = 0$$

$$u = 4 \quad \text{or} \quad u = 20$$

$$x = \pm 2 \quad \text{or} \quad x = \pm 2\sqrt{5}$$

$$y\text{-intercept: } y = (0)^4 - 24(0)^2 + 80 = 80$$



8. $f(x) = -x^4 + 6x^2$

Domain is $(-\infty, \infty)$.

$$f(-x) = -(-x)^4 + 6(-x)^2 = -x^4 + 6x^2 = f(x)$$

The graph is symmetric about the y -axis.

$$f'(x) = -4x^3 + 12x$$

$$-4x^3 + 12x = 0$$

$$-4x(x^2 - 3) = 0$$

$$-4x(x - \sqrt{3})(x + \sqrt{3}) = 0$$

Critical numbers: $-\sqrt{3}$, 0, and $\sqrt{3}$

Critical points: $(-\sqrt{3}, 9)$, $(0, 0)$ and $(\sqrt{3}, 9)$

$$f''(x) = -12x^2 + 12$$

$$f''(-\sqrt{3}) = -12(\sqrt{3})^2 + 12 = -24 < 0$$

$$f''(0) = -12(0)^2 + 12 = 12 > 0$$

$$f''(\sqrt{3}) = -12(\sqrt{3})^2 + 12 = -24 < 0$$

Relative minimum at 0, relative maxima at $-\sqrt{3}$ and $\sqrt{3}$

Increasing on $(\infty, -\sqrt{3})$ and $(0, \sqrt{3})$

Decreasing on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$

$$-12x^2 + 12 = 0$$

$$-12(x^2 - 1) = 0$$

$$x = \pm 1$$

Points of inflection at $(-1, 5)$ and $(1, 5)$

Concave upward on $(-1, 1)$

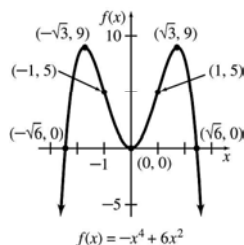
Concave downward on $(-\infty, -1)$ and $(1, \infty)$

$$x\text{-intercepts: } -x^4 + 6x^2 = 0$$

$$-x^2(x^2 - 6) = 0$$

$$x = 0 \text{ or } x = \pm\sqrt{6}$$

$$y\text{-intercept: } y = -(0)^4 + 6(0)^2 = 0$$



9. $f(x) = x^4 - 4x^3$

Domain is $(-\infty, \infty)$.

$$f(-x) = (-x)^4 - 4(-x)^3$$

$$= x^4 + 4x^3 \neq f(x) \text{ or } -f(x)$$

The graph is not symmetric about the y -axis or the origin.

$$f'(x) = 4x^3 - 12x^2$$

$$4x^3 - 12x^2 = 0$$

$$4x^2(x - 3) = 0$$

Critical numbers: 0 and 3

Critical points: $(0, 0)$ and $(3, -27)$

$$f''(x) = 12x^2 - 24x$$

$$f''(0) = 12(0)^2 - 24(0) = 0$$

$$f''(3) = 12(3)^2 - 24(3) = 36 > 0$$

Second derivative test fails for 0. Use first derivative test.

$$f'(-1) = 4(-1)^3 - 12(-1)^2 = -16 < 0$$

$$f'(1) = 4(1)^3 - 12(1)^2 = -8 < 0$$

Neither a relative minimum nor maximum at 0

Relative minimum at 3

Increasing on $(3, \infty)$

Decreasing on $(-\infty, 3)$

$$12x^2 - 24x = 0$$

$$12x(x - 2) = 0$$

$$x = 0 \text{ or } x = 2$$

Points of inflection at $(0, 0)$ and $(2, -16)$

Concave upward on $(-\infty, 0)$ and $(2, \infty)$

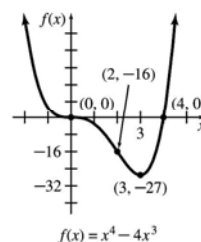
Concave downward on $(0, 2)$

$$x\text{-intercepts: } x^4 - 4x^3 = 0$$

$$x^3(x - 4) = 0$$

$$x = 0 \text{ or } x = 4$$

$$y\text{-intercepts: } y = (0)^4 - 4(0)^3 = 0$$



10. $f(x) = x^5 - 15x^3$

Domain is $(-\infty, \infty)$.

$$\begin{aligned} f(-x) &= (-x)^5 - 15(-x)^3 \\ &= -x^5 + 15x^3 = -f(x) \end{aligned}$$

The graph is symmetric about the origin.

$$f'(x) = 5x^4 - 45x^2 = 0$$

$$5x^2(x^2 - 9) = 0$$

$$5x^2(x + 3)(x - 3) = 0$$

Critical numbers: 0, -3, and 3

Critical points: (0, 0), (-3, 162), and (3, -162)

$$f''(x) = 20x^3 - 90x = 10x(x^2 - 9)$$

$$f''(0) = 0$$

$$f''(-3) = -270 < 0$$

$$f''(3) = 270 > 0$$

Relative maximum at -3

Relative minimum at 3

No relative extremum at 0

Increasing on $(-\infty, -3)$ and $(3, \infty)$

Decreasing on $(-3, 3)$

$$f''(x) = 20x^3 - 90x = 0$$

$$10x(2x^2 - 9) = 0$$

$$x = 0 \text{ or } x = \pm \frac{3}{\sqrt{2}}$$

Points of inflection at (0, 0), $\left(-\frac{3}{\sqrt{2}}, 100.23\right)$,

and $\left(\frac{3}{\sqrt{2}}, -100.23\right)$

Concave upward on $\left(-\frac{3}{\sqrt{2}}, 0\right)$ and $\left(\frac{3}{\sqrt{2}}, \infty\right)$

Concave downward on $\left(-\infty, -\frac{3}{\sqrt{2}}\right)$ and

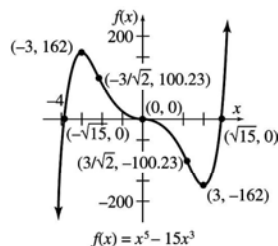
$\left(0, \frac{3}{\sqrt{2}}\right)$

$$x\text{-intercepts: } 0 = x^5 - 15x^3$$

$$0 = x^3(x^2 - 15)$$

$$x = 0, x = \pm\sqrt{15}$$

$$y\text{-intercepts: } y = 0^5 - 15(0)^3 = 0$$



11. $f(x) = 2x + \frac{10}{x}$
 $= 2x + 10x^{-1}$

Since $f(x)$ does not exist when $x = 0$, the domain is $(-\infty, 0) \cup (0, \infty)$.

$$\begin{aligned} f(-x) &= 2(-x) + 10(-x)^{-1} \\ &= -(2x + 10x^{-1}) \\ &= -f(x) \end{aligned}$$

The graph is symmetric about the origin.

$$f'(x) = 2 - 10x^{-2}$$

$$2 - \frac{10}{x^2} = 0$$

$$\frac{2(x^2 - 5)}{x^2} = 0$$

$$x = \pm\sqrt{5}$$

Critical numbers: $-\sqrt{5}$ and $\sqrt{5}$

Critical points: $(-\sqrt{5}, -4\sqrt{5})$ and $(\sqrt{5}, 4\sqrt{5})$

Test a point in the intervals $(-\infty, -\sqrt{5})$,

$(-\sqrt{5}, 0)$, $(0, \sqrt{5})$, and $(\sqrt{5}, \infty)$.

$$f'(-3) = 2 - 10(-3)^{-2} = \frac{8}{9} > 0$$

$$f'(-1) = 2 - 10(-1)^{-2} = -8 < 0$$

$$f'(1) = 2 - 10(1)^{-2} = -8 < 0$$

$$f'(3) = 2 - 10(3)^{-2} = \frac{8}{9} > 0$$

Relative maximum at $-\sqrt{5}$

Relative minimum at $\sqrt{5}$

Increasing on $(-\infty, -\sqrt{5})$ and $(\sqrt{5}, \infty)$

Decreasing on $(-\sqrt{5}, 0)$ and $(0, \sqrt{5})$

(Recall that $f(x)$ does not exist at $x = 0$.)

$$f''(x) = 20x^{-3} = \frac{20}{x^3}$$

$$f''(x) = \frac{20}{x^3} \text{ is never equal to zero.}$$

There are no inflection points.

Test a point in the intervals $(-\infty, 0)$ and $(0, \infty)$.

$$f''(-1) = \frac{20}{(-1)^3} = -20 < 0$$

$$f''(1) = \frac{20}{(1)^3} = 20 > 0$$

Concave upward on $(0, \infty)$

Concave downward on $(-\infty, 0)$

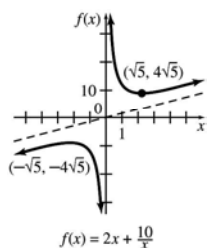
$f(x)$ is never zero, so there are no x -intercepts.

$f(x)$ does not exist for $x = 0$, so there is no y -intercept.

Vertical asymptote at $x = 0$

$y = 2x$ is an oblique asymptote.

$$\begin{aligned} f(-x) &= 2(-x) + 10(-x)^{-1} \\ &= -(2x + 10x^{-1}) \\ &= -f(x) \end{aligned}$$



$$\begin{aligned} 12. \quad f(x) &= 16x + \frac{1}{x^2} \\ &= 16x + x^{-2} \end{aligned}$$

Since $f(x)$ does not exist when $x = 0$, the domain is $(-\infty, 0) \cup (0, \infty)$.

$$f(-x) = 16(-x) + (-x)^{-2} = -16x + x^{-2}$$

The graph is not symmetric about the y -axis or the origin.

$$f'(x) = 16 - 2x^{-3} = 16 - \frac{2}{x^3}$$

$$16 - \frac{2}{x^3} = 0$$

$$\frac{2(8x^3 - 1)}{x^3} = 0$$

$$x = \frac{1}{2}$$

Critical number: $\frac{1}{2}$

Critical point: $(\frac{1}{2}, 12)$

Test a point in the intervals $(-\infty, 0)$, $(0, \frac{1}{2})$, and $(\frac{1}{2}, \infty)$.

$$f'(-1) = 16 - 2(-1)^{-3} = 18 > 0$$

$$f'\left(\frac{1}{3}\right) = 16 - 2\left(\frac{1}{3}\right)^{-3} = -38 < 0$$

$$f'(1) = 16 - 2(1)^{-3} = 14 > 0$$

Relative minimum at $\frac{1}{2}$

Increasing on $(-\infty, 0)$ and $(\frac{1}{2}, \infty)$

Decreasing on $(0, \frac{1}{2})$

$$f''(x) = 6x^{-4} = \frac{6}{x^4}$$

$$f''(x) = \frac{6}{x^4} > 0 \text{ and is never zero.}$$

There are no inflection points.

Concave upward on $(-\infty, 0)$ and $(0, \infty)$

$$x\text{-intercept: } 16x + x^{-2} = 0$$

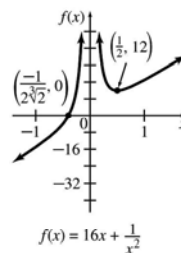
$$\frac{16x^3 + 1}{x^2} = 0$$

$$x = -\frac{1}{2\sqrt[3]{2}}$$

$f(x)$ does not exist for $x = 0$, so there is no y -intercept.

Vertical asymptote at $x = 0$

$y = 16x$ is an oblique asymptote.



$$13. \quad f(x) = \frac{-x + 4}{x + 2}$$

Since $f(x)$ does not exist when $x = -2$, the domain is $(-\infty, -2) \cup (-2, \infty)$.

$$f(-x) = \frac{-(-x) + 4}{(-x) + 2} = \frac{x + 4}{-x + 2}$$

The graph is not symmetric about the y -axis or the origin.

$$\begin{aligned} f'(-x) &= \frac{(x+2)(-1) - (-x+4)(1)}{(x+2)^2} \\ &= \frac{-6}{(x+2)^2} \end{aligned}$$

$f'(x) < 0$ and is never zero. $f'(x)$ fails to exist for $x = -2$.

No critical numbers; no relative extrema

Decreasing on $(-\infty, -2)$ and $(-2, \infty)$

$$f''(x) = \frac{12}{(x+2)^3}$$

$f''(x)$ fails to exist for $x = -2$.

No points of inflection

Test a point in the intervals $(-\infty, -2)$ and $(-2, \infty)$.

$$f'''(-3) = -12 < 0$$

$$f'''(-1) = 12 > 0$$

Concave upward on $(-2, \infty)$

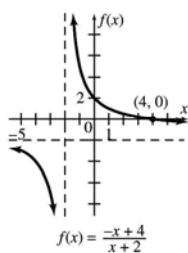
Concave downward on $(-\infty, -2)$

$$\begin{aligned} \text{x-intercept: } \frac{-x+4}{x+2} &= 0 \\ x &= 4 \end{aligned}$$

$$\text{y-intercept: } y = \frac{-0+4}{0+2} = 2$$

Vertical asymptote at $x = -2$

Horizontal asymptote at $y = -1$



14. $f(x) = \frac{3x}{x-2}$

Since $f(x)$ does not exist when $x = 2$, the domain is $(-\infty, 2) \cup (2, \infty)$.

$$f(-x) = \frac{3(-x)}{(-x)-2} = \frac{-3x}{-x-2}$$

The graph is not symmetric about the y -axis or the origin.

$$\begin{aligned} f'(x) &= \frac{(x-2)(3) - (3x)(1)}{(x-2)^2} \\ &= \frac{-6}{(x-2)^2} \end{aligned}$$

$f'(x) < 0$ is never zero. $f'(x)$ fails to exist for $x = 2$.

No critical numbers; no relative extrema

Decreasing on $(-\infty, 2)$ and $(2, \infty)$

$$f''(x) = \frac{12}{(x-2)^3}$$

$f''(x)$ fails to exist for $x = 2$.

No points of inflection

Test a point in the intervals $(-\infty, 2)$ and $(2, \infty)$.

$$f''(1) = -12 < 0$$

$$f''(3) = 12 > 0$$

Concave upward on $(2, \infty)$

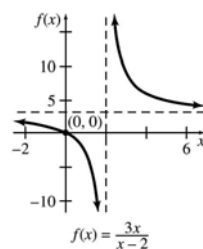
Concave downward on $(-\infty, 2)$

$$\begin{aligned} \text{x-intercept: } \frac{3x}{x-2} &= 0 \\ x &= 0 \end{aligned}$$

$$\text{y-intercept: } y = \frac{3(0)}{0-2} = 0$$

Vertical asymptote at $x = 2$

Horizontal asymptote at $y = 3$



15. $f(x) = \frac{1}{x^2 + 4x + 3}$

$$= \frac{1}{(x+3)(x+1)}$$

Since $f(x)$ does not exist when $x = -3$ and $x = -1$, the domain is $(-\infty, -3) \cup (-3, -1) \cup (-1, \infty)$.

$$f(-x) = \frac{1}{(-x)^2 + 4(-x) + 3} = \frac{1}{x^2 - 4x + 3}$$

The graph is not symmetric about the y -axis or the origin.

$$f'(x) = \frac{0 - (2x + 4)}{(x^2 + 4x + 3)^2} = \frac{-2(x + 2)}{[(x + 3)(x + 1)]^2}$$

Critical number: -2

Test a point in the intervals $(-\infty, -3)$,
 $(-3, -2)$, $(-2, -1)$, and $(-1, \infty)$.

$$f'(-4) = \frac{-2(-4 + 2)}{[(-4 + 3)(-4 + 1)]^2} = \frac{4}{9} > 0$$

$$f'\left(-\frac{5}{2}\right) = \frac{-2\left(-\frac{5}{2} + 2\right)}{\left[\left(-\frac{5}{2} + 3\right)\left(-\frac{5}{2} + 1\right)\right]^2} = \frac{16}{9} > 0$$

$$f'\left(-\frac{3}{2}\right) = \frac{-2\left(-\frac{3}{2} + 2\right)}{\left[\left(-\frac{3}{2} + 3\right)\left(-\frac{3}{2} + 1\right)\right]^2} = -\frac{16}{9} < 0$$

$$f'(0) = \frac{-2(0 + 2)}{[(0 + 3)(0 + 1)]^2} = -\frac{4}{9} < 0$$

$$f(-2) = \frac{1}{(-2 + 3)(-2 + 1)} = -1$$

Relative maximum at $(-2, -1)$

Increasing on $(-\infty, -3)$ and $(-3, -2)$

Decreasing on $(-2, -1)$ and $(-1, \infty)$

$$\begin{aligned} f''(x) &= \frac{(x^2 + 4x + 3)^2(-2) - (-2x - 4)(2)(x^2 + 4x + 3)(2x + 4)}{(x^2 + 4x + 3)^4} \\ &= \frac{-2(x^2 + 4x + 3)[(x^2 + 4x + 3) + (-2x - 4)(2x + 4)]}{(x^2 + 4x + 3)^4} \\ &= \frac{-2(x^2 + 4x + 3 - 4x^2 - 16x - 16)}{(x^2 + 4x + 3)^3} \\ &= \frac{-2(-3x^2 - 12x - 13)}{(x^2 + 4x + 3)^3} \\ &= \frac{2(3x^2 + 12x + 13)}{[(x + 3)(x + 1)]^3} \end{aligned}$$

Since $3x^2 + 12x + 13 = 0$ has no real solutions, there are no x -values where $f''(x) = 0$. $f''(x)$ does not exist where $x = -3$ and $x = -1$. Since $f(x)$ does not exist at these x -values, there are no points of inflection.

Test a point in the intervals $(-\infty, -3)$, $(-3, -1)$, and $(-1, \infty)$.

$$f''(-4) = \frac{2[3(-4)^2 + 12(-4) + 13]}{[(-4 + 3)(-4 + 1)]^3} = \frac{26}{27} > 0$$

$$f''(-2) = \frac{2[3(-2)^2 + 12(-2) + 13]}{[(-2 + 3)(-2 + 1)]^3} = -2 < 0$$

$$f''(0) = \frac{2[3(0)^2 + 12(0) + 13]}{[(0 + 3)(0 + 1)]^3} = \frac{26}{27} > 0$$

Concave upward on $(-\infty, -3)$ and $(-1, \infty)$

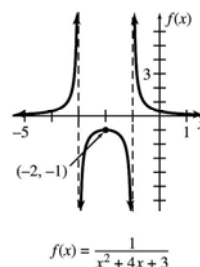
Concave downward on $(-3, -1)$

$f(x)$ is never zero, so there are no x -intercepts.

$$y\text{-intercept: } y = \frac{1}{(0 + 3)(0 + 1)} = \frac{1}{3}$$

Vertical asymptotes where $f(x)$ is undefined at $x = -3$ and $x = -1$.

Horizontal asymptote at $y = 0$



$$16. \quad f(x) = \frac{-8}{x^2 - 6x - 7} = \frac{-8}{(x + 1)(x - 7)}$$

Since $f(x)$ does not exist when $x = -1$ and $x = 7$, the domain is $(-\infty, -1) \cup (-1, 7) \cup (7, \infty)$.

$$f(-x) = \frac{8}{(-x)^2 - 6(-x) - 7} = \frac{8}{x^2 + 6x - 7}$$

The graph is not symmetric about the y -axis or the origin.

$$\begin{aligned} f'(x) &= \frac{0 - (-8)(2x - 6)}{(x^2 - 6x - 7)^2} \\ &= \frac{16(x - 3)}{[(x + 1)(x - 7)]^2} \end{aligned}$$

Critical number: 3

Test a point in the intervals $(-\infty, -1)$, $(-1, 3)$, $(3, 7)$, and $(7, \infty)$.

$$f'(-2) = \frac{16(-2-3)}{[(-2+1)(-2-7)]^2} = -\frac{80}{81} < 0$$

$$f'(0) = \frac{16(0-3)}{[(0+1)(0-7)]^2} = -\frac{48}{49} < 0$$

$$f'(6) = \frac{16(6-3)}{[(6+1)(6-7)]^2} = \frac{48}{49} > 0$$

$$f'(8) = \frac{16(8-3)}{[(8+1)(8-7)]^2} = \frac{80}{81} > 0$$

$$f(3) = \frac{-8}{(3+1)(3-7)} = \frac{1}{2}$$

Relative minimum at $(3, \frac{1}{2})$

Increasing on $(3, 7)$ and $(7, \infty)$

Decreasing on $(-\infty, -1)$ and $(-1, 3)$

$$\begin{aligned} f''(x) &= \frac{(x^2 - 6x - 7)^2(16) - 16(x-3)(2)(x^2 - 6x - 7)(2x-6)}{(x^2 - 6x - 7)^4} \\ &= \frac{16(x^2 - 6x - 7)[(x^2 - 6x - 7) - 4(x-3)(x-3)]}{(x^2 - 6x - 7)^4} \\ &= \frac{16(x^2 - 6x - 7 - 4x^2 + 24x - 36)}{(x^2 - 6x - 7)^3} \\ &= \frac{16(-3x^2 + 18x - 43)}{(x^2 - 6x - 7)^3} \\ &= \frac{-16(3x^2 - 18x + 43)}{[(x+1)(x-7)]^3} \end{aligned}$$

Since $3x^2 - 18x + 43 = 0$ has no real solutions, there are no x -values where $f''(x) = 0$.

$f''(x)$ does not exist where $x = -1$ and $x = 7$. Since $f(x)$ does not exist at these x -values, there are no points of inflection.

Test a point in the intervals $(-\infty, -1)$, $(-1, 7)$, and $(7, \infty)$.

$$f''(-2) = \frac{-16[3(-2)^2 - 18(-2) + 43]}{[(-2+1)(-2-7)]^3} = -\frac{1456}{729} < 0$$

$$f''(0) = \frac{-16[3(0)^2 - 18(0) + 43]}{[(0+1)(0-7)]^3} = \frac{688}{343} > 0$$

$$f''(8) = \frac{-16[3(8)^2 - 18(8) + 43]}{[(8+1)(8-7)]^3} = -\frac{1456}{729} < 0$$

Concave upward on $(-1, 7)$

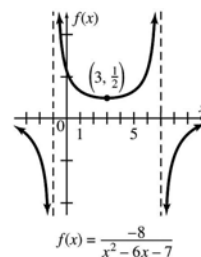
Concave downward on $(-\infty, -1)$ and $(7, \infty)$

$f(x)$ is never zero, so there are no x -intercepts.

$$y\text{-intercept: } y = \frac{-8}{(0+1)(0-7)} = \frac{8}{7}$$

Vertical asymptotes where $f(x)$ is undefined at $x = -1$ and $x = 7$.

Horizontal asymptote at $y = 0$



$$17. \quad f(x) = \frac{x}{x^2 + 1}$$

Domain is $(-\infty, \infty)$

$$f(-x) = \frac{-x}{(-x)^2 + 1} = -\frac{x}{x^2 + 1} = -f(x)$$

The graph is symmetric about the origin.

$$\begin{aligned} f'(x) &= \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} \\ &= \frac{1 - x^2}{(x^2 + 1)^2} \\ 1 - x^2 &= 0 \end{aligned}$$

Critical numbers: 1 and -1

Critical points: $(1, \frac{1}{2})$ and $(-1, -\frac{1}{2})$

$$\begin{aligned} f''(x) &= \frac{(x^2 + 1)^2(-2x) - (1 - x^2)(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} \\ &= \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2 + 1)^3} = \frac{2x^3 - 6x}{(x^2 + 1)^3} \end{aligned}$$

$$f''(1) = -\frac{1}{2} < 0$$

$$f''(-1) = \frac{1}{2} > 0$$

Relative maximum at 1

Relative minimum at -1

Increasing on $(-1, 1)$

Decreasing on $(-\infty, -1)$ and $(1, \infty)$

$$f''(x) = \frac{2x^3 - 6x}{(x^2 + 1)^3} = 0$$

$$2x^3 - 6x = 0$$

$$2x(x^2 - 3) = 0$$

$$x = 0, x = \pm\sqrt{3}$$

Inflection points at $(0, 0)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$ and $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$

Concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$

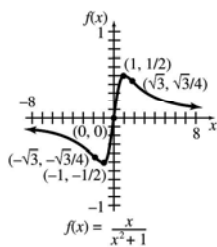
Concave downward on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$

$$x\text{-intercept: } 0 = \frac{x}{x^2 + 1}$$

$$0 = x$$

$$y\text{-intercept: } y = \frac{0}{0^2 + 1} = 0$$

Horizontal asymptote at $y = 0$



18. $f(x) = \frac{1}{x^2 + 4}$

Domain is $(-\infty, \infty)$.

$$f(-x) = \frac{1}{(-x)^2 + 4} = \frac{1}{x^2 + 4} = f(x)$$

The graph is symmetric about the y-axis.

$$f'(x) = \frac{-2x}{(x^2 + 4)^2}$$

Critical number: 0

Critical point: $\left(0, \frac{1}{4}\right)$

$$\begin{aligned} f''(x) &= \frac{(x^2 + 4)^2(-2) - (-2x)(2)(x^2 + 4)(2x)}{(x^2 + 4)^4} \\ &= \frac{-2(x^2 + 4)[(x^2 + 4) + (-2x)(2x)]}{(x^2 + 4)^4} \\ &= \frac{-2(x^2 + 4 - 4x^2)}{(x^2 + 4)^3} \\ &= \frac{-2(-3x^2 + 4)}{(x^2 + 4)^3} \\ &= \frac{2(3x^2 - 4)}{(x^2 + 4)^3} \end{aligned}$$

$$f''(0) = \frac{2[3(0)^2 - 4]}{[(0)^2 + 4]^3} = -\frac{1}{8} < 0$$

Relative maximum at $\left(0, \frac{1}{4}\right)$

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

$$f(0) = \frac{1}{(0)^2 + 4} = \frac{1}{4}$$

$$f''(x) = 0 \text{ when}$$

$$2(3x^2 - 4) = 0$$

$$x^2 = \frac{4}{3}$$

$$x = \pm \frac{2}{\sqrt{3}}$$

Inflection points at $\left(-\frac{2}{\sqrt{3}}, \frac{3}{16}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{3}{16}\right)$

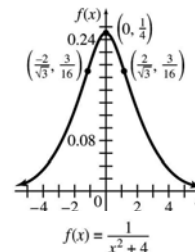
Concave upward on $(-\infty, -\frac{2}{\sqrt{3}})$ and $(\frac{2}{\sqrt{3}}, \infty)$

Concave downward on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

$f(x)$ is never zero, so there are no x-intercepts.

$$y\text{-intercept: } y = \frac{1}{0^2 + 4} = \frac{1}{4}$$

Horizontal asymptote at $y = 0$



19. $f(x) = \frac{1}{x^2 - 9}$

$$= \frac{1}{(x + 3)(x - 3)}$$

Since $f(x)$ does not exist when $x = -3$ and $x = 3$, the domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

$$f(-x) = \frac{1}{(-x)^2 - 9} = \frac{1}{x^2 - 9} = f(x)$$

The graph is symmetric about the y-axis.

$$f'(x) = \frac{-2x}{(x^2 - 9)^2}$$

Critical number: 0

Critical point: $(0, -\frac{1}{9})$

Test a point in the intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$, and $(3, \infty)$.

$$f'(-4) = \frac{-2(-4)}{[(-4)^2 - 9]^2} = \frac{8}{49} > 0$$

$$f'(-1) = \frac{-2(-1)}{[(-1)^2 - 9]^2} = \frac{1}{32} > 0$$

$$f'(1) = \frac{-2(1)}{[(1)^2 - 9]^2} = -\frac{1}{32} < 0$$

$$f'(4) = \frac{-2(4)}{[(4)^2 - 9]^2} = -\frac{8}{49} < 0$$

Relative maximum at $(0, -\frac{1}{9})$

Increasing on $(-\infty, -3)$ and $(-3, 0)$

Decreasing on $(0, 3)$ and $(3, \infty)$

$$\begin{aligned} f''(x) &= \frac{(x^2 - 9)^2(-2) - (-2x)(2)(x^2 - 9)(2x)}{(x^2 - 9)^4} \\ &= \frac{-2(x^2 - 9)[(x^2 - 9) + (-2x)(2x)]}{(x^2 - 9)^4} \\ &= \frac{-2(x^2 - 9 - 4x^2)}{(x^2 - 9)^3} \\ &= \frac{-2(-3x^2 - 9)}{(x^2 - 9)^3} \\ &= \frac{6(x^2 + 3)}{[(x + 3)(x - 3)]^3} \end{aligned}$$

Since $x^2 + 3 = 0$ has no solutions, there are no x -values where $f''(x) = 0$. $f''(x)$ does not exist where $x = -3$ and $x = 3$. Since $f(x)$ does not exist at these x -values, there are no points of inflection.

Test a point in the intervals $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$.

$$f''(-4) = \frac{6[(-4)^2 + 3]}{[(-4 + 3)(-4 - 3)]^3} = \frac{114}{343} > 0$$

$$f''(0) = \frac{6[(0)^2 + 3]}{[(0 + 3)(0 - 3)]^3} = -\frac{2}{81} < 0$$

$$f''(4) = \frac{6[(4)^2 + 3]}{[(4 + 3)(4 - 3)]^3} = \frac{114}{343} > 0$$

Concave upward on $(-\infty, -3)$ and $(3, \infty)$

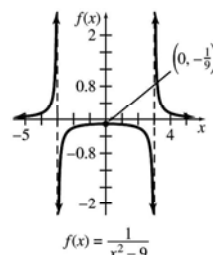
Concave downward on $(-3, 3)$

$f(x)$ is never zero, so there are no x -intercepts.

$$y\text{-intercept: } y = \frac{1}{0^2 - 9} = -\frac{1}{9}$$

Vertical asymptotes where $f(x)$ is undefined at $x = -3$ and $x = 3$.

Horizontal asymptote at $y = 0$



$$\begin{aligned} 20. \quad f(x) &= \frac{-2x}{x^2 - 4} \\ &= \frac{-2x}{(x + 2)(x - 2)} \end{aligned}$$

Since $f(x)$ does not exist when $x = -2$ and $x = 2$, the domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

$$f(-x) = \frac{-2(-x)}{(-x)^2 - 4} = \frac{2x}{x^2 - 4} = -f(x)$$

The graph is symmetric about the origin.

$$\begin{aligned} f'(x) &= \frac{(x^2 - 4)(-2) - (-2x)(2x)}{(x^2 - 4)^2} \\ &= \frac{2x^2 + 8}{(x^2 - 4)^2} \end{aligned}$$

$f'(x) > 0$ and is never zero.

No critical points, no relative extrema

Increasing on $(-\infty, -2)$, and $(-2, 2)$ and $(2, \infty)$

$$\begin{aligned} f''(x) &= \frac{(x^2 - 4)^2(4x) - (2x^2 + 8)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4} \\ &= \frac{4x(x^2 - 4)[(x^2 - 4) - (2x^2 + 8)]}{(x^2 - 4)^4} \\ &= \frac{4x(-x^2 - 12)}{(x^2 - 4)^3} \\ &= \frac{-4x(x^2 + 12)}{(x^2 - 4)^3} \\ f''(x) &= 0 \text{ when } \end{aligned}$$

$$-4x(x^2 + 12) = 0$$

$$x = 0$$

Test a point in the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$.

$$f''(-3) = \frac{-4(-3)[(-3)^2 + 12]}{[(-3)^2 - 4]^3} = \frac{252}{125} > 0$$

$$f''(-1) = \frac{-4(-1)[(-1)^2 + 12]}{[(-1)^2 - 4]^3} = -\frac{52}{27} < 0$$

$$f''(1) = \frac{-4(1)[(1)^2 + 12]}{[(1)^2 - 4]^3} = \frac{52}{27} > 0$$

$$f''(3) = \frac{-4(3)[(3)^2 + 12]}{[(3)^2 - 4]^3} = -\frac{252}{125} < 0$$

Inflection at $(0, 0)$

Concave upward on $(-\infty, -2)$ and $(0, 2)$

Concave downward on $(-2, 0)$ and $(2, \infty)$

$$x\text{-intercept: } \frac{-2x}{x^2 - 4} = 0$$

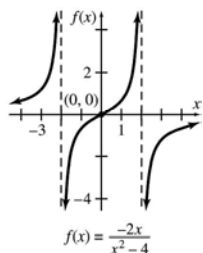
$$-2x = 0$$

$$x = 0$$

$$y\text{-intercept: } y = \frac{-2(0)}{0^2 - 4} = 0$$

Vertical asymptotes where $f(x)$ is undefined at $x = -2$ and $x = 2$.

Horizontal asymptote at $y = 0$



21. $f(x) = x \ln |x|$

The domain of this function is $(-\infty, 0) \cup (0, \infty)$.

$$f(-x) = -x \ln |-x|$$

$$= -x \ln |x| = -f(x)$$

The graph is symmetric about the origin.

$$f'(x) = x \cdot \frac{1}{x} + \ln |x|$$

$$= 1 + \ln |x|$$

$$f'(x) = 0 \text{ when}$$

$$0 = 1 + \ln |x|$$

$$-1 = \ln |x|$$

$$e^{-1} = |x|$$

$$x = \pm \frac{1}{e} \approx \pm 0.37.$$

Critical numbers: $\pm \frac{1}{e} \approx \pm 0.37$.

$$f'(-1) = 1 + \ln |-1| = 1 > 0$$

$$f'(-0.1) = 1 + \ln |-0.1| \approx -1.3 < 0$$

$$f'(0.1) = 1 + \ln |0.1| \approx -1.3 < 0$$

$$f'(1) = 1 + \ln |1| = 1 > 0$$

$$f\left(\frac{1}{e}\right) = \frac{1}{e} \ln \left|\frac{1}{e}\right| = -\frac{1}{e}$$

$$f\left(-\frac{1}{e}\right) = -\frac{1}{e} \ln \left|-\frac{1}{e}\right| = \frac{1}{e}$$

Relative maximum of $\left(-\frac{1}{e}, \frac{1}{e}\right)$; relative minimum of $\left(\frac{1}{e}, -\frac{1}{e}\right)$.

Increasing on $\left(-\infty, -\frac{1}{e}\right)$ and $\left(\frac{1}{e}, \infty\right)$ and decreasing on $\left(-\frac{1}{e}, 0\right)$ and $\left(0, \frac{1}{e}\right)$.

$$f''(x) = \frac{1}{x}$$

$$f''(-1) = \frac{1}{-1} = -1 < 0$$

$$f''(1) = \frac{1}{1} = 1 > 0$$

Concave downward on $(-\infty, 0)$;

Concave upward on $(0, \infty)$.

There is no y -intercept.

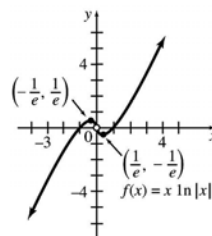
$$x\text{-intercept: } 0 = x \ln |x|$$

$$x = 0 \text{ or } \ln |x| = 0$$

$$|x| = e^0 = 1$$

$$x = \pm 1$$

Since 0 is not in the domain, the only x -intercepts are -1 and 1 .



22. $f(x) = x - \ln |x|$

Domain is $(-\infty, 0) \cup (0, \infty)$

$$\begin{aligned} f(-x) &= -x - \ln |-x| \\ &= -x - \ln |x| \end{aligned}$$

The graph has no symmetry.

$$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$$

Critical number: $x = 1$

(Recall that $f(x)$ does not exist at $x = 0$.)

$$f'(-1) = \frac{-1-1}{-1} = 2 > 0$$

$$f'\left(\frac{1}{2}\right) = \frac{\frac{1}{2}-1}{\frac{1}{2}} = -1 < 0$$

$$f'(2) = \frac{2-1}{2} = \frac{1}{2} > 0$$

$$f(1) = 1 - \ln |1| = 1$$

Relative minimum at $(1, 1)$

Increasing on $(-\infty, 0)$ and $(1, \infty)$

Decreasing on $(0, 1)$

$$f''(x) = \frac{x(1) - (x-1)(1)}{x^2} = \frac{1}{x^2}$$

No inflection point.

$f''(x)$ is positive everywhere it is defined,

Concave upward on $(-\infty, 0)$ and $(0, \infty)$

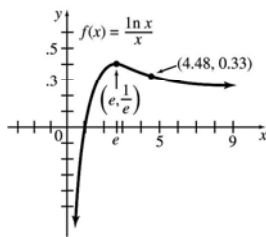
There is no y -intercept

x -intercept: $x - \ln |x| = 0$

$$x = \ln |x|$$

$$x \approx -0.567$$

Vertical asymptote at $x = 0$



23. $f(x) = \frac{\ln x}{x}$

Note that the domain of this function is $(0, \infty)$.

$$f(-x) = \frac{\ln(-x)}{-x} \text{ does not exist when } x \geq 0,$$

no symmetry.

$$\begin{aligned} f'(x) &= \frac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2} \\ &= \frac{1 - \ln x}{x^2} \end{aligned}$$

Critical numbers:

$$1 - \ln x = 0$$

$$1 = \ln x$$

$$e^1 = x$$

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Critical points: $\left(e, \frac{1}{e}\right)$

$$f'(1) = \frac{1 - \ln 1}{1^2} = \frac{1}{1} = 1 > 0$$

$$f'(3) = \frac{1 - \ln 3}{3^2} = -0.01 < 0$$

There is a relative maximum at $\left(e, \frac{1}{e}\right)$.

The function is increasing on $(0, e)$ and decreasing on (e, ∞) .

$$\begin{aligned} f''(x) &= \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{x^4} \\ &= \frac{-x - 2x(1 - \ln x)}{x^4} \\ &= \frac{-x[1 + 2(1 - \ln x)]}{x^4} \\ &= \frac{-(1 + 2 - 2\ln x)}{x^3} \\ &= \frac{-3 + 2\ln x}{x^3} \end{aligned}$$

$$f''(x) = 0 \text{ when } -3 + 2\ln x = 0$$

$$2\ln x = 3$$

$$\ln x = \frac{3}{2} = 1.5$$

$$x = e^{1.5} \approx 4.48.$$

$$f''(1) = \frac{-3 + 2\ln 1}{1^3} = -3 < 0$$

$$f''(5) = \frac{-3 + 2\ln 5}{5^3} \approx 0.0018 > 0$$

Inflection point at $\left(e^{1.5}, \frac{1.5}{e^{1.5}}\right) \approx (4.48, 0.33)$

Concave downward on $(0, e^{1.5})$; concave upward on $(e^{1.5}, \infty)$

$$f(e^{1.5}) = \frac{\ln e^{1.5}}{e^{1.5}} = \frac{1.5}{e^{1.5}}$$

$$= \frac{3}{2e^{1.5}} \approx 0.33$$

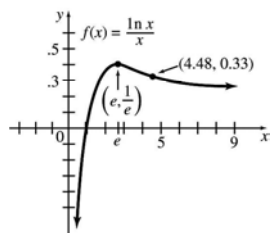
Since $x \neq 0$, there is no y -intercept.

x -intercept: $f(x) = 0$ when $\ln x = 0$

$$x = e^0 = 1$$

Vertical asymptote at $x = 0$

Horizontal asymptote at $y = 0$



24. $f(x) = \frac{\ln x^2}{x^2}$

Domain is $(-\infty, 0) \cup (0, \infty)$

$$f(-x) = \frac{\ln(-x)^2}{(-x)^2}$$

$$= \frac{\ln x^2}{x^2} = f(x)$$

The graph is symmetric about the y -axis.

$$f'(x) = \frac{x^2 \cdot \frac{1}{x^2} \cdot 2x - (\ln x^2)2x}{x^4}$$

$$= \frac{2x(1 - \ln x^2)}{x^4} = \frac{2(1 - \ln x^2)}{x^3}$$

$$f'(x) = 0 \text{ when } 1 - \ln x^2 = 0$$

$$\ln x^2 = 1$$

$$x^2 = e$$

$$x = \pm\sqrt{e}.$$

Critical numbers: $-\sqrt{e}, \sqrt{e}$

Critical points: $(-\sqrt{e}, \frac{1}{e}), (\sqrt{e}, \frac{1}{e})$

$$f''(x) = \frac{x^3 \left(-2 \cdot \frac{1}{x^2} \cdot 2x \right) - 2(1 - \ln x^2)3x^2}{x^6}$$

$$= \frac{x^2[-4 - 6(1 - \ln x^2)]}{x^6}$$

$$= \frac{-10 + 6 \ln x^2}{x^4}$$

$$f''(x)(\pm\sqrt{e}) = \frac{-10 + 6 \ln e}{e^2} = \frac{-4}{e^2} < 0$$

There are relative maxima at $(\pm\sqrt{e}, \frac{1}{e})$.

Increasing on $(-\infty, -\sqrt{e}) \cup (0, \sqrt{e})$

Decreasing on $(-\sqrt{e}, 0) \cup (\sqrt{e}, \infty)$

$$f''(x) = 0 \text{ when } -10 + 6 \ln x^2 = 0$$

$$6 \ln x^2 = 10$$

$$\ln x^2 = \frac{5}{3}$$

$$x^2 = e^{5/3}$$

$$x = \pm\sqrt{e^{5/3}} = \pm e^{5/6}$$

$$f''(3) = \frac{-10 + 6 \ln 9}{3^4} = 0.0393 > 0$$

$$f''(1) = \frac{-10 + 6 \ln 1^2}{1^4} = -10 < 0$$

$$f''(-1) = \frac{-10 + 6 \ln(-1)^2}{(-1)^4} = -10 < 0$$

$$f''(-3) = \frac{-10 + 6 \ln(-3)^2}{(-3)^4} = 0.0393 > 0$$

Inflection points at $(\pm e^{5/6}, \frac{5}{3e^{5/3}})$

Concave upward on $(-\infty, -e^{5/6})$ and

$(e^{5/6}, \infty)$

Concave downward on $(-e^{5/6}, 0)$ and $(0, e^{5/6})$

There is a vertical asymptote at $x = 0$.

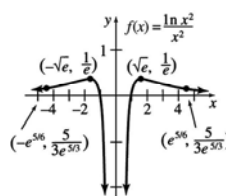
There is no y -intercept.

There are x -intercepts when $f(x) = 0$:

$$\ln x^2 = 0$$

$$x^2 = 1$$

$$x = \pm 1.$$



25. $f(x) = xe^{-x}$

Domain is $(-\infty, \infty)$.

$$f(-x) = -xe^x$$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= -xe^{-x} + e^{-x} \\ &= e^{-x}(1 - x) \end{aligned}$$

$$\begin{aligned} f'(x) = 0 \text{ when } e^{-x}(1 - x) &= 0 \\ x &= 1 \end{aligned}$$

Critical numbers: 1

Critical points: $(1, \frac{1}{e})$

$$f'(0) = e^{-0}(1 - 0) = 1 > 0$$

$$f'(2) = e^{-2}(1 - 2) = -\frac{1}{e^2} < 0$$

Relative maximum at $(1, \frac{1}{e})$

Increasing on $(-\infty, 1)$; decreasing on $(1, \infty)$

$$\begin{aligned} f''(x) &= e^{-x}(-1) + (1 - x)(-e^{-x}) \\ &= -e^{-x}(1 + 1 - x) \\ &= -e^{-x}(2 - x) \end{aligned}$$

$$\begin{aligned} f'' = 0 \text{ when } -e^{-x}(2 - x) &= 0 \\ x &= 2. \end{aligned}$$

$$f''(0) = -e^{-0}(2 - 0) = -2 < 0$$

$$f''(3) = -e^{-3}(2 - 3) = \frac{1}{e^3} > 0$$

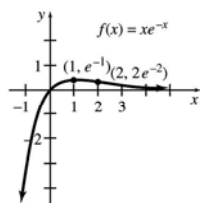
Inflection point at $(2, \frac{2}{e^2})$

Concave downward on $(-\infty, 2)$, concave upward on $(2, \infty)$

$$\begin{aligned} x\text{-intercept: } 0 &= xe^{-x} \\ x &= 0 \end{aligned}$$

$$y\text{-intercept: } y = 0 \cdot e^{-0} = 0$$

Horizontal asymptote at $y = 0$



26. $f(x) = x^2e^{-x}$

Domain is $(-\infty, \infty)$

$$\begin{aligned} f(-x) &= (-x)^2e^x \\ &= x^2e^x \end{aligned}$$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= x^2e^{-x}(-1) + e^{-x}(2x) \\ &= xe^{-x}(-x + 2) \end{aligned}$$

$$\begin{aligned} \text{Critical numbers } x = 0 \text{ or } -x + 2 &= 0 \\ x &= 2. \end{aligned}$$

$$f'(-1) = (-1)e^{-(-1)}[-(-1) + 2] = -3e < 0$$

$$f'(1) = (1)e^{-1}(-1 + 2) = \frac{1}{e} > 0$$

$$f'(3) = 3e^{-3}(-3 + 2) = -\frac{3}{e^3} < 0$$

The function is decreasing on $(-\infty, 0)$ and $(2, \infty)$ and increasing on $(0, 2)$.

$$f(0) = 0^2e^{-0} = 0$$

$$f(2) = 2^2e^{-2} = \frac{4}{e^2}$$

Relative minimum at $(0, 0)$; relative maximum at

$$\left(2, \frac{4}{e^2}\right)$$

$$\begin{aligned} f''(x) &= xe^{-x}(-1) + (-x + 2)(-xe^{-x} + e^{-x}) \\ &= -xe^{-x} + x^2e^{-x} - xe^{-x} - 2xe^{-x} + 2e^{-x} \\ &= x^2e^{-x} - 4xe^{-x} + 2e^{-x} \\ &= e^{-x}(x^2 - 4x + 2) \end{aligned}$$

$$f''(x) = 0 \text{ when}$$

$$x^2 - 4x + 2 = 0$$

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2} \\ &= \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}. \end{aligned}$$

$$x = 2 + \sqrt{2} \approx 3.4 \text{ or } 2 - \sqrt{2} \approx 0.6$$

$$f''(0) = e^{-0}[0^2 - 4(0) + 2] = 2 > 0$$

$$f''(1) = e^{-1}(1^2 - 4(1) + 2) = -\frac{1}{e} < 0$$

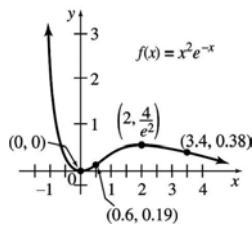
$$f''(4) = e^{-4}[4^2 - 4(4) + 2] = \frac{2}{e^4} > 0$$

Inflection points at $(0.6, 0.19)$ and $(3.4, 0.38)$

Concave upward on $(-\infty, 0.6)$ and $(3.4, \infty)$; concave downward on $(0.6, 3.4)$

$$y\text{-intercept: } y = 0^2e^{-0} = 0$$

x -intercept: 0



27. $f(x) = (x-1)e^{-x}$

Domain is $(-\infty, \infty)$

$$f(-x) = (-x-1)e^x$$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= -(x-1)e^{-x} + e^{-x}(1) \\ &= e^{-x}[-(x-1) + 1] \\ &= e^{-x}(2-x) \end{aligned}$$

$$f'(x) = 0 \text{ when } e^{-x}(2-x) = 0$$

$$x = 2.$$

Critical number: 2

Critical point: $\left(2, \frac{1}{e^2}\right)$

$$\begin{aligned} f''(x) &= -e^{-x} + (2-x)(-e^{-x}) \\ &= -e^{-x}[1 + (2-x)] \\ &= -e^{-x}(3-x) \end{aligned}$$

$$f''(2) = -e^{-2}(3-2) = \frac{-1}{e^2} < 0$$

Relative maximum at $\left(2, \frac{1}{e^2}\right)$

$$f'(0) = e^{-0}(2-0) = 2 > 0$$

$$f'(3) = e^{-3}(2-3) = \frac{-1}{e^3} < 0$$

Increasing on $(-\infty, 2)$; decreasing on $(2, \infty)$.

$$f''(x) = 0 \text{ when } -e^{-x}(3-x) = 0$$

$$x = 3.$$

$$f''(0) = -e^{-0}(3-0) = -3 < 0$$

$$f''(4) = -e^{-4}(3-4) = \frac{1}{e^4} > 0$$

Inflection point at $\left(3, \frac{2}{e^3}\right)$

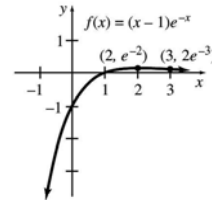
Concave downward on $(-\infty, 3)$; concave upward on $(3, \infty)$

$$f(3) = (3-1)e^{-3} = \frac{2}{e^3}$$

$$\begin{aligned} y\text{-intercept: } y &= (0-1)e^{-0} \\ &= (-1)(1) = -1 \end{aligned}$$

$$\begin{aligned} x\text{-intercept: } 0 &= (x-1)e^{-x} \\ x-1 &= 0 \\ x &= 1 \end{aligned}$$

Horizontal asymptote at $y = 0$



28. $f(x) = e^x + e^{-x}$

Domain is $(-\infty, \infty)$

$$f(-x) = e^{-x} + e^x = f(x)$$

The graph has symmetry about the y -axis.

$$f'(x) = e^x - e^{-x}$$

Critical numbers:

$$e^x - e^{-x} = 0$$

$$e^x - \frac{1}{e^x} = 0$$

$$\frac{e^{2x} - 1}{e^x} = 0$$

$$e^{2x} - 1 = 0$$

$$e^{2x} = 1$$

$$2x = \ln 1$$

$$2x = 0$$

$$x = 0$$

The only critical number is 0.

The only critical point is $(0, 2)$.

$$f'(-1) = e^{-1} - e^{-(-1)} = \frac{1}{e} - e \approx -2.35 < 0$$

$$f'(1) = e^1 - e^{-1} \approx 2.35 > 0$$

Relative minimum at $(0, 2)$

Decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$

$$f''(x) = e^x + e^{-x} = \frac{e^{2x} + 1}{e^x}$$

Since $f''(x) \neq 0$ ($e^{2x} + 1$ is always positive), there is no inflection point.

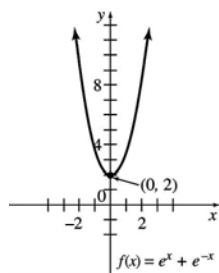
$$f''(0) = e^0 + e^{-0} = 2 > 0$$

The entire graph is concave upward on $(-\infty, \infty)$.

y-intercept: $y = e^0 + e^{-0} = 1 + 1 = 2$

The y-intercept is 2.

There is no x-intercept.



29. $f(x) = x^{2/3} - x^{5/3}$

Domain is $(-\infty, \infty)$.

$$f(-x) = x^{2/3} + x^{5/3}$$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} \\ &= \frac{2 - 5x}{3x^{1/3}} \end{aligned}$$

$$f'(x) = 0 \text{ when } 2 - 5x = 0$$

$$\text{Critical number: } x = \frac{2}{5}$$

$$\begin{aligned} f\left(\frac{2}{5}\right) &= \left(\frac{2}{5}\right)^{2/3} - \left(\frac{2}{5}\right)^{5/3} \\ &= \frac{3 \cdot 2^{2/3}}{5^{5/3}} \approx 0.326 \end{aligned}$$

Critical point: $(0.4, 0.326)$

$$\begin{aligned} f''(x) &= \frac{3x^{1/3}(-5) - (2 - 5x)(\frac{1}{3})x^{-2/3}}{(3x^{1/3})^2} \\ &= \frac{-15x^{1/3} - (2 - 5x)x^{-2/3}}{9x^{2/3}} \\ &= \frac{-15x - (2 - 5x)}{9x^{4/3}} \\ &= \frac{-10x - 2}{9x^{4/3}} \end{aligned}$$

$$\begin{aligned} f''\left(\frac{2}{5}\right) &= \frac{-10\left(\frac{2}{5}\right) - 2}{9\left(\frac{2}{5}\right)^{4/3}} \\ &\approx -2.262 < 0 \end{aligned}$$

Relative maximum at $\left(\frac{2}{5}, \frac{3 \cdot 2^{2/3}}{5^{5/3}}\right) \approx (0.4, 0.326)$

$f'(x)$ does not exist when $x = 0$

Since $f''(0)$ is undefined, use the first derivative test.

$$f'(-1) = \frac{2 - 5(-1)}{3(-1)^{1/3}} = \frac{7}{-3} < 0$$

$$f'\left(\frac{1}{8}\right) = \frac{2 - 5\left(\frac{1}{8}\right)}{3\left(\frac{1}{8}\right)^{1/3}} = \frac{11}{12} > 0$$

$$f'(1) = \frac{2 - 5}{3 \cdot 1^{1/3}} = -1 < 0$$

Relative minimum at $(0, 0)$

f increases on $\left(0, \frac{2}{5}\right)$.

f decreases on $(-\infty, 0)$ and $\left(\frac{2}{5}, \infty\right)$.

$$\begin{aligned} f''(x) = 0 \text{ when } -10x - 2 &= 0 \\ x &= -\frac{1}{5} \end{aligned}$$

$$\begin{aligned} f''(x) \text{ undefined when } 9x^{4/3} &= 0 \\ x &= 0 \end{aligned}$$

$$f''(-1) = \frac{-10(-1) - 2}{9(-1)^{4/3}} = \frac{8}{9} > 0$$

$$f''\left(-\frac{1}{8}\right) = \frac{-10\left(-\frac{1}{8}\right) - 2}{9\left(-\frac{1}{8}\right)^{4/3}} = -\frac{4}{3} < 0$$

$$f''(1) = \frac{-10(1) - 2}{9(1)^{4/3}} = -\frac{4}{3} < 0$$

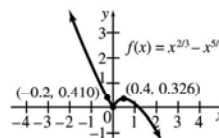
Concave upward on $(-\infty, -\frac{1}{5})$

Concave upward on $(-\frac{1}{5}, \infty)$

Inflection point at $\left(-\frac{1}{5}, \frac{6}{5^{5/3}}\right) \approx (-0.2, 0.410)$

$$\text{y-intercept: } y = 0^{2/3} - 0^{5/3} = 0$$

$$\begin{aligned} \text{x-intercept: } 0 &= x^{2/3} - x^{5/3} \\ &= x^{2/3}(1 - x) \\ x &= 0 \text{ or } x = 1 \end{aligned}$$



30. $f(x) = x^{1/3} + x^{4/3}$

Domain is $(-\infty, \infty)$.

The graph has no symmetry.

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3} + \frac{4}{3}x^{1/3} \\ &= \frac{1 + 4x}{3x^{2/3}} \end{aligned}$$

$$f'(x) = 0 \text{ when } 1 + 4x = 0$$

$$x = -\frac{1}{4}$$

Critical number: $x = -\frac{1}{4}$ and $x = 0$

$$\begin{aligned} f\left(-\frac{1}{4}\right) &= \left(-\frac{1}{4}\right)^{1/3} + \left(-\frac{1}{4}\right)^{4/3} \\ &= -\frac{3}{4^{4/3}} \approx -0.472 \end{aligned}$$

Critical points: $\left(-\frac{1}{4}, -0.472\right), (0, 0)$

$$\begin{aligned} f''(x) &= \frac{3x^{2/3}(4) - (1 + 4x)3\left(\frac{2}{3}\right)x^{-1/3}}{(3x^{2/3})^2} \\ &= \frac{12x^{2/3} - 2(1 + 4x)x^{-1/3}}{9x^{4/3}} \\ &= \frac{12x - 2(1 + 4x)}{9x^{5/3}} \\ &= \frac{4x - 2}{9x^{5/3}} \end{aligned}$$

$$\begin{aligned} f''\left(-\frac{1}{4}\right) &= \frac{4\left(-\frac{1}{4}\right) - 2}{9\left(-\frac{1}{4}\right)^{5/3}} = \frac{4^{5/3}}{3} \\ &\approx 3.360 > 0 \end{aligned}$$

Relative minimum at $\left(-\frac{1}{4}, -\frac{3}{4^{4/3}}\right)$
 $\approx (-0.25, -0.472)$

$$f'(x) = \frac{1 + 4x}{3x^{2/3}} \text{ undefined at } x = 0.$$

Test sign of $f'(x)$ on intervals defined by

$$x = -\frac{1}{4}, x = 0.$$

$$f'(-1) = \frac{-3}{3} = -1 < 0$$

$$f'\left(-\frac{1}{8}\right) = \frac{1 - \frac{1}{2}}{\frac{3}{4}} > 0$$

$$f'(1) = \frac{5}{3} > 0$$

f increases on $\left(-\frac{1}{4}, \infty\right)$, f decreases on $\left(-\infty, -\frac{1}{4}\right)$

No extreme point at $(0, f(0)) = (0, 0)$

$$f''(x) = 0 \text{ when } 4x - 2 = 0$$

$$x = \frac{1}{2}$$

$f''(x)$ is undefined when $9x^{5/3} = 0$
 $x = 0$

$$f''(-1) = \frac{4(-1) - 2}{9(-1)^{5/3}} = \frac{2}{3} > 0$$

$$f''\left(\frac{1}{8}\right) = \frac{4\left(\frac{1}{8}\right) - 2}{9\left(\frac{1}{8}\right)^{5/3}} = -\frac{16}{3} < 0$$

$$f''(1) = \frac{4(1) - 2}{9(1)^{5/3}} = \frac{2}{9} > 0$$

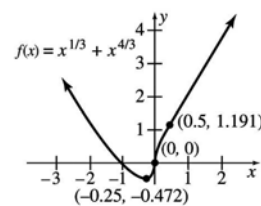
Inflection points at $(0, 0)$ and $\left(\frac{1}{2}, \frac{3}{2^{4/3}}\right)$
 $\approx (0.5, 1.191)$

Concave upward on $(-\infty, 0)$ and $\left(\frac{1}{2}, \infty\right)$

Concave downward on $\left(0, \frac{1}{2}\right)$

$$y\text{-intercept: } y = 0^{1/3} + 0^{4/3} = 0$$

$$\begin{aligned} x\text{-intercept: } 0 &= x^{1/3} + x^{4/3} \\ &= x^{1/3}(1 + x) \\ x &= 0 \text{ or } x = -1 \end{aligned}$$



31. For Exercises 3, 7, and 9, the relative maxima or minima are outside the vertical window of $-10 \leq y \leq 10$.

For Exercise 11, the default window shows only a small portion of the graph.

For Exercise 15, the default window does not allow the graph to properly display the vertical asymptotes.

32. For Exercise 6, the y -intercept is outside the vertical window of $-10 \leq y \leq 10$.

For Exercises 4, 10, and 12, the relative maxima or minima are outside the vertical window of $-10 \leq y \leq 10$.

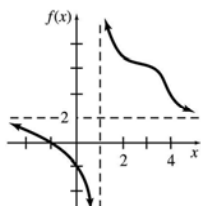
For Exercise 16, the default window does not allow the graph to properly display the vertical asymptotes.

33. For Exercises 17, 19, 23, 25, and 27, the y -coordinate of the relative minimum, relative maximum, or inflection points is so small, it may be hard to distinguish.
34. For Exercises 18 and 24, the y -coordinate of the relative minimum, relative maximum, or inflection point is so small, it may be hard to distinguish.
- For Exercise 20, the default window does not allow the graph to properly display the vertical asymptotes.

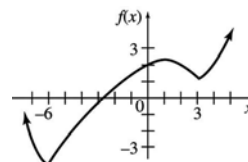
For Exercises 35–39 other graphs are possible.

35. (a) indicates a smooth, continuous curve except where there is a vertical asymptote.
- (b) indicates that the function decreases on both sides of the asymptote, so there are no relative extrema.
- (c) gives the horizontal asymptote $y = 2$.
- (d) and (e) indicate that concavity does not change left of the asymptote, but that the right portion of the graph changes concavity at $x = 2$ and $x = 4$.

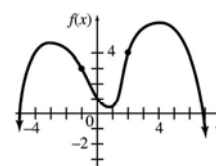
There are inflection points at 2 and 4.



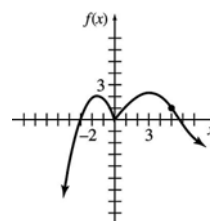
36. (a) indicates that the curve may not contain breaks.
- (b) and (c) indicate relative minima at -6 and 3 and a relative maximum at 1 .
- (d) and (e), when combined with (b) and (c) show that concavity does not change between relative extrema.
- (f) gives the y -intercept.



37. (a) indicates that there can be no asymptotes, sharp “corners”, holes, or jumps. The graph must be one smooth curve.
- (b) and (c) indicate relative maxima at -3 and 4 and a relative minimum at 1 .
- (d) and (e) are consistent with (g).
- (f) indicates turning points at the critical numbers -3 and 4 .

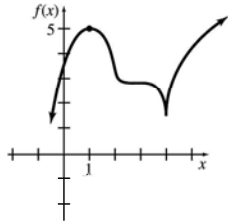


38. (a) indicates that the curve may not contain breaks.
- (b) and (c) indicate relative maxima at -2 and 3 and a relative minimum at 0 .
- (d) shows that concavity does not change at 0 .
- (d) and (e) are consistent with (i).
- (f) shows critical values.
- (g) and (h) indicate that the function is not differentiable at 0 , and is differentiable everywhere else.
- Thus, a sharp corner must exist at 0 .
- (i) indicates that concavity changes just once, at $(5, 1)$.



39. (a) indicates that the curve may not contain breaks.
- (b) indicates that there is a sharp “corner” at 4 .
- (c) gives a point at $(1, 5)$.
- (d) shows critical numbers.
- (e) and (f) indicate (combined with (c) and (d)) a relative maximum at $(1, 5)$, and (combined with (b)) a relative minimum at 4 .

- (g) is consistent with (b).
 (h) indicates the curve is concave upward on $(2, 3)$.
 (i) indicates the curve is concave downward on $(-\infty, 2)$, $(3, 4)$ and $(4, \infty)$.



Chapter 5 Review Exercises

- True
- False: The function is increasing on this interval.
- False: The function could have neither a minimum nor a maximum; consider $f(x) = x^3$ at $c = 0$.
- True
- False: Consider $f(x) = x^{3/2}$ at $c = 0$.
- True
- False: The function is concave upward.
- False: Consider $f(x) = x^4$ at $c = 0$.
- False: Consider $f(x) = x^4$, which has a relative minimum at $c = 0$.
- False: Polynomials are rational functions, and nonconstant polynomials have neither vertical nor horizontal asymptotes.
- True
- False: Consider $f(x) = x^2$ on $(-1, 1)$ with $c = 0$.
- $f(x) = x^2 + 9x + 8$
 $f'(x) = 2x + 9$
 $f'(x) = 0$ when $x = -\frac{9}{2}$ and f' exists everywhere.
 Critical number: $-\frac{9}{2}$
 Test an x -value in the intervals $(-\infty, -\frac{9}{2})$ and $(-\frac{9}{2}, \infty)$.
 $f'(-5) = -1 < 0$
 $f'(-4) = 1 > 0$
 f is increasing on $(-\frac{9}{2}, \infty)$ and decreasing on $(-\infty, -\frac{9}{2})$.
- $f(x) = -2x^2 + 7x + 14$
 $f'(x) = -4x + 7$
 $f'(x) = 0$ when $x = \frac{7}{4}$ and f' exists everywhere.
 Critical number: $\frac{7}{4}$
 Test an x -value in the intervals $(-\infty, \frac{7}{4})$ and $(\frac{7}{4}, \infty)$.
 $f'(0) = 7 > 0$
 $f'(2) = -1 < 0$
 f is increasing on $(-\infty, \frac{7}{4})$ and decreasing on $(\frac{7}{4}, \infty)$.
- $f(x) = -x^3 + 2x^2 + 15x + 16$
 $f'(x) = -3x^2 + 4x + 15$
 $= -(3x^2 - 4x - 15)$
 $= -(3x + 5)(x - 3)$
 $f'(x) = 0$ when $x = -\frac{5}{3}$ or $x = 3$ and f' exists everywhere.
 Critical numbers: $-\frac{5}{3}$ and 3
 Test an x -value in the intervals $(-\infty, -\frac{5}{3})$, $(-\frac{5}{3}, 3)$, and $(3, \infty)$.
 $f'(-2) = -5 < 0$
 $f'(0) = 15 > 0$
 $f'(4) = -17 < 0$

f is increasing on $\left(-\frac{5}{3}, 3\right)$ and decreasing on $\left(-\infty, -\frac{5}{3}\right)$ and $(3, \infty)$.

20. $f(x) = 4x^3 + 8x^2 - 16x + 11$

$$f'(x) = 12x^2 + 16x - 16$$

$$= 4(3x^2 + 4x - 4)$$

$$= 4(3x - 2)(x + 2)$$

$$f'(x) = 0 \text{ when } x = \frac{2}{3} \text{ or } x = -2 \text{ and}$$

f' exists everywhere.

Critical numbers: -2 and $\frac{2}{3}$

Test an x -value in the intervals $(-\infty, -2)$,

$\left(-2, \frac{2}{3}\right)$, and $\left(\frac{2}{3}, \infty\right)$.

$$f'(-3) = 44 > 0$$

$$f'(0) = -16 < 0$$

$$f'(1) = 12 > 0$$

f is increasing on $(-\infty, -2)$ and $\left(\frac{2}{3}, \infty\right)$ and decreasing on $\left(-2, \frac{2}{3}\right)$.

21. $f(x) = \frac{16}{9 - 3x}$

$$f'(x) = \frac{16(-1)(-3)}{(9 - 3x)^2} = \frac{48}{(9 - 3x)^2}$$

$f'(x) > 0$ for all x ($x \neq 3$), and f is not defined for $x = 3$.

f is increasing on $(-\infty, 3)$ and $(3, \infty)$ and never decreasing.

22. $f(x) = \frac{15}{2x + 7}$

$$f'(x) = \frac{15(-1)(2)}{(2x + 7)^2} = \frac{-30}{(2x + 7)^2}$$

$f'(x) < 0$ for all x ($x \neq -\frac{7}{2}$), and f is not defined for $x = -\frac{7}{2}$.

f is never increasing, and decreasing on $\left(-\infty, -\frac{7}{2}\right)$ and $\left(-\frac{7}{2}, \infty\right)$.

23. $f(x) = \ln|x^2 - 1|$

$$f'(x) = \frac{2x}{x^2 - 1}$$

f is not defined for $x = -1$ and $x = 1$.

$$f'(x) = 0 \text{ when } x = 0.$$

Test an x -value in the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

$$f'(-2) = -\frac{4}{3} < 0$$

$$f'\left(-\frac{1}{2}\right) = \frac{4}{3} > 0$$

$$f'\left(\frac{1}{2}\right) = -\frac{4}{3} < 0$$

$$f'(2) = \frac{4}{3} > 0$$

f is increasing on $(-1, 0)$ and $(1, \infty)$ and decreasing on $(-\infty, -1)$ and $(0, 1)$.

24. $f(x) = 8xe^{-4x}$

$$f'(x) = 8(e^{-4x}) + 8x(-4e^{-4x})$$

$$= 8e^{-4x} - 32xe^{-4x}$$

$$= 8e^{-4x}(1 - 4x)$$

$$f'(x) = 0 \text{ when } x = \frac{1}{4}.$$

Test an x -value in the intervals $(-\infty, \frac{1}{4})$ and $\left(\frac{1}{4}, \infty\right)$.

$$f'(0) = 8 > 0$$

$$f'(1) = -24e^{-4} < 0$$

f is increasing on $(-\infty, \frac{1}{4})$ and decreasing on $\left(\frac{1}{4}, \infty\right)$.

25. $f(x) = -x^2 + 4x - 8$

$$f'(x) = -2x + 4 = 0$$

Critical number: $x = 2$

$f''(x) = -2 < 0$ for all x , so $f(2)$ is a relative maximum.

$$f(2) = -4$$

Relative maximum of -4 at 2

26. $f(x) = x^2 - 6x + 4$

$$f'(x) = 2x - 6$$

$$f'(x) = 0 \text{ when } x = 3.$$

Critical number: 3

$f''(x) = 2 > 0$ for all x , so $f(3)$ is a relative minimum.

$$f(3) = -5$$

Relative minimum of -5 at 3

27. $f(x) = 2x^2 - 8x + 1$

$$f'(x) = 4x - 8 = 0$$

Critical number: $x = 2$

$f''(x) = 4 > 0$ for all x , so $f(2)$ is a relative minimum.

$$f(2) = -7$$

Relative minimum of -7 at 2

28. $f(x) = -3x^2 + 2x - 5$

$$f'(x) = -6x + 2$$

$$f'(x) = 0 \text{ when } x = \frac{1}{3}.$$

Critical number: $\frac{1}{3}$

Since $f''(x) = -6 < 0$ for all x , $f\left(\frac{1}{3}\right)$ is a relative maximum.

$$f\left(\frac{1}{3}\right) = -\frac{14}{3}$$

Relative maximum of $-\frac{14}{3}$ at $\frac{1}{3}$

29. $f(x) = 2x^3 + 3x^2 - 36x + 20$

$$f'(x) = 6x^2 + 6x - 36 = 0$$

$$6(x^2 + x - 6) = 0$$

$$(x + 3)(x - 2) = 0$$

Critical numbers: -3 and 2

$$f''(x) = 12x + 6$$

$f''(-3) = -30 < 0$, so a maximum occurs at $x = -3$.

$f''(2) = 30 > 0$, so a minimum occurs at $x = 2$.

$$f(-3) = 101$$

$$f(2) = -24$$

Relative maximum of 101 at -3

Relative minimum of -24 at 2

30. $f(x) = 2x^3 + 3x^2 - 12x + 5$

$$f'(x) = 6x^2 + 6x - 12$$

$$= 6(x^2 + x - 2)$$

$$= 6(x + 2)(x - 1)$$

$$f'(x) = 0 \text{ when } x = -2 \text{ or } x = 1.$$

Critical numbers: $-2, 1$

$$f''(x) = 12x + 6$$

$f''(-2) = 18 < 0$, so a maximum occurs at $x = -2$.

$f''(1) = 18 > 0$, so a minimum occurs at $x = 1$.

$$f(-2) = 25$$

$$f(1) = -2$$

Relative maximum of 25 at -2

Relative minimum of -2 at 1

31. $f(x) = \frac{xe^x}{x-1}$

$$f'(x) = \frac{(x-1)(xe^x + e^x) - xe^x(1)}{(x-1)^2}$$

$$= \frac{x^2e^x - xe^x - xe^x - e^x - xe^x}{(x-1)^2}$$

$$= \frac{x^2e^x - xe^x - e^x}{(x-1)^2}$$

$$= \frac{e^x(x^2 - x - 1)}{(x-1)^2}$$

$f'(x)$ is undefined at $x = 1$, but 1 is not in the domain of $f(x)$.

$$f'(x) = 0 \text{ when } x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{1 + \sqrt{5}}{2} \approx 1.618 \text{ or } \frac{1 - \sqrt{5}}{2} = -0.618$$

Critical numbers are -0.618 and 1.618 .

$$f'(1.4) = \frac{e^{1.4}(1.4^2 - 1.4 - 1)}{(1.4 - 1)^2} \approx -11.15 < 0$$

$$f'(2) = \frac{e^2(2^2 - 2 - 1)}{(2 - 1)^2} = e^2 \approx 7.39 > 0$$

$$f'(-1) = \frac{e^{-1}[(-1)^2 - (-1) - 1]}{(-1 - 1)^2} \approx 0.09 > 0$$

$$f'(0) = \frac{e^0(0^2 - 0 - 1)}{(0 - 1)^2} = -1 < 0$$

There is a relative maximum at $(-0.618, 0.206)$ and a relative minimum at $(1.618, 13.203)$.

$$\begin{aligned}
 32. \quad y &= \frac{\ln(3x)}{2x^2} \\
 y' &= \frac{2x^2 \cdot \frac{1}{x} - (\ln 3x) \cdot 4x}{4x^4} \\
 &= \frac{2x(1 - 2\ln 3x)}{4x^4} \\
 &= \frac{1 - 2\ln 3x}{2x^3}
 \end{aligned}$$

$$y' = 0 \text{ when}$$

$$1 - 2\ln 3x = 0$$

$$1 = 2\ln 3x$$

$$\frac{1}{2} = \ln 3x$$

$$e^{1/2} = 3x$$

$$\frac{e^{1/2}}{3} = x$$

$$\frac{\sqrt{e}}{3} \approx 0.55$$

$$f'(1) = \frac{1 - 2\ln 3}{2} \approx -0.6 < 0$$

$$f'\left(\frac{1}{3}\right) = \frac{1 - 2\ln 1}{2\left(\frac{1}{3}\right)^3} = \frac{1}{2 \cdot \frac{1}{27}} = \frac{27}{2} > 0$$

$$f\left(\frac{\sqrt{e}}{3}\right) \approx 0.83$$

Relative maximum at $\left(\frac{\sqrt{e}}{3}, 0.83\right)$ or $(0.55, 0.83)$

$$\begin{aligned}
 33. \quad f(x) &= 3x^4 - 5x^2 - 11x \\
 f'(x) &= 12x^3 - 10x - 11 \\
 f''(x) &= 36x^2 - 10 \\
 f''(1) &= 36(1)^2 - 10 = 26 \\
 f''(-3) &= 36(-3)^2 - 10 = 314
 \end{aligned}$$

$$\begin{aligned}
 34. \quad f(x) &= 9x^3 + \frac{1}{x} = 9x^3 + x^{-1} \\
 f'(x) &= 27x^2 - x^{-2} \\
 f''(x) &= 54x + 2x^{-3} = 54x + \frac{2}{x^3} \\
 f''(1) &= 54(1) + \frac{2}{(1)^3} = 56 \\
 f''(-3) &= 54(-3) + \frac{2}{(-3)^3} \\
 &= -162 - \frac{2}{27} = -\frac{4376}{27}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad f(x) &= \frac{4x + 2}{3x - 6} \\
 f'(x) &= \frac{(3x - 6)(4) - (4x + 2)(3)}{(3x - 6)^2} \\
 &= \frac{12x - 24 - 12x - 6}{(3x - 6)^2} = \frac{-30}{(3x - 6)^2} \\
 &= -30(3x - 6)^{-2}
 \end{aligned}$$

$$\begin{aligned}
 f''(x) &= -30(-2)(3x - 6)^{-3}(3) \\
 &= 180(3x - 6)^{-3} \text{ or } \frac{180}{(3x - 6)^3}
 \end{aligned}$$

$$f''(1) = 180[3(1) - 6]^{-3} = -\frac{20}{3}$$

$$f''(-3) = 180[3(-3) - 6]^{-3} = -\frac{4}{75}$$

$$\begin{aligned}
 36. \quad f(x) &= \frac{1 - 2x}{4x + 5} \\
 f'(x) &= \frac{(4x + 5)(-2) - (1 - 2x)(4)}{(4x + 5)^2} \\
 &= \frac{-8x - 10 - 4 + 8x}{(4x + 5)^2} \\
 &= \frac{-14}{(4x + 5)^2} = -14(4x + 5)^{-2}
 \end{aligned}$$

$$\begin{aligned}
 f''(x) &= -14(-2)(4x + 5)^{-3}(4) \\
 &= 112(4x + 5)^{-3} \text{ or } \frac{112}{(4x + 5)^3}
 \end{aligned}$$

$$f''(1) = \frac{112}{[4(1) + 5]^3} = \frac{112}{729}$$

$$f''(-3) = \frac{112}{[4(-3) + 5]^3} = -\frac{16}{49}$$

$$\begin{aligned}
 37. \quad f(t) &= \sqrt{t^2 + 1} = (t^2 + 1)^{1/2} \\
 f'(t) &= \frac{1}{2}(t^2 + 1)^{-1/2}(2t) = t(t^2 + 1)^{-1/2} \\
 f''(t) &= (t^2 + 1)^{-1/2}(1) \\
 &\quad + t\left[\left(-\frac{1}{2}\right)(t^2 + 1)^{-3/2}(2t)\right] \\
 &= (t^2 + 1)^{-1/2} - t^2(t^2 + 1)^{-3/2} \\
 &= \frac{1}{(t^2 + 1)^{1/2}} - \frac{t^2}{(t^2 + 1)^{3/2}} = \frac{t^2 + 1 - t^2}{(t^2 + 1)^{3/2}} \\
 &= (t^2 + 1)^{-3/2} \text{ or } \frac{1}{(t^2 + 1)^{3/2}}
 \end{aligned}$$

$$f''(1) = \frac{1}{(1+1)^{3/2}} = \frac{1}{2^{3/2}} \approx 0.354$$

$$f''(-3) = \frac{1}{(9+1)^{3/2}} = \frac{1}{10^{3/2}} \approx 0.032$$

$$38. \quad f(t) = -\sqrt{5-t^2} = -(5-t^2)^{1/2}$$

$$f'(t) = -\frac{1}{2}(5-t^2)^{-1/2}(-2t) = t(5-t^2)^{-1/2}$$

$$\begin{aligned} f''(t) &= (1)(5-t^2)^{-1/2} + t\left[-\frac{1}{2}(5-t^2)^{-3/2}(-2t)\right] \\ &= (5-t^2)^{-1/2} + t[t(5-t^2)^{-3/2}] \\ &= (5-t^2)^{-3/2}(5-t^2+t^2) = \frac{5}{(5-t^2)^{3/2}} \end{aligned}$$

$$f''(1) = \frac{5}{(5-1)^{3/2}} = \frac{5}{8}$$

$$f''(-3) = \frac{5}{(5-9)^{3/2}}$$

This value does not exist since $(-4)^{3/2}$ does not exist. (In fact, f is undefined at $t = -4$.)

$$39. \quad f(x) = -2x^3 - \frac{1}{2}x^2 + x - 3$$

Domain is $(-\infty, \infty)$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= -6x^2 - x + 1 = 0 \\ (3x-1)(2x+1) &= 0 \end{aligned}$$

Critical numbers: $\frac{1}{3}$ and $-\frac{1}{2}$

Critical points: $\left(\frac{1}{3}, -2.80\right)$ and $\left(-\frac{1}{2}, -3.375\right)$

$$f''(x) = -12x - 1$$

$$f''\left(\frac{1}{3}\right) = -5 < 0$$

$$f''\left(-\frac{1}{2}\right) = 5 > 0$$

Relative maximum at $\frac{1}{3}$

Relative minimum at $-\frac{1}{2}$

Increasing on $\left(-\frac{1}{2}, \frac{1}{3}\right)$

Decreasing on $\left(-\infty, -\frac{1}{2}\right)$ and $\left(\frac{1}{3}, \infty\right)$

$$f''(x) = -12x - 1 = 0$$

$$x = -\frac{1}{12}$$

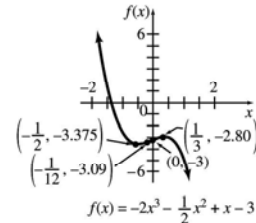
Point of inflection at $\left(-\frac{1}{12}, -3.09\right)$

Concave upward on $\left(-\infty, -\frac{1}{12}\right)$

Concave downward on $\left(-\frac{1}{12}, \infty\right)$

y-intercept:

$$y = -2(0)^3 - \frac{1}{2}(0)^2 + (0) - 3 = -3$$



$$40. \quad f(x) = -\frac{4}{3}x^3 + x^2 + 30x - 7$$

Domain is $(-\infty, \infty)$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= -4x^2 + 2x + 30 \\ &= -2(2x^2 - x - 15) \\ &= -2(2x+5)(x-3) = 0 \end{aligned}$$

Critical numbers: $-\frac{5}{2}$ and 3

Critical points: $\left(-\frac{5}{2}, -54.91\right)$ and $(3, 56)$

$$f''(x) = -8x + 2$$

$$f''\left(-\frac{5}{2}\right) = 22 > 0$$

$$f''(3) = -22 < 0$$

Relative maximum at 3

Relative minimum at $-\frac{5}{2}$

Increasing on $\left(-\frac{5}{2}, 3\right)$

Decreasing on $\left(-\infty, -\frac{5}{2}\right)$ and $(3, \infty)$

$$f''(x) = -8x + 2 = 0$$

$$x = \frac{1}{4}$$

Point of inflection at $\left(\frac{1}{4}, 0.54\right)$

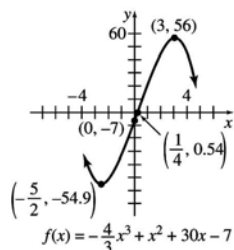
Concave upward on $\left(-\infty, \frac{1}{4}\right)$

Concave downward on $\left(\frac{1}{4}, \infty\right)$

y-intercept:

$$y = -\frac{4}{3}(0)^3 + (0)^3 + 30(0)^2 - 7$$

$$= -7$$



41. $f(x) = x^4 - \frac{4}{3}x^3 - 4x^2 + 1$

Domain is $(-\infty, \infty)$

The graph has no symmetry.

$$f'(x) = 4x^3 - 4x^2 - 8x = 0$$

$$4x(x^2 - x - 2) = 0$$

$$4x(x - 2)(x + 1) = 0$$

Critical numbers: 0, 2, and -1

Critical points: $(0, 1)$, $(2, -\frac{29}{3})$ and $(-1, -\frac{2}{3})$

$$f''(x) = 12x^2 - 8x - 8$$

$$= 4(3x^2 - 2x - 2)$$

$$f''(-1) = 12 > 0$$

$$f''(0) = -8 < 0$$

$$f''(2) = 24 > 0$$

Relative maximum at 0

Relative minima at -1 and 2

Increasing on $(-1, 0)$ and $(2, \infty)$

Decreasing on $(-\infty, -1)$ and $(0, 2)$

$$f''(x) = 4(3x^2 - 2x - 2) = 0$$

$$x = \frac{2 \pm \sqrt{4 - (-24)}}{6}$$

$$= \frac{1 \pm \sqrt{7}}{3}$$

Points of inflection at $(\frac{1 \pm \sqrt{7}}{3}, -5.12)$ and

$$(\frac{1 - \sqrt{7}}{3}, 0.11)$$

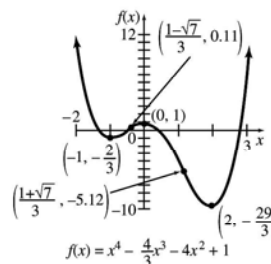
Concave upward on $(-\infty, \frac{1 - \sqrt{7}}{3})$ and

$$(\frac{1 + \sqrt{7}}{3}, \infty)$$

Concave downward on $(\frac{1 - \sqrt{7}}{3}, \frac{1 + \sqrt{7}}{3})$

y-intercept:

$$y = (0)^4 - \frac{4}{3}(0)^3 - 4(0)^2 + 1 = 1$$



42. $f(x) = -\frac{2}{3}x^3 + \frac{9}{2}x^2 + 5x + 1$

Domain is $(-\infty, \infty)$

The graph has no symmetry.

$$f'(x) = -2x^2 + 9x + 5 = 0$$

$$-(2x + 1)(x - 5) = 0$$

Critical numbers: $-\frac{1}{2}$ and 5

Critical points: $(-\frac{1}{2}, -0.29)$ and $(5, 55.17)$

$$f''(x) = -4x + 9$$

$$f''(-\frac{1}{2}) = 11 > 0$$

$$f''(5) = -11 < 0$$

Relative maximum at 5

Relative minimum at $-\frac{1}{2}$

Increasing on $(-\frac{1}{2}, 5)$

Decreasing on $(-\infty, -\frac{1}{2})$ and $(5, \infty)$

$$f''(x) = -4x + 9 = 0$$

$$x = \frac{9}{4}$$

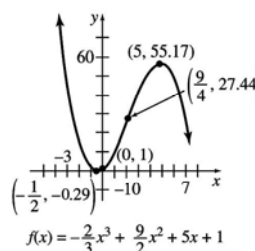
Point of inflection at $(\frac{9}{4}, 27.44)$

Concave upward on $(-\infty, \frac{9}{4})$

Concave downward on $(\frac{9}{4}, \infty)$

y-intercept:

$$y = -\frac{2}{3}(0)^3 + \frac{9}{2}(0)^2 + 5(0) + 1 = 1$$



43. $f(x) = \frac{x-1}{2x+1}$

Domain is $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

The graph has no symmetry.

$$f'(x) = \frac{(2x+1)(1) - (x-1)(2)}{(2x+1)^2}$$

$$= \frac{3}{(2x+1)^2}$$

f' is never zero.

$f'(-\frac{1}{2})$ does not exist, but $-\frac{1}{2}$ is not a critical number because $-\frac{1}{2}$ is not in the domain of f .

Thus, there are no critical numbers, so $f(x)$ has no relative extrema.

Increasing on $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$

$$f''(x) = \frac{-12}{(2x+1)^3}$$

$$f''(0) = -12 < 0$$

$$f''(-1) = 12 > 0$$

No inflection points

Concave upward on $(-\infty, -\frac{1}{2})$

Concave downward on $(-\frac{1}{2}, \infty)$

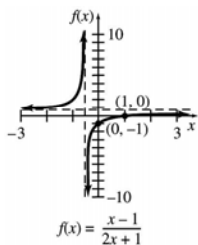
$$x\text{-intercept: } \frac{x-1}{2x+1} = 0$$

$$x = 1$$

$$y\text{-intercept: } y = \frac{0-1}{2(0)+1} = -1$$

Vertical asymptote at $x = -\frac{1}{2}$

Horizontal asymptote at $y = \frac{1}{2}$



44. $f(x) = \frac{2x-5}{x+3}$

Domain is $(-\infty, -3) \cup (-3, \infty)$

$$f'(x) = \frac{2(x+3) - (2x-5)}{(x+3)^2}$$

$$= \frac{11}{(x+3)^2}$$

f' is never zero.

$f(x)$ has no extrema.

$$f''(x) = \frac{-22}{(x+3)^3}$$

$$f''(-4) = 22 > 0$$

$$f''(-2) = -22 < 0$$

Increasing on $(-\infty, -3) \cup (-3, \infty)$

Concave upward on $(-\infty, -3)$

Concave downward on $(-3, \infty)$

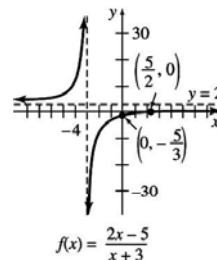
$$x\text{-intercept: } \frac{2x-5}{x+3} = 0$$

$$x = \frac{5}{2}$$

$$y\text{-intercept: } \frac{2(0)-5}{0+3} = -\frac{5}{3}$$

Vertical asymptote at $x = -3$

Horizontal asymptote at $y = 2$



45. $f(x) = -4x^3 - x^2 + 4x + 5$

Domain is $(-\infty, \infty)$

The graph has no symmetry.

$$f'(x) = -12x^2 - 2x + 4$$

$$= -2(6x^2 + x - 2) = 0$$

$$(3x+2)(2x-1) = 0$$

Critical numbers: $-\frac{2}{3}$ and $\frac{1}{2}$

Critical points: $(-\frac{2}{3}, 3.07)$ and $(\frac{1}{2}, 6.25)$

$$\begin{aligned} f''(x) &= -24x - 2 \\ &= -2(12x + 1) \end{aligned}$$

$$f''\left(-\frac{2}{3}\right) = 14 > 0$$

$$f''\left(\frac{1}{2}\right) = -14 < 0$$

Relative maximum at $\frac{1}{2}$

Relative minimum at $-\frac{2}{3}$

Increasing on $\left(-\frac{2}{3}, \frac{1}{2}\right)$

Decreasing on $\left(-\infty, -\frac{2}{3}\right)$ and $\left(\frac{1}{2}, \infty\right)$

$$f''(x) = -2(12x + 1) = 0$$

$$x = -\frac{1}{12}$$

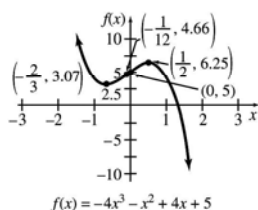
Point of inflection at $\left(-\frac{1}{12}, 4.66\right)$

Concave upward on $\left(-\infty, -\frac{1}{12}\right)$

Concave downward on $\left(-\frac{1}{12}, \infty\right)$

y-intercept:

$$y = -4(0)^3 - (0)^2 + 4(0) + 5 = 5$$



46. $f(x) = x^3 + \frac{5}{2}x^2 - 2x - 3$

Domain is $(-\infty, \infty)$

The graph has no symmetry.

$$f'(x) = 3x^2 + 5x - 2 = (3x - 1)(x + 2) = 0$$

Critical numbers: $\frac{1}{3}$ and -2

Critical points: $\left(\frac{1}{3}, -3.35\right)$ and $(-2, 3)$

$$f''(x) = 6x + 5$$

$$f''\left(\frac{1}{3}\right) = 7 > 0$$

$$f''(-2) = -7 < 0$$

Relative maximum at -2

Relative minimum at $\frac{1}{3}$

Increasing on $(-\infty, -2)$ and $\left(\frac{1}{3}, \infty\right)$

Decreasing on $\left(-2, \frac{1}{3}\right)$

$$f''(x) = 6x + 5 = 0$$

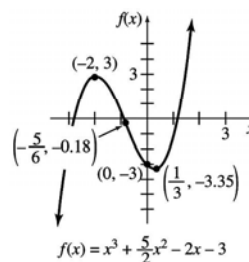
$$x = -\frac{5}{6}$$

Point of inflection at $\left(-\frac{5}{6}, -0.18\right)$

Concave upward on $\left(-\frac{5}{6}, \infty\right)$

Concave downward on $(-\infty, -\frac{5}{6})$

y-intercept: -3



47. $f(x) = x^4 + 2x^2$

Domain is $(-\infty, \infty)$

$$\begin{aligned} f(-x) &= (-x)^4 + 2(-x)^2 \\ &= x^4 + 2x^2 = f(x) \end{aligned}$$

The graph is symmetric about the y-axis.

$$\begin{aligned} f'(x) &= 4x^3 + 4x \\ &= 4x(x^2 + 1) = 0 \end{aligned}$$

Critical number: 0

Critical point: (0, 0)

$$f''(x) = 12x^2 + 4 = 4(3x^2 + 1)$$

$$f''(0) = 4 > 0$$

Relative minimum at 0

Increasing on $(0, \infty)$

Decreasing on $(-\infty, 0)$

$$f''(x) = 4(3x^2 + 1) \neq 0 \text{ for any } x$$

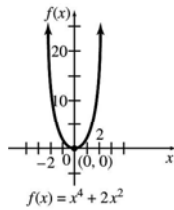
No points of inflection

$$f''(-1) = 16 > 0$$

$$f''(1) = 16 > 0$$

Concave upward on $(-\infty, \infty)$

x-intercept: 0; y-intercept: 0



48. $f(x) = 6x^3 - x^4$

Domain is $(-\infty, \infty)$

The graph has no symmetry.

$$f'(x) = 18x^2 - 4x^3 = 2x^2(9 - 2x) = 0$$

Critical number: 0 and $\frac{9}{2}$

Critical points: $(0, 0)$ and $(\frac{9}{2}, 136.7)$

$$f''(x) = 36x - 12x^2 = 12x(3 - x)$$

$$f''(0) = 0$$

$$f''\left(\frac{9}{2}\right) = -81 < 0$$

Relative maximum at $\frac{9}{2}$

No relative extrema at 0

Increasing on $(-\infty, \frac{9}{2})$

Decreasing on $(\frac{9}{2}, \infty)$

$$f''(x) = 12x(3 - x) = 0$$

$$x = 0 \text{ or } x = 3$$

Points of inflection at $(0, 0)$ and $(3, 81)$

Concave upward on $(0, 3)$

Concave downward on $(-\infty, 0)$ and $(3, \infty)$

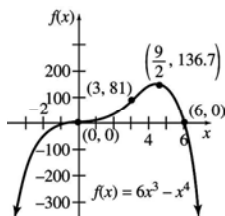
$$x\text{-intercept: } 6x^3 - x^4 = 0$$

$$x^3(6 - x) = 0$$

$$x = 0, x = 6$$

The x -intercepts are 0 and 6.

y -intercept: 0



49. $f(x) = \frac{x^2 + 4}{x}$

Domain is $(-\infty, 0) \cup (0, \infty)$

$$\begin{aligned} f(-x) &= \frac{(-x)^2 + 4}{-x} \\ &= \frac{x^2 + 4}{-x} = -f(x) \end{aligned}$$

The graph is symmetric about the origin.

$$\begin{aligned} f'(x) &= \frac{x(2x) - (x^2 + 4)}{x^2} \\ &= \frac{x^2 - 4}{x^2} = 0 \end{aligned}$$

Critical numbers: -2 and 2

Critical points: $(-2, -4)$ and $(2, 4)$

$$f''(x) = \frac{8}{x^3}$$

$$f''(-2) = -1 < 0$$

$$f''(2) = 1 > 0$$

Relative maximum at -2

Relative minimum at 2

Increasing on $(-\infty, -2)$ and $(2, \infty)$

Decreasing on $(-2, 0)$ and $(0, 2)$

$$f''(x) = \frac{8}{x^3} > 0 \text{ for all } x.$$

No inflection points

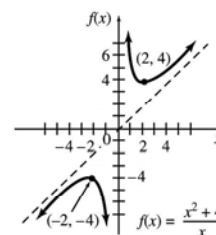
Concave upward on $(0, \infty)$

Concave downward on $(-\infty, 0)$

No x - or y -intercepts

Vertical asymptote at $x = 0$

Oblique asymptote at $y = x$



50. $f(x) = x + \frac{8}{x}$

Domain is $(-\infty, 0) \cup (0, \infty)$

$$\begin{aligned} f(-x) &= -x + \frac{8}{-x} \\ &= -\left(x + \frac{8}{x}\right) = -f(x) \end{aligned}$$

The graph is symmetric about the origin.

$$\begin{aligned} f'(x) &= 1 - \frac{8}{x^2} \\ &= \frac{x^2 - 8}{x^2} = 0 \end{aligned}$$

Critical numbers: $x = \pm 2\sqrt{2}$

Critical points: $(2\sqrt{2}, 4\sqrt{2}), (-2\sqrt{2}, -4\sqrt{2})$

$$f''(x) = \frac{16}{x^3}$$

$$f''(-2\sqrt{2}) = -\frac{\sqrt{2}}{2} < 0$$

$$f''(2\sqrt{2}) = \frac{\sqrt{2}}{2} > 0$$

Relative maximum at $-2\sqrt{2}$

Relative minimum at $2\sqrt{2}$

Increasing on $(-\infty, -2\sqrt{2})$ and $(2\sqrt{2}, \infty)$

Decreasing on $(-2\sqrt{2}, 0)$ and $(0, 2\sqrt{2})$

$$f''(x) = \frac{16}{x^3} > 0 \text{ for all } x.$$

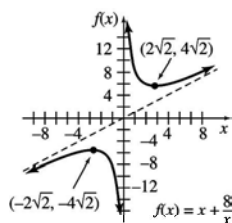
No inflection points

Concave upward on $(0, \infty)$

Concave downward on $(-\infty, 0)$

Vertical asymptote at $x = 0$

Oblique asymptote at $y = x$



51. $f(x) = \frac{2x}{3-x}$

Domain is $(-\infty, 3) \cup (3, \infty)$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= \frac{(3-x)(2) - (2x)(-1)}{(3-x)^2} \\ &= \frac{6}{(3-x)^2} \end{aligned}$$

$f'(x)$ is never zero. $f'(3)$ does not exist, but since 3 is not in the domain of f , it is not a critical number.

No critical numbers, so no relative extrema

$$f'(0) = \frac{2}{3} > 0$$

$$f'(4) = 6 > 0$$

Increasing on $(-\infty, 3)$ and $(3, \infty)$

$$f''(x) = \frac{12}{(3-x)^3}$$

$f''(x)$ is never zero. $f''(3)$ does not exist, but since 3 is not in the domain of f , there is no inflection point at $x = 3$.

$$f''(0) = \frac{12}{27} > 0$$

$$f''(4) = -12 < 0$$

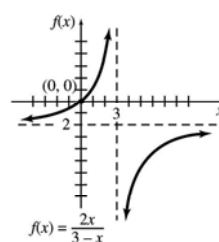
Concave upward on $(-\infty, 3)$

Concave downward on $(3, \infty)$

x -intercept: 0; y -intercept: 0

Vertical asymptote at $x = 3$

Horizontal asymptote at $y = -2$



52. $f(x) = \frac{-4x}{1+2x}$

Domain is $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= \frac{-4(1+2x) - 2(-4x)}{(1+2x)^2} \\ &= \frac{-4 - 8x + 8x}{(1+2x)^2} \\ &= \frac{-4}{(1+2x)^2} \end{aligned}$$

$f'(x)$ is never zero.

No critical values; no relative extrema

$$f'(0) = -4 < 0$$

$$f'(-1) = -4 < 0$$

Decreasing on $(-\infty, -\frac{1}{2})$ and $(-\frac{1}{2}, \infty)$

$$f''(x) = \frac{16}{(1+2x)^3}$$

$f''(x)$ is never zero; no points of inflection.

$$f''(0) = 16 > 0$$

$$f''(-1) = -16 < 0$$

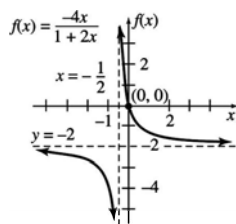
Concave upward on $(-\frac{1}{2}, \infty)$

Concave downward on $(-\infty, -\frac{1}{2})$

x-intercept: 0; y-intercept: 0

Vertical asymptote at $x = -\frac{1}{2}$

Horizontal asymptote at $y = -2$



53. $f(x) = xe^{2x}$

Domain is $(-\infty, \infty)$.

$$f(-x) = -xe^{-2x}$$

The graph has no symmetry.

$$\begin{aligned} f'(x) &= (1)(e^{2x}) + (x)(2e^{2x}) \\ &= e^{2x}(2x + 1) \end{aligned}$$

$$f'(x) = 0 \text{ when } x = -\frac{1}{2}.$$

Critical number: $-\frac{1}{2}$

Critical point: $(-\frac{1}{2}, -\frac{1}{2e})$

$$f'(-1) = e^{2(-1)}[2(-1) + 1] = -e^{-2} < 0$$

$$f'(0) = e^{2(0)}[2(0) + 1] = 1 > 0$$

No relative maximum

Relative minimum at $(-\frac{1}{2}, -\frac{1}{2e})$

Decreasing on $(-\infty, -\frac{1}{2})$ and increasing on $(-\frac{1}{2}, \infty)$

$$\begin{aligned} f''(x) &= 2e^{2x}(2x + 1) + e^{2x}(2) \\ &= 4e^{2x}(x + 1) \end{aligned}$$

$$f''(x) = 0 \text{ when } x = -1.$$

$$f''(-2) = 4e^{2(-2)}[(-2) + 1] = -4e^{-4} < 0$$

$$f''(0) = 4e^{2(0)}[(0) + 1] = 4 > 0$$

Inflection point at $(-1, -e^{-2})$

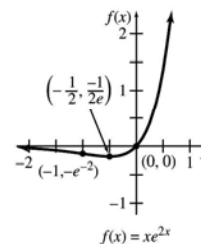
Concave upward on $(-1, \infty)$

Concave downward on $(-\infty, -1)$

$$\begin{aligned} \text{x-intercept: } xe^{2x} &= 0 \\ x &= 0 \end{aligned}$$

$$\text{y-intercept: } y = (0)e^{2(0)} = 0$$

Since $\lim_{x \rightarrow -\infty} xe^{2x} = 0$, there is a horizontal asymptote at $y = 0$.



54. $f(x) = x^2e^{2x}$

Domain is $(-\infty, \infty)$.

$$f(-x) = (-x)^2e^{2(-x)} = x^2e^{-2x}$$

The graph is not symmetric about the y-axis or origin.

$$\begin{aligned} f'(x) &= (2x)(e^{2x}) + (x^2)(2e^{2x}) \\ &= 2(x^2 + x)e^{2x} \\ &= 2x(x + 1)e^{2x} \end{aligned}$$

$$f'(x) = 0 \text{ when } x = -1 \text{ and } x = 0.$$

Critical numbers: -1 and 0

Critical points: $(-1, e^{-2})$ and $(0, 0)$

$$f'(-2) = 2[(-2)^2 + (-2)]e^{2(-2)} = 4e^{-4} > 0$$

$$\begin{aligned} f'\left(-\frac{1}{2}\right) &= 2\left[\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)\right]e^{2(-1/2)} \\ &= -\frac{1}{2}e^{-1} < 0 \end{aligned}$$

$$f'(1) = 2[(1)^2 + (1)]e^{2(1)} = 4e^2 > 0$$

Relative maximum at $(-1, e^{-2})$

Relative minimum at $(0, 0)$

Increasing on $(-\infty, -1)$ and $(0, \infty)$

Decreasing on $(-1, 0)$

$$\begin{aligned} f''(x) &= 2(2x + 1)e^{2x} + 2(x^2 + x)e^{2x}(2) \\ &= 2(2x^2 + 4x + 1)e^{2x} \end{aligned}$$

$$f''(x) = 0 \text{ when}$$

$$2x^2 + 4x + 1 = 0$$

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{16 - 4(2)(1)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{8}}{4} \\ &= -1 \pm \frac{\sqrt{2}}{2} \end{aligned}$$

$$f''(-2) = 2[2(-2)^2 + 4(-2) + 1]e^{2(-2)} = 2e^{-4} > 0$$

$$f''(-1) = 2[2(-1)^2 + 4(-1) + 1]e^{2(-1)} = -2e^{-2} < 0$$

$$f''(0) = 2[2(0)^2 + 4(0) + 1]e^{2(0)} = 2 > 0$$

Inflection points at

$$\left(-1 - \frac{\sqrt{2}}{2}, \left(\frac{3}{2} + \sqrt{2}\right)e^{-2-\sqrt{2}}\right) \approx (-1.707, 0.09588)$$

$$\left(-1 + \frac{\sqrt{2}}{2}, \left(\frac{3}{2} - \sqrt{2}\right)e^{-2+\sqrt{2}}\right) \approx (-0.293, 0.04775)$$

Concave upward on

$$\left(-\infty, -1 - \frac{\sqrt{2}}{2}\right) \text{ and } \left(-1 + \frac{\sqrt{2}}{2}, \infty\right)$$

Concave downward on

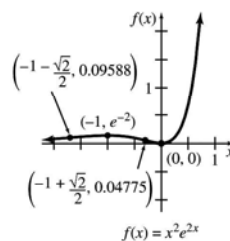
$$\left(-1 - \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}\right)$$

$$x\text{-intercept: } x^2 e^{2x} = 0$$

$$x = 0$$

$$y\text{-intercept: } y = (0)^2 e^{2(0)} = 0$$

Since $\lim_{x \rightarrow -\infty} x^2 e^{2x} = 0$, horizontal asymptote at $y = 0$.



55. $f(x) = \ln(x^2 + 4)$

Domain is $(-\infty, \infty)$.

$$f(-x) = \ln[(-x)^2 + 4] = \ln(x^2 + 4) = f(x)$$

The graph is symmetric about the y -axis.

$$f'(x) = \frac{2x}{x^2 + 4}$$

$$f'(x) = 0 \text{ when } x = 0.$$

Critical number: 0

Critical point: $(0, \ln 4)$

$$f'(-1) = \frac{2(-1)}{(-1)^2 + 4} = -\frac{2}{5} < 0$$

$$f'(1) = \frac{2(1)}{(1)^2 + 4} = \frac{2}{5} > 0$$

No relative maximum

Relative minimum at $(0, \ln 4)$

Increasing on $(0, \infty)$

Decreasing on $(-\infty, 0)$

$$\begin{aligned} f''(x) &= \frac{(x^2 + 4)(2) - (2x)(2x)}{(x^2 + 4)^2} \\ &= \frac{-2(x^2 - 4)}{(x^2 + 4)^2} \end{aligned}$$

$$f''(x) = 0 \text{ when}$$

$$x^2 - 4 = 0$$

$$x = \pm 2$$

$$f''(-3) = \frac{-2[(-3)^2 - 4]}{[(-3)^2 + 4]^2} = -\frac{10}{169} < 0$$

$$f''(0) = \frac{-2[(0)^2 - 4]}{[(0)^2 + 4]^2} = \frac{1}{2} > 0$$

$$f''(3) = \frac{-2[(3)^2 - 4]}{[(3)^2 + 4]^2} = -\frac{10}{169} < 0$$

Inflection points at $(-2, \ln 8)$ and $(2, \ln 8)$

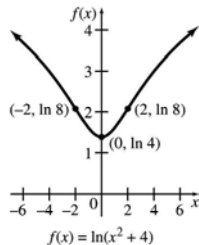
Concave upward on $(-2, 2)$

Concave downward on $(-\infty, -2)$ and $(2, \infty)$

Since $f(x)$ never equals zero, there are no x -intercepts.

y -intercept: $y = \ln[(0)^2 + 4] = \ln 4$

No horizontal or vertical asymptotes.



56. $f(x) = x^2 \ln x$

Domain is $(0, \infty)$.

The graph is not symmetric about the y -axis or origin since $f(x)$ is not defined for $x \leq 0$.

$$\begin{aligned} f'(x) &= 2x(\ln x)x^2 \cdot \frac{1}{x} \\ &= 2x \ln x + x \\ &= x(2 \ln x + 1) \end{aligned}$$

$f'(x) = 0$ when

$$2 \ln x + 1 = 0$$

$$\begin{aligned} \ln x &= -\frac{1}{2} \\ x &= e^{-1/2} \end{aligned}$$

Critical number: $e^{-1/2}$

Critical point: $(e^{-1/2}, -\frac{1}{2e})$

$$f'\left(\frac{1}{2}\right) = \frac{1}{2}\left(2 \ln \frac{1}{2} + 1\right) \approx -0.1931 < 0$$

$$f'(1) = 1(2 \ln 1 + 1) = 1 > 0$$

No relative maximum

Relative minimum at $(e^{-1/2}, -\frac{1}{2e})$

Increasing on $(e^{-1/2}, \infty)$

Decreasing on $(0, e^{-1/2})$

$$\begin{aligned} f''(x) &= (1)(2 \ln x + 1) + x\left(\frac{2}{x}\right) \\ &= 2 \ln x + 3 \\ f''(x) &= 0 \text{ when} \end{aligned}$$

$$2 \ln x + 3 = 0$$

$$\begin{aligned} \ln x &= -\frac{3}{2} \\ x &= e^{-3/2} \end{aligned}$$

$$f''(e^{-3/2}) = 2 \ln e^{-3/2} + 3 = -1 < 0$$

$$f''(1) = 2 \ln 1 + 3 = 3 > 0$$

Inflection point at $(e^{-3/2}, -\frac{3}{2e^3})$

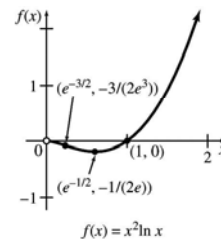
Concave upward on $(e^{-3/2}, \infty)$

Concave downward on $(0, e^{-3/2})$

$$\begin{aligned} x\text{-intercept: } x^2 \ln x &= 0 \\ \ln x &= 0 \\ x &= 1 \end{aligned}$$

Since $f(x)$ is not defined at $x = 0$, there is no y -intercept.

No horizontal or vertical asymptotes



57. $f(x) = 4x^{1/3} + x^{4/3}$

Domain is $(-\infty, \infty)$.

$$f(-x) = 4(-x)^{1/3} + (-x)^{4/3} = -4x^{1/3} + x^{4/3}$$

The graph is not symmetric about the y -axis or origin.

$$f'(x) = \frac{4}{3}x^{-2/3} + \frac{4}{3}x^{1/3}$$

$f'(x) = 0$ when

$$\frac{4}{3}x^{-2/3} + \frac{4}{3}x^{1/3} = 0$$

$$\begin{aligned} \frac{4}{3}x^{-2/3}(1 + x) &= 0 \\ x &= -1 \end{aligned}$$

$f'(x)$ is not defined when $x = 0$

Critical numbers: -1 and 0

Critical points: $(-1, -3)$ and $(0, 0)$

$$f'(-8) = \frac{4}{3}(-8)^{-2/3} + \frac{4}{3}(-8)^{1/3} = -\frac{7}{3} < 0$$

$$f'\left(-\frac{1}{8}\right) = \frac{4}{3}\left(-\frac{1}{8}\right)^{-2/3} + \frac{4}{3}\left(-\frac{1}{8}\right)^{1/3} = \frac{14}{3} > 0$$

$$f'(1) = \frac{4}{3}(1)^{-2/3} + \frac{4}{3}(1)^{1/3} = \frac{8}{3} > 0$$

No relative maximum

Relative minimum at $(-1, -3)$

Increasing on $(-1, \infty)$

Decreasing on $(-\infty, -1)$

$$f''(x) = -\frac{8}{9}x^{-5/3} + \frac{4}{9}x^{-2/3}$$

$f''(x) = 0$ when

$$-\frac{8}{9}x^{-5/3} + \frac{4}{9}x^{-2/3} = 0$$

$$\frac{4}{9}x^{-5/3}(-2 + x) = 0$$

$$x = 2$$

$f''(x)$ is not defined when $x = 0$

$$f''(-1) = -\frac{8}{9}(-1)^{-5/3} + \frac{4}{9}(-1)^{-2/3} = \frac{4}{3} > 0$$

$$f''(1) = -\frac{8}{9}(1)^{-5/3} + \frac{4}{9}(1)^{-2/3} = -\frac{4}{9} < 0$$

$$f''(8) = -\frac{8}{9}(8)^{-5/3} + \frac{4}{9}(8)^{-2/3} = \frac{1}{12} > 0$$

Inflection points at $(0, 0)$ and $(2, 6 \cdot 2^{1/3})$

Concave upward on $(-\infty, 0)$ and $(2, \infty)$

Concave downward on $(0, 2)$

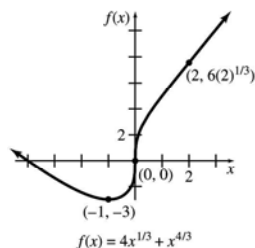
x -intercept: $4x^{1/3} + x^{4/3} = 0$

$$x^{1/3}(4 + x) = 0$$

$$x = 0 \text{ or } x = -4$$

y -intercept: $y = 4(0)^{1/3} + (0)^{4/3} = 0$

No horizontal or vertical asymptotes



58. $f(x) = 5x^{2/3} + x^{5/3}$

Domain is $(-\infty, \infty)$.

$$f(-x) = 5(-x)^{2/3} + (-x)^{5/3} = 5x^{2/3} - x^{5/3}$$

The graph is not symmetric about the y -axis or origin.

$$f'(x) = \frac{10}{3}x^{-1/3} + \frac{5}{3}x^{2/3}$$

$f'(x) = 0$ when

$$\frac{10}{3}x^{-1/3} + \frac{5}{3}x^{2/3} = 0$$

$$\frac{5}{3}x^{-1/3}(2 + x) = 0$$

$$x = -2$$

$f'(x)$ is not defined when $x = 0$

Critical numbers: -2 and 0

Critical points: $(0, 0)$ and $(-2, 3 \cdot 2^{2/3})$

$$\begin{aligned} f'(-8) &= \frac{10}{3}(-8)^{-1/3} + \frac{5}{3}(-8)^{2/3} \\ &= 5 > 0 \end{aligned}$$

$$\begin{aligned} f'(-1) &= \frac{10}{3}(-1)^{-1/3} + \frac{5}{3}(-1)^{2/3} \\ &= -\frac{5}{3} < 0 \end{aligned}$$

$$\begin{aligned} f'(1) &= \frac{10}{3}(1)^{-1/3} + \frac{5}{3}(1)^{2/3} \\ &= 5 > 0 \end{aligned}$$

Relative maximum at $(-2, 3 \cdot 2^{2/3})$

Relative minimum at $(0, 0)$

Increasing on $(-\infty, -2)$ and $(0, \infty)$

Decreasing on $(-2, 0)$

$$f''(x) = -\frac{10}{9}x^{-4/3} + \frac{10}{9}x^{-1/3}$$

$f''(x) = 0$ when

$$-\frac{10}{9}x^{-4/3} + \frac{10}{9}x^{-1/3} = 0$$

$$\frac{10}{9}x^{-4/3}(-1 + x) = 0$$

$$x = 1$$

$f''(x)$ is not defined when $x = 0$

$$\begin{aligned} f''(-1) &= -\frac{10}{9}(-1)^{-4/3} + \frac{10}{9}(-1)^{-1/3} \\ &= -\frac{20}{9} < 0 \end{aligned}$$

$$f''\left(\frac{1}{8}\right) = -\frac{10}{9}\left(\frac{1}{8}\right)^{-4/3} + \frac{10}{9}\left(\frac{1}{8}\right)^{-1/3}$$

$$= -\frac{140}{9} < 0$$

$$f''(8) = -\frac{10}{9}(8)^{-4/3} + \frac{10}{9}(8)^{-1/3}$$

$$= \frac{35}{72} > 0$$

Concave upward on $(1, \infty)$

Concave downward on $(-\infty, 0)$ and $(0, 1)$

Inflection point at $(1, 6)$

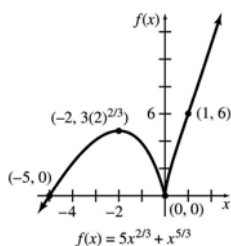
$$x\text{-intercept: } 5x^{2/3} + x^{5/3} = 0$$

$$x^{2/3}(5 + x) = 0$$

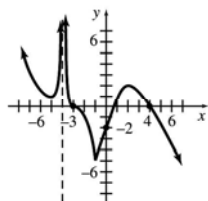
$$x = 0 \text{ or } x = -5$$

$$y\text{-intercept: } y = 5(0)^{2/3} + (0)^{5/3} = 0$$

No horizontal or vertical asymptotes.

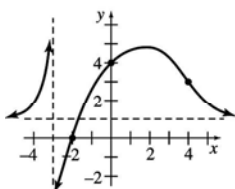


59.



Other graphs are possible.

60.



Other graphs are possible.

61. (a)-(b) If the price of the stock is falling faster and faster, $P(t)$ would be decreasing, so $P'(t)$ would be negative. $P(t)$ would be concave downward, so $P''(t)$ would also be negative.

62. (a)-(b) When a stock reaches its highest price of the day, $P(t)$ is at a maximum. Maxima and minima occur when $P'(t) = 0$. Since this is a maximum, the graph would be concave down. Therefore, $P''(t) < 0$.

63. (a) Profit = Income - Cost

$$P(q) = qp - C(q)$$

$$= q(-q^2 - 3q + 299)$$

$$= -q^3 - 3q^2 + 299q$$

$$= -q^3 + 7q^2 + 49q$$

$$(b) P'(q) = -3q^2 + 14q + 49$$

$$= (-3q - 7)(q - 7)$$

$$= -(3q + 7)(q - 7)$$

$$q = \frac{7}{3} \text{ (nonsensical) or } q = 7$$

$$P''(q) = -6q + 14$$

$$P''(7) = -6 \cdot 7 + 14 = -28 < 0$$

(indicates a maximum)

7 brushes would produce the maximum profit.

$$(c) p = -7^2 - 3(7) + 299$$

$$= -49 - 21 + 299 = 229$$

\$229 is the price that produces the maximum profit.

$$(d) P(7) = -7^3 + 7(7^2) + 49(7) = 343$$

The maximum profit is \$343.

$$(e) P''(q) = 0 \text{ when } -6q + 14 = 0$$

$$q = \frac{7}{3}$$

$$P''(2) = -6(2) + 14 = 2 > 0$$

$$P''(3) = -6 \cdot 3 + 14 = -4 < 0$$

The point of diminishing returns is $q = \frac{7}{3}$ (between 2 and 3 brushes).

64. $p(t) = -0.614t^3 + 6.26t^2 + 1.94t + 297$
for $0 < t < 12$, t in months and p in cents

$$p'(t) = -1.842t^2 + 12.5x + 1.94$$

Find the critical numbers.

$$p'(t) = -1.842t^2 + 12.5x + 1.94 = 0$$

Using a graphing calculator, we find that the only zero of p' in the interval $0 < t < 12$ is $t \approx 6.94$.

Evaluate $p'(t)$ on either side of this critical number.

$$p'(5) = 18.39$$

$$p'(10) = -57.26$$

- (a) The price of gasoline increased on the interval $(0, 6.94)$, that is, up until July.
- (b) The price of gasoline decreased on the interval $(6.94, 12)$, that is, after July.
- (c) The price had a relative maximum in July of $p(6.94) \approx 406.3$, or 406.3 cents.

65. (a) Since the second derivative has many sign changes, the graph continually changes from concave upward to concave downward. Since there is a nonlinear decline, the graph must be one that declines, levels off, declines, levels off, etc. Therefore, the first derivative has many critical numbers where the first derivative is zero.
- (b) The curve is always decreasing except at frequent points of inflection.

66. (a) $Y(M) = Y_0 M^b$
 $Y'(M) = bY_0 M^{b-1}$
 $Y''(M) = b(b-1)Y_0 M^{b-2}$

When $b > 0$, $Y' > 0$, so metabolic rate and life span are increasing function of mass.

When $b < 0$, $Y' < 0$, so heartbeat is a decreasing function of mass.

When $0 < b < 1$, $b(b-1) < 0$, so metabolic rate and life span have graphs that are concave downward.

When $b < 0$, $b(b-1) > 0$, so heartbeat has a graph that is concave upward.

(b) $\frac{dY}{dM} = bY_0 M^{b-1} = \frac{b}{M} Y_0 M^b$
 $= \frac{b}{M} Y$

67. Sketch the curve for $l_1(v) = 0.08e^{0.33v}$

$$l_1'(v) = 0.0264e^{0.33v}$$

$$e^{0.33v} \neq 0$$

$l_1(v)$ has no critical points.

$$l_1''(v) = 0.008712e^{0.33v}$$

$$e^{0.33v} \neq 0$$

$l_1(v)$ has no inflection points.

Sketch the curve for $l_2 = -0.87v^2 + 28.17v - 211.41$

$$l_2'(v) = -1.74v + 28.17$$

$$-1.74v + 28.17 = 0$$

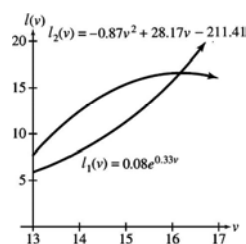
$$v \approx 16.19$$

Critical point: $(16.19, 16.62)$

$$l_2''(v) = -1.74$$

$l_2(v)$ has no inflection points.

$l_2(v)$ has a relative maximum at $(16.19, 16.62)$.



68. Let $a = 0.25$.

$$f(v) = v(0.25 - v)(v - 1)$$

$$f'(v) = -v^3 + 1.25v^2 - 0.25v$$

$$= -3v^2 + 2.5v - 0.25$$

By the quadratic formula, $f'(v) = 0$ when $v = 0.12$ and $v \approx 0.72$.

Critical numbers $v \approx 0.12$ and $v \approx 0.72$.

$$f'(0.1) = -0.03 < 0$$

$$f'(0.5) = 0.25 > 0$$

$$f'(1) = -0.75 < 0$$

The function is decreasing on $(0, 0.12)$ and $(0.72, 1)$ and increasing on $(0.12, 0.72)$.

$$f(0.12) \approx -0.01$$

$$f(0.72) \approx 0.09$$

Relative minimum at $(0.12, -0.01)$, relative maximum at $(0.72, 0.09)$.

$$f''(v) = -6v + 2.5$$

$$f''(v) = 0 \text{ when}$$

$$-6v + 2.5 = 0$$

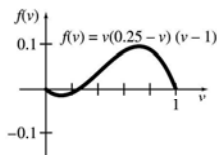
$$v \approx 0.42$$

$$f''(0.1) = 1.9 > 0$$

$$f''(0.5) = -0.5 < 0$$

Concave upward on $(0, 0.42)$

Concave downward on $(0.42, 1)$



69. $y = 34.7(1.0186)^{-x}(x^{-0.658})$

In chapter 4, the function was originally defined as

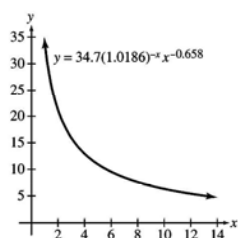
$$\log y = 1.54 - 0.008x - 0.658 \log x$$

$$\text{so, } 0 < x < \infty, \text{ and } 0 < y < \infty.$$

The function will have a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$.

$$\frac{dy}{dx} = -34.7(1.0186)^{-x}x^{-0.658} \left[\ln(1.0186) + \frac{0.658}{x} \right]$$

For every value in the domain, $\frac{dy}{dx} < 0$, so y has no critical points and is decreasing on $(0, \infty)$.



70. (a) Set the two formulas equals to each other.

$$1486S^2 - 4106S + 4514 = 1486S - 825$$

$$1486S^2 - 5592S + 5339 = 0$$

Take the derivative.

$$2972S - 5592 = 0$$

Solve for S .

$$S \approx 1.88$$

For males with 1.88 square meters of surface area, the red cell volume increases approximately 1486 ml for each additional square meter of surface area.

- (b) Set the formulas equal to each other.

$$995e^{0.6085S} = 1578S$$

$$995e^{0.6085S} - 1578S = 0$$

Take the derivative.

Chapter 5 GRAPHS AND THE DERIVATIVE

$$605.4575e^{0.6085S} - 1578 = 0$$

$$e^{0.6085S} \approx 2.6063$$

$$0.6085S \approx \ln 2.6063$$

$$S \approx 1.57 \text{ square meters}$$

By plugging the exact value of S into the two formulas given for PV , we get about 2593 ml (Hurley) and 2484 for Pearson et al.

- (c) For males with 1.57 square meters of surface area, the red cell volume increases approximately 1578 ml for each additional square meter of surface area.

- (d) When f and g are closest together, their absolute difference is minimized.

$$\left. \frac{d}{dx} [f(x) - g(x)] \right|_{x=x_0} = 0$$

$$f'(x_0) - g'(x_0) = 0$$

$$f'(x_0) = g'(x_0)$$

71. $y(t) = A^{c^t}$

$$y'(t) = (\ln A)A^{c^t} \cdot \frac{d}{dt}c^t$$

$$= (\ln A)(\ln c)c^t A^{c^t}$$

$$y''(t) = (\ln A)(\ln c) \cdot [(\ln c)c^t A^{c^t}$$

$$+ c^t(\ln A)(\ln c)c^t A^{c^t}]$$

$$= (\ln A)(\ln c)^2 c^t A^{c^t} [1 + (\ln A)c^t]$$

$$y''(t) = 0 \text{ when } 1 + \ln(A)c^t = 0$$

$$c^t = -\frac{1}{\ln A}$$

$$t \ln c = \ln \left(-\frac{1}{\ln A} \right)$$

$$t = -\frac{\ln(-\ln A)}{\ln c}$$

$$= -\frac{\ln[-\ln(0.3982 \cdot 10^{-291})]}{\ln 0.4152}$$

By properties of logarithms,

$$-\ln(0.3982 \cdot 10^{-291}) = -[\ln(0.3982) + \ln(10^{-291})]$$

$$= -[\ln(0.3982) - 291 \ln(10)]$$

$$= -\ln(0.3982) + 291 \ln(10)$$

So,

$$t = -\frac{\ln[-\ln(0.3982) + 291 \ln(10)]}{\ln(0.4152)}$$

$$\approx 7.405$$

At about 7.405 years the rate of learning to pass the test begins to slow down.

72. (a) $P(t) = 325 + 7.475(t + 10)e^{-(t+10)/20}$

$$\begin{aligned} P'(t) &= 7.475 \left[1 \cdot e^{-(t+10)/20} + (t + 10)e^{-(t+10)/20} \cdot \frac{-1}{20} \right] \\ &= 7.475 \left[1 - \frac{1}{20}(t + 10) \right] e^{-(t+10)/20} \\ &= 7.475 \left(\frac{1}{2} - \frac{t}{20} \right) e^{-(t+10)/20} \end{aligned}$$

$P'(t)$ is zero when

$$\begin{aligned} \frac{1}{2} - \frac{t}{20} &= 0 \\ t &= 10 \end{aligned}$$

$$p'(9) \approx 0.39$$

$$p'(11) \approx -0.36$$

$P(t)$ is increasing at 9 and decreasing at 11. So, a relative maximum occurs at $t = 10$. The population is largest in the year 2010.

$$\begin{aligned} P(10) &= 325 + 7.475(20)e^{-1} \\ &\approx 380.0 \end{aligned}$$

The population is predicted to be 380 million in 2010.

(b) $P'(t) = 7.475 \left(\frac{1}{2} - \frac{t}{20} \right) e^{-(t+10)/20}$

$$\begin{aligned} P''(t) &= 7.475 \left[-\frac{1}{20} e^{-(t+10)/20} + \left(\frac{1}{2} - \frac{t}{20} \right) e^{-(t+10)/20} \cdot \frac{-1}{20} \right] \\ &= 7.475 \left(-\frac{1}{20} - \frac{1}{40} + \frac{t}{400} \right) e^{-(t+10)/20} \\ &= 7.475 \left(\frac{t}{400} - \frac{3}{40} \right) e^{-(t+10)/20} \end{aligned}$$

$P'(t)$ is zero when

$$\begin{aligned} \frac{t}{400} - \frac{3}{40} &= 0 \\ t &= 30 \end{aligned}$$

$$P'(29) = -0.0027$$

$$P'(31) = 0.0024$$

P' is decreasing at 29, and increasing at 31. So, a relative minimum occurs at $t = 30$. The population is declining most rapidly in the year 2030.

(c) As time t approaches infinity, the population P approaches

$$\begin{aligned} \lim_{t \rightarrow \infty} P(t) &= \lim_{t \rightarrow \infty} \left(325 + 7.475(t + 10)e^{-(t+10)/20} \right) \\ &= 325 + 7.475 \lim_{t \rightarrow \infty} \frac{t + 10}{e^{(t+10)/20}} \\ &= 325 + 7.475(0) \\ &= 325. \end{aligned}$$

The population is approaching 325 million.

73. (a) The U.S. total inventory was at a relative maximum in 1965, 1973, 1976, 1983, 1986, and 1988.
- (b) The U.S. total inventory was at its largest relative maximum from 1965 to 1967. During this period, the Soviet total inventory was concave upward. This means that the total inventory was increasing at an increasingly rapid rate.

74. (a) $s(t) = 512t - 16t^2$
 $v(t) = s'(t) = 512 - 32t$
 $a(t) = v'(t) = s''(t) = -32$

(b) The maximum height is attained when $v(t) = 0$.

$$512 - 32t = 0$$

$$t = 16$$

$$v(0) = 512 > 0$$

$$v(20) = 512 - 640 = -128 < 0$$

The height reaches a maximum when $t = 16$.

$$s(16) = 512 \cdot 16 - 16(16^2) = 4096$$

The maximum height is 4096 ft.

(c) The projectile hits the ground when $s(t) = 0$.

$$512t - 16t^2 = 0$$

$$16t(32 - t) = 0$$

$$t = 0 \text{ or } t = 32$$

$$v(32) = 512 - 32(32) = -512$$

The projectile hits the ground after 32 seconds with a velocity of -512 ft/sec.

Extended Application: A Drug Concentration Model for Orally Administered Medications

- Let $C(t)$ stand for the steady-state concentration in mcg/mL at time t measured in hours, with $0 \leq t \leq 12$.

$$C(t) = 1.99(500)e^{-0.14t} - 0.162(500)e^{-2.08t}$$

$$C'(t) = -139.3e^{-0.14t} + 1685e^{-2.08t}$$

Using a graphing calculator we find that

$C'(t) = 0$ when $t \approx 1.28$. Testing values of C' on either side of this time, we find

$$C'(0.5) = 465.62$$

$$C'(2) = -78.99$$

Since C is increasing before $t = 1.28$ and decreasing after, $t = 1.28$ represents a local maximum. The value of C at this time is $C(1.28) \approx 775.2$ so the maximum steady-state concentration is 775.2 mcg/mL.

The minimum concentration will occur at one of the endpoints, so we evaluate $C(0)$ and $C(12)$.

$$C(0) = 185$$

$$C(12) \approx 185$$

Thus the minimum steady state concentration of 185 mcg/mL occurs immediately after a dose and immediately before the next dose.

- If the dose is increased to 1500 mg, the factor of 500 in the formula for C changes to 1500. This does not affect the zero of the derivative, but it multiplies all the values of C by a factor of 3. Thus the maximum concentration is now $(775.2)(3) = 2325.6$ at $t = 1.28$, and the minimum concentration is $(185)(3) = 555$ at $t = 0$ and $t = 12$.
- $\frac{400}{775.2} \approx 0.52$ and $\frac{80}{185} \approx 0.43$. So if we multiply the original 500 mg dose by a fraction between these two values, we will keep the concentration within the required range. For example, if we give a dose of $(0.5)(500) = 250$ mg, the maximum concentration will be $(0.5)(775.2) = 387.6$ mcg/mL and the minimum concentration will be $(0.5)(185) = 92.5$ mcg/mL.

APPLICATIONS OF THE DERIVATIVE

6.1 Absolute Extrema

Your Turn 1

To find the absolute extrema of $f(x) = 3x^{2/3} - 3x^{5/3}$ on $[0, 8]$, first find the critical numbers in the interval $(0, 8)$.

$$\begin{aligned} f'(x) &= 2x^{-1/3} - 5x^{2/3} \\ &= \frac{2}{x^{1/3}} - 5x^{2/3} \\ &= \frac{2}{x^{1/3}} - \frac{5x}{x^{1/3}} \\ &= \frac{1}{x^{1/3}}(2 - 5x) \end{aligned}$$

$f'(x)$ is 0 when $2 - 5x = 0$, that is, when $x = \frac{2}{5} = 0.4$. Now evaluate the function f at the critical value 0.4 and at the endpoints 0 and 8.

x	$f(x)$
0	0
0.4	0.977
8	-84

The absolute maximum of approximately 0.977 occurs when $x = 2/5$; the absolute minimum of -84 occurs when $x = 8$.

Your Turn 2

The domain of the function

$$f(x) = -x^4 - 4x^3 + 8x^2 + 20$$

is the open interval $(-\infty, \infty)$. As x approaches $+\infty$ and $-\infty$ the dominant term is $-x^4$, and this approaches $-\infty$. Thus the function has no absolute minimum. To find the absolute maximum, we evaluate f at the critical points.

$$\begin{aligned} f'(x) &= -4x^3 - 12x^2 + 16x \\ &= -4x(x^2 + 3x - 4) \\ &= -4x(x + 4)(x - 1) \end{aligned}$$

Setting $f'(x)$ equal to 0, we have $x = 0$, $x = -4$, or $x = 1$.

x	$f(x)$
-4	148
0	20
1	23

The absolute maximum is 148, at $x = -4$. As noted above, there is no absolute minimum.

6.1 Exercises

- As shown on the graph, the absolute maximum occurs at x_3 ; there is no absolute minimum. (There is no functional value that is less than all others.)
- As shown on the graph, the absolute minimum occurs at x_1 ; there is no absolute maximum. (There is no functional value that is greater than all others.)
- As shown on the graph, there are no absolute extrema.
- As shown on the graph, there are no absolute extrema.
- As shown on the graph, the absolute minimum occurs at x_1 ; there is no absolute maximum.
- As shown on the graph, the absolute maximum occurs at x_1 ; there is no absolute minimum.
- As shown on the graph, the absolute maximum occurs at x_1 ; the absolute minimum occurs at x_2 .
- As shown on the graph, the absolute maximum occurs at x_2 ; the absolute minimum occurs at x_1 .

11. $f(x) = x^3 - 6x^2 + 9x - 8; [0, 5]$

Find critical numbers:

$$\begin{aligned} f'(x) &= 3x^2 - 12x + 9 = 0 \\ x^2 - 4x + 3 &= 0 \\ (x - 3)(x - 1) &= 0 \\ x &= 1 \text{ or } x = 3 \end{aligned}$$

x	$f(x)$	
0	-8	Absolute minimum
1	-4	
3	-8	Absolute minimum
5	12	

12. $f(x) = x^3 - 3x^2 - 24x + 5; [-3, 6]$

Find critical numbers:

$$\begin{aligned} f'(x) &= 3x^2 - 6x - 24 = 0 \\ 3(x^2 - 2x - 8) &= 0 \\ 3(x + 2)(x - 4) &= 0 \\ x &= -2 \text{ or } x = 4 \end{aligned}$$

x	$f(x)$	
-3	23	Absolute maximum
-2	33	
4	-75	Absolute minimum
6	-31	

13. $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x + 1; [-5, 2]$

Find critical numbers:

$$\begin{aligned} f'(x) &= x^2 + 3x - 4 = 0 \\ (x + 4)(x - 1) &= 0 \\ x &= -4 \text{ or } x = 1 \end{aligned}$$

x	$f(x)$	
-4	$\frac{59}{3} \approx 19.67$	Absolute maximum
1	$-\frac{7}{6} \approx -1.17$	
-5	$\frac{101}{6} \approx 16.83$	Absolute minimum
2	$\frac{5}{3} \approx 1.67$	

14. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 3; [-4, 4]$

Find critical numbers:

$$\begin{aligned} f'(x) &= x^2 - x - 6 = 0 \\ (x + 2)(x - 3) &= 0 \\ x &= -2 \text{ or } x = 3 \end{aligned}$$

x	$f(x)$	
-4	$-\frac{7}{3} \approx -2.3$	Absolute maximum
-2	$\frac{31}{3} \approx 10.33$	
3	$-\frac{21}{2} \approx -10.5$	Absolute minimum
4	$-\frac{23}{3} \approx -7.7$	

15. $f(x) = x^4 - 18x^2 + 1; [-4, 4]$

$$\begin{aligned} f'(x) &= 4x^3 - 36x = 0 \\ 4x(x^2 - 9) &= 0 \\ 4x(x + 3)(x - 3) &= 0 \\ x &= 0 \text{ or } x = -3 \text{ or } x = 3 \end{aligned}$$

x	$f(x)$	
-4	-31	Absolute minimum
-3	-80	
0	1	Absolute maximum
3	-80	
4	-31	Absolute minimum

16. $f(x) = x^4 - 32x^2 - 7; [-5, 6]$

$$\begin{aligned} f'(x) &= 4x^3 - 64x = 0 \\ 4x(x^2 - 16) &= 0 \\ 4x(x - 4)(x + 4) &= 0 \\ x &= 0 \text{ or } x = 4 \text{ or } x = -4 \end{aligned}$$

x	$f(x)$	
-5	-182	Absolute minimum
-4	-263	
0	-7	Absolute maximum
4	-263	
6	137	Absolute minimum

17. $f(x) = \frac{1-x}{3+x}; [0, 3]$

$$f'(x) = \frac{-4}{(3+x)^2}$$

No critical numbers

x	$f(x)$	
0	$\frac{1}{3}$	Absolute maximum
3	$-\frac{1}{3}$	Absolute minimum

18. $f(x) = \frac{8+x}{8-x}; [4, 6]$

$$f'(x) = \frac{(8-x)(1) - (8+x)(-1)}{(8-x)^2}$$

$$= \frac{16}{(8-x)^2}$$

$f'(x)$ is never zero. Although $f'(x)$ fails to exist if $x = 8$, 8 is not in the given interval.

x	$f(x)$	
4	3	Absolute minimum
6	7	Absolute maximum

19. $f(x) = \frac{x-1}{x^2+1}; [1, 5]$

$$f'(x) = \frac{-x^2 + 2x + 1}{(x^2 + 1)^2}$$

$$f'(x) = 0 \text{ when}$$

$$-x^2 + 2x + 1 = 0$$

$$x = 1 \pm \sqrt{2},$$

but $1 - \sqrt{2}$ is not in $[1, 5]$.

x	$f(x)$	
1	0	Absolute minimum
5	$\frac{2}{13} \approx 0.15$	
$1 + \sqrt{2}$	$\frac{\sqrt{2}-1}{2} \approx 0.21$	Absolute maximum

20. $f(x) = \frac{x}{x^2+2}; [0, 4]$

$$f'(x) = \frac{(x^2+2)1 - x(2x)}{(x^2+2)^2}$$

$$= \frac{-x^2+2}{(x^2+2)^2} = 0$$

$$-x^2 + 2 = 0$$

$$x^2 = 2$$

$$x = \sqrt{2} \text{ or } x = -\sqrt{2}, \text{ but } -\sqrt{2} \text{ is not in } [0, 4].$$

$f'(x)$ is defined for all x .

x	$f(x)$	
0	0	Absolute minimum
$\sqrt{2}$	$\frac{\sqrt{2}}{4} \approx 0.35$	Absolute maximum
4	$\frac{2}{9} \approx 0.22$	

21. $f(x) = (x^2 - 4)^{1/3}; [-2, 3]$

$$f'(x) = \frac{1}{3}(x^2 - 4)^{-2/3}(2x)$$

$$= \frac{2x}{3(x^2 - 4)^{2/3}}$$

$$f'(x) = 0 \text{ when } 2x = 0$$

$$x = 0$$

$f'(x)$ is undefined at $x = -2$ and $x = 2$, but $f(x)$ is defined there, so -2 and 2 are also critical numbers:

x	$f(x)$	
-2	0	
0	$(-4)^{1/3} \approx -1.587$	Absolute minimum
2	0	
3	$5^{1/3} \approx 1.710$	Absolute maximum

22. $f(x) = (x^2 - 16)^{2/3}; [-5, 8]$

$$f'(x) = \frac{2}{3}(x^2 - 16)^{-1/3}(2x) = \frac{4x}{3(x^2 - 16)^{1/3}}$$

$$f'(x) = 0 \text{ when } 4x = 0$$

$$x = 0$$

$f'(x)$ is undefined at $x = -4$ and $x = 4$; but $f(x)$ is defined there, so -4 and 4 are also critical numbers.

x	$f(x)$	
-5	$9^{2/3} \approx 4.327$	
-4	0	Absolute minimum
0	$(-16)^{2/3} \approx 6.350$	
4	0	Absolute minimum
8	$48^{2/3} \approx 13.208$	Absolute maximum

23. $f(x) = 5x^{2/3} + 2x^{5/3}; [-2, 1]$

$$\begin{aligned} f'(x) &= \frac{10}{3}x^{-1/3} + \frac{10}{3}x^{2/3} \\ &= \frac{10}{3x^{1/3}} + \frac{10x^{2/3}}{3} \\ &= \frac{10x + 10}{3x^{1/3}} \\ &= \frac{10(x+1)}{3\sqrt[3]{x}} \end{aligned}$$

$$\begin{aligned} f'(x) = 0 \text{ when } 10(x+1) &= 0 \\ x+1 &= 0 \\ x &= -1. \end{aligned}$$

$f'(x)$ is undefined at $x = 0$, but $f(x)$ is defined at $x = 0$, so 0 is also a critical number.

x	$f(x)$	
-2	1.587	
-1	3	
0	0	Absolute minimum
1	7	Absolute maximum

24. $f(x) = x + 3x^{2/3}; [-10, 1]$

$$\begin{aligned} f'(x) &= 1 + 2x^{-1/3} \\ &= 1 + \frac{2}{\sqrt[3]{x}} \\ &= \frac{\sqrt[3]{x} + 2}{\sqrt[3]{x}} \end{aligned}$$

$$\begin{aligned} f'(x) = 0 \text{ when } \sqrt[3]{x} + 2 &= 0 \\ \sqrt[3]{x} &= -2 \\ x &= -8 \end{aligned}$$

$f'(x)$ is undefined at $x = 0$, but $f(x)$ is defined at $x = 0$, so 0 is also a critical number.

x	$f(x)$	
-10	3.925	
-8	4	Absolute maximum
0	0	Absolute minimum
1	4	Absolute maximum

25. $f(x) = x^2 - 8 \ln x; [1, 4]$

$$f'(x) = 2x - \frac{8}{x}$$

$$f'(x) = 0 \text{ when } 2x - \frac{8}{x} = 0$$

$$2x = \frac{8}{x}$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = -2 \text{ or } x = 2$$

but $x = -2$ is not in the given interval.

Although $f'(x)$ fails to exist at $x = 0$, 0 is not in the specified domain for $f(x)$, so 0 is not a critical number.

x	$f(x)$	
1	1	
2	-1.545	Absolute minimum
4	4.910	Absolute maximum

26. $f(x) = \frac{\ln x}{x^2}; [1, 4]$

$$\begin{aligned} f'(x) &= \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{x^4} \\ &= \frac{x - 2x \ln x}{x^4} \\ &= \frac{x(1 - 2 \ln x)}{x^4} \\ &= \frac{1 - 2 \ln x}{x^3} \end{aligned}$$

$$f'(x) = 0 \text{ when } 1 - 2 \ln x = 0$$

$$-2 \ln x = -1$$

$$\ln x = \frac{1}{2}$$

$$x = e^{1/2}$$

Although $f'(x)$ fails to exist at $x = 0$, 0 is not in the specified domain for $f(x)$, so 0 is not a critical number.

x	$f(x)$	
0	0	Absolute minimum
$e^{1/2}$	0.1839	Absolute maximum
4	0.0866	

27. $f(x) = x + e^{-3x}; [-1, 3]$
 $f'(x) = 1 - 3e^{-3x}$

$$\begin{aligned} f'(x) = 0 \text{ when } 1 - 3e^{-3x} &= 0 \\ -3e^{-3x} &= -1 \\ e^{-3x} &= \frac{1}{3} \\ -3x &= \ln \frac{1}{3} \\ x &= \frac{\ln 3}{3} \end{aligned}$$

x	$f(x)$	
-1	19.09	Absolute maximum
$\frac{\ln 3}{3}$	0.6995	Absolute minimum
3	3.000	

28. $f(x) = x^2 e^{-0.5x}; [2, 5]$
 $f'(x) = x^2 \left(-\frac{1}{2} e^{-0.5x} \right) + 2x e^{-0.5x}$
 $= e^{-0.5x} \left(2x - \frac{1}{2} x^2 \right)$
 $f'(x) = 0 \text{ when}$
 $e^{-0.5x} \left(2x - \frac{1}{2} x^2 \right) = 0$
 $x = 0 \text{ or } x = 4$

but $x = 0$ is not in the given interval.

x	$f(x)$	
2	1.472	Absolute minimum
4	2.165	Absolute maximum
5	2.052	

29. $f(x) = \frac{-5x^4 + 2x^3 + 3x^2 + 9}{x^4 - x^3 + x^2 + 7}; [-1, 1]$

The indicated domain tells us the x -values to use for the viewing window, but we must experiment to find a suitable range for the y -values. In order to show the absolute extrema on $[-1, 1]$, we find that a suitable window is $[-1, 1]$ by $[0, 1.5]$ with Xscl = 0.1, Yscl = 0.1.

From the graph, we see that on $[-1, 1]$, f has an absolute maximum of 1.356 at about 0.6085 and an absolute minimum of 0.5 at -1 .

30. $f(x) = \frac{x^3 + 2x + 5}{x^4 + 3x^3 + 10}; [-3, 0]$

The indicated domain tells us the x -values to use for the viewing window, but we must experiment to find a suitable range for the y -values. In order to show the absolute extrema on $[-3, 0]$, we find that a suitable window is $[-3, 0]$ by $[-9, 1]$.

From the graph, we see that on $[-3, 0]$, f has an absolute maximum of 0.5 at 0 and an absolute minimum of 8.10 at about -2.35 .

31. $f(x) = 2x + \frac{8}{x^2} + 1, x > 0$
 $f'(x) = 2 - \frac{16}{x^3}$
 $= \frac{2x^3 - 16}{x^3}$
 $= \frac{2(x - 2)(x^2 + 2x + 4)}{x^3}$

Since the specified domain is $(0, \infty)$, a critical number is $x = 2$.

x	$f(x)$
2	7

There is an absolute minimum of 7 at $x = 2$; there is no absolute maximum, as can be seen by looking at the graph of f .

32. $f(x) = 12 - x - \frac{9}{x}, x > 0$
 $f'(x) = -1 + \frac{9}{x^2}$
 $= \frac{9 - x^2}{x^2}$
 $= \frac{(3 + x)(3 - x)}{x^2}$

$f'(x) = 0$ when $x = -3$ or $x = 3$, and $f'(x)$ does not exist when $x = 0$. However, the specified domain for f is $(0, \infty)$. Since -3 and 0 are not in the domain of f , the only critical number is 3.

x	$f(x)$
3	6

There is an absolute maximum of 6 at $x = 3$. There is no absolute minimum, as can be seen by looking at the graph of f .

33. $f(x) = -3x^4 + 8x^3 + 18x^2 + 2$

$$f'(x) = -12x^3 + 24x^2 + 36x$$

$$= -12x(x^2 - 2x - 3)$$

$$= -12x(x - 3)(x + 1)$$

Critical numbers are 0, 3, and -1 .

x	$f(x)$
-1	9
0	2
3	137

There is an absolute maximum of 137 at $x = 3$; there is no absolute minimum, as can be seen by looking at the graph of f .

34. $f(x) = x^4 - 4x^3 + 4x^2 + 1$

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$= 4x(x^2 - 3x + 2)$$

$$= 4x(x - 2)(x - 1)$$

The critical numbers are 0, 1, and 2.

x	$f(x)$
0	1
1	2
2	1

There is no absolute maximum, as can be seen by looking at the graph of f . There is an absolute minimum of 1 at $x = 0$ and $x = 2$.

35. $f(x) = \frac{x - 1}{x^2 + 2x + 6}$

$$f'(x) = \frac{(x^2 + 2x + 6)(1) - (x - 1)(2x + 2)}{(x^2 + 2x + 6)^2}$$

$$= \frac{x^2 + 2x + 6 - 2x^2 + 2}{(x^2 + 2x + 6)^2}$$

$$= \frac{-x^2 + 2x + 8}{(x^2 + 2x + 6)^2}$$

$$= \frac{-(x^2 - 2x - 8)}{(x^2 + 2x + 6)^2}$$

$$= \frac{-(x - 4)(x + 2)}{(x^2 + 2x + 6)^2}$$

Critical numbers are 4 and -2 .

x	$f(x)$
-2	$-\frac{1}{2}$
4	0.1

There is an absolute maximum of 0.1 at $x = 4$ and an absolute minimum of 0.5 at $x = -2$. This can be verified by looking at the graph of f .

36. $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}$$

$$= \frac{1 - x^2}{(x^2 + 1)^2}$$

$$= \frac{(1 + x)(1 - x)}{(x^2 + 1)^2}$$

The critical numbers are -1 and 1.

x	$f(x)$
-1	-0.5
1	0.5

There is an absolute maximum of 0.5 at $x = 1$ and an absolute minimum of -0.5 at $x = -1$. This can be verified by looking at the graph of f .

37. $f(x) = \frac{\ln x}{x^3}$

$$f'(x) = \frac{x^3 \cdot \frac{1}{x} - 3x^2 \ln x}{x^6}$$

$$= \frac{x^2 - 3x^2 \ln x}{x^6}$$

$$= \frac{x^2(1 - 3 \ln x)}{x^6}$$

$$= \frac{1 - 3 \ln x}{x^4}$$

$f'(x) = 0$ when $x = e^{1/3}$, and $f'(x)$ does not exist when $x \leq 0$. The only critical number is $e^{1/3}$.

x	$f(x)$
$e^{1/3}$	$\frac{1}{3}e^{-1} \approx 0.1226$

There is an absolute maximum of 0.1226 at $x = e^{1/3}$. There is no absolute minimum, as can be seen by looking at the graph of f .

38. $f(x) = x \ln x$

$$f'(x) = x \cdot \frac{1}{x} + 1 \cdot \ln x \\ = 1 + \ln x$$

$f'(x) = 0$ when $x = e^{-1}$, and $f'(x)$ does not exist when $x \leq 0$. The only critical number is e^{-1} .

x	$f(x)$
e^{-1}	$-e^{-1} \approx -0.3679$

There is an absolute minimum of -0.3679 at $x = e^{-1}$. There is no absolute maximum, as can be seen by looking at the graph of f .

39. $f(x) = 2x - 3x^{2/3}$

$$f'(x) = 2 - 2x^{-1/3} = 2 - \frac{2}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x} - 2}{\sqrt[3]{x}}$$

$$f'(x) = 0 \text{ when } 2\sqrt[3]{x} - 2 = 0 \\ 2\sqrt[3]{x} = 2 \\ \sqrt[3]{x} = 1 \\ x = 1$$

$f'(x)$ is undefined at $x = 0$, but $f(x)$ is defined at $x = 0$. So the critical numbers are 0 and 1.

(a) On $[-1, 0.5]$

x	$f(x)$
-1	-5
0	0
1	-1
0.5	-0.88988

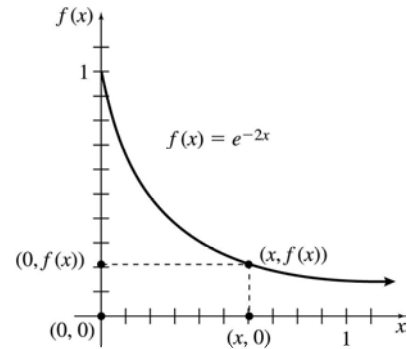
Absolute minimum of -5 at $x = -1$;
absolute maximum of 0 at $x = 0$

(b) On $[0.5, 2]$

x	$f(x)$
0.5	-0.88988
1	-1
2	-0.7622

Absolute maximum of about -0.76 at $x = 2$; absolute minimum of -1 at $x = 1$.

40. Let $P(x)$ be the perimeter of the rectangle with vertices $(0, 0)$, $(x, 0)$, $(x, f(x))$, and $(0, f(x))$ for $x > 0$ when $f(x) = e^{-2x}$.



The length of the rectangle is x and the width is given by e^{-2x} . Therefore, an equation for the perimeter is

$$P(x) = x + e^{-2x} + x + e^{-2x} \\ = 2(x + e^{-2x}).$$

$$P'(x) = 2 - 4e^{-2x}$$

$$P'(x) = 0 \text{ when } 2 - 4e^{-2x} = 0 \\ -4e^{-2x} = -2 \\ e^{-2x} = \frac{1}{2} \\ e^{2x} = 2 \\ 2x = \ln 2 \\ x = \frac{\ln 2}{2}$$

x	$P(x)$
$\frac{\ln 2}{2}$	$1 + \ln 2 \approx 1.693$

There is an absolute minimum of 1.693 at $x = \frac{\ln 2}{2}$. There is no absolute maximum, as can be seen by looking at the graph of P . Therefore, the correct statement is a.

41. (a) Looking at the graph we see that there are relative maxima of 8496 in 2001, 7556 in 2004, 6985 in 2006, and 6700 in 2008. There are relative minima of 7127 in 2000, 7465 in 2003, 6748 in 2005, 5933 in 2007, and 5943 in 2009.
- (b) The absolute maximum is 8496 (in 2001) and the absolute minimum is 5933 (in 2007).

42. (a) Looking at the graph we see that there are relative maxima of 341 in 2000 and 218 in 2007. There are relative minima of 13 in 2003 and 100 in 2009.
- (b) The absolute maximum is 341 (in 2000) and the absolute minimum is 100 (in 2009).

43. $P(x) = -x^3 + 9x^2 + 120x - 400, x \geq 5$

$$\begin{aligned} P'(x) &= -3x^2 + 18x + 120 \\ &= -3(x^2 - 6x - 40) \\ &= -3(x - 10)(x + 4) = 0 \\ x &= 10 \quad \text{or} \quad x = -4 \end{aligned}$$

-4 is not relevant since $x \geq 5$, so the only critical number is 10.

The graph of $P'(x)$ is a parabola that opens downward, so $P'(x) > 0$ on the interval $[5, 10]$ and $P'(x) < 0$ on the interval $(10, \infty)$. Thus, $P(x)$ is a maximum at $x = 10$.

Since x is measured in hundreds thousands, 10 hundred thousand or 1,000,000 tires must be sold to maximize profit.

Also,

$$\begin{aligned} P(10) &= -(10)^3 + 9(10)^2 + 120(10) - 400 \\ &= 700. \end{aligned}$$

The maximum profit is \$700 thousand or \$700,000.

44. $P(x) = -0.02x^3 + 600x - 20,000, [50, 150]$

$$\begin{aligned} P'(x) &= -0.06x^2 + 600 \\ &= -0.06(x^2 - 10,000) \\ &= -0.06(x + 100)(x - 100) \end{aligned}$$

There are critical numbers at $x = -100$ and $x = 100$. Only $x = 100$ is in the domain of $P(x)$.

x	$P(x)$
50	7500
100	20,000
150	2500

The maximum profit is \$20,000, which occurs when 100 units per week are made.

45. $C(x) = x^3 + 37x + 250$

(a) $1 \leq x \leq 10$

$$\begin{aligned} \bar{C}(x) &= \frac{C(x)}{x} = \frac{x^3 + 37x + 250}{x} \\ &= x^2 + 37 + \frac{250}{x} \end{aligned}$$

$$\begin{aligned} \bar{C}'(x) &= 2x - \frac{250}{x^2} \\ &= \frac{2x^3 - 250}{x^2} = 0 \quad \text{when} \end{aligned}$$

$$2x^3 = 250$$

$$x^3 = 125$$

$$x = 5.$$

Test for relative minimum.

$$\bar{C}'(4) = -7.625 < 0$$

$$\bar{C}'(6) \approx 5.0556 > 0$$

$$\bar{C}(5) = 112$$

$$\bar{C}(1) = 1 + 37 + 250 = 288$$

$$\bar{C}(10) = 100 + 37 + 25 = 162$$

The minimum for $1 \leq x \leq 10$ is 112.

(b) $10 \leq x \leq 20$

There are no critical values in this interval. Check the endpoints.

$$\bar{C}(10) = 162$$

$$\bar{C}(20) = 400 + 37 + 12.5 = 449.5$$

The minimum for $10 \leq x \leq 20$ is 162.

46. $C(x) = 81x^2 + 17x + 324$

(a) $1 \leq x \leq 10$

$$\begin{aligned} \bar{C}(x) &= \frac{C(x)}{x} = \frac{81x^2 + 17x + 324}{x} \\ &= 81x + 17 + \frac{324}{x} \end{aligned}$$

$$\bar{C}'(x) = 81 - \frac{324}{x^2} = 0 \quad \text{when}$$

$$81 = \frac{324}{x^2}$$

$$x = \pm 2.$$

-2 is not meaningful in this application and is not in the given domain.

Test 2 for minimum.

$$\bar{C}'(1) = -243 < 0$$

$$\bar{C}'(3) = 45 > 0$$

$\bar{C}(x)$ is minimum when $x = 2$.

$$\bar{C}(2) = 81(2) + 17 + \frac{324}{2} = 341$$

The minimum on the interval $1 \leq x \leq 10$ is 341.

(b) $10 \leq x \leq 20$

There are no critical numbers in this interval. Test the endpoints.

$$\bar{C}(10) = 859.4$$

$$\begin{aligned}\bar{C}(20) &= 1620 + 17 + 16.2 \\ &= 1653.2\end{aligned}$$

The minimum on the interval $10 \leq x \leq 20$ is 859.4.

47. The value $x = 11.5$ minimizes $\frac{f(x)}{x}$ because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 11.5 units results in the minimum cost per unit.

48. The value $x = 20$ minimizes $\frac{f(x)}{x}$ because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 20 units results in the minimum cost per unit.

49. The value $x = 100$ maximizes $\frac{f(x)}{x}$ because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 100 units results in the maximum profit per item produced.

50. The value $x = 280$ maximizes $\frac{f(x)}{x}$ because this is the point where the line from the origin to the curve is tangent to the curve.
A production level of 280 units results in the maximum profit per item produced.

$$51. f(x) = \frac{x^2 + 36}{2x}, 1 \leq x \leq 12$$

$$\begin{aligned}f'(x) &= \frac{2x(2x) - (x^2 + 36)(2)}{(2x)^2} \\ &= \frac{4x^2 - 2x^2 - 72}{4x^2} \\ &= \frac{2x^2 - 72}{4x^2} = \frac{2(x^2 - 36)}{4x^2} \\ &= \frac{(x + 6)(x - 6)}{2x^2}\end{aligned}$$

$$f'(x) = 0 \text{ when } x = 6 \text{ and when } x = -6.$$

Only 6 is in the interval $1 \leq x \leq 12$.

Test for relative maximum or minimum.

$$f'(5) = \frac{(11)(-1)}{50} < 0$$

$$f'(7) = \frac{(13)(1)}{98} > 0$$

The minimum occurs at $x = 6$, or at 6 months.
Since $f(6) = 6$, $f(1) = 18.5$, and $f(12) = 7.5$, the minimum percent is 6%.

$$52. S(x) = -x^3 + 3x^2 + 360x + 5000; 6 \leq x \leq 20$$

$$\begin{aligned}S'(x) &= -3x^2 + 6x + 360 \\ &= -3(x^2 - 2x - 120)\end{aligned}$$

$$S'(x) = -3(x - 12)(x + 10) = 0$$

$$x = 12 \text{ or } x = -10 \text{ (not in the interval)}$$

x	$f(x)$
6	7052
12	8024
10	7900

12° is the temperature that produces the maximum number of salmon.

53. Since we are only interested in the length during weeks 22 through 28, the domain of the function for this problem is $[22, 28]$. We now look for any critical numbers in this interval. We find

$$L'(t) = 0.788 - 0.02t$$

There is a critical number at $t = \frac{0.788}{0.02} = 39.4$, which is not in the interval. Thus, the maximum value will occur at one of the endpoints.

t	$L(t)$
22	5.4
28	7.2

The maximum length is about 7.2 millimeters.

54. The function is defined on the interval $[15, 46]$. We look first for critical numbers in the interval. We find

$$R'(T) = -0.00021T^2 + 0.0802T - 1.6572$$

Using our graphing calculator, we find one critical number in the interval at about 21.92

T	$R(T)$
15	81.01
21.92	79.29
46	98.89

The relative humidity is minimized at about 21.92°C.

55. $M(x) = -\frac{1}{45}x^2 + 2x - 20, 30 \leq x \leq 65$

$$M'(x) = -\frac{1}{45}(2x) + 2 = -\frac{2x}{45} + 2$$

When $M'(x) = 0$,

$$\begin{aligned} -\frac{2x}{45} + 2 &= 0 \\ 2 &= \frac{2x}{45} \\ 45 &= x. \end{aligned}$$

x	$M(x)$
30	20
45	25
65	$\frac{145}{9} \approx 16.1$

The absolute maximum miles per gallon is 25 at 45 mph and the absolute minimum miles per gallon is about 16.1 at 65 mph.

56. $M(x) = -0.015x^2 + 1.31x - 7.3, 30 \leq x \leq 60$

$$M'(x) = -0.03x + 1.31 = 0$$

$$x \approx 43.7$$

x	$M(x)$
30	18.5
43.7	21.3
60	17.3

The absolute maximum of 21.3 mpg occurs at 43.7 mph. The absolute minimum of 17.3 mpg occurs at 60 mph.

57. Total area $A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{12-x}{4}\right)^2$
- $$= \frac{x^2}{4\pi} + \frac{(12-x)^2}{16}$$

$$A'(x) = \frac{x}{2\pi} - \frac{12-x}{8} = 0$$

$$\frac{4x - \pi(12-x)}{8\pi} = 0$$

$$x = \frac{12\pi}{4+\pi} \approx 5.28$$

x	Area
0	9
5.28	5.04
12	11.46

The total area is minimized when the piece used to form the circle is $\frac{12\pi}{4+\pi}$ feet, or about 5.28 feet long.

58. Total area $= A(x)$

$$\begin{aligned} &= \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{12-x}{4}\right)^2 \\ &= \frac{x^2}{4\pi} + \frac{(12-x)^2}{16} \end{aligned}$$

$$A'(x) = \frac{x}{2\pi} - \frac{12-x}{8} = 0$$

$$\frac{4x - \pi(12-x)}{8\pi} = 0$$

$$x = \frac{12\pi}{4+\pi} \approx 5.28$$

x	Area
0	9
5.28	5.04
12	11.46

The total area is maximized when all 12 feet of wire are used to form the circle.

59. For the solution to Exercise 57, the piece of length x used to form the circle is $\frac{12\pi}{4+\pi}$ feet. The circle can be inscribed inside the square if the side of the square equals the diameter of the circle (that is, twice the radius).

side of the square $= 2$ (radius)

$$\begin{aligned}\frac{12-x}{4} &= 2\left(\frac{x}{2\pi}\right) \\ \frac{12-x}{4} &= \frac{x}{\pi} \\ 4x &= 12\pi - \pi x \\ x(4+\pi) &= 12\pi \\ x &= \frac{12\pi}{4+\pi}\end{aligned}$$

Therefore, the circle formed by piece of length $x = \frac{12\pi}{4+\pi}$ can be inscribed inside the square.

$$\begin{aligned}60. \text{ (a) } I(p) &= -p \ln p - (1-p) \ln(1-p) \\ I'(p) &= -p \left(\frac{1}{p}\right) + (\ln p)(-1) \\ &\quad - \left[(1-p) \frac{-1}{1-p} + [\ln(1-p)](-1)\right] \\ &= -1 - \ln p + 1 + \ln(1-p) \\ &= -\ln p + \ln(1-p) \\ \text{(b) } -\ln p + \ln(1-p) &= 0 \\ \ln(1-p) &= \ln p \\ 1-p &= p \\ 1 &= 2p \\ \frac{1}{2} &= p \\ I'(0.25) &= 1.0986 \\ I'(0.75) &= -1.099\end{aligned}$$

There is a relative maximum of 0.693 at $p = \frac{1}{2}$.

6.2 Applications of Extrema

Your Turn 1

Assign a variable to the quantity to be minimized:

$$M = x^2 y$$

Use the given condition to express M in terms of one variable, say x :

$$\begin{aligned}x + 3y &= 30 \\ y &= \frac{30-x}{3} \\ M &= x^2 \left(\frac{30-x}{3}\right) = 10x^2 - \frac{1}{3}x^3\end{aligned}$$

Find the domain of M :

Since x and y must be nonnegative, we have $x \geq 0$ and $\frac{30-x}{3} \geq 0$ which requires $x \leq 30$. Thus the domain of M is $[0, 30]$.

Find the critical numbers:

$$\frac{dM}{dx} = 20x - x^2 = x(20-x)$$

$\frac{dM}{dx}$ is 0 when $x = 0$ (already found as an endpoint) and when $x = 20$.

Evaluate M at the critical numbers and endpoints:

x	M
0	0
20	$\frac{4000}{3}$
30	0

The maximum of M occurs at $x = 20$.

The corresponding value of y is $\frac{30-20}{3}$ or $\frac{10}{3}$.

Your Turn 2

The first steps of the solution follow Example 2. As in Example 2, $300 - x$ will be the distance the professor must run along the trail. The first change in the model occurs when we compute the professor's total time. Since his speed through the woods is now 40 m/min rather than 70 m/min, his total time $T(x)$ is now

$$T(x) = \frac{300-x}{160} + \frac{\sqrt{800^2 + x^2}}{40}.$$

As before, the domain of T is $[0, 300]$. The derivative of T is

$$T'(x) = -\frac{1}{160} + \frac{1}{40} \left(\frac{1}{2}\right) (800^2 + x^2)^{-1/2} (2x)$$

Now find the critical numbers by setting the derivative of T equal to 0.

$$-\frac{1}{160} + \frac{1}{40} \left(\frac{1}{2}\right) (800^2 + x^2)^{-1/2} (2x) = 0$$

$$\begin{aligned}\frac{x}{40\sqrt{800^2 + x^2}} &= \frac{1}{160} \\ 4x &= \sqrt{800^2 + x^2} \\ 16x^2 &= 800^2 + x^2 \\ 15x^2 &= 800^2 \\ x &= \frac{800}{\sqrt{15}} \\ x &\approx 206.56\end{aligned}$$

Calculate the total time at this critical number and at the endpoints of the domain.

x	T
0	21.875
206.56	21.24
300	21.36

The minimum travel time occurs for $x = 206.56$. The professor runs $300 - 206.56$ meters or about 93 meters along the path and then heads into the woods.

Your Turn 3

We follow the procedure of Example 3, but now our volume function is

$$V(x) = x(8 - 2x)^2 = 4(16x - 8x^2 + x^3).$$

For nonnegative side lengths we require $x \geq 0$ and $8 - 2x \geq 0$ or $x \leq 4$; the domain of V is thus $[0, 4]$.

Set the derivative of V equal to 0 and solve for x .

$$V'(x) = 4(16 - 16x + 3x^2) = 4(3x - 4)(x - 4)$$

This derivative has two positive roots, $x = 4$ and $x = 4/3$. We have already identified one of these as an endpoint of the domain and the other lies within the domain. Evaluating V at 0, $4/3$ and 4 gives the following table.

x	V
0	0
$4/3$	$\frac{1024}{27}$
4	0

The maximum volume is $1024/27 \text{ m}^3$ and occurs when $x = 4/3 \text{ m}$.

Note that we can solve this problem efficiently using a scaling argument. Since the proportions of the largest-volume box should not depend on the linear scale we adopt (that is, on the units in which we measure length), we can just note that our piece of metal is $8/12$ or the $2/3$ size of the one in Example 3. Our minimizing length x will be $2/3$ of the value we found in Example 3, or $(2/3)(2) = 4/3 \text{ m}$.

Your Turn 4

Follow the procedure of Example 4 with a volume of 500 cm^3 instead of 1000 cm^3 .

$$V = \pi r^2 h = 500$$

so

$$h = \frac{500}{\pi r^2}$$

and

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r \frac{500}{\pi r^2} \\ &= 2\pi r^2 + \frac{1000}{r}. \end{aligned}$$

Excluding a radius of 0 gives a domain for S of $(0, \infty)$. Now find the critical numbers of S .

$$\begin{aligned} S' &= 4\pi r - \frac{1000}{r^2} \\ 4\pi r &= \frac{1000}{r^2} \\ r^3 &= \frac{250}{\pi} \\ r &= \left(\frac{250}{\pi} \right)^{1/3} \approx 4.301 \end{aligned}$$

We can verify that S' is negative to the left of 4.3 and positive to the right (for example, $S'(4) \approx -12.2$ and $S'(5) \approx 22.8$). Thus the function S is decreasing as we move toward 4.3 from left and increasing as we move past 4.3 to the right, so there is a relative minimum at 4.3. Since there is only one critical number, the critical point theorem tells us that it corresponds to an absolute minimum of the area function.

Then $h \approx \frac{500}{\pi(4.301)^2} \approx 8.604$. The minimum surface area is obtained with a radius of 4.3 cm and a height of 8.6 cm.

As with the box in Your Turn 3, we could obtain this answer directly from Example 4 using a scaling argument. Changing the volume by a factor of $1/2$ (from 1000 to 500) scales all linear measures by a factor of $(1/2)^{1/3}$, so we can find the new r and h by dividing the values found in Example 4 by $(1/2)^{1/3}$:

$$\begin{aligned} r &= \frac{5.419}{2^{1/3}} \approx 4.301 \\ h &= \frac{10.84}{2^{1/3}} \approx 8.604 \end{aligned}$$

6.2 Exercises

1. $x + y = 180, P = xy$

(a) $y = 180 - x$

(b) $P = xy = x(180 - x)$

(c) Since $y = 180 - x$ and x and y are nonnegative numbers, $x \geq 0$ and $180 - x \geq 0$ or $x \leq 180$. The domain of P is $[0, 180]$.

(d) $P'(x) = 180 - 2x$
 $180 - 2x = 0$
 $2(90 - x) = 0$
 $x = 90$

(e)

x	P
0	0
90	8100
180	0

(f) From the chart, the maximum value of P is 8100; this occurs when $x = 90$ and $y = 90$.

2. $x + y = 140$

Minimize $x^2 + y^2$.

(a) $y = 140 - x$

(b) Let
 $P = x^2 + y^2 = x^2 + (140 - x)^2$
 $= x^2 + 19,600 - 280x + x^2$
 $= 2x^2 - 280x + 19,600$.

(c) Since $y = 140 - x$ and x and y are nonnegative numbers, the domain of P is $[0, 140]$.

(d) $P' = 4x - 280$
 $4x - 280 = 0$
 $4x = 280$
 $x = 70$

(e)

x	P
0	19,600
70	9800
140	19,600

(f) The minimum value of $x^2 + y^2$ occurs when $x = 70$ and $y = 140 - x = 140 - 70 = 70$. The minimum value is 9800.

3. $x + y = 90$

Minimize x^2y .

(a) $y = 90 - x$

(b) Let $P = x^2y = x^2(90 - x)$
 $= 90x^2 - x^3$.

(c) Since $y = 90 - x$ and x and y are nonnegative numbers, the domain of P is $[0, 90]$.

(d) $P' = 180x - 3x^2$
 $180x - 3x^2 = 0$
 $3x(60 - x) = 0$
 $x = 0$ or $x = 60$

(e)

x	P
0	0
60	108,000
90	0

(f) The maximum value of x^2y occurs when $x = 60$ and $y = 30$. The maximum value is 108,000.

4. $x + y = 105$

Maximize xy^2 .

(a) $y = 105 - x$

(b) Let $P = xy^2 = x(105 - x)^2$
 $= x(11,025 - 210x + x^2)$
 $= 11,025x - 210x^2 + x^3$.

(c) Since $y = 105 - x$ and x and y are nonnegative numbers, the domain of P is $[0, 105]$.

(d) $P' = 11,025 - 420x + 3x^2$
 $3x^2 - 420x + 11,025 = 0$
 $3(x^2 - 140x + 3675) = 0$
 $3(x - 35)(x - 105) = 0$
 $x = 35$ or $x = 105$

(e)

x	P
0	0
35	171,500
105	0

- (f) The maximum value of xy^2 occurs when $x = 35$ and $y = 70$. The maximum value is 171,500.

5. $C(x) = \frac{1}{2}x^3 + 2x^2 - 3x + 35$

The average cost function is

$$\begin{aligned} A(x) &= \bar{C}(x) = \frac{C(x)}{x} \\ &= \frac{\frac{1}{2}x^3 + 2x^2 - 3x + 35}{x} \\ &= \frac{1}{2}x^2 + 2x - 3 + \frac{35}{x} \\ &\text{or } \frac{1}{2}x^2 + 2x - 3 + 35x^{-1}. \end{aligned}$$

Then

$$\begin{aligned} A'(x) &= x + 2 - 35x^{-2} \\ &\text{or } x + 2 - \frac{35}{x^2}. \end{aligned}$$

Graph $y = A'(x)$ on a graphing calculator. A suitable choice for the viewing window is $[0, 10]$ by $[-10, 10]$. (Negative values of x are not meaningful in this application.) Using the calculator, we see that the graph has an x -intercept at $x \approx 2.722$. Thus, 2.722 is a critical number.

Now graph $y = A(x)$ and use this graph to confirm that a minimum occurs at $x \approx 2.722$. Thus, the average cost is smallest at $x \approx 2.722$. At this value of x , $A \approx 19.007$.

6. $C(x) = 10 + 20x^{1/2} + 16x^{3/2}$

The average cost function is

$$\begin{aligned} A(x) &= \bar{C}(x) = \frac{C(x)}{x} \\ &= \frac{10 + 20x^{1/2} + 16x^{3/2}}{x} \\ &= \frac{10}{x} + 20x^{-1/2} + 16x^{1/2} \\ &\text{or } 10x^{-1} + 20x^{-1/2} + 16x^{1/2}. \end{aligned}$$

Then

$$A'(x) = -10x^{-2} - 10x^{-3/2} + 8x^{-1/2}.$$

Graph $y = A'(x)$ on a graphing calculator. A suitable choice for the viewing window is $[0, 10]$ by $[-10, 10]$. (Negative values of x are not meaningful in this application.) We see that this graph has one x -intercept. Using the calculator, we

find that this x -value is about 2.110, which shows that 2.110 is the only critical number of A .

Now graph $y = A(x)$ and use this graph to confirm that a minimum occurs at $x \approx 2.110$. Thus, the average cost is smallest at $x \approx 2.110$. At this value of x , $A \approx 41.749$.

7. $p(x) = 160 - \frac{x}{10}$

- (a) Revenue from sale of x thousand candy bars:

$$\begin{aligned} R(x) &= 1000xp \\ &= 1000x\left(160 - \frac{x}{10}\right) \\ &= 160,000x - 100x^2 \end{aligned}$$

(b) $R'(x) = 160,000 - 200x$

$$\begin{aligned} 160,000 - 200x &= 0 \\ 800 &= x \end{aligned}$$

The maximum revenue occurs when 800 thousand bars are sold.

(c) $R(800) = 160,000(800) - 100(800)^2$
 $= 64,000,000$

The maximum revenue is 64,000,000 cents or \$640,000.

8. $p(x) = 12 - \frac{x}{8}$

- (a) Revenue from x thousand compact discs:

$$\begin{aligned} R(x) &= 1000xp \\ &= 1000x\left(12 - \frac{x}{8}\right) \\ &= 12,000x - 125x^2 \end{aligned}$$

(b) $R'(x) = 12,000 - 250x$

$$\begin{aligned} 12,000 - 250x &= 0 \\ 12,000 &= 250x \\ 48 &= x \end{aligned}$$

The maximum revenue occurs when 48 thousand compact discs are sold.

(c) $R(48) = 12,000(48) - 125(48)^2$
 $= 288,000$

The maximum revenue is \$288,000.

9. Let $x =$ the width
 and $y =$ the length.

- (a) The perimeter is

$$\begin{aligned} P &= 2x + y \\ &= 1400, \end{aligned}$$

so

$$y = 1400 - 2x.$$

$$(b) \text{ Area} = xy = x(1400 - 2x)$$

$$A(x) = 1400x - 2x^2$$

$$(c) \quad A' = 1400 - 4x$$

$$1400 - 4x = 0$$

$$1400 = 4x$$

$$350 = x$$

$A'' = -4$, which implies that $x = 350$ m leads to the maximum area.

$$(d) \text{ If } x = 350,$$

$$y = 1400 - 2(350) = 700.$$

The maximum area is $(350)(700)$

$$= 245,000 \text{ m}^2.$$

10. Let x = length of field

y = width of field.

Perimeter:

$$P = 2x + 2y = 300$$

$$x + y = 150$$

$$y = 150 - x$$

Area:

$$A = xy$$

$$= x(150 - x)$$

$$= 150x - x^2$$

Thus,

$$A(x) = 150x - x^2$$

$$A'(x) = 150 - 2x.$$

$$A'(x) = 0 \text{ when}$$

$$150 - 2x = 0$$

$$x = 75.$$

$A''(x) = -2$, so $A''(75) = -2 < 0$, which confirms that a maximum value occurs at $x = 75$.

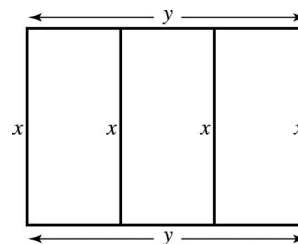
If $x = 75$,

$$y = 150 - x = 150 - 75 = 75.$$

A maximum area occurs when the length is 75 m and the width is 75 m.

11. Let x = the width of the rectangle

y = the total length of the rectangle.



An equation for the fencing is

$$3600 = 4x + 2y$$

$$2y = 3600 - 4x$$

$$y = 1800 - 2x.$$

$$\text{Area} = xy = x(1800 - 2x)$$

$$A(x) = 1800x - 2x^2$$

$$A' = 1800 - 4x$$

$$1800 - 4x = 0$$

$$1800 = 4x$$

$$450 = x$$

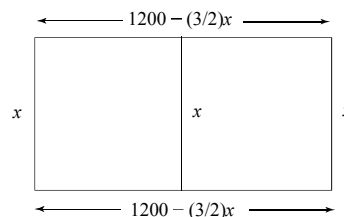
$A'' = -4$, which implies that $x = 450$ is the location of a maximum.

$$\text{If } x = 450, y = 1800 - 2(450) = 900.$$

The maximum area is

$$(450)(900) = 405,000 \text{ m}^2.$$

12.



There are three fence pieces of length x , which leaves $2400 - 3x$ for the two remaining sides. Each of these remaining sides thus has length $1200 - (3/2)x$. The area enclosed is

$$A(x) = x[1200 - (3/2)x]$$

$$= 1200x - \frac{3x^2}{2},$$

measured in square meters. Both x and $1200 - (3/2)x$ must be nonnegative, so the domain of A is $[0, 800]$. Now set the derivative equal to zero to solve for any critical numbers:

$$\begin{aligned}
 A'(x) &= 1200 - 3x \\
 1200 - 3x &= 0 \\
 1200 &= 3x \\
 x &= 400
 \end{aligned}$$

Evaluate A at the endpoints of the domain and at the single critical value $x = 400$.

x	A
0	0
400	240,000
800	0

The maximum area enclosed by the pen is $240,000 \text{ m}^2$. To achieve this, the farmer will use three fence sections of length 400 m and two of length $1200 - (3/2)(400) = 600 \text{ m}$.

13. Let x = length at \$1.50 per meter
 y = width at \$3 per meter.

$$\begin{aligned}
 xy &= 25,600 \\
 y &= \frac{25,600}{x}
 \end{aligned}$$

$$\text{Perimeter} = x + 2y = x + \frac{51,200}{x}$$

$$\begin{aligned}
 \text{Cost} = C(x) &= x(1.5) + \frac{51,200}{x}(3) \\
 &= 1.5x + \frac{153,600}{x}
 \end{aligned}$$

Minimize cost:

$$\begin{aligned}
 C'(x) &= 1.5 - \frac{153,600}{x^2} \\
 1.5 - \frac{153,600}{x^2} &= 0 \\
 1.5 &= \frac{153,600}{x^2} \\
 1.5x^2 &= 153,600 \\
 x^2 &= 102,400 \\
 x &= 320 \\
 y &= \frac{25,600}{320} = 80
 \end{aligned}$$

320 m at \$1.50 per meter will cost \$480. 160 m at \$3 per meter will cost \$480. The total cost will be \$960.

14. Let x = the length at \$2.50 per foot
 y = the width at \$3.20 per foot.
 $xy = 20,000$
 $y = \frac{20,000}{x}$

$$\text{Perimeter} = 2x + 2y = 2x + \frac{40,000}{x}$$

$$\begin{aligned}
 \text{Cost} = C(x) &= 2x(2.5) + \frac{40,000}{x}(3.2) \\
 &= 5x + \frac{128,000}{x}
 \end{aligned}$$

Minimize cost:

$$\begin{aligned}
 C'(x) &= 5 - \frac{128,000}{x^2} \\
 5 - \frac{128,000}{x^2} &= 0 \\
 5 &= \frac{128,000}{x^2} \\
 5x^2 &= 128,000 \\
 x^2 &= 25,600 \\
 x &= 160 \\
 y &= \frac{20,000}{160} = 125
 \end{aligned}$$

320 ft at \$2.50 per foot will cost \$800. 250 ft at \$3.20 per foot will cost \$800. The entire cost will be \$1600.

15. Let x = the number of refunds.
 Then $90 + x$ = the number of passengers.

$$\begin{aligned}
 \text{(a) Revenue} = R(x) &= (90 + x)(1600 - 10x) \\
 &= 144,000 + 700x - 10x^2
 \end{aligned}$$

Assume that the number of refunds is nonnegative and that the number of refunds is limited to 160 so that the revenue will be nonnegative. Thus the domain of R is $[0, 160]$. Now set the derivative of R equal to 0 and solve.

$$\begin{aligned}
 R' &= 700 - 20x = 0 \\
 x &= 35
 \end{aligned}$$

Check the value of R at this critical number and at the endpoints of the domain:

x	R
0	144,000
35	156,250
160	0

Thus the maximum revenue is obtained with 35 refunds, which happens when there are 125 passengers.

(b) The maximum revenue is \$156,250.

16. Let x = the number of seats.

Profit is 8 dollars per seat for $0 \leq x \leq 50$.

Profit (in dollars) is $8 - 0.1(x - 50)$ per seat for $x > 50$.

We expect that the total number of seats which makes the total profit a maximum will be greater than 50 because after 50 the profit is still increasing, though at a slower rate. (Thus we know the function is concave down and its one extremum will be a maximum.)

(a) The total profit for x seats is

$$\begin{aligned} P(x) &= [8 - 0.1(x - 50)]x \\ &= (8 - 0.1x + 5)x \\ &= (13 - 0.1x)x \\ &= (13x - 0.1x^2) \end{aligned}$$

$$P'(x) = 13 - 0.2x$$

$$13 - 0.2x = 0$$

$$x = 65$$

65 seats will produce the maximum profit.

$$\begin{aligned} \text{(b)} \quad P(65) &= 13(65) - 0.1(65^2) \\ &= 845 - 422.5 \\ &= 422.5 \end{aligned}$$

The maximum profit is \$422.50.

17. Let x = the number of days to wait.

$$\frac{12,000}{100} = 120 = \text{the number of 100-lb groups collected already.}$$

Then $7.5 - 0.15x$ = the price per 100 lb;

$4x$ = the number of 100-lb groups collected per day;

$120 + 4x$ = total number of 100-lb groups collected.

$$\begin{aligned} \text{Revenue} &= R(x) \\ &= (7.5 - 0.15x)(120 + 4x) \\ &= 900 + 12x - 0.6x^2 \end{aligned}$$

$$\begin{aligned} R'(x) &= 12 - 1.2x = 0 \\ x &= 10 \end{aligned}$$

$R''(x) = -1.2 < 0$ so $R(x)$ is maximized at $x = 10$.

The scouts should wait 10 days at which time their income will be maximized at

$$R(10) = 900 + 12(10) - 0.6(10)^2 = \$960.$$

18. Let x = the number of additional tables.

Then $160 - 0.50x$ = the cost per table

and $250 + x$ = the number of tables ordered.

$$\begin{aligned} R &= (160 - 0.50x)(250 + x) \\ &= 40,000 + 35x - 0.50x^2 \\ R' &= 35 - x = 0 \\ x &= 35 \end{aligned}$$

$R'' = -1 < 0$, so when $250 + 35 = 285$ tables are ordered, revenue is maximum.

Thus, the maximum revenue is

$$\begin{aligned} R(35) &= 40,000 + 35(35) - 0.50(35)^2 \\ &= 40,612.5 \end{aligned}$$

The maximum revenue is \$40,612.50.

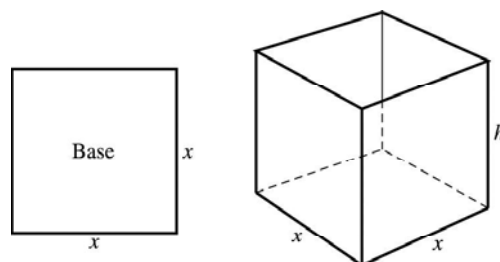
Minimum revenue is found by letting $R = 0$.

$$\begin{aligned} (160 - 0.50x)(250 + x) &= 0 \\ 160 - 0.50x &= 0 \quad \text{or} \quad 250 + x = 0 \\ x &= 320 \quad \text{or} \quad x = -250 \\ &\hspace{15em} \text{(impossible)} \end{aligned}$$

So when $250 + 320 = 570$ tables are ordered, revenue is 0, that is, each table is free.

I would fire the assistant.

19. Let x = a side of the base
 h = the height of the box.



An equation for the volume of the box is

$$\begin{aligned} V &= x^2 h, \\ \text{so } 32 &= x^2 h \\ h &= \frac{32}{x^2}. \end{aligned}$$

The box is open at the top so the area of the surface material $m(x)$ in square inches is the area of the base plus the area of the four sides.

$$\begin{aligned} m(x) &= x^2 + 4xh \\ &= x^2 + 4x\left(\frac{32}{x^2}\right) \\ &= x^2 + \frac{128}{x} \\ m'(x) &= 2x - \frac{128}{x^2} \end{aligned}$$

$$\begin{aligned} \frac{2x^3 - 128}{x^2} &= 0 \\ 2x^3 - 128 &= 0 \\ 2(x^3 - 64) &= 0 \\ x &= 4 \end{aligned}$$

$$m'(x) = 2 + \frac{256}{x^3} > 0 \text{ since } x > 0.$$

So, $x = 4$ minimizes the surface material.

If $x = 4$,

$$h = \frac{32}{x^2} = \frac{32}{16} = 2.$$

The dimensions that will minimize the surface material are 4 in. by 4 in. by 2 in.

20. Let x = the width. Then $2x$ = the length and h = the height.

An equation for volume is

$$\begin{aligned} V &= (2x)(x)h = 2x^2h \\ 36 &= 2x^2h. \end{aligned}$$

$$\text{So, } h = \frac{18}{x^2}.$$

The surface area $S(x)$ is the sum of the areas of the base and the four sides.

$$\begin{aligned} S(x) &= (2x)(x) + 2xh + 2(2x)h \\ &= 2x^2 + 6xh \\ &= 2x^2 + 6x\left(\frac{18}{x^2}\right) \\ &= 2x^2 + \frac{108}{x} \end{aligned}$$

$$S'(x) = 4x - \frac{108}{x^2}$$

$$\frac{4x^3 - 108}{x^2} = 0$$

$$\begin{aligned} 4(x^3 - 27) &= 0 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} S''(x) &= 4 + \frac{108(2)}{x^3} \\ &= 4 + \frac{216}{x^3} > 0 \text{ since } x > 0 \end{aligned}$$

So $x = 3$ minimizes the surface material.

If $x = 3$,

$$h = \frac{18}{x^2} = \frac{18}{9} = 2.$$

The dimensions are 3 ft by 6 ft by 2 ft.

21. Let x = the length of the side of the cutout square.

Then $3 - 2x$ = the width of the box
and $8 - 2x$ = the length of the box.

$$\begin{aligned} V(x) &= x(3 - 2x)(8 - 2x) \\ &= 4x^3 - 22x^2 + 24x \end{aligned}$$

The domain of V is $(0, \frac{3}{2})$.

Maximize the volume.

$$\begin{aligned} V'(x) &= 12x^2 - 44x + 24 \\ 12x^2 - 44x + 24 &= 0 \\ 4(3x^2 - 11x + 6) &= 0 \\ 4(3x - 2)(x - 3) &= 0 \\ x &= \frac{2}{3} \text{ or } x = 3 \end{aligned}$$

3 is not in the domain of V .

$$\begin{aligned} V''(x) &= 24x - 44 \\ V''\left(\frac{2}{3}\right) &= -28 < 0 \end{aligned}$$

This implies that V is maximized when $x = \frac{2}{3}$.

The box will have maximum volume when $x = \frac{2}{3}$ ft or 8 in.

- 22. (a)** From Example 3, the area of the base is $(12 - 2x)(12 - 2x) = 4x^2 - 48x + 144$ and the total area of all four walls is $4x(12 - 2x) = -8x^2 + 48x$. Since the box has maximum volume when $x = 2$, the area of the base is $4(2)^2 - 48(2) + 144 = 64$ square inches and the total area of all four walls is $-8(2)^2 + 48(2) = 64$ square inches. So, both are 64 square inches.

- (b)** From Exercise 21, the area of the base is $(3 - 2x)(8 - 2x) = 4x^2 - 22x + 24$ and the total area of all four walls is $2x(3 - 2x) + 2x(8 - 2x) = -8x^2 + 22x$. Since the box has maximum volume when $x = \frac{2}{3}$, the area of the base is $4\left(\frac{2}{3}\right)^2 - 22\left(\frac{2}{3}\right) + 24 = \frac{100}{9}$ square feet and the total area of all four walls is $-8\left(\frac{2}{3}\right)^2 + 22\left(\frac{2}{3}\right) = \frac{100}{9}$ square feet. So, both are $\frac{100}{9}$ square feet.

- (c)** Based on the results from parts (a) and (b), it appears that the area of the base and the total area of the walls for the box with maximum volume are equal. (This conjecture is true.)

- 23.** Let x = the length of a side of the top and bottom.
 Then x^2 = the area of the top and bottom
 and $(3)(2x^2)$ = the cost for the top and bottom
 Let y = depth of box.
 Then xy = the area of one side,
 $4xy$ = the total area of the sides.
 and $(1.50)(4xy)$ = The cost of the sides

The total cost is

$$C(x) = (3)(2x^2) + (1.50)(4xy) = 6x^2 + 6xy.$$

The volume is

$$V = 16,000 = x^2y.$$

$$y = \frac{16,000}{x^2}$$

$$C(x) = 6x^2 + 6x\left(\frac{16,000}{x^2}\right) = 6x^2 + \frac{96,000}{x}$$

$$C'(x) = 12x - \frac{96,000}{x^2} = 0$$

$$x^3 = 8000$$

$$x = 20$$

$$C''(x) = 12 + \frac{192,000}{x^3} > 0 \text{ at } x = 20, \text{ which}$$

implies that $C(x)$ is minimized when $x = 20$.

$$y = \frac{16,000}{(20)^2} = 40$$

So the dimensions of the box are x by x by y , or 20 cm by 20 cm by 40 cm.

$$C(20) = 6(20)^2 + \frac{96,000}{20} = 7200$$

The minimum total cost is \$7200.

- 24.** Let x = the width of printed material and y = the length of printed material. Then, the area of the printed material is

$$xy = 36,$$

$$\text{so } y = \frac{36}{x}.$$

Also, $x + 2$ = the width of a page and $y + 3$ = the length of a page.

The area of a page is

$$A = (x + 2)(y + 3)$$

$$= xy + 2y + 3x + 6$$

$$= 36 + 2\left(\frac{36}{x}\right) + 3x + 6$$

$$= 42 + \frac{72}{x} + 3x.$$

$$A' = -\frac{72}{x^2} + 3 = 0$$

$$x^2 = 24$$

$$x = \sqrt{24}$$

$$= 2\sqrt{6}$$

(We discard $x = -2\sqrt{6}$ once we must have $x > 0$.) $A'' = \frac{216}{x^3} > 0$ when $x = 2\sqrt{6}$, which

implies that A is minimized when $x = 2\sqrt{6}$.

$$y = \frac{36}{x} = \frac{36}{2\sqrt{6}} = \frac{18}{\sqrt{6}} = \frac{18\sqrt{6}}{6} = 3\sqrt{6}$$

The width of a page is

$$\begin{aligned}x + 2 &= 2\sqrt{6} + 2 \\ &\approx 6.9 \text{ in.}\end{aligned}$$

The length of a page is

$$\begin{aligned}y + 3 &= 3\sqrt{6} + 3 \\ &\approx 10.3 \text{ in.}\end{aligned}$$

25. (a) $S = 2\pi r^2 + 2\pi rh$, $V = \pi r^2 h$

$$S = 2\pi r^2 + \frac{2V}{r}$$

Treat V as a constant.

$$S' = 4\pi r - \frac{2V}{r^2}$$

$$4\pi r - \frac{2V}{r^2} = 0$$

$$\frac{4\pi r^3 - 2V}{r^2} = 0$$

$$4\pi r^3 - 2V = 0$$

$$2\pi r^3 - V = 0$$

$$2\pi r^3 = V$$

$$2\pi r^3 = \pi r^2 h$$

$$2r = h$$

26. $V = \pi r^2 h = 16$

$$h = \frac{16}{\pi r^2}$$

The total cost is the sum of the cost of the top and bottom and the cost of the sides.

$$\begin{aligned}C &= 2(2)(\pi r^2) + 1(2\pi rh) \\ &= 4(\pi r^2) + 1(2\pi r)\left(\frac{16}{\pi r^2}\right) \\ &= 4\pi r^2 + \frac{32}{r}\end{aligned}$$

Minimize cost.

$$C' = 8\pi r - \frac{32}{r^2}$$

$$8\pi r - \frac{32}{r^2} = 0$$

$$8\pi r^3 = 32$$

$$\pi r^3 = 4$$

$$\begin{aligned}r &= \sqrt[3]{\frac{4}{\pi}} \\ &\approx 1.08\end{aligned}$$

$$h = \frac{16}{\pi(1.08)^2} \approx 4.34$$

The radius should be 1.08 ft and the height should be 4.34 ft. If these rounded values for the height and radius are used, the cost is

$$\begin{aligned}&\$2(2)(\pi r^2) + \$1(2\pi rh) \\ &= 4\pi(1.08)^2 + 2\pi(1.08)(4.34) \\ &= \$44.11.\end{aligned}$$

27. From Example 4, we know that the surface area of the can is given by

$$S = 2\pi r^2 + \frac{2000}{r}.$$

Aluminum costs $3\text{¢}/\text{cm}^2$, so the cost of the aluminum to make the can is

$$0.03\left(2\pi r^2 + \frac{2000}{r}\right) = 0.06\pi r^2 + \frac{60}{r}.$$

The perimeter (or circumference) of the circular top is $2\pi r$. Since there is a $2\text{¢}/\text{cm}$ charge to seal the top and bottom, the sealing cost is

$$0.02(2)(2\pi r) = 0.08\pi r.$$

Thus, the total cost is given by the function

$$\begin{aligned}C(r) &= 0.06\pi r^2 + \frac{60}{r} + 0.08\pi r \\ &= 0.06\pi r^2 + 60r^{-1} + 0.08\pi r.\end{aligned}$$

Then

$$\begin{aligned}C'(r) &= 0.12\pi r - 60r^{-2} + 0.08\pi \\ &= 0.12\pi r - \frac{60}{r^2} + 0.08\pi.\end{aligned}$$

Graph

$$y = 0.12\pi x - \frac{60}{x^2} + 0.08\pi$$

on a graphing calculator. Since r must be positive in this application, our window should not include negative values of x . A suitable choice for the viewing window is $[0, 10]$ by $[-10, 10]$. From the graph, we find that $C'(x) = 0$ when $x \approx 5.206$.

Thus, the cost is minimized when the radius is about 5.206 cm.

We can find the corresponding height by using the equation

$$h = \frac{1000}{\pi r^2}$$

from Example 4.

If $r = 5.206$.

$$h = \frac{1000}{\pi(5.206)^2} \approx 11.75.$$

To minimize cost, the can should have radius 5.206 cm and height 11.75 cm.

- 28.** In Exercise 28, we found that the cost of the aluminum to make the can is

$$0.03\left(2\pi r^2 + \frac{2000}{r}\right) = 0.06\pi r^2 + \frac{60}{r}.$$

The cost for the vertical seam is $0.01h$. From Example 4, we see that h and r are related by the equation

$$h = \frac{1000}{\pi r^2},$$

so the sealing cost is

$$\begin{aligned} 0.01h &= 0.01\left(\frac{1000}{\pi r^2}\right) \\ &= \frac{10}{\pi r^2}. \end{aligned}$$

Thus, the total cost is given by the function

$$\begin{aligned} C(r) &= 0.06\pi r^2 + \frac{60}{r} + \frac{10}{\pi r^2} \\ \text{or } 0.06\pi r^2 &+ 60r^{-1} + \frac{10}{\pi}r^{-2}. \end{aligned}$$

Then

$$\begin{aligned} C'(x) &= 0.12\pi r - 60r^{-2} - \frac{20}{\pi}r^{-3} \\ \text{or } 0.12\pi r &- \frac{60}{r^2} - \frac{20}{\pi r^3}. \end{aligned}$$

Graph

$$y = 0.12\pi r - \frac{60}{r^2} - \frac{20}{\pi r^3}$$

on a graphing calculator. Since r must be positive, our window should not include negative values of x . A suitable choice for the viewing window is $[0, 10]$ by $[-10, 10]$. From the graph, we find that $C'(x) = 0$ when $x \approx 5.454$.

Thus, the cost is minimized when the radius is about 5.454 cm.

We can find the corresponding height by using the equation.

$$h = \frac{1000}{\pi r^2}$$

$$h = \frac{1000}{\pi(5.454)^2} \approx 10.70.$$

To minimize cost, the can should have radius 5.454 cm and height 10.70 cm.

- 29.** In Exercise 27 and 28, we found that the cost of the aluminum to make the can is $0.06\pi r^2 + \frac{60}{r}$, the cost to seal the top and bottom is $0.08\pi r$, and the cost to seal the vertical seam is $\frac{10}{\pi r^2}$. Thus, the total cost is now given by the function

$$\begin{aligned} C(r) &= 0.06\pi r^2 + \frac{60}{r} + 0.08\pi r + \frac{10}{\pi r^2} \\ \text{or } 0.06\pi r^2 &+ 60r^{-1} + 0.08\pi r + \frac{10}{\pi}r^{-2}. \end{aligned}$$

Then

$$\begin{aligned} C'(r) &= 0.12\pi r - 60r^{-2} + 0.08\pi + \frac{20}{\pi}r^{-3} \\ \text{or } 0.12\pi r &- \frac{60}{r^2} + 0.08\pi - \frac{20}{\pi r^3}. \end{aligned}$$

Graph

$$y = 0.12\pi r - \frac{60}{r^2} + 0.08\pi - \frac{20}{\pi r^3}$$

on a graphing calculator. A suitable choice for the viewing window is $[0, 10]$ by $[-10, 10]$. From the graph, we find that $C'(x) = 0$ when $x \approx 5.242$.

Thus, the cost is minimized when the radius is about 5.242 cm. To find the corresponding height, use the equation

$$h = \frac{1000}{\pi r^2}$$

from Example 4.

If $r = 5.242$,

$$h = \frac{1000}{\pi(5.242)^2} \approx 11.58.$$

To minimize cost, the can should have radius 5.242 cm and height 11.58 cm.

30. 120 centimeters of ribbon are available; it will cover 4 heights and 8 radii.

$$4h + 8r = 120$$

$$h + 2r = 30$$

$$h = 30 - 2r$$

$$V = \pi r^2 h$$

$$\begin{aligned} V &= \pi r^2 (30 - 2r) \\ &= 30\pi r^2 - 2\pi r^3 \end{aligned}$$

Maximize volume.

$$V' = 60\pi r - 6\pi r^2$$

$$60\pi r - 6\pi r^2 = 0$$

$$6\pi r(10 - r) = 0$$

$$r = 0 \text{ or } r = 10$$

If $r = 0$, there is no box, so we discard this value.

$V'' = 60\pi - 12\pi r < 0$ for $r = 10$, which implies that $r = 10$ gives maximum volume.

When $r = 10$, $h = 30 - 2(10) = 10$.

The volume is maximum when the radius and height are both 10 cm.

31. Distance on shore: $9 - x$ miles

Cost on shore: \$400 per mile

Distance underwater: $\sqrt{x^2 + 36}$

Cost underwater: \$500 per mile

Find the distance from A , that is, $(9 - x)$, to minimize cost, $C(x)$.

$$\begin{aligned} C(x) &= (9 - x)(400) + \left(\sqrt{x^2 + 36}\right)(500) \\ &= 3600 - 400x + 500(x^2 + 36)^{1/2} \end{aligned}$$

$$\begin{aligned} C'(x) &= -400 + 500\left(\frac{1}{2}\right)(x^2 + 36)^{-1/2}(2x) \\ &= -400 + \frac{500x}{\sqrt{x^2 + 36}} \end{aligned}$$

If $C'(x) = 0$,

$$\frac{500x}{\sqrt{x^2 + 36}} = 400$$

$$\frac{5x}{4} = \sqrt{x^2 + 36}$$

$$\frac{25}{16}x^2 = x^2 + 36$$

$$\frac{9}{16}x^2 = 36$$

$$x = \frac{6 \cdot 4}{3} = 8.$$

(Discard the negative solution.)

Then the distance should be

$$\begin{aligned} 9 - x &= 9 - 8 \\ &= 1 \text{ mile from point } A. \end{aligned}$$

32. Distance on shore: $7 - x$ miles

Cost on shore: \$400 per mile

Distance underwater: $\sqrt{x^2 + 36}$

Cost underwater: \$500 per mile

Find the distance from A , that is, $7 - x$, to minimize cost, $C(x)$.

$$\begin{aligned} C(x) &= (7 - x)(400) + (\sqrt{x^2 + 36})(500) \\ &= 2800 - 400x + 500(x^2 + 36)^{1/2} \end{aligned}$$

$$\begin{aligned} C'(x) &= -400 + 500\left(\frac{1}{2}\right)(x^2 + 36)^{1/2}(2x) \\ &= -400 + \frac{500x}{\sqrt{x^2 + 36}} \end{aligned}$$

If $C'(x) = 0$,

$$\frac{500x}{\sqrt{x^2 + 36}} = 400$$

$$\frac{5x}{4} = \sqrt{x^2 + 36}$$

$$\frac{25}{16}x^2 = x^2 + 36$$

$$\frac{9}{16}x^2 = 36$$

$$x^2 = \frac{36 \cdot 16}{9}$$

$$x = \frac{6 \cdot 4}{3} = 8.$$

(Discard the negative solution.)

$x = 8$ is impossible since Point A is only 7 miles from point C .

Check the endpoints.

x	$C(x)$
0	5800
7	4610

The cost is minimized when $x = 7$.

$7 - x = 7 - 7 = 0$, so the company should angle the cable at Point A .

33. $p(t) = \frac{20t^3 - t^4}{1000}, [0, 20]$

(a) $p'(t) = \frac{3}{50}t^2 - \frac{1}{250}t^3$
 $= \frac{1}{50}t^2 \left[3 - \frac{1}{5}t \right]$

Critical numbers:

$$\frac{1}{50}t^2 = 0 \text{ or } 3 - \frac{1}{5}t = 0$$

$$t = 0 \text{ or } t = 15$$

t	$p(t)$
0	0
15	16.875
20	0

The number of people infected reaches a maximum in 15 days.

(b) $P(15) = 16.875\%$

34. (a) $p(t) = 10te^{-t/8}, [0, 40]$

$$p'(t) = 10te^{-t/8} \left(-\frac{1}{8} \right) + e^{-t/8}(10)$$

$$= 10e^{-t/8} \left(-\frac{t}{8} + 1 \right)$$

Critical numbers:

$p'(t) = 0$ when

$$-\frac{t}{8} + 1 = 0$$

$$t = 8.$$

t	$p(t)$
0	0
8	29.43
40	2.6952

The percent of the population infected reaches a maximum in 8 days.

(b) $P(8) = 29.43\%$

35. $H(S) = f(S) - S$

$$f(S) = 12S^{0.25}$$

$$H(S) = 12S^{0.25} - S$$

$$H'(S) = 3S^{-0.75} - 1$$

$$H'(S) = 0 \text{ when}$$

$$3S^{-0.75} - 1 = 0$$

$$S^{-0.75} = \frac{1}{3}$$

$$\frac{1}{S^{0.75}} = \frac{1}{3}$$

$$S^{3/4} = 3$$

$$S = 3^{4/3}$$

$$S = 4.327.$$

The number of creatures needed to sustain the population is $S_0 = 4.327$ thousand.

$H''(S) = \frac{-2.25}{S^{1.75}} < 0$ when $S = 4.327$, so $H(S)$ is maximized.

$$H(4.327) = 12(4.327)^{0.25} - 4.327$$

$$\approx 12.98$$

The maximum sustainable harvest is 12.98 thousand.

36. $H(S) = f(S) - S$

$$f(S) = \frac{25S}{S+2}$$

$$H'(S) = \frac{(S+2)(25) - 25S}{(S+2)^2} - 1$$

$$= \frac{25S + 50 - 25S - (S+2)^2}{(S+2)^2}$$

$$= \frac{50 - (S^2 + 4S + 4)}{(S+2)^2}$$

$$= \frac{-S^2 - 4S + 46}{(S+2)^2}$$

$H'(S) = 0$ when

$$S^2 + 4S - 46 = 0$$

$$S = \frac{-4 \pm \sqrt{16 + 184}}{2}$$

$$= 5.071.$$

(Discard the negative solution.)

The number of creatures needed to sustain the population is $S_0 = 5.071$ thousand.

$$H'' = \frac{(S+2)^2(-2S-4) - (S^2-4S+46)(2S+4)}{(S+2)^4} < 0,$$

so H is a maximum at $S_0 = 5.071$.

$$H(S_0) = \frac{25(5.071)}{7.071} - 5.071$$

$$\approx 12.86$$

The maximum sustainable harvest is 12.86 thousand.

$$37. N(t) = 20 \left(\frac{t}{12} - \ln \left(\frac{t}{12} \right) \right) + 30;$$

$$1 \leq t \leq 15$$

$$N'(t) = 20 \left[\frac{1}{12} - \frac{12}{t} \left(\frac{1}{12} \right) \right]$$

$$= 20 \left(\frac{1}{12} - \frac{1}{t} \right)$$

$$= \frac{20(t-12)}{12t}$$

$N'(t) = 0$ when

$$t - 12 = 0$$

$$t = 12.$$

$N''(t)$ does not exist at $t = 0$, but 0 is not in the domain of N .

Thus, 12 is the only critical number.

To find the absolute extrema on $[1, 15]$, evaluate N at the critical number and at the endpoints.

t	$N(t)$
1	81.365
12	50
15	50.537

Use this table to answer the questions in (a)-(d).

- (a) The number of bacteria will be a minimum at $t = 12$, which represents 12 days.
- (b) The minimum number of bacteria is given by $N(12) = 50$, which represents 50 bacteria per ml.
- (c) The number of bacteria will be a maximum at $t = 1$, which represents 1 day.
- (d) The maximum number of bacteria is given by $N(1) = 81.365$, which represents 81.365 bacteria per ml.

$$38. (a) H(S) = f(S) - S$$

$$= Se^{r(1-S/P)} - S$$

$$H'(S) = Se^{r(1-S/P)} \left(-\frac{r}{P} \right) + e^{r(1-S/P)} - 1$$

Note that

$$f(S) = Se^{r(1-S/P)}$$

$$f'(S) = Se^{r(1-S/P)} \left(-\frac{r}{P} \right) + e^{r(1-S/P)}.$$

$$H'(S) = Se^{r(1-S/P)} \left(-\frac{r}{P} \right) + e^{r(1-S/P)} - 1 = 0$$

$$Se^{r(1-S/P)} \left(-\frac{r}{P} \right) + e^{r(1-S/P)} = 1$$

$$f'(S) = 1$$

$$(b) f(S) = Se^{r(1-S/P)}$$

$$f'(S) = \left(-\frac{r}{P} \right) Se^{r(1-S/P)} + e^{r(1-S/P)}$$

Set $f'(S_0) = 1$

$$e^{r(1-S_0/P)} \left[\frac{-rS_0}{P} + 1 \right] = 1$$

$$e^{r(1-S_0/P)} = \frac{1}{\frac{-rS_0}{P} + 1}$$

Using $H(S)$ from part (a), we get

$$H(S_0) = S_0 e^{r(1-S_0/P)} - S_0$$

$$= S_0 \left(e^{r(1-S_0/P)} - 1 \right)$$

$$= S_0 \left(\frac{1}{1 - \frac{rS_0}{P}} - 1 \right).$$

$$39. r = 0.1, P = 100$$

$$f(S) = Se^{r(1-S/P)}$$

$$f'(S) = -\frac{1}{1000} \cdot Se^{0.1(1-S/100)} + e^{0.1(1-S/100)}$$

$$f'(S_0) = -0.001S_0 e^{0.1(1-S_0/100)} + e^{0.1(1-S_0/100)}$$

Graph

$$Y_1 = -0.001xe^{0.1(1-x/100)} + e^{0.1(1-x/100)}$$

and

$$Y_2 = 1$$

on the same screen. A suitable choice for the viewing window is $[0, 60]$ by $[0.5, 1.5]$ with $Xscl = 10$ and $Yscl = 0.5$. By zooming or using the “intersect” option, we find the graphs intersect when $x \approx 49.37$.

The maximum sustainable harvest is 49.37.

$$40. r = 0.4, P = 500$$

$$f(S) = Se^{r(1-S/P)}$$

$$f'(S) = -\frac{0.4}{500} Se^{0.4(1-S/500)} + e^{0.4(1-S/500)}$$

$$f'(S_0) = -0.0008S_0 e^{0.4(1-S_0/500)} + e^{0.4(1-S_0/500)}$$

Graph

$$Y_1 = -0.0008x e^{0.4(1-x/500)} + e^{0.4(1-x/500)}$$

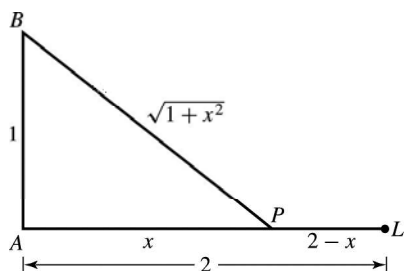
and

$$Y_2 = 1$$

on the same screen. A suitable choice for the viewing window is $[0, 300]$ by $[0.5, 1.5]$ with $Xscl = 50$, $Yscl = 0.5$. By zooming or using the “intersect” option, we find that the graphs intersect when $x \approx 237.10$.

The maximum sustainable harvest is 237.10.

41. Let x = distance from P to A .



Energy used over land: 1 unit per mile

Energy used over water: $\frac{4}{3}$ units per mile

Distance over land: $(2 - x)$ mi

Distance over water: $\sqrt{1 + x^2}$ mi

Find the location of P to minimize energy used.

$$E(x) = 1(2 - x) + \frac{4}{3}\sqrt{1 + x^2}, \text{ where } 0 \leq x \leq 2.$$

$$E'(x) = -1 + \frac{4}{3}\left(\frac{1}{2}\right)(1 + x^2)^{-1/2}(2x)$$

If $E'(x) = 0$,

$$\frac{4}{3}x(1 + x^2)^{-1/2} = 1$$

$$\frac{4x}{3(1 + x^2)^{1/2}} = 1$$

$$\frac{4}{3}x = (1 + x^2)^{1/2}$$

$$\frac{16}{9}x^2 = 1 + x^2$$

$$\frac{7}{9}x^2 = 1$$

$$x^2 = \frac{9}{7}$$

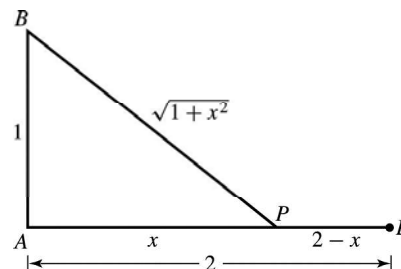
$$x = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}.$$

x	$E(x)$
0	3.3333
1.134	2.8819
2	2.9814

The absolute minimum occurs at $x \approx 1.134$.

Point P is $\frac{3\sqrt{7}}{7} \approx 1.134$ mi from Point A .

42. Let x = distance from P to A .



Energy used over land: 1 unit per mile

Energy used over water: $\frac{10}{9}$ units per mile

Distance over land: $(2 - x)$ mi

Distance over water: $\sqrt{1 + x^2}$ mi

Find the location of P to minimize energy used.

$$E(x) = 1(2 - x) + \frac{10}{9}\sqrt{1 + x^2}, \text{ where } 0 \leq x \leq 2.$$

$$E'(x) = -1 + \frac{10}{9}\left(\frac{1}{2}\right)(1 + x^2)^{-1/2}(2x)$$

If $E'(x) = 0$,

$$\frac{10}{9}x(1 + x^2)^{-1/2} = 1$$

$$\frac{10x}{9(1 + x^2)^{1/2}} = 1$$

$$\frac{10}{9}x = (1 + x^2)^{1/2}$$

$$\frac{100}{81}x^2 = 1 + x^2$$

$$\frac{19}{81}x^2 = 1$$

$$x^2 = \frac{81}{19}$$

$$x = \frac{9}{\sqrt{19}}$$

$$= \frac{9\sqrt{19}}{19}$$

$$\approx 2.06.$$

This value cannot give the absolute maximum since the total distance from A to L is just 2 miles. Test the endpoints of the domain.

x	$E(x)$
0	$3\frac{1}{9} \approx 3.1111$
2	2.4845

Point P must be at Point L .

$$\begin{aligned}
 43. \text{ (a) } f(S) &= aSe^{-bS} & f(S) &= Se^{r(1-S/P)} \\
 & & &= Se^{r-rS/P} \\
 & & &= Se^r e^{-rS/P} \\
 & & &= e^r Se^{-(r/P)S}
 \end{aligned}$$

Comparing the two terms, replace a with e^r and b with r/P .

(b) Shepherd:

$$\begin{aligned}
 f(S) &= \frac{aS}{1 + (S/b)^c} \\
 f'(S) &= \frac{[1 + (S/b)^c](a) - (aS)[c(S/b)^{c-1}(1/b)]}{[1 + (S/b)^c]^2} \\
 &= \frac{a + a(S/b)^c - (acS/b)(S/b)^{c-1}}{[1 + (S/b)^c]^2} \\
 &= \frac{a + a(S/b)^c - ac(S/b)^c}{[1 + (S/b)^c]^2} \\
 &= \frac{a[1 + (1 - c)(S/b)^c]}{[1 + (S/b)^c]^2}
 \end{aligned}$$

Ricker:

$$\begin{aligned}
 f(S) &= aSe^{-bS} \\
 f'(S) &= ae^{-bS} + aSe^{-bS}(-b) \\
 &= ae^{-bS}(1 - bS)
 \end{aligned}$$

Berverton-Holt:

$$\begin{aligned}
 f(S) &= \frac{aS}{1 + (S/b)} \\
 f'(S) &= \frac{[1 + (S/b)](a) - aS(1/b)}{[1 + (S/b)]^2} \\
 &= \frac{a + a(S/b) - a(S/b)}{[1 + (S/b)]^2} \\
 &= \frac{a}{[1 + (S/b)]^2}
 \end{aligned}$$

(c) Shepherd:

$$f'(0) = \frac{a[1 + (1 - c)(0/b)^c]}{[1 + (0/b)^c]^2} = a$$

Ricker:

$$f'(0) = ae^{-b(0)}[1 - b(0)] = a$$

Berverton-Holt:

$$f'(0) = \frac{a}{[1 + (0/b)]^2} = a$$

The constant a represents the slope of the graph of $f(S)$ at $S = 0$.

(d) First find the critical numbers by solving $f'(S) = 0$.

Shepherd:

$$\begin{aligned}
 f'(S) &= 0 \\
 a[1 + (1 - c)(S/b)^c] &= 0 \\
 (1 - c)(S/b)^c &= -1 \\
 (c - 1)(S/b)^c &= 1
 \end{aligned}$$

Substitute $b = 248.72$ and $c = 3.24$ and solve for S .

$$\begin{aligned}
 (3.24 - 1)(S/248.72)^{3.24} &= 1 \\
 \left(\frac{S}{248.72}\right)^{3.24} &= \frac{1}{2.24} \\
 \frac{S}{248.72} &= \left(\frac{1}{2.24}\right)^{1/3.24} \\
 S &= 248.72 \left(\frac{1}{2.24}\right)^{1/3.24} \approx 193.914
 \end{aligned}$$

Using the Shepherd model, next year's population is maximized when this year's population is about 194,000 tons. This can be verified by examining the graph of $f(S)$.

(e) First find the critical numbers by solving $f'(S) = 0$.

Ricker:

$$\begin{aligned}
 f'(S) &= 0 \\
 ae^{-bS}(1 - bS) &= 0 \\
 1 - bS &= 0 \\
 S &= \frac{1}{b}
 \end{aligned}$$

Substitute $b = 0.0039$ and solve for S .

$$\begin{aligned}
 S &= \frac{1}{0.0039} \\
 S &\approx 256.410
 \end{aligned}$$

Using the Ricker model, next year's population is maximized when this year's population is about 256,000 tons. This can be verified by examining the graph of $f(S)$.

44. (a) Solve the given equation for effective power for T , time.

$$\frac{kE}{T} = aSv^3 + I$$

$$\frac{kE}{aSv^3 + I} = T$$

Since distance is velocity, v , times time, T , we have

$$D(v) = v \frac{kE}{aSv^3 + I}$$

$$= \frac{kEv}{aSv^3 + I}.$$

$$(b) \quad D'(v) = \frac{(aSv^3 + I)kE - kEv(3aSv^2)}{(aSv^3 + I)^2}$$

$$= \frac{kE(aSv^3 + I - 3aSv^3)}{(aSv^3 + I)^2}$$

$$= \frac{kE(I - 2aSv^3)}{(aSv^3 + I)^2}$$

Find the critical numbers by solving

$$D'(v) = 0 \text{ for } v.$$

$$I - 2aSv^3 = 0$$

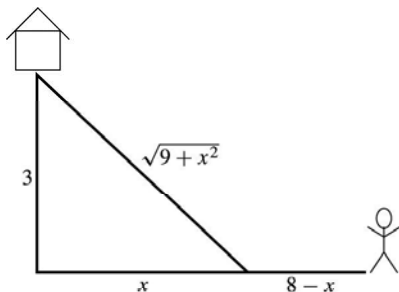
$$2aSv^3 = I$$

$$v^3 = \frac{I}{2aS}$$

$$v = \left(\frac{I}{2aS} \right)^{1/3}$$

45. Let $8 - x$ = the distance the hunter will travel on the river.

Then $\sqrt{9 + x^2}$ = the distance he will travel on land.



The rate on the river is 5 mph, the rate on land is 2 mph. Using $t = \frac{d}{r}$,

$$\frac{8 - x}{5} = \text{the time on the river,}$$

$$\frac{\sqrt{9 + x^2}}{2} = \text{the time on land.}$$

The total time is

$$T(x) = \frac{8 - x}{5} + \frac{\sqrt{9 + x^2}}{2}$$

$$= \frac{8}{5} - \frac{1}{5}x + \frac{1}{2}(9 + x^2)^{1/2}.$$

$$T' = -\frac{1}{5} + \frac{1}{4} \cdot 2x(9 + x^2)^{-1/2}$$

$$-\frac{1}{5} + \frac{x}{2(9 + x^2)^{1/2}} = 0$$

$$\frac{1}{5} = \frac{x}{2(9 + x^2)^{1/2}}$$

$$2(9 + x^2)^{1/2} = 5x$$

$$4(9 + x^2) = 25x^2$$

$$36 + 4x^2 = 25x^2$$

$$36 = 21x^2$$

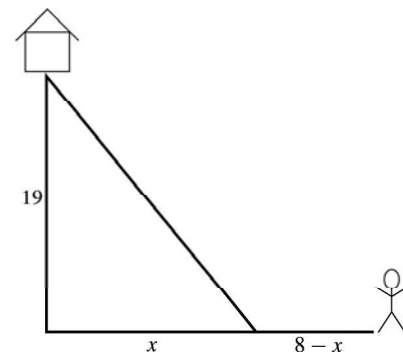
$$\frac{6}{\sqrt{21}} = x$$

$$\frac{6\sqrt{21}}{21} = \frac{2\sqrt{21}}{7} = x$$

x	$T(x)$
0	3.1
$\frac{2\sqrt{21}}{7}$	2.98
8	4.27

Since the maximum time is 2.98 hr, the hunter should travel $8 - \frac{2\sqrt{21}}{7} = \frac{56 - 2\sqrt{21}}{7}$ or about 6.7 miles along the river.

46. Let $8 - x$ = the distance the hunter will travel on the river.



Then $\sqrt{19^2 + x^2}$ = the distance he will travel on land.

Since the rate on the river is 5 mph, the rate on land is 2 mph, and $t = \frac{d}{r}$,

$$\frac{8-x}{5} = \text{the time on the river}$$

$$\frac{\sqrt{361+x^2}}{2} = \text{the time on the land.}$$

The total time is

$$T(x) = \frac{8-x}{5} + \frac{\sqrt{361+x^2}}{2}$$

$$= \frac{8}{5} - \frac{1}{5}x + \frac{1}{2}(361+x^2)^{-1/2}$$

$$T'(x) = -\frac{1}{5} + \frac{1}{4} \cdot 2x(361+x^2)^{-3/2}$$

$$-\frac{1}{5} + \frac{x}{2(361+x^2)^{3/2}} = 0$$

$$\frac{1}{5} = \frac{x}{2(361+x^2)^{3/2}}$$

$$2(361+x^2)^{3/2} = 5x$$

$$4(361+x^2) = 25x^2$$

$$1444 + 4x^2 = 25x^2$$

$$1444 = 21x^2$$

$$\frac{38}{\sqrt{21}} = x$$

$$8.29 = x$$

8.29 is not possible, since the cabin is only 8 miles west. Check the endpoints.

x	$T(x)$
0	11.1
8	10.3

$T(x)$ is minimized when $x = 8$.

The distance along the river is given by $8 - x$, so the hunter should travel $8 - 8 = 0$ miles along the river. He should complete the entire trip on land.

47. Let x = width.

Then x = height

and $108 - 4x$ = length.

(since length plus girth = 108)

$$V(x) = l \cdot w \cdot h$$

$$= (108 - 4x)x \cdot x$$

$$= 108x^2 - 4x^3$$

$$V'(x) = 216x - 12x^2$$

Set $V'(x) = 0$, and solve for x .

$$216x - 12x^2 = 0$$

$$12x(18 - x) = 0$$

$$x = 0 \text{ or } x = 18$$

0 is not in the domain, so the only critical number is 18.

Width = 18

Height = 18

Length = $108 - 4(18) = 36$

The dimensions of the box with maximum volume are 36 inches by 18 inches by 18 inches.

6.3 Further Business Applications: Economic Lot Size; Economic Order Quantity; Elasticity of Demand

Your Turn 1

Use Equation (3) with $k = 3$, $M = 18,000$ and $f = 750$.

$$q = \sqrt{\frac{2fM}{k}}$$

$$= \sqrt{\frac{2(750)(18,000)}{3}}$$

$$= \sqrt{(900)(10,000)}$$

$$= 3000$$

To minimize production costs there should be 3000 cans per batch, requiring $18,000/3,000 = 6$ batches per year.

Your Turn 2

Use Equation (3) with $k = 2$, $M = 320$ and $f = 30$.

$$q = \sqrt{\frac{2(30)(320)}{2}}$$

$$= \sqrt{9600}$$

$$\approx 97.98$$

Since q is very close to 98 we expect a q of 98 to minimize costs, but we will check both 97 and 98 using $T(q) = \frac{fM}{q} + \frac{kq}{2}$. $T(97) \approx 195.969$ and $T(98) \approx 195.959$, so the company should order 98 units in each batch. The time between orders will be about $12\left(\frac{98}{320}\right) = 3.675$ months.

Your Turn 3

$$q = 24,000 - 3p^2$$

$$\frac{dq}{dp} = -6p$$

$$E = -\frac{p}{q} \frac{dq}{dp} = \frac{6p^2}{24,000 - 3p^2}$$

When $p = 50$,

$$E = \frac{6(50^2)}{24,000 - 3(50^2)}$$

$$\approx 0.909,$$

which corresponds to inelastic demand.

Your Turn 4

$$q = 200e^{-0.4p}$$

$$\frac{dq}{dp} = -80e^{-0.4p}$$

$$E = -\frac{p}{q} \frac{dq}{dp}$$

$$= \frac{80pe^{-0.4p}}{200e^{-0.4p}}$$

$$= 0.4p$$

For $p = 100$ we have $E = 0.4(100) = 40$, which corresponds to elastic demand.

Your Turn 5

$$q = 3600 - 3p^2$$

$$\frac{dq}{dp} = -6p$$

$$E = -\frac{p}{q} \frac{dq}{dp}$$

$$= \frac{6p^2}{3600 - 3p^2}$$

$$= \frac{2p^2}{1200 - p^2}$$

The demand has unit elasticity when $E = 1$.

$$\frac{2p^2}{1200 - p^2} = 1$$

$$2p^2 = 1200 - p^2$$

$$3p^2 = 1200$$

$$p^2 = 400$$

$$p = 20$$

Testing value for p smaller and larger than 20 (say 15 and 25) we find

$$E(15) < 1$$

$$E(25) > 1$$

Thus demand is inelastic when $p < 20$ and elastic when $p > 20$.

Revenue is maximized at the price corresponding to unit elasticity, which is $p = \$20$. At this price the revenue is

$$pq = (20)[3600 - 3(20^2)]$$

$$= (20)(2400)$$

$$= 48,000$$

or \$48,000.

6.3 Exercises

- When $q < \sqrt{\frac{2fM}{k}}$, $T'(q) < -\frac{k}{2} + \frac{k}{2} = 0$; and when $q > \sqrt{\frac{2fM}{k}}$, $T'(q) > -\frac{k}{2} + \frac{k}{2} = 0$. Since the function $T(q)$ is decreasing before $q = \sqrt{\frac{2fM}{k}}$ and increasing after $q = \sqrt{\frac{2fM}{k}}$, there must be a relative minimum at $q = \sqrt{\frac{2fM}{k}}$. By the critical point theorem, there is an absolute minimum there.
- The economic order quantity formula assumes that M , the total units needed per year, is known. Thus, c is the correct answer.
- The demand function $q(p)$ is positive and increasing, so $\frac{dq}{dp}$ is positive. Since p_0 and q_0 are also positive, the elasticity $E = -\frac{p_0}{q_0} \cdot \frac{dq}{dp}$ is negative.

6. $E = -\frac{p}{q} \cdot \frac{dq}{dp}$

Since $p \neq 0$, $E = 0$ when $\frac{dq}{dp} = 0$. The derivative is zero, which implies that the demand function has a horizontal tangent line at the value of p where $E = 0$.

7. $q = m - np$ for $0 \leq p \leq \frac{m}{n}$

$$\frac{dq}{dp} = -n$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$E = -\frac{p}{m - np}(-n)$$

$$E = \frac{pn}{m - np} = 1$$

$$pn = m - np$$

$$2np = m$$

$$p = \frac{m}{2n}$$

Thus, $E = 1$ when $p = \frac{m}{2n}$, or at the midpoint of the demand curve on the interval $0 \leq p \leq \frac{m}{n}$.

8. (a) $q = Cp^{-k}$

$$\frac{dq}{dp} = -Ckp^{-k-1}$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$= \frac{-p}{Cp^{-k}}(-Ckp^{-k-1})$$

$$= \frac{kp^{-k}}{p^{-k}} = k$$

9. Use equation (3) with $k = 1$, $M = 100,000$, and $f = 500$.

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(500)(100,000)}{1}}$$

$$= \sqrt{100,000,000} = 10,000$$

10,000 lamps should be made in each batch to minimize production costs.

10. Use equation (3) with $k = 9$, $M = 13,950$, and $f = 31$.

$$q = \sqrt{\frac{2fM}{k}}$$

$$= \sqrt{\frac{2(31)(13,950)}{9}}$$

$$= \sqrt{96,100} = 310$$

310 cases should be made in each batch to minimize production costs.

11. From Exercise 9, $M = 100,000$, and $q = 10,000$. The number of batches per year is

$$\frac{M}{q} = \frac{100,000}{10,000} = 10.$$

12. From Exercise 10, $M = 13,950$ and $q = 310$. The number of batches per year is

$$\frac{M}{q} = \frac{13,950}{310} = 45$$

13. Here $k = 0.50$, $M = 100,000$, and $f = 60$. We have

$$q = \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(60)(100,000)}{0.50}}$$

$$= \sqrt{24,000,000} \approx 4898.98$$

$T(4898) = 2449.489792$ and $T(4899) = 2449.489743$, so ordering 4899 copies per order minimizes the annual costs.

14. Here $k = 1$, $M = 900$, and $f = 5$. We have

$$q = \sqrt{\frac{2fM}{k}}$$

$$= \sqrt{\frac{2(5)(900)}{1}}$$

$$= \sqrt{9000} \approx 94.9$$

$T(94) \approx 94.872$ and $T(95) \approx 94.868$, so ordering 95 bottles per order minimizes the annual costs.

15. Using maximum inventory size,

$$T(q) = \frac{fM}{q} + gM + kq; (0, \infty)$$

$$T'(q) = \frac{-fM}{q^2} + k$$

Set the derivative equal to 0.

$$\begin{aligned}\frac{-fM}{q^2} + k &= 0 \\ k &= \frac{fM}{q^2} \\ q^2 k &= fM \\ q^2 &= \frac{fM}{k} \\ q &= \sqrt{\frac{fM}{k}}\end{aligned}$$

Since $\lim_{q \rightarrow 0} T(q) = \infty$,

$\lim_{q \rightarrow \infty} T(q) = \infty$, and

$q = \sqrt{\frac{fM}{k}}$ is the only critical value in $(0, \infty)$,

$q = \sqrt{\frac{fM}{k}}$ is the number of unit that should be ordered or manufactured to minimize total costs.

16. Use $q = \sqrt{\frac{fM}{k}}$ from Exercise 15 with $k = 6$, $M = 5000$, and $f = 1000$.

$$q = \sqrt{\frac{fM}{k}} = \sqrt{\frac{(1000)(5000)}{6}} \approx 912.9$$

with $T(q) = \frac{fM}{q} + kq$ (assume $g = 0$ since the subsequent cost per book is so low that it can be ignored), $T(912) \approx 10,954.456$ and $T(913) \approx 10,954.451$. So, 913 books should be printed in each print run.

17. Assuming an annual cost, k_1 , for storing a single unit, plus an annual cost per unit, k_2 , that must be paid for each unit up to the maximum number of units stored, we have

$$T(q) = \frac{fM}{q} + gM + \frac{k_1 q}{2} + k_2 q; (0, \infty)$$

$$T'(q) = \frac{-fM}{q^2} + \frac{k_1}{2} + k_2$$

Set the derivative equal to 0.

$$\begin{aligned}\frac{-fM}{q^2} + \frac{k_1}{2} + k_2 &= 0 \\ \frac{k_1 + 2k_2}{2} &= \frac{fM}{q^2} \\ \frac{q^2(k_1 + 2k_2)}{2} &= fM \\ q^2 &= \frac{2fM}{k_1 + 2k_2} \\ q &= \sqrt{\frac{2fM}{k_1 + 2k_2}}\end{aligned}$$

Since $\lim_{q \rightarrow 0} T(q) = \infty$, $\lim_{q \rightarrow \infty} T(q) = \infty$, and

$q = \sqrt{\frac{2fM}{k_1 + 2k_2}}$ is the only critical value in $(0, \infty)$,

$q = \sqrt{\frac{2fM}{k_1 + 2k_2}}$ is the number of units that should be ordered or manufactured to minimize the total cost in this case.

18. Use $q = \sqrt{\frac{2fM}{k_1 + 2k_2}}$ from Exercise 17 with $k_1 = 1$, $k_2 = 2$, $M = 30,000$, and $f = 750$. Also, note that $g = 8$.

$$\begin{aligned}q &= \sqrt{\frac{2fM}{k_1 + 2k_2}} \\ &= \sqrt{\frac{2(750)(30,000)}{1 + 2(2)}} \\ &= \sqrt{9,000,000} = 3000\end{aligned}$$

The number of production runs each year to minimize her total costs is

$$\frac{M}{q} = \frac{30,000}{3000} = 10.$$

19. $q = 50 - \frac{p}{4}$

$$(a) \quad \frac{dq}{dp} = -\frac{1}{4}$$

$$\begin{aligned}E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ &= -\frac{p}{50 - \frac{p}{4}} \left(-\frac{1}{4} \right)\end{aligned}$$

$$= -\frac{p}{\frac{200-p}{4}} \left(-\frac{1}{4} \right)$$

$$= \frac{p}{200 - p}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$.

Since $q = 0$ means no revenue, set

$$1 - E = 0, \text{ so } E = 1.$$

From (a),

$$\frac{p}{200 - p} = 1$$

$$p = 200 - p$$

$$p = 100.$$

$$q = 50 - \frac{p}{4}$$

$$= 50 - \frac{100}{4}$$

$$= 25$$

Total revenue is maximized if $q = 25$.

20. $q = 25,000 - 50p$

(a) $\frac{dq}{dp} = -50$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$= -\frac{p}{25,000 - 50p}(-50)$$

$$= \frac{p}{500 - p}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$.

Since $q = 0$ means no revenue, set

$$1 - E = 0.$$

$$E = 1$$

From part (a),

$$\frac{p}{500 - p} = 1$$

$$p = 500 - p$$

$$p = 250.$$

$$q = 25,000 - 50p$$

$$= 25,000 - 50(250)$$

$$= 12,500$$

Total revenue is maximized if $q = 12,500$.

21. (a) $q = 37,500 - 5p^2$

$$\frac{dq}{dp} = -10p$$

$$E = \frac{-p}{q} \cdot \frac{dq}{dp}$$

$$= \frac{-p}{37,500 - 5p^2}(-10p)$$

$$= -\frac{10p^2}{37,500 - 5p^2}$$

$$= \frac{2p^2}{7500 - p^2}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$. Since

$q = 0$ means no revenue, set $1 - E = 0$.

$$E = 1$$

From (a),

$$\frac{2p^2}{7500 - p^2} = 1$$

$$2p^2 = 7500 - p^2$$

$$3p^2 = 7500$$

$$p^2 = 2500$$

$$p = \pm 50.$$

Since p must be positive, $p = 50$.

$$q = 37,500 - 5p^2$$

$$= 37,500 - 5(50)^2$$

$$= 37,500 - 5(2500)$$

$$= 37,500 - 12,500$$

$$= 25,000.$$

22. $q = 48,000 - 10p^2$

(a) $\frac{dq}{dp} = -20p$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$= \frac{-p}{48,000 - 10p^2}(-20p)$$

$$= -\frac{20p^2}{48,000 - 10p^2}$$

$$= \frac{2p^2}{4800 - p^2}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$. Since $q = 0$ means no revenue, set $1 - E = 0$.

$$E = 1$$

From part (a),

$$\begin{aligned}\frac{2p^2}{4800 - p^2} &= 1 \\ 2p^2 &= 4800 - p^2 \\ 3p^2 &= 4800 \\ p^2 &= 1600 \\ p &= \pm 40.\end{aligned}$$

Since p must be positive, $p = 40$.

$$\begin{aligned}q &= 48,000 - 10p^2 \\ &= 48,000 - 10(40)^2 \\ &= 48,000 - 10(1600) \\ &= 48,000 - 16,000 \\ &= 32,000\end{aligned}$$

23. $p = 400e^{-0.2q}$

In order to find the derivative $\frac{dq}{dp}$, we first need to solve for q in the equation $p = 400e^{-0.2q}$.

(a)
$$\frac{p}{400} = e^{-0.2q}$$

$$\ln\left(\frac{p}{400}\right) = \ln(e^{-0.2q}) = -0.2q$$

$$q = \frac{\ln\frac{p}{400}}{-0.2} = -5 \ln\left(\frac{p}{400}\right)$$

Now

$$\begin{aligned}\frac{dq}{dp} &= -5 \frac{1}{\frac{p}{400}} \cdot \frac{1}{400} = \frac{-5}{p}, \text{ and} \\ E &= -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p}{q} \cdot \frac{-5}{p} = \frac{5}{q}.\end{aligned}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$. Since $q = 0$ means no revenue, set $1 - E = 0$.

$$E = 1$$

From part (a),

$$\begin{aligned}\frac{5}{q} &= 1 \\ 5 &= q\end{aligned}$$

24. $q = 10 - \ln p$

(a)
$$\frac{dq}{dp} = -\frac{1}{p}$$

$$\begin{aligned}E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ &= \frac{-p}{10 - \ln p} \left(-\frac{1}{p}\right) \\ &= \frac{1}{10 - \ln p}\end{aligned}$$

(b) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$. Since $q = 0$ means no revenue, set $1 - E = 0$.

$$E = 1$$

From part (a),

$$\begin{aligned}\frac{1}{10 - \ln p} &= 1 \\ 1 &= 10 - \ln p \\ \ln p &= 9 \\ p &= e^9 \\ q &= 10 - \ln p \\ &= 10 - \ln e^9 \\ &= 10 - 9 \\ &= 1\end{aligned}$$

Note that $E = \frac{1}{q}$, thus we would expect E to be maximum when $q = 1$.

25. $q = 400 - 0.2p^2$

$$\begin{aligned}\frac{dq}{dp} &= 0 - 0.4p \\ E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ E &= -\frac{p}{400 - 0.2p^2} (-0.4p) \\ &= \frac{0.4p^2}{400 - 0.2p^2}\end{aligned}$$

- (a) If
- $p = \$20$
- ,

$$E = \frac{(0.4)(20)^2}{400 - 0.2(20)^2} = 0.5.$$

Since $E < 1$, demand is inelastic. This indicates that total revenue increases as price increases.

- (b) If
- $p = \$40$
- ,

$$E = \frac{(0.4)(40)^2}{400 - 0.2(40)^2} = 8.$$

Since $E > 1$, demand is elastic. This indicates that total revenue decreases as price increases.

- 26.
- $q = 300 - 2p$

$$\frac{dq}{dp} = -2$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$E = -\frac{p(-2)}{300 - 2p} = \frac{2p}{300 - 2p}$$

- (a) When
- $p = \$100$
- ,

$$E = \frac{200}{300 - 200} = 2.$$

Since $E > 1$, demand is elastic. This indicates that a percentage increase in price will result in a greater percentage decrease in demand.

- (b) When
- $p = \$50$
- ,

$$E = \frac{100}{300 - 100} = \frac{1}{2} < 1.$$

Since $E < 1$, supply is inelastic. This indicates that a percentage change in price will result in a smaller percentage change in demand.

- 27.
- $q = 2,431,129p^{-0.06}$

$$\frac{dq}{dp} = (-0.06)[2,431,129p^{-1.06}]$$

$$E = -\frac{p}{q} \frac{dq}{dp}$$

$$= -\frac{p}{2,431,129p^{-0.06}}(-0.06)[2,431,129p^{-1.06}] = 0.06$$

At any price (including \$40/barrel) the elasticity is 0.06 and the demand is inelastic.

- 28.
- $q = Ap^{-0.13}$

$$\frac{dq}{dp} = (-0.13)[Ap^{-1.13}]$$

$$E = -\frac{p}{q} \frac{dq}{dp}$$

$$= -\frac{p}{Ap^{-0.13}}(-0.13)[Ap^{-1.13}]$$

$$= 0.13$$

At any price the elasticity is 0.13 and the demand for rice is inelastic.

- 29.
- $q = 3,751,000p^{-2.826}$

$$\frac{dq}{dp} = (-2.826)[3,751,000p^{-3.826}]$$

$$E = -\frac{p}{q} \frac{dq}{dp}$$

$$= -\frac{p}{3,751,000p^{-2.826}}(-2.826)[3,751,000p^{-3.826}] = 2.826$$

At any price the elasticity is 2.826 and the demand is elastic.

- 30.
- $q = 342.5p^{-0.5314}$

$$\frac{dq}{dp} = 342.5(-0.5314)p^{-0.5314-1}$$

$$= \frac{-182}{p^{1.5314}}$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$= \frac{-p}{342.5p^{-0.5314}} \cdot \frac{-182}{p^{1.5314}}$$

$$= \frac{182}{342.5}$$

$$\approx 0.5314$$

Since $E < 1$, the demand is inelastic.

31. (a) $q = 55.2 - 0.022p$

$$\frac{dq}{dp} = -0.022$$

$$\begin{aligned} E &= -\frac{p}{q} \cdot \frac{dq}{dp} \\ &= \frac{-p}{55.2 - 0.022p} \cdot (-0.022) \\ &= \frac{0.022p}{55.2 - 0.022p} \end{aligned}$$

When $p = \$166.10$,

$$E = \frac{3.6542}{55.2 - 3.6542} \approx 0.071.$$

(b) Since $E < 1$, the demand for airfare is inelastic at this price.

(c) $R = pq$

$$\frac{dR}{dp} = q(1 - E)$$

When R is maximum, $q(1 - E) = 0$.

Since $q = 0$ means no revenue, set

$$1 - E = 0.$$

$$E = 1$$

From (a),

$$\begin{aligned} \frac{0.022p}{55.2 - 0.022p} &= 1 \\ 0.022p &= 55.2 - 0.022p \\ 0.044p &= 55.2 \\ p &\approx 1255 \end{aligned}$$

Total revenue is maximized if $p \approx \$1255$.

32. $p = 0.604q^2 - 20.16q + 263.067$

$$\frac{dp}{dq} = 1.208q - 20.16$$

$$\begin{aligned} E &= -\frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{-q \left(\frac{dp}{dq} \right)} \\ &= \frac{0.604q^2 - 20.16q + 263.067}{-q(1.208q - 20.16)} \end{aligned}$$

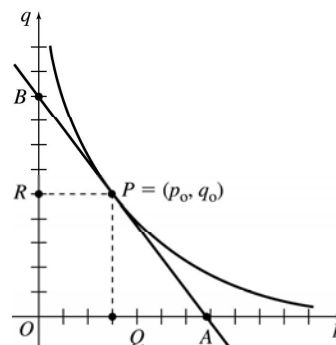
(a) Since $q = 11$,

$$E = \frac{114.391}{-11(-6.872)} \approx 1.51$$

(b) Since $E > 1$, demand is elastic.

(c) As q approaches 16.6887, the denominator of E approaches zero, so E approaches infinity.

33.



In the figure, we label P_0 as P . The slope of the tangent line is

$$\frac{dq}{dp} = -\frac{BR}{RP} = \frac{OB - OR}{-RP} = \frac{OB - q_0}{-p_0}$$

or

$$\begin{aligned} -p_0 \frac{dq}{dp} &= OB - q_0 \\ -\frac{p_0}{q_0} \cdot \frac{dq}{dp} &= \frac{OB}{q_0} - 1 \\ &= \frac{OB}{OR} - 1 \end{aligned}$$

Because triangles AOB and PRB are similar,

$$\begin{aligned} -\frac{p_0}{q_0} \cdot \frac{dq}{dp} &= \frac{AB}{AP} - 1 \\ &= \frac{AB - AP}{AP} \\ &= \frac{PB}{PA} \end{aligned}$$

But $E = -\frac{p_0}{q_0} \cdot \frac{dq}{dp}$ so the ratio $\frac{PB}{PA}$ equals the elasticity E .

6.4 Implicit Differentiation

Your Turn 1

$$x^2 + y^2 = xy$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}xy$$

$$2x + 2y \frac{dy}{dx} = x \frac{dy}{dx} + y$$

$$(2y - x) \frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

Your Turn 2

$$xe^y + x^2 = \ln y$$

$$\frac{d}{dx}(xe^y + x^2) = \frac{d}{dx} \ln y$$

$$\frac{d}{dx} xe^y + \frac{d}{dx} x^2 = \frac{d}{dx} \ln y$$

$$e^y + xe^y \frac{dy}{dx} + 2x = \frac{1}{y} \frac{dy}{dx}$$

$$\left(xe^y - \frac{1}{y}\right) \frac{dy}{dx} = -2x - e^y$$

$$(xye^y - 1) \frac{dy}{dx} = -2xy - ye^y$$

$$\frac{dy}{dx} = \frac{-2xy - ye^y}{xye^y - 1} = \frac{ye^y - 2xy}{1 - xye^y}$$

Your Turn 3

$$y^4 - x^4 - y^2 + x^2 = 0$$

$$\frac{d}{dx}(y^4 - x^4 - y^2 + x^2) = \frac{d}{dx}(0)$$

$$4y^3 \frac{dy}{dx} - 4x^3 - 2y \frac{dy}{dx} + 2x = 0$$

$$(4y^3 - 2y) \frac{dy}{dx} = 4x^3 - 2x$$

At the point (1, 1), $4y^3 - 2y \neq 0$ so we can divide both sides of the equation above by this factor.

$$\frac{dy}{dx} = \frac{4x^3 - 2x}{4y^3 - 2y}$$

At (1, 1),

$$\frac{dy}{dx} = \frac{4x^3 - 2x}{4y^3 - 2y} = \frac{4 - 2}{4 - 2} = 1$$

so the slope of the tangent line is $m = 1$. Use the point-slope form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 1(x - 1)$$

$$y = x$$

The equation of the tangent line at the point (1, 1) is $y = x$.

Your Turn 4

$$p = \frac{100,000}{q^2 + 100q} = 100,000(q^2 + 100q)^{-1}$$

$$\frac{d}{dp} p = \frac{d}{dp} [100,000(q^2 + 100q)^{-1}]$$

$$1 = -100,000 \frac{2q + 100}{(q^2 + 100q)^2} \frac{dq}{dp}$$

$$\frac{dq}{dp} = -\frac{(q^2 + 100q)^2}{100,000(2q + 100)}$$

Substitute $q = 200$ in this expression for the derivative.

$$\begin{aligned} \frac{dq}{dp} &= -\frac{(q^2 + 100q)^2}{100,000(2q + 100)} \\ &= -\frac{[200^2 + 100(200)]^2}{(100,000)[2(200) + 100]} \\ &= -72 \end{aligned}$$

When the price is 200, the rate of change of demand with respect to price is -72 units per unit change in price.

6.4 Exercises

1. $6x^2 + 5y^2 = 36$

$$\frac{d}{dx}(6x^2 + 5y^2) = \frac{d}{dx}(36)$$

$$\frac{d}{dx}(6x^2) + \frac{d}{dx}(5y^2) = \frac{d}{dx}(36)$$

$$12x + 5 \cdot 2y \frac{dy}{dx} = 0$$

$$10y \frac{dy}{dx} = -12x$$

$$\frac{dy}{dx} = -\frac{6x}{5y}$$

2. $7x^2 - 4y^2 = 24$

$$\frac{d}{dx}(7x^2 - 4y^2) = \frac{d}{dx}(24)$$

$$\frac{d}{dx}(7x^2) - \frac{d}{dx}(4y^2) = \frac{d}{dx}(24)$$

$$14x - 8y \frac{dy}{dx} = 0$$

$$8y \frac{dy}{dx} = 14x$$

$$\frac{dy}{dx} = \frac{7x}{4y}$$

3. $8x^2 - 10xy + 3y^2 = 26$

$$\frac{d}{dx}(8x^2 - 10xy + 3y^2) = \frac{d}{dx}(26)$$

$$16x - \frac{d}{dx}(10xy) + \frac{d}{dx}(3y^2) = 0$$

$$16x - 10x \frac{dy}{dx} - y \frac{d}{dx}(10x) + 6y \frac{dy}{dx} = 0$$

$$16x - 10x \frac{dy}{dx} - 10y + 6y \frac{dy}{dx} = 0$$

$$(-10x + 6y) \frac{dy}{dx} = -16x + 10y$$

$$\frac{dy}{dx} = \frac{-16x + 10y}{-10x + 6y}$$

$$\frac{dy}{dx} = \frac{8x - 5y}{5x - 3y}$$

4. $7x^2 = 5y^2 + 4xy + 1$

$$\frac{d}{dx}(7x^2) = \frac{d}{dx}(5y^2 + 4xy + 1)$$

$$14x = \frac{d}{dx}(5y^2 + 4xy + 1)$$

$$14x = 10y \frac{dy}{dx} + 4x \frac{d}{dx}(y) + y \frac{d}{dx}(4x)$$

$$14x = 10y \frac{dy}{dx} + 4x \frac{dy}{dx}(1) + 4y$$

$$14x - 4y = \frac{dy}{dx}(10y + 4x)$$

$$\frac{14x - 4y}{10y + 4x} = \frac{dy}{dx}$$

$$\frac{7x - 2y}{2x + 5y} = \frac{dy}{dx}$$

5. $5x^3 = 3y^2 + 4y$

$$\frac{d}{dx}(5x^3) = \frac{d}{dx}(3y^2 + 4y)$$

$$15x^2 = \frac{d}{dx}(3y^2) + \frac{d}{dx}(4y)$$

$$15x^2 = 6y \frac{dy}{dx} + 4 \frac{dy}{dx}$$

$$\frac{15x^2}{6y + 4} = \frac{dy}{dx}$$

6. $3x^3 - 8y^2 = 10y$

$$\frac{d}{dx}(3x^3 - 8y^2) = \frac{d}{dx}(10y)$$

$$\frac{d}{dx}(3x^3) - \frac{d}{dx}(8y^2) = 10 \frac{dy}{dx}$$

$$9x^2 - 16y \frac{dy}{dx} = 10 \frac{dy}{dx}$$

$$9x^2 = (16y + 10) \frac{dy}{dx}$$

$$\frac{9x^2}{16y + 10} = \frac{dy}{dx}$$

7. $3x^2 = \frac{2 - y}{2 + y}$

$$\frac{d}{dx}(3x^2) = \frac{d}{dx}\left(\frac{2 - y}{2 + y}\right)$$

$$6x = \frac{(2 + y) \frac{d}{dx}(2 - y) - (2 - y) \frac{d}{dx}(2 + y)}{(2 + y)^2}$$

$$6x = \frac{(2 + y)\left(-\frac{dy}{dx}\right) - (2 - y) \frac{dy}{dx}}{(2 + y)^2}$$

$$6x = \frac{-4 \frac{dy}{dx}}{(2 + y)^2}$$

$$6x(2 + y)^2 = -4 \frac{dy}{dx}$$

$$-\frac{3x(2 + y)^2}{2} = \frac{dy}{dx}$$

8. $2y^2 = \frac{5 + x}{5 - x}$

$$\frac{d}{dx}(2y^2) = \frac{d}{dx}\left(\frac{5 + x}{5 - x}\right)$$

$$4y \frac{dy}{dx} = \frac{(1)(5 - x) - (-1)(5 + x)}{(5 - x)^2}$$

$$4y \frac{dy}{dx} = \frac{5 - x + 5 + x}{(5 - x)^2}$$

$$= \frac{10}{(5 - x)^2}$$

$$\frac{dy}{dx} = \frac{10}{4y(5 - x)^2}$$

$$= \frac{5}{2y(5 - x)^2}$$

9. $2\sqrt{x} + 4\sqrt{y} = 5y$

$$\begin{aligned}\frac{d}{dx}(2x^{1/2} + 4y^{1/2}) &= \frac{d}{dx}(5y) \\ x^{-1/2} + 2y^{-1/2} \frac{dy}{dx} &= 5 \frac{dy}{dx} \\ (2y^{-1/2} - 5) \frac{dy}{dx} &= -x^{-1/2} \\ \frac{dy}{dx} &= \frac{x^{-1/2}}{5 - 2y^{-1/2}} \left(\frac{x^{1/2}y^{1/2}}{x^{1/2}y^{1/2}} \right) \\ &= \frac{y^{1/2}}{x^{1/2}(5y^{1/2} - 2)} \\ &= \frac{\sqrt{y}}{\sqrt{x}(5\sqrt{y} - 2)}\end{aligned}$$

10. $4\sqrt{x} - 8\sqrt{y} = 6y^{3/2}$

$$\begin{aligned}\frac{d}{dx}(4x^{1/2} - 8y^{1/2}) &= 6 \frac{d}{dx}(y^{3/2}) \\ 2x^{-1/2} - 4y^{-1/2} \frac{dy}{dx} &= 6 \cdot \frac{3}{2} y^{1/2} \frac{dy}{dx} \\ 2x^{-1/2} &= (9y^{1/2} + 4y^{-1/2}) \frac{dy}{dx} \\ \frac{2x^{-1/2}}{9y^{1/2} + 4y^{-1/2}} &= \frac{dy}{dx} \\ \frac{2\sqrt{y}}{\sqrt{x}(9y + 4)} &= \frac{dy}{dx}\end{aligned}$$

11. $x^4y^3 + 4x^{3/2} = 6y^{3/2} + 5$

$$\begin{aligned}\frac{d}{dx}(x^4y^3 + 4x^{3/2}) &= \frac{d}{dx}(6y^{3/2} + 5) \\ \frac{d}{dx}(x^4y^3) + \frac{d}{dx}(4x^{3/2}) &= \frac{d}{dx}(6y^{3/2}) + \frac{d}{dx}(5) \\ 4x^3y^3 + x^4 \cdot 3y^2 \frac{dy}{dx} + 6x^{1/2} &= 9y^{1/2} \frac{dy}{dx} + 0 \\ 4x^3y^3 + 6x^{1/2} &= 9y^{1/2} \frac{dy}{dx} - 3x^4y^2 \frac{dy}{dx} \\ 4x^3y^3 + 6x^{1/2} &= (9y^{1/2} - 3x^4y^2) \frac{dy}{dx} \\ \frac{4x^3y^3 + 6x^{1/2}}{9y^{1/2} - 3x^4y^2} &= \frac{dy}{dx}\end{aligned}$$

12. $(xy)^{4/3} + x^{1/3} = y^6 + 1$

$$\begin{aligned}\frac{d}{dx}[(xy)^{4/3} + x^{1/3}] &= \frac{d}{dx}(y^6 + 1) \\ \frac{d}{dx}(x^{4/3}y^{4/3}) + \frac{d}{dx}(x^{1/3}) &= \frac{d}{dx}(y^6) + \frac{d}{dx}(1) \\ x^{4/3} \cdot \frac{4}{3}y^{1/3} \frac{dy}{dx} + \frac{4}{3}x^{1/3}y^{4/3} + \frac{1}{3}x^{-2/3} &= 6y^5 \frac{dy}{dx} + 0 \\ \frac{4}{3}x^{1/3}y^{4/3} + \frac{1}{3}x^{-2/3} &= 6y^5 \frac{dy}{dx} - \frac{4}{3}x^{4/3}y^{1/3} \frac{dy}{dx} \\ 4x^{1/3}y^{4/3} + x^{-2/3} &= 18y^5 \frac{dy}{dx} - 4x^{4/3}y^{1/3} \frac{dy}{dx} \\ 4x^{1/3}y^{4/3} + x^{-2/3} &= (18y^5 - 4x^{4/3}y^{1/3}) \cdot \frac{dy}{dx} \\ \frac{4x^{1/3}y^{4/3} + x^{-2/3}}{18y^5 - 4x^{4/3}y^{1/3}} &= \frac{dy}{dx} \\ \frac{x^{2/3}}{x^{2/3}} \cdot \frac{4x^{1/3}y^{4/3} + x^{-2/3}}{18y^5 - 4x^{4/3}y^{1/3}} &= \frac{dy}{dx} \\ \frac{4xy^{4/3} + 1}{18x^{2/3}y^5 - 4x^2y^{1/3}} &= \frac{dy}{dx}\end{aligned}$$

13. $e^{x^2y} = 5x + 4y + 2$

$$\begin{aligned}\frac{d}{dx}(e^{x^2y}) &= \frac{d}{dx}(5x + 4y + 2) \\ e^{x^2y} \frac{d}{dx}(x^2y) &= \frac{d}{dx}(5x) + \frac{d}{dx}(4y) + \frac{d}{dx}(2) \\ e^{x^2y} \left(2xy + x^2 \frac{dy}{dx} \right) &= 5 + 4 \frac{dy}{dx} + 0 \\ 2xye^{x^2y} + x^2e^{x^2y} \frac{dy}{dx} &= 5 + 4 \frac{dy}{dx} \\ x^2e^{x^2y} \frac{dy}{dx} - 4 \frac{dy}{dx} &= 5 - 2xye^{x^2y} \\ (x^2e^{x^2y} - 4) \frac{dy}{dx} &= 5 - 2xye^{x^2y} \\ \frac{dy}{dx} &= \frac{5 - 2xye^{x^2y}}{x^2e^{x^2y} - 4}\end{aligned}$$

14. $x^2e^y + y = x^3$

$$\begin{aligned}\frac{d}{dx}(x^2e^y + y) &= \frac{d}{dx}(x^3) \\ \frac{d}{dx}(x^2e^y) + \frac{d}{dx}(y) &= 3x^2 \\ 2xe^y + x^2e^y \frac{dy}{dx} + \frac{dy}{dx} &= 3x^2 \\ x^2e^y \frac{dy}{dx} + \frac{dy}{dx} &= 3x^2 - 2xe^y \\ (x^2e^y + 1) \frac{dy}{dx} &= 3x^2 - 2xe^y \\ \frac{dy}{dx} &= \frac{3x^2 - 2xe^y}{x^2e^y + 1}\end{aligned}$$

15. $x + \ln y = x^2y^3$

$$\begin{aligned}\frac{d}{dx}(x + \ln y) &= \frac{d}{dx}(x^2y^3) \\ 1 + \frac{1}{y} \frac{dy}{dx} &= 2xy^3 + 3x^2y^2 \frac{dy}{dx} \\ \frac{1}{y} \frac{dy}{dx} - 3x^2y^2 \frac{dy}{dx} &= 2xy^3 - 1 \\ \left(\frac{1}{y} - 3x^2y^2 \right) \frac{dy}{dx} &= 2xy^3 - 1 \\ \frac{dy}{dx} &= \frac{2xy^3 - 1}{\frac{1}{y} - 3x^2y^2} \\ &= \frac{y(2xy^3 - 1)}{1 - 3x^2y^3}\end{aligned}$$

16. $y \ln x + 2 = x^{3/2}y^{5/2}$

$$\begin{aligned}\frac{d}{dx}(y \ln x + 2) &= \frac{d}{dx}(x^{3/2}y^{5/2}) \\ \ln x \frac{dy}{dx} + \frac{y}{x} + 0 &= \frac{3}{2}x^{1/2}y^{5/2} + \frac{5}{2}x^{3/2}y^{3/2} \frac{dy}{dx} \\ \ln x \frac{dy}{dx} - \frac{5}{2}x^{3/2}y^{3/2} \frac{dy}{dx} &= \frac{3}{2}x^{1/2}y^{5/2} - \frac{y}{x} \\ \frac{dy}{dx} \left(\ln x - \frac{5}{2}x^{3/2}y^{3/2} \right) &= \frac{3}{2}x^{1/2}y^{5/2} - \frac{y}{x} \\ \frac{dy}{dx} &= \frac{\frac{3}{2}x^{1/2}y^{5/2} - \frac{y}{x}}{\ln x - \frac{5}{2}x^{3/2}y^{3/2}} \\ &= \frac{3x^{3/2}y^{5/2} - 2y}{x(2 \ln x - 5x^{3/2}y^{3/2})}\end{aligned}$$

17. $x^2 + y^2 = 25$; tangent at $(-3, 4)$

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25) \\ 2x + 2y \frac{dy}{dx} &= 0 \\ 2y \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y} \\ m = -\frac{x}{y} &= -\frac{-3}{4} = \frac{3}{4} \\ y - y_1 &= m(x - x_1) \\ y - 4 &= \frac{3}{4}[x - (-3)] \\ 4y - 16 &= 3x + 9 \\ 4y &= 3x + 25 \\ y &= \frac{3}{4}x + \frac{25}{4}\end{aligned}$$

18. $x^2 + y^2 = 100$; tangent at $(8, -6)$

$$\begin{aligned}\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(100) \\ 2x + 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \\ m = -\frac{x}{y} &= -\frac{8}{-6} = \frac{4}{3} \\ y - y_1 &= m(x - x_1) \\ y + 6 &= \frac{4}{3}(x - 8) \\ 3y + 18 &= 4x - 32 \\ 3y &= 4x - 50 \\ y &= \frac{4}{3}x - \frac{50}{3}\end{aligned}$$

19. $x^2y^2 = 1$; tangent at $(-1, 1)$

$$\begin{aligned}\frac{d}{dx}(x^2y^2) &= \frac{d}{dx}(1) \\ x^2 \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x^2) &= 0 \\ x^2(2y) \frac{dy}{dx} + y^2(2x) &= 0 \\ 2x^2y \frac{dy}{dx} &= -2xy^2 \\ \frac{dy}{dx} &= \frac{-2xy^2}{2x^2y} = -\frac{y}{x} \\ m &= -\frac{y}{x} = -\frac{1}{-1} = 1 \\ y - 1 &= 1[x - (-1)] \\ y &= x + 1 + 1 \\ y &= x + 2\end{aligned}$$

20. $x^2y^3 = 8$; tangent at $(-1, 2)$

$$\begin{aligned}\frac{d}{dx}(x^2y^3) &= \frac{d}{dx}(8) \\ 2xy^3 + 3x^2y^2 \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{2xy^3}{3x^2y^2} \\ m &= -\frac{2xy^3}{3x^2y^2} = -\frac{2(-1)(2)^3}{3(-1)^2(2)^2} \\ &= \frac{16}{12} = \frac{4}{3} \\ y - 2 &= \frac{4}{3}(x + 1) \\ 3y - 6 &= 4x + 4 \\ 3y &= 4x + 10\end{aligned}$$

21. $2y^2 - \sqrt{x} = 4$; tangent at $(16, 2)$

$$\begin{aligned}\frac{d}{dx}(2y^2 - \sqrt{x}) &= \frac{d}{dx}(4) \\ 4y \frac{dy}{dx} - \frac{1}{2}x^{-1/2} &= 0 \\ 4y \frac{dy}{dx} &= \frac{1}{2x^{1/2}} \\ \frac{dy}{dx} &= \frac{1}{8yx^{1/2}}\end{aligned}$$

$$\begin{aligned}m &= \frac{1}{8yx^{1/2}} = \frac{1}{8(2)(16)^{1/2}} \\ &= \frac{1}{8(2)(4)} = \frac{1}{64}\end{aligned}$$

$$\begin{aligned}y - 2 &= -\frac{1}{4}(x - 4) \\ y &= -\frac{1}{4}x + 3 \\ y - 2 &= \frac{1}{64}(x - 16) \\ 64y - 128 &= x - 16 \\ 64y &= x + 112 \\ y &= \frac{x}{64} + \frac{7}{4}\end{aligned}$$

22. $y + \frac{\sqrt{x}}{y} = 3$; tangent at $(4, 2)$

$$\begin{aligned}\frac{d}{dx}\left(y + \frac{\sqrt{x}}{y}\right) &= \frac{d}{dx}(3) \\ \frac{dy}{dx} + \frac{d}{dx}\left(\frac{\sqrt{x}}{y}\right) &= 0 \\ \frac{dy}{dx} + \frac{y\left(\frac{1}{2}\right)x^{-1/2} - \sqrt{x} \frac{dy}{dx}}{y^2} &= 0 \\ \frac{dy}{dx} &= \frac{-\frac{1}{2}yx^{-1/2} + \sqrt{x} \frac{dy}{dx}}{y^2} \\ y^2 \frac{dy}{dx} &= -\frac{1}{2}yx^{-1/2} + \sqrt{x} \frac{dy}{dx} \\ (y^2 - \sqrt{x}) \frac{dy}{dx} &= -\frac{1}{2}yx^{-1/2} \\ \frac{dy}{dx} &= \frac{-y}{2x^{1/2}(y^2 - \sqrt{x})} \\ m &= \frac{-y}{2x^{1/2}(y^2 - \sqrt{x})} \\ &= \frac{-2}{2(2)(4 - 2)} \\ &= -\frac{1}{4}\end{aligned}$$

23. $e^{x^2+y^2} = xe^{5y} - y^2e^{5x/2}$; tangent at (2,1)

$$\begin{aligned}\frac{d}{dx}(e^{x^2+y^2}) &= \frac{d}{dx}(xe^{5y} - y^2e^{5x/2}) \\ e^{x^2+y^2} \cdot \frac{d}{dx}(x^2 + y^2) &= e^{5y} + x \frac{d}{dx}(e^{5y}) - \left[2y \frac{dy}{dx} e^{5x/2} + y^2 e^{5x/2} \frac{d}{dx} \left(\frac{5x}{2} \right) \right] \\ e^{x^2+y^2} \left(2x + 2y \frac{dy}{dx} \right) &= e^{5y} + x \cdot 5e^{5y} \frac{dy}{dx} - 2ye^{5x/2} \frac{dy}{dx} - \frac{5}{2} y^2 e^{5x/2} \\ \left(2ye^{x^2+y^2} - 5xe^{5y} + 2ye^{5x/2} \right) \frac{dy}{dx} &= -2xe^{x^2+y^2} + e^{5y} - \frac{5}{2} y^2 e^{5x/2} \\ \frac{dy}{dx} &= \frac{-2xe^{x^2+y^2} + e^{5y} - \frac{5}{2} y^2 e^{5x/2}}{2ye^{x^2+y^2} - 5xe^{5y} + 2ye^{5x/2}} \\ m &= \frac{-4e^5 + e^5 - \frac{5}{2}e^5}{2e^5 - 10e^5 + 2e^5} = \frac{-\frac{11}{2}e^5}{-6e^5} = \frac{11}{12} \\ y - 1 &= \frac{11}{12}(x - 2) \\ y &= \frac{11}{12}x - \frac{5}{6}\end{aligned}$$

24. $2xe^{xy} = e^{x^3} + ye^{x^2}$; tangent at (1,1)

$$\begin{aligned}\frac{d}{dx}(2xe^{xy}) &= \frac{d}{dx}(e^{x^3} + ye^{x^2}) \\ 2e^{xy} + 2x \cdot e^{xy} \frac{d}{dx}(xy) &= e^{x^3} \frac{d}{dx}(x^3) + \frac{dy}{dx} \cdot 2x^2 + ye^{x^2} \cdot \frac{d}{dx}(x^2) \\ 2e^{xy} + 2xe^{xy} \left(y + x \frac{dy}{dx} \right) &= e^{x^3} \cdot 3x^2 + \frac{dy}{dx} e^{x^2} + ye^{x^2} \cdot 2x \\ \left(2x^2e^{xy} - e^{x^2} \right) \frac{dy}{dx} &= -2e^{xy} - 2xye^{xy} + 2xye^{x^2} + 3x^2e^{x^3} \\ \frac{dy}{dx} &= \frac{-2e^{xy} - 2xye^{xy} + 2xye^{x^2} + 3x^2e^{x^3}}{2x^2e^{xy} - e^{x^2}} \\ m &= \frac{-2e - 2e + 2e + 3e}{2e - e} = \frac{e}{e} = 1 \\ y - 1 &= 1(x - 1) \\ y &= x\end{aligned}$$

25. $\ln(x + y) = x^3y^2 + \ln(x^2 + 2) - 4$; tangent at $(1, 2)$

$$\begin{aligned}\frac{d}{dx}[\ln(x + y)] &= \frac{d}{dx}[x^3y^2 + \ln(x^2 + 2) - 4] \\ \frac{1}{x + y} \cdot \frac{d}{dx}(x + y) &= 3x^2y^2 + x^3 \cdot 2y \frac{dy}{dx} + \frac{1}{x^2 + 2} \cdot \frac{d}{dx}(x^2 + 2) - \frac{d}{dx}(4) \\ \left(\frac{1}{x + y} - 2x^3y \right) \frac{dy}{dx} &= 3x^2y^2 + \frac{2x}{x^2 + 2} - \frac{1}{x + y} \\ \frac{dy}{dx} &= \frac{3x^2y^2 + \frac{2x}{x^2 + 2} - \frac{1}{x + y}}{\frac{1}{x + y} - 2x^3y} \\ m &= \frac{3 \cdot 1 \cdot 4 + \frac{2 \cdot 1}{3} - \frac{1}{3}}{\frac{1}{3} - 2 \cdot 1 \cdot 2} = \frac{\frac{37}{3}}{\frac{-11}{3}} = -\frac{37}{11} \\ y - 2 &= -\frac{37}{11}(x - 1) \\ y &= -\frac{37}{11}x + \frac{59}{11}\end{aligned}$$

26. $\ln(x^2 + y^2) = \ln(5x) + \frac{y}{x} - 2$; tangent at $(1, 2)$

$$\begin{aligned}\frac{d}{dx}[\ln(x^2 + y^2)] &= \frac{d}{dx}\left[\ln(5x) + \frac{y}{x} - 2\right] \\ \frac{1}{x^2 + y^2} \cdot \frac{d}{dx}(x^2 + y^2) &= \frac{1}{5x} \cdot \frac{d}{dx}(5x) + \frac{x \frac{dy}{dx} - y}{x^2} - \frac{d}{dx}(2) \\ \frac{1}{x^2 + y^2} \left(2x + 2y \frac{dy}{dx} \right) &= \frac{1}{x} + \frac{1}{x} \cdot \frac{dy}{dx} - \frac{y}{x^2} \\ \left(\frac{2y}{x^2 + y^2} - \frac{1}{x} \right) \frac{dy}{dx} &= \frac{1}{x} - \frac{y}{x^2} - \frac{2x}{x^2 + y^2} \\ \frac{dy}{dx} &= \frac{\frac{1}{x} - \frac{y}{x^2} - \frac{2x}{x^2 + y^2}}{\frac{2y}{x^2 + y^2} - \frac{1}{x}} \\ m &= \frac{1 - 2 - \frac{2}{5}}{\frac{4}{5} - 1} = \frac{\frac{-7}{5}}{\frac{-1}{5}} = 7 \\ y - 2 &= 7(x - 1) \\ y &= 7x - 5\end{aligned}$$

27. $y^3 + xy - y = 8x^4; x = 1$

First, find the y -value of the point.

$$\begin{aligned} y^3 + (1)y - y &= 8(1)^4 \\ y^3 &= 8 \\ y &= 2 \end{aligned}$$

The point is (1, 2).

Find $\frac{dy}{dx}$.

$$\begin{aligned} 3y^2 \frac{dy}{dx} + x \frac{dy}{dx} + y - \frac{dy}{dx} &= 32x^3 \\ (3y^2 + x - 1) \frac{dy}{dx} &= 32x^3 - y \\ \frac{dy}{dx} &= \frac{32x^3 - y}{3y^2 + x - 1} \end{aligned}$$

At (1, 2),

$$\frac{dy}{dx} = \frac{32(1)^3 - 2}{3(2)^2 + 1 - 1} = \frac{30}{12} = \frac{5}{2}.$$

$$y - 2 = \frac{5}{2}(x - 1)$$

$$y - 2 = \frac{5}{2}x - \frac{5}{2}$$

$$y = \frac{5}{2}x - \frac{1}{2}$$

28. $y^3 + 2x^2y - 8y = x^3 + 19, x = 2$

$$\begin{aligned} 3y^2 \frac{dy}{dx} + 2x^2 \frac{dy}{dx} + 4xy - 8 \frac{dy}{dx} &= 3x^2 \\ (3y^2 + 2x^2 - 8) \frac{dy}{dx} &= 3x^2 - 4xy \\ \frac{dy}{dx} &= \frac{3x^2 - 4xy}{3y^2 + 2x^2 - 8} \end{aligned}$$

Find y when $x = 2$.

$$\begin{aligned} y^3 + 8y - 8y &= 8 + 19 \\ y^3 &= 27 \\ y &= 3 \\ \frac{dy}{dx} &= \frac{12 - 24}{27 + 8 - 8} = \frac{-12}{27} = -\frac{4}{9} \\ y - 3 &= -\frac{4}{9}(x - 2) \\ y - 3 &= -\frac{4}{9}x + \frac{8}{9} \\ y &= -\frac{4}{9}x + \frac{35}{9} \end{aligned}$$

29. $y^3 + xy^2 + 1 = x + 2y^2; x = 2$

Find the y -value of the point.

$$\begin{aligned} y^3 + 2y^2 + 1 &= 2 + 2y^2 \\ y^3 + 1 &= 2 \\ y^3 &= 1 \\ y &= 1 \end{aligned}$$

The point is (2, 1).

Find $\frac{dy}{dx}$.

$$\begin{aligned} 3y^2 \frac{dy}{dx} + x2y \frac{dy}{dx} + y^2 &= 1 + 4y \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} + 2xy \frac{dy}{dx} - 4y \frac{dy}{dx} &= 1 - y^2 \\ (3y^2 + 2xy - 4y) \frac{dy}{dx} &= 1 - y^2 \\ \frac{dy}{dx} &= \frac{1 - y^2}{3y^2 + 2xy - 4y} \end{aligned}$$

At (2, 1),

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 - 1^2}{3(1)^2 + 2(2)(1) - 4(1)} = 0. \\ y - 0 &= 0(x - 2) \\ y &= 1 \end{aligned}$$

30. $y^4(1 - x) + xy = 2, x = 1$

Find the y -value.

$$\begin{aligned} y^4(1 - 1) + y &= 2 \\ y &= 2 \end{aligned}$$

The point is (1, 2).

Find $\frac{dy}{dx}$.

$$\begin{aligned} y^4(-1) + 4y^3(1 - x) \frac{dy}{dx} + x \frac{dy}{dx} + y &= 0 \\ -y^4 + 4y^3 \frac{dy}{dx} - 4xy^3 \frac{dy}{dx} + x \frac{dy}{dx} + y &= 0 \\ (4y^3 - 4xy^3 + x) \frac{dy}{dx} &= y^4 - y \\ \frac{dy}{dx} &= \frac{y^4 - y}{4y^3 - 4xy^3 + x} \end{aligned}$$

$\frac{dy}{dx}$ at (1, 2) is

$$\frac{2^4 - 2}{4 \cdot 2^3 - 1(1)2^3 + 1} = \frac{16 - 2}{32 - 32 + 1} = 14.$$

$$\begin{aligned}y - 2 &= 14(x - 1) \\y - 2 &= 14x - 14 \\y &= 14x - 12\end{aligned}$$

31. $2y^3(x - 3) + x\sqrt{y} = 3; x = 3$

Find the y -value of the point.

$$\begin{aligned}2y^3(3 - 3) + 3\sqrt{y} &= 3 \\3\sqrt{y} &= 3 \\\sqrt{y} &= 1 \\y &= 1\end{aligned}$$

The point is (3, 1)

Find $\frac{dy}{dx}$.

$$\begin{aligned}2y^3(1) + 6y^2(x - 3)\frac{dy}{dx} \\+ x\left(\frac{1}{2}\right)y^{-1/2}\frac{dy}{dx} + \sqrt{y} &= 0 \\6y^2(x - 3)\frac{dy}{dx} + \frac{x}{2\sqrt{y}}\frac{dy}{dx} &= -2y^3 - \sqrt{y} \\\left[6y^2(x - 3) + \frac{x}{2\sqrt{y}}\right]\frac{dy}{dx} &= -2y^3 - \sqrt{y} \\\frac{dy}{dx} &= \frac{-2y^3 - \sqrt{y}}{6y^2(x - 3) + \frac{x}{2\sqrt{y}}} \\&= \frac{-4y^{7/2} - 2y}{12y^{5/2}(x - 3) + x}\end{aligned}$$

At (3, 1),

$$\begin{aligned}\frac{dy}{dx} &= \frac{-4(1) - 2}{12(1)(3 - 3) + 3} = \frac{-6}{3} = -2. \\y - 1 &= -2(x - 3) \\y - 1 &= -2x + 6 \\y &= -2x + 7\end{aligned}$$

32. $\frac{y}{18}(x^2 - 64) + x^{2/3}y^{1/3} = 12, x = 8$

Find the y -value of the point.

$$\begin{aligned}\frac{y}{18}(64 - 64) + 8^{2/3}y^{1/3} &= 12 \\4y^{1/3} &= 12 \\y^{1/3} &= 3 \\y &= 27\end{aligned}$$

The point is (8, 27).

Find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{y}{18}(2x) + \frac{1}{18}(x^2 - 64)\frac{dy}{dx} + \frac{1}{3}x^{2/3}y^{-2/3}\frac{dy}{dx} \\+ \frac{2}{3}x^{-1/3}y^{1/3} &= 0\end{aligned}$$

$$\left(\frac{x^2 - 64}{18} + \frac{x^{2/3}y^{-2/3}}{3}\right)\frac{dy}{dx} = \frac{-2xy}{18} - \frac{2x^{-1/3}y^{1/3}}{3}$$

$$\frac{dy}{dx} = \frac{\frac{-2xy}{18} - \frac{2x^{-1/3}y^{1/3}}{3}}{\frac{x^2 - 64}{18} + \frac{x^{2/3}y^{-2/3}}{3}} = \frac{-2xy - 12x^{-1/3}y^{1/3}}{x^2 - 64 + 6x^{2/3}y^{-2/3}}$$

$\frac{dy}{dx}$ at (8, 27) is

$$\begin{aligned}\frac{-2(8)(27) - 12(8)^{-1/3}(27)^{1/3}}{64 - 64 + 6(8)^{2/3}(27)^{-2/3}} &= \frac{-432 - 18}{\frac{24}{9}} \\&= (-450)\left(\frac{9}{24}\right) \\&= \frac{-675}{4}\end{aligned}$$

$$y - 27 = -\frac{675}{4}(x - 8)$$

$$y - 27 = -\frac{675}{4}x + 1350$$

$$y = -\frac{675}{4}x + 1377$$

33. $x^{2/3} + y^{2/3} = 2; (1, 1)$

Find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} &= 0 \\\frac{2}{3}y^{-1/3}\frac{dy}{dx} &= -\frac{2}{3}x^{-1/3} \\\frac{dy}{dx} &= \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} \\&= -\frac{y^{1/3}}{x^{1/3}}\end{aligned}$$

At (1, 1)

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1^{1/3}}{1^{1/3}} = -1 \\y - 1 &= -1(x - 1) \\y - 1 &= -x + 1 \\y &= -x + 2\end{aligned}$$

34. $3(x^2 + y^2)^2 = 25(x^2 - y^2); (2, 1)$

Find $\frac{dy}{dx}$.

$$\begin{aligned} 6(x^2 + y^2) \frac{d}{dx}(x^2 + y^2) &= 25 \frac{d}{dx}(x^2 - y^2) \\ 6(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) &= 25 \left(2x - 2y \frac{dy}{dx} \right) \\ 12x^3 + 12x^2y \frac{dy}{dx} + 12xy^2 + 12y^3 \frac{dy}{dx} &= 50x - 50y \frac{dy}{dx} \\ 12x^2y \frac{dy}{dx} + 12y^3 \frac{dy}{dx} + 50y \frac{dy}{dx} &= -12x^3 - 12xy^2 + 50x \\ (12x^2y + 12y^3 + 50y) \frac{dy}{dx} &= -12x^3 - 12xy^2 + 50x \\ \frac{dy}{dx} &= \frac{-12x^3 - 12xy^2 + 50x}{12x^2y + 12y^3 + 50y} \end{aligned}$$

At $(2, 1)$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{-12(2)^3 - 12(2)(1)^2 + 50(2)}{12(2)^2 + 12(1)^3 + 50(1)} \\ &= \frac{-20}{110} \\ &= -\frac{2}{11} \\ y - 1 &= -\frac{2}{11}(x - 2) \\ y - 1 &= -\frac{2}{11}x + \frac{4}{11} \\ y &= -\frac{2}{11}x + \frac{15}{11} \end{aligned}$$

35. $y^2(x^2 + y^2) = 20x^2; (1, 2)$

Find $\frac{dy}{dx}$.

$$\begin{aligned} 2y(x^2 + y^2) \frac{dy}{dx} + y^2 \left(2x + 2y \frac{dy}{dx} \right) &= 40x \\ 2x^2y \frac{dy}{dx} + 2y^3 \frac{dy}{dx} + 2xy^2 + 2y^3 \frac{dy}{dx} &= 40x \\ 2x^2y \frac{dy}{dx} + 4y^3 \frac{dy}{dx} &= -2xy^2 + 40x \\ (2x^2y + 4y^3) \left(\frac{dy}{dx} \right) &= -2xy^2 + 40x \\ \frac{dy}{dx} &= \frac{-2xy^2 + 40x}{2x^2y + 4y^3} \end{aligned}$$

At (1, 2),

$$\begin{aligned}\frac{dy}{dx} &= \frac{-2(1)(2)^2 + 40(1)}{2(1)^2(2) + 4(2)^3} \\ &= \frac{32}{36} = \frac{8}{9} \\ y - 2 &= \frac{8}{9}(x - 1) \\ y - 2 &= \frac{8}{9}x - \frac{8}{9} \\ y &= \frac{8}{9}x + \frac{10}{9}\end{aligned}$$

36. $2(x^2 + y^2)^2 = 25xy^2$; (2, 1)

Find $\frac{dy}{dx}$.

$$\begin{aligned}4(x^2 + y^2)\frac{d}{dx}(x^2 + y^2) &= 25\frac{d}{dx}(xy^2) \\ 4(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) &= 25\left(y^2 + 2xy\frac{dy}{dx}\right) \\ 8x^3 + 8x^2y\frac{dy}{dx} + 8xy^2 + 8y^3\frac{dy}{dx} &= 25y^2 + 50xy\frac{dy}{dx} \\ 8x^2y\frac{dy}{dx} + 8y^3\frac{dy}{dx} - 50xy\frac{dy}{dx} &= -8x^3 - 8xy^2 + 25y^2 \\ (8x^2y + 8y^3 - 50xy)\left(\frac{dy}{dx}\right) &= -8x^3 - 8xy^2 + 25y^2 \\ \frac{dy}{dx} &= \frac{-8x^3 - 8xy^2 + 25y^2}{8x^2y + 8y^3 - 50xy}\end{aligned}$$

At (2, 1),

$$\begin{aligned}\frac{dy}{dx} &= \frac{-8(2)^3 - 8(2)(1)^2 + 25(1)^2}{8(2)^2(1) + 8(1)^3 - 50(2)(1)} \\ &= \frac{-55}{-60} = \frac{11}{12} \\ y - 1 &= \frac{11}{12}(x - 2) \\ y - 1 &= \frac{11}{12}x - \frac{11}{6} \\ y &= \frac{11}{12}x - \frac{5}{6}\end{aligned}$$

37. $x^2 + y^2 = 100$

- (a) Lines are tangent at points where $x = 6$.
By substituting $x = 6$ in the equation, we find that the points are $(6, 8)$ and $(6, -8)$.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(100)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$dy = -\frac{x}{y}$$

$$m_1 = -\frac{x}{y} = -\frac{6}{8} = -\frac{3}{4}$$

$$m_2 = -\frac{x}{y} = -\frac{6}{-8} = \frac{3}{4}$$

First tangent:

$$y - 8 = -\frac{3}{4}(x - 6)$$

$$y = -\frac{3}{4}x + \frac{25}{2}$$

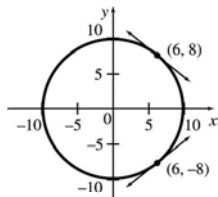
Second tangent:

$$y - (-8) = \frac{3}{4}(x - 6)$$

$$y + 8 = \frac{3}{4}x - \frac{18}{4}$$

$$y = \frac{3}{4}x - \frac{25}{2}$$

(b)



38. $y^2 = x^3 + ax + b$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 + ax + b)$$

$$2y \frac{dy}{dx} = 3x^2 + a$$

$$\frac{dy}{dx} = \frac{3x^2 + a}{2y}$$

39. (a) $\sqrt{u} + \sqrt{2v+1} = 5$

$$\frac{du}{dv}(\sqrt{u} + \sqrt{2v+1}) = \frac{du}{dv}(5)$$

$$\frac{1}{2}u^{-1/2} \frac{du}{dv} + \frac{1}{2}(2v+1)^{-1/2}(2) = 0$$

$$\frac{1}{2}u^{-1/2} \frac{du}{dv} = -\frac{1}{(2v+1)^{1/2}}$$

$$\frac{du}{dv} = -\frac{2u^{1/2}}{(2v+1)^{1/2}}$$

(b) $\sqrt{u} + \sqrt{2v+1} = 5$

$$\frac{dv}{du}(\sqrt{u} + \sqrt{2v+1}) = \frac{dv}{du}(5)$$

$$\frac{1}{2}u^{-1/2} + \frac{1}{2}(2v+1)^{-1/2}(2) \frac{dv}{du} = 0$$

$$(2v+1)^{-1/2} \frac{dv}{du} = -\frac{1}{2}u^{-1/2}$$

$$\frac{dv}{du} = -\frac{(2v+1)^{1/2}}{2u^{1/2}}$$

The derivatives are reciprocals.

40. (a) $e^{u^2-v} - v = 1$

$$\frac{d}{dv}(e^{u^2-v} - v) = \frac{d}{dv}(1)$$

$$\frac{d}{dv}(e^{u^2-v}) - \frac{d}{dv}v = \frac{d}{dv}(1)$$

$$e^{u^2-v} \frac{d}{dv}(u^2 - v) - 1 = 0$$

$$e^{u^2-v} \left(2u \frac{du}{dv} - 1 \right) = 1$$

$$2u \frac{du}{dv} - 1 = \frac{1}{e^{u^2-v}}$$

$$\frac{du}{dv} = \frac{1}{2u} \left(\frac{1}{e^{u^2-v}} + 1 \right)$$

$$\frac{du}{dv} = \frac{1 + e^{u^2-v}}{2ue^{u^2-v}}$$

(b) $e^{u^2-v} - v = 1$

$$\frac{d}{du}(e^{u^2-v} - v) = \frac{d}{du}(1)$$

$$\frac{d}{du}(e^{u^2-v}) - \frac{d}{du}v = \frac{d}{du}(1)$$

$$e^{u^2-v} \frac{d}{du}(u^2 - v) - \frac{dv}{du} = 0$$

$$e^{u^2-v} \left(2u - \frac{dv}{du} \right) - \frac{dv}{du} = 0$$

$$e^{u^2-v} \frac{dv}{du} - \frac{dv}{du} = -2ue^{u^2-v}$$

$$-\frac{dv}{du} \left(e^{u^2-v} + 1 \right) = -2ue^{u^2-v}$$

$$\frac{dv}{du} = \frac{2ue^{u^2-v}}{e^{u^2-v} + 1}$$

41. $x^2 + y^2 + 1 = 0$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(-1)$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

If x and y are real numbers, x^2 and y^2 are nonnegative; 1 plus a nonnegative number cannot equal zero, so there is no function $y = f(x)$ that satisfies $x^2 + y^2 + 1 = 0$.

42. $2p^2 + q^2 = 1600$

(a) $4p + 2q \frac{dq}{dp} = 0$

$$4p = -2q \frac{dq}{dp}$$

$$-\frac{2p}{q} = \frac{dq}{dp}$$

This is the rate of change of demand with respect to price.

(b) $4p \frac{dp}{dq} + 2q = 0$

$$\frac{dp}{dq} = -\frac{q}{2p}$$

This is the rate of change of price with respect to demand.

43. $C^2 = x^2 + 100\sqrt{x} + 50$

(a) $2C \frac{dC}{dx} = 2x + \frac{1}{2}(100)x^{-1/2}$

$$\frac{dC}{dx} = \frac{2x + 50x^{-1/2}}{2C}$$

$$\frac{dC}{dx} = \frac{x + 25x^{-1/2}}{C} \cdot \frac{x^{1/2}}{x^{1/2}}$$

$$\frac{dC}{dx} = \frac{x^{3/2} + 25}{Cx^{1/2}}$$

When $x = 5$, the approximate increase in cost of an additional unit is

$$\frac{(5)^{3/2} + 25}{(5^2 + 100\sqrt{5} + 50)^{1/2}(5)^{1/2}} = \frac{36.18}{(17.28)\sqrt{5}} \approx 0.94.$$

(b) $900(x - 5)^2 + 25R^2 = 22,500$

$$R^2 = 900 - 36(x - 5)^2$$

$$2R \frac{dR}{dx} = -72(x - 5)$$

$$\frac{dR}{dx} = \frac{-36(x - 5)}{R} = \frac{180 - 36x}{R}$$

When $x = 5$, the approximate change in revenue for a unit increase in sales is

$$\frac{180 - 36(5)}{R} = \frac{0}{R} = 0.$$

44. (a) $\ln q = C - 0.678 \ln p$

$$\frac{1}{q} \frac{dq}{dp} = -\frac{0.678}{p}$$

$$\frac{dp}{dq} = -0.678 \frac{q}{p}$$

$$E = -\frac{p}{q} \frac{dq}{dp}$$

$$= -\frac{p}{q} \left(-0.678 \frac{q}{p} \right)$$

$$= 0.678$$

E is less than 1, so the demand is inelastic.

(b) Solving for q first:

$$\ln q = C - 0.678 \ln p$$

$$e^{\ln q} = e^{C-0.678 \ln p}$$

$$q = e^C p^{-0.678}$$

$$\frac{dq}{dp} = e^C (-0.678) p^{-1.678}$$

$$\begin{aligned}
 E &= -\frac{p}{q} \frac{dq}{dp} \\
 &= \left(-\frac{p}{e^C p^{-0.678}} \right) (e^C (-0.678) p^{-1.678}) \\
 &= 0.678
 \end{aligned}$$

This is the same answer as found in part (a).

45. (a) $\ln q = D - 0.44 \ln p$

$$\frac{1}{q} \frac{dq}{dp} = -\frac{0.44}{p}$$

$$\frac{dp}{dq} = -0.44 \frac{q}{p}$$

$$\begin{aligned}
 E &= -\frac{p}{q} \frac{dq}{dp} \\
 &= -\frac{p}{q} \left(-0.44 \frac{q}{p} \right) \\
 &= 0.44
 \end{aligned}$$

E is less than 1, so the demand is inelastic.

(b) Solving for q first:

$$\ln q = D - 0.44 \ln p$$

$$e^{\ln q} = e^{D-0.44 \ln p}$$

$$q = e^D p^{-0.44}$$

$$\frac{dq}{dp} = e^D (-0.678) p^{-1.44}$$

$$\begin{aligned}
 E &= -\frac{p}{q} \frac{dq}{dp} \\
 &= \left(-\frac{p}{e^D p^{-0.44}} \right) (e^D (-0.44) p^{-1.44}) \\
 &= 0.44
 \end{aligned}$$

This is the same answer as found in part (a).

46. First note that

if $\log R(w) = 1.83 - 0.43 \log(w)$

then
$$\begin{aligned}
 R(w) &= 10^{1.83-0.43 \log(w)} \\
 &= 10^{1.83} 10^{-0.43 \log(w)} \\
 &= 10^{1.83} [10^{\log(w)}]^{-0.43} \\
 &= 10^{1.83} w^{-0.43}
 \end{aligned}$$

(a) $\frac{d}{dw} [\log R(w)] = \frac{d}{dw} [1.83 - 0.43 \log(w)]$

$$\frac{1}{\ln 10} \frac{1}{R(w)} \frac{dR}{dw} = 0 - 0.43 \frac{1}{\ln 10} \frac{1}{w}$$

$$\frac{dR}{dw} = -0.43 \frac{R(w)}{w}$$

$$= -0.43 \frac{10^{1.83} w^{-0.43}}{w}$$

$$\approx -29.0716 w^{-1.43}$$

(b) $R(w) = 10^{1.83} w^{-0.43}$

$$\frac{d}{dw} [R(w)] = \frac{d}{dw} [10^{1.83} w^{-0.43}]$$

$$\frac{dR}{dw} = 10^{1.83} (-0.43) w^{-1.43}$$

$$\approx -29.0716 w^{-1.43}$$

47. $b - a = (b + a)^3$

$$\frac{d}{db} (b - a) = \frac{d}{db} [(b + a)^3]$$

$$1 - \frac{da}{db} = 3(b + a)^2 \frac{d}{db} (b + a)$$

$$1 - \frac{da}{db} = 3(b + a)^2 \left(1 + \frac{da}{db} \right)$$

$$1 - \frac{da}{db} = 3(b + a)^2 + 3(b + a)^2 \frac{da}{db}$$

$$-\frac{da}{db} - 3(b + a)^2 \frac{da}{db} = 3(b + a)^2 - 1$$

$$[-1 - 3(b + a)^2] \frac{da}{db} = 3(b + a)^2 - 1$$

$$\frac{da}{db} = \frac{3(b + a)^2 - 1}{-1 - 3(b + a)^2}$$

$$\frac{da}{db} = 0$$

$$3(b + a)^2 - 1 = 0$$

$$b + a = \frac{1}{\sqrt{3}}$$

Since $b - a = (b + a)^3 = \left(\frac{1}{\sqrt{3}} \right)^3 = \frac{1}{3\sqrt{3}}$.

$$b + a = \frac{1}{\sqrt{3}}$$

$$-(b - a) = -\frac{1}{3\sqrt{3}}$$

$$2a = \frac{2}{3\sqrt{3}}$$

$$a = \frac{1}{3\sqrt{3}}$$

48.

$$\begin{aligned}
 xy^a &= k \\
 \frac{d}{dx}(xy^a) &= \frac{d}{dx}(k) \\
 x \frac{d}{dx}(y^a) + y^a(1) &= 0 \\
 x \left(ay^{a-1} \frac{dy}{dx} \right) + y^a &= 0 \\
 axy^{a-1} \frac{dy}{dx} &= -y^a \\
 \frac{dy}{dx} &= -\frac{y^a}{axy^{a-1}} \\
 \frac{dy}{dx} &= -\frac{y}{ax}
 \end{aligned}$$

$$49. \quad s^3 - 4st + 6t^3 - 5t = 0$$

$$\begin{aligned}
 3s^2 \frac{ds}{dt} - \left(4t \frac{ds}{dt} + 4s \right) + 6t^2 - 5 &= 0 \\
 3s^2 \frac{ds}{dt} - 4t \frac{ds}{dt} - 4s + 6t^2 - 5 &= 0 \\
 \frac{ds}{dt}(3s^2 - 4t) &= 4s - 6t^2 + 5 \\
 \frac{ds}{dt} &= \frac{4s - 6t^2 + 5}{3s^2 - 4t}
 \end{aligned}$$

$$50. \quad 2s^2 + \sqrt{st} - 4 = 3t$$

$$\begin{aligned}
 4s \frac{ds}{dt} + \frac{1}{2}(st)^{-1/2} \left(s + t \frac{ds}{dt} \right) &= 3 \\
 4s \frac{ds}{dt} + \frac{s + t \frac{ds}{dt}}{2\sqrt{st}} &= 3 \\
 \frac{8s(\sqrt{st}) \frac{ds}{dt} + s + t \frac{ds}{dt}}{2\sqrt{st}} &= 3 \\
 \frac{(8s\sqrt{st} + t) \frac{ds}{dt} + s}{2\sqrt{st}} &= 3 \\
 (8s\sqrt{st} + t) \frac{ds}{dt} &= 6\sqrt{st} - s \\
 \frac{ds}{dt} &= \frac{-s + 6\sqrt{st}}{8s\sqrt{st} + t}
 \end{aligned}$$

6.5 Related Rates

Your Turn 1

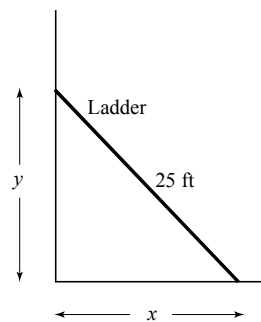
$x^3 + 2xy + y^2 = 1$, where both x and y are functions of t . Given $x = 1$, $y = -2$, and $dx/dt = 6$, find dy/dt .

$$\begin{aligned}
 \frac{d}{dt}(x^3 + 2xy + y^2) &= \frac{d}{dt}(1) \\
 3x^2 \frac{dx}{dt} + 2x \frac{dy}{dt} + 2y \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0 \\
 52x \frac{dy}{dt} + 2y \frac{dy}{dt} &= -\left(3x^2 \frac{dx}{dt} + 2y \frac{dx}{dt} \right) \\
 \frac{dy}{dt} &= \frac{dx}{dt} \left(-\frac{3x^2 + 2y}{2x + 2y} \right)
 \end{aligned}$$

Now substitute the given values.

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{dx}{dt} \left(-\frac{3x^2 + 2y}{2x + 2y} \right) \\
 &= 6 \left(-\frac{3(1^2) + 2(-2)}{2(1) + 2(-2)} \right) \\
 &= 6 \left(-\frac{-1}{-2} \right) \\
 &= -3
 \end{aligned}$$

Your Turn 2



$x^2 + y^2 = 25^2$, where both x and y are functions of t . We are interested in what happens when $x = 7$ ft. At this time, $y = \sqrt{25^2 - 7^2} = 24$, and since the bottom of the ladder is slipping away from the building at 3 ft/min, $dx/dt = 3$.

$$\begin{aligned}
 \frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}(25^2) \\
 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0
 \end{aligned}$$

Now substitute the known values.

$$\begin{aligned}
 2(7)(3) + 2(24)\frac{dy}{dt} &= 0 \\
 \frac{dy}{dt} &= \frac{-2(7)(3)}{2(24)} \\
 &= -\frac{7}{8}
 \end{aligned}$$

The latter is sliding *down* the side of the building at 7/8 ft/min.

Your Turn 3

In Example 4, differentiating the formula for the volume of a cone gives the following result:

$$\frac{dV}{dt} = \frac{1}{3}\pi \left[r^2 \frac{dh}{dt} + (h)(2r) \frac{dr}{dt} \right]$$

For this problem,

$$\frac{dV}{dt} = -10, \quad \frac{dr}{dt} = -0.4, \quad r = 4, \quad \text{and} \quad h = 20.$$

Substitute these values above and solve for dh/dt .

$$\begin{aligned}
 \frac{dV}{dt} &= \frac{1}{3}\pi \left[r^2 \frac{dh}{dt} + (h)(2r) \frac{dr}{dt} \right] \\
 -10 &= \frac{1}{3}\pi \left[(4^2) \frac{dh}{dt} + (20)(2)(4)(-0.4) \right] \\
 -\frac{30}{\pi} + 64 &= 16 \frac{dh}{dt} \\
 \frac{dh}{dt} &= \frac{-\frac{30}{\pi} + 64}{16} \\
 \frac{dh}{dt} &\approx 3.4
 \end{aligned}$$

The length increases at a rate of 3.4 cm per hour.

Your Turn 4

The revenue equation is

$$R = qp = q \left(2000 - \frac{q^2}{100} \right) = 2000q - \frac{q^3}{100}.$$

Differentiate with respect to t .

$$\begin{aligned}
 \frac{dR}{dt} &= 2000 \frac{dq}{dt} - \frac{3q^2}{100} \frac{dq}{dt} \\
 &= \left(2000 - \frac{3q^2}{100} \right) \frac{dq}{dt}
 \end{aligned}$$

Substitute the know values for q and dq/dt , which are the same as in Example 5: $q = 200$, $dq/dt = 50$.

$$\begin{aligned}
 \frac{dR}{dt} &= \left(2000 - \frac{3q^2}{100} \right) \frac{dq}{dt} \\
 &= \left(2000 - \frac{3(200^2)}{100} \right) (50) \\
 &= (800)(50) = 40,000
 \end{aligned}$$

Revenue is increasing at the rate of \$40,000 per day.

6.5 Exercises

$$1. \quad y^2 - 8x^3 = -55; \quad \frac{dx}{dt} = -4, \quad x = 2, \quad y = 3$$

$$\begin{aligned}
 2y \frac{dy}{dt} - 24x^2 \frac{dx}{dt} &= 0 \\
 y \frac{dy}{dt} &= 12x^2 \frac{dx}{dt} \\
 3 \frac{dy}{dt} &= 48(-4) \\
 \frac{dy}{dt} &= -64
 \end{aligned}$$

$$2. \quad 8y^3 + x^2 = 1; \quad \frac{dx}{dt} = 2, \quad x = 3, \quad y = -1$$

$$\begin{aligned}
 24y^2 \frac{dy}{dt} + 2x \frac{dx}{dt} &= 0 \\
 \frac{dy}{dt} &= \frac{-2x \frac{dx}{dt}}{24y^2} = -\frac{x \frac{dx}{dt}}{12y^2} \\
 &= -\frac{(3)(2)}{12(-1)^2} = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad 2xy - 5x + 3y^3 &= -51; \quad \frac{dx}{dt} = -6, \\
 x &= 3, \quad y = -2
 \end{aligned}$$

$$\begin{aligned}
 2x \frac{dy}{dt} + 2y \frac{dx}{dt} - 5 \frac{dx}{dt} + 9y^2 \frac{dy}{dt} &= 0 \\
 (2x + 9y^2) \frac{dy}{dt} + (2y - 5) \frac{dx}{dt} &= 0 \\
 (2x + 9y^2) \frac{dy}{dt} &= (5 - 2y) \frac{dx}{dt} \\
 \frac{dy}{dt} &= \frac{5 - 2y}{2x + 9y^2} \cdot \frac{dx}{dt} \\
 &= \frac{5 - 2(-2)}{2(3) + 9(-2)^2} \cdot (-6) \\
 &= \frac{9}{42} \cdot (-6) = \frac{-54}{42} \\
 &= -\frac{9}{7}
 \end{aligned}$$

$$4. \quad 4x^3 - 6xy^2 + 3y^2 = 228; \frac{dx}{dt} = 3, \\ x = -3, y = 4$$

$$\begin{aligned} 12x^2 \frac{dx}{dt} - \left(6y^2 \frac{dx}{dt} + 12xy \frac{dy}{dt} \right) + 6y \frac{dy}{dt} &= 0 \\ 12x^2 \frac{dx}{dt} - 6y^2 \frac{dx}{dt} - 12xy \frac{dy}{dt} + 6y \frac{dy}{dt} &= 0 \\ (6y - 12xy) \frac{dy}{dt} &= (6y^2 - 12x^2) \frac{dx}{dt} \\ \frac{dy}{dt} &= \frac{y^2 - 2x^2}{y - 2xy} \cdot \frac{dx}{dt} \\ &= \frac{4^2 - 2 \cdot (-3)^2}{4 - 2 \cdot (-3) \cdot 4} \cdot 3 \\ &= -\frac{2}{28} \cdot 3 = -\frac{6}{28} \\ &= -\frac{3}{14} \end{aligned}$$

$$5. \quad \frac{x^2 + y}{x - y} = 9; \frac{dx}{dt} = 2, x = 4, y = 2$$

$$\begin{aligned} \frac{(x - y) \left(2x \frac{dx}{dt} + \frac{dy}{dt} \right) - (x^2 + y) \left(\frac{dx}{dt} - \frac{dy}{dt} \right)}{(x - y)^2} &= 0 \\ \frac{2x(x - y) \frac{dx}{dt} + (x - y) \frac{dy}{dt} - (x^2 + y) \frac{dx}{dt} + (x^2 + y) \frac{dy}{dt}}{(x - y)^2} &= 0 \\ [2x(x - y) - (x^2 + y)] \frac{dx}{dt} + [(x - y) + (x^2 + y)] \frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= \frac{[(x^2 + y) - 2x(x - y)] \frac{dx}{dt}}{(x - y) + (x^2 + y)} \\ \frac{dy}{dt} &= \frac{(-x^2 + y + 2xy) \frac{dx}{dt}}{x + x^2} \\ &= \frac{[-(4)^2 + 2 + 2(4)(2)](2)}{4 + 4^2} \\ &= \frac{4}{20} = \frac{1}{5} \end{aligned}$$

$$6. \quad \frac{y^3 - 4x^2}{x^3 + 2y} = \frac{44}{31}; \frac{dx}{dt} = 5, \\ x = -3, y = -2$$

$$\begin{aligned} 31(y^3 - 4x^2) &= 44(x^3 + 2y) \\ 31y^3 - 124x^2 &= 44x^3 + 88y \\ 93y^2 \frac{dy}{dt} - 248x \frac{dx}{dt} &= 132x^2 \frac{dx}{dt} + 88 \frac{dy}{dt} \\ (93y^2 - 88) \frac{dy}{dt} &= (132x^2 + 248x) \frac{dx}{dt} \\ \frac{dy}{dt} &= \frac{132x^2 + 248x}{93y^2 - 88} \cdot \frac{dx}{dt} \\ &= \frac{132(-3)^2 + 248(-3)}{93(-2)^2 - 88} \cdot 5 \\ &= \frac{444}{284} \cdot 5 = \frac{2220}{284} = \frac{555}{71} \end{aligned}$$

$$7. \quad xe^y = 3 + \ln x; \frac{dx}{dt} = 6, x = 2, y = 0$$

$$\begin{aligned} e^y \frac{dx}{dt} + xe^y \frac{dy}{dt} &= 0 + \frac{1}{x} \frac{dx}{dt} \\ xe^y \frac{dy}{dt} &= \left(\frac{1}{x} - e^y \right) \frac{dx}{dt} \\ \frac{dy}{dt} &= \frac{\left(\frac{1}{x} - e^y \right) \frac{dx}{dt}}{xe^y} \\ &= \frac{(1 - xe^y) \frac{dx}{dt}}{x^2 e^y} \\ &= \frac{[1 - (2)e^0](6)}{2^2 e^0} \\ &= \frac{-6}{4} = -\frac{3}{2} \end{aligned}$$

$$8. \quad y \ln x + xe^y = 1; \frac{dx}{dt} = 5, x = 1, y = 0$$

$$\begin{aligned} \frac{d}{dt}(y \ln x) + \frac{d}{dx}(xe^y) &= \frac{d}{dt}(1) \\ \ln x \frac{dy}{dt} + \frac{y}{x} \frac{dx}{dt} + e^y \frac{dx}{dt} + x \cdot e^y \frac{dy}{dt} &= 0 \\ (\ln x + xe^y) \frac{dy}{dt} + \left(\frac{y}{x} + e^y \right) \frac{dx}{dt} &= 0 \end{aligned}$$

$$\begin{aligned} (\ln x + xe^y) \frac{dy}{dt} &= - \left(\frac{y}{x} + e^y \right) \frac{dx}{dt} \\ \frac{dy}{dt} &= - \frac{\left(\frac{y}{x} + e^y \right) \frac{dx}{dt}}{\ln x + xe^y} \\ &= - \frac{(y + xe^y) \frac{dx}{dt}}{x \ln x + x^2 e^y} \\ &= - \frac{[0 + (1)e^0](5)}{(1) \ln 1 + 1^2 e^0} = -5 \end{aligned}$$

$$9. \quad C = 0.2x^2 + 10,000; x = 80, \frac{dx}{dt} = 12$$

$$\frac{dC}{dt} = 0.2(2x) \frac{dx}{dt} = 0.2(160)(12) = 384$$

The cost is changing at a rate of \$384 per month.

$$10. \quad C = \frac{R^2}{450,000} + 12,000; \frac{dC}{dx} = 15 \\ R = 25,000$$

$$\begin{aligned} \frac{dC}{dx} &= \frac{R}{225,000} \cdot \frac{dR}{dx} \\ 15 &= \frac{25,000}{225,000} \cdot \frac{dR}{dx} \\ 15 &= \frac{1}{9} \cdot \frac{dR}{dx} \\ 135 &= \frac{dR}{dx} \end{aligned}$$

Revenue is changing at a rate of \$135 per month.

$$11. \quad R = 50x - 0.4x^2; C = 5x + 15; \\ x = 40; \frac{dx}{dt} = 10$$

$$\begin{aligned} \text{(a)} \quad \frac{dR}{dt} &= 50 \frac{dx}{dt} - 0.8x \frac{dx}{dt} \\ &= 50(10) - 0.8(40)(10) \\ &= 500 - 320 \\ &= 180 \end{aligned}$$

Revenue is increasing at a rate of \$180 per day.

$$\text{(b)} \quad \frac{dC}{dt} = 5 \frac{dx}{dt} = 5(10) = 50$$

Cost is increasing at a rate of \$50 per day.

$$\text{(c)} \quad \text{Profit} = \text{Revenue} - \text{Cost}$$

$$P = R - C$$

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt} = 180 - 50 = 130$$

Profit is increasing at a rate of \$130 per day.

12. $R = 50x - 0.4x^2$, $C = 5x + 15$;

$$x = 80, \frac{dx}{dt} = 12$$

$$\begin{aligned} \text{(a)} \quad \frac{dR}{dt} &= 50 \frac{dx}{dt} - 0.8x \frac{dx}{dt} \\ &= 50(12) - 0.8(80)(12) \\ &= 600 - 768 \\ &= -168 \end{aligned}$$

Revenue is decreasing at a rate of \$168 per day.

$$\text{(b)} \quad \frac{dC}{dt} = (5) \frac{dx}{dt} = (5)(12) = 60$$

Cost is increasing at a rate of \$60 per day.

$$\begin{aligned} \text{(c)} \quad P &= R - C \\ \frac{dP}{dt} &= \frac{dR}{dt} - \frac{dC}{dt} \\ &= -168 - 60 \\ &= -228 \end{aligned}$$

Profit is decreasing at a rate of \$228 per day.

13. $pq = 8000$; $p = 3.50$, $\frac{dp}{dt} = 0.15$

$$pq = 8000$$

$$p \frac{dq}{dt} + q \frac{dp}{dt} = 0$$

$$\begin{aligned} \frac{dq}{dt} &= \frac{-q \frac{dp}{dt}}{p} \\ &= \frac{-\left(\frac{8000}{3.50}\right)(0.15)}{3.50} \\ &\approx -98 \end{aligned}$$

Demand is decreasing at a rate of approximately 98 units per unit time.

14. $R = pq$; $\frac{dq}{dt} = 25$

Find the relationship between p and q by finding the equation of the line through $(0, 70)$, and $(100, 60)$.

$$m = \frac{70 - 60}{6 - 100} = \frac{10}{-100} = -\frac{1}{10}$$

$$p - 70 = -\frac{1}{10}(q - 0)$$

$$p - 70 = -\frac{1}{10}q$$

$$p = -\frac{1}{10}q + 70$$

$$R = \left(-\frac{1}{10}q + 70\right)q$$

$$= -\frac{1}{10}q^2 + 70q$$

$$\frac{dR}{dt} = -\frac{1}{5}q \frac{dq}{dt} + 70 \frac{dq}{dt}$$

$$= -\frac{1}{5}(20)(25) + 70(25)$$

$$= -100 + 1750$$

$$= \$1650$$

Revenue is increasing at a rate of \$1650 per day.

15. $V = k(R^2 - r^2)$; $k = 555.6$, $R = 0.02$ mm, $\frac{dR}{dt} = 0.003$ mm per minute; r is constant.

$$V = k(R^2 - r^2)$$

$$V = 555.6(R^2 - r^2)$$

$$\begin{aligned} \frac{dV}{dt} &= 555.6 \left(2R \frac{dR}{dt} - 0 \right) \\ &= 555.6(2)(0.02)(0.003) \\ &= 0.067 \text{ mm/min} \end{aligned}$$

16. $y = nx^m$

Note that n is a constant.

$$\ln y = \ln(nx^m)$$

$$\ln y = \ln n + \ln x^m$$

$$\ln y = \ln n + m \ln x$$

Now take the derivative of both sides with respect to t .

$$\frac{1}{y} \frac{dy}{dt} = 0 + m \frac{1}{x} \frac{dx}{dt}$$

$$\frac{1}{y} \frac{dy}{dt} = m \frac{1}{x} \frac{dx}{dt}$$

17. $b = 0.22m^{0.87}$

$$\frac{db}{dt} = 0.22(0.87)m^{-0.13} \frac{dm}{dt}$$

$$= 0.1914m^{-0.13} \frac{dm}{dt}$$

$$\frac{dm}{dt} = \frac{m^{0.13}}{0.1914} \frac{db}{dt}$$

$$= \frac{25^{0.13}}{0.1914} (0.25)$$

$$\approx 1.9849$$

The rate of change of the total weight is about 1.9849 g/day.

18. $E = 429m^{-0.35}$

$$\begin{aligned}\frac{dE}{dt} &= 429(-0.35)m^{-1.35} \frac{dm}{dt} \\ &= -150.15m^{-1.35} \frac{dm}{dt} \\ &= -150.15(10)^{-1.35}(0.001) \\ &\approx -0.0067\end{aligned}$$

The rate of change of the energy expenditure is about -0.0067 cal/g/hr².

19. $r = 140.2m^{0.75}$

(a) $\frac{dr}{dt} = 140.2(0.75)m^{-0.25} \frac{dm}{dt}$
 $= 105.15m^{-0.25} \frac{dm}{dt}$

(b) $\frac{dr}{dt} = 105.15(250)^{-0.25}(2)$
 ≈ 52.89

The rate of change of the average daily metabolic rate is about 52.89 kcal/day².

20. $E = 26.5w^{-0.34}$

$$\begin{aligned}\frac{dE}{dw} &= 26.5(-0.34)w^{-1.34} \frac{dw}{dt} \\ &= -9.01w^{-1.34} \frac{dw}{dt} \\ &= -9.01(5)^{-1.34}(0.05) \\ &\approx -0.0521\end{aligned}$$

The rate of change of the energy expenditure is about -0.0521 kcal/kg/km/day.

21. $C = \frac{1}{10}(T - 60)^2 + 100$

$$\frac{dC}{dT} = \frac{1}{5}(T - 60) \frac{dT}{dt}$$

If $T = 76^\circ$ and $\frac{dT}{dt} = 8$,

$$\begin{aligned}\frac{dC}{dT} &= \frac{1}{5}(76 - 60)(8) = \frac{1}{5}(16)(8) \\ &= 25.6.\end{aligned}$$

The crime rate is rising at the rate of 25.6 crimes/month.

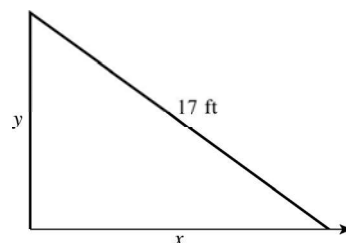
22. $W(t) = \frac{-0.02t^2 + t}{t + 1}$

$$\frac{dW}{dt} = \frac{(-0.04t + 1)(t + 1) - (1)(-0.02t^2 + t)}{(t + 1)^2}$$

If $t = 5$,

$$\begin{aligned}\frac{dW}{dt} &= \frac{(-0.2 + 1)(6) - (-0.5 + 5)}{6^2} \\ &= \frac{4.8 - 4.5}{36} \\ &= 0.008.\end{aligned}$$

23. Let x = The distance of the base of the ladder from the base of the building
 y = The distance up the side of the building to the top of the ladder



Find $\frac{dy}{dt}$ when $x = 8$ ft and $\frac{dx}{dt} = 9$ ft/min.

Since $y = \sqrt{17^2 - x^2}$, when $x = 8$,
 $y = 15$.

By the Pythagorean theorem,

$$x^2 + y^2 = 17^2.$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(17^2)$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$2y \frac{dy}{dt} = -2x \frac{dx}{dt}$$

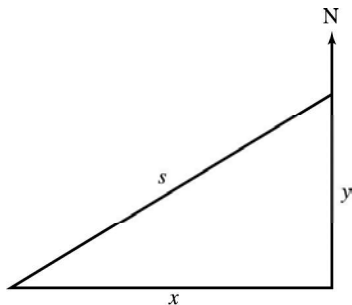
$$\frac{dy}{dt} = \frac{-2x}{2y} \cdot \frac{dx}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

$$= -\frac{8}{15}(9)$$

$$= -\frac{24}{5}$$

The ladder is sliding down the building at the rate of $\frac{24}{5}$ ft/min.

24. (a) Let x = the distance one car travels west;
 y = the distance the other car travels north
 s = the distance between the two cars



$$\begin{aligned}s^2 &= x^2 + y^2 \\ 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \\ s \frac{ds}{dt} &= x \frac{dx}{dt} + y \frac{dy}{dt}\end{aligned}$$

Use $d = rt$ to find x and y .

$$\begin{aligned}x &= (40)(2) = 80 \text{ mi} \\ y &= (30)(2) = 60 \text{ mi} \\ s &= \sqrt{x^2 + y^2} \\ &= \sqrt{(80)^2 + (60)^2} \\ &= 100\end{aligned}$$

The distance between the cars after 2 hours is 100 mi.

$$\frac{dx}{dt} = 40 \text{ mph and } \frac{dy}{dt} = 30 \text{ mph}$$

$$\begin{aligned}(100) \frac{ds}{dt} &= (80)(40) + (60)(30) \\ \frac{ds}{dt} &= \frac{5000}{100} = 50\end{aligned}$$

The distance between the two cars is changing at the rate of 50 mph.

- (b) From part (a), we have

$$s \frac{ds}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

Use $d = rt$ to find x and y . When the second car has traveled 1 hour, the first car has traveled 2 hours.

$$\begin{aligned}x &= 40(1) = 40 \text{ mi} \\ y &= 30(2) = 60 \text{ mi} \\ s &= \sqrt{x^2 + y^2} \\ &= \sqrt{(40)^2 + (60)^2} \\ &= 72.11\end{aligned}$$

The distance between the cars after the second car has traveled 1 hour is about 72.11 mi.

$$\begin{aligned}72.11 \frac{ds}{dt} &= (40)(40) + (60)(30) \\ \frac{ds}{dt} &= \frac{3400}{72.11} \approx 47.15\end{aligned}$$

The distance between the two cars is changing at a rate of about 47.15 mph.

25. Let r = the radius of the circle formed by the ripple.

Find $\frac{dA}{dt}$ when $r = 4$ ft and $\frac{dr}{dt} = 2$ ft/min.

$$\begin{aligned}A &= \pi r^2 \\ \frac{dA}{dt} &= 2\pi r \frac{dr}{dt} \\ &= 2\pi(4)(2) = 16\pi\end{aligned}$$

The area is changing at the rate of 16π ft²/min.

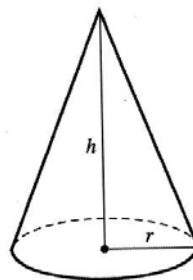
26. $V = \frac{4}{3}\pi r^3$, $r = 4$ in, and $\frac{dr}{dt} = -\frac{1}{4}$ in/hr

$$\begin{aligned}\frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ &= 4\pi(4)^2 \left(-\frac{1}{4}\right) = -16\pi \text{ in}^3/\text{hr}\end{aligned}$$

27. $V = x^3$, $x = 3$ cm, and $\frac{dV}{dt} = 2$ cm³/min

$$\begin{aligned}\frac{dV}{dt} &= 3x^2 \frac{dx}{dt} \\ \frac{dx}{dt} &= \frac{1}{3x^2} \frac{dV}{dt} \\ &= \frac{1}{3 \cdot 3^2} (2) = \frac{2}{27} \text{ cm/min}\end{aligned}$$

28. Let r = the radius of the base of the conical pile.



Find $\frac{dV}{dt}$ when $r = 6$ in, and $\frac{dr}{dt} = 0.75$ in/min.
 $h = 2r$ for all t .

$$V = \frac{\pi}{3} r^2 h$$

$$V = \frac{\pi}{3} r^2 (2r)$$

$$= \frac{2\pi}{3} r^3$$

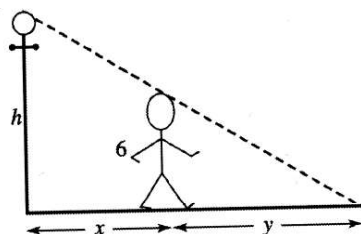
$$\frac{dV}{dt} = \frac{3 \cdot 2\pi r^2}{3} \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 2\pi(6)^2(0.75)$$

$$= 54\pi$$

The volume is changing at the rate of $54\pi \text{ in}^3/\text{min}$.

29. Let y = the length of the man's shadow;
 x = the distance of the man from the lamp post;
 h = the height of the lamp post.



$$\frac{dx}{dt} = 50 \text{ ft/min}$$

Find $\frac{dy}{dt}$ when $x = 25$ ft.

Now $\frac{h}{x+y} = \frac{6}{y}$, by similar triangles.

When $x = 8$, $y = 10$,

$$\frac{h}{18} = \frac{6}{10}$$

$$h = 10.8.$$

$$\frac{10.8}{x+y} = \frac{6}{y},$$

$$10.8y = 6x + 6y$$

$$4.8y = 6x$$

$$y = 1.25x$$

$$\frac{dy}{dt} = 1.25 \frac{dx}{dt}$$

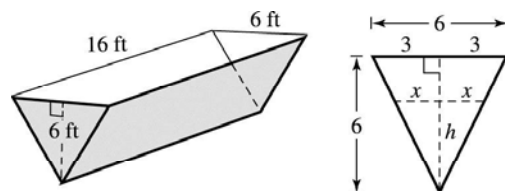
$$= 1.25(50)$$

$$\frac{dy}{dt} = 62.5$$

The length of the shadow is increasing at the rate of 62.5 ft/min .

30. Let x = one-half the width of the triangular cross section;
 h = the height of the water;
 V = the volume of the water.

$$\frac{dV}{dt} = 4 \text{ cu ft per min}$$



Find $\frac{dh}{dt}$ when $h = 4$.

$$V = \left(\begin{array}{c} \text{Area of} \\ \text{triangular} \\ \text{cross section} \end{array} \right) \cdot (\text{length})$$

Area of triangular cross section

$$= \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(2x)(h) = xh$$

By similar triangles,

$$\frac{6}{2x} = \frac{6}{h},$$

$$\text{so } x = \frac{h}{2}.$$

$$V = (xh)(16)$$

$$= \left(\frac{h}{2} h \right) 16$$

$$= 8h^2$$

$$\frac{dV}{dt} = 16h \frac{dh}{dt}$$

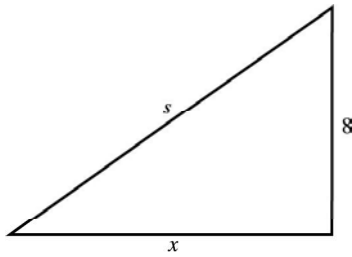
$$\frac{1}{16h} \frac{dV}{dt} = \frac{dh}{dt}$$

$$\frac{1}{16(4)}(4) = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{16}$$

The height of the water is increasing at a rate of $\frac{1}{16} \text{ ft/min}$.

31. Let x = the distance from the docks
 s = the length of the rope.



$$\begin{aligned}\frac{ds}{dt} &= 1 \text{ ft/sec} \\ s^2 &= x^2 + (8)^2 \\ 2s \frac{ds}{dt} &= 2x \frac{dx}{dt} + 0 \\ s \frac{ds}{dt} &= x \frac{dx}{dt}\end{aligned}$$

If $x = 8$,

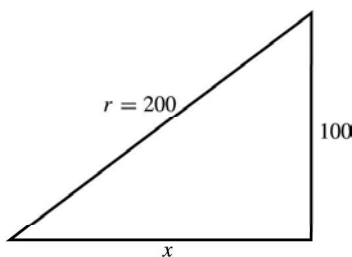
$$s = \sqrt{(8)^2 + (8)^2} = \sqrt{128} = 8\sqrt{2}.$$

Then,

$$\begin{aligned}8\sqrt{2}(1) &= 8 \frac{dx}{dt} \\ \frac{dx}{dt} &= \sqrt{2} \approx 1.41\end{aligned}$$

The boat is approaching the dock at $\sqrt{2} \approx 1.41$ ft/sec.

32. Let x = the horizontal length;
 r = the rope length.



$$\frac{dx}{dt} = 50 \text{ ft/min}$$

By the Pythagorean theorem,

$$\begin{aligned}x^2 + 100^2 &= 200^2 \\ x &= \sqrt{30,000} = 100\sqrt{3} \\ r^2 &= x^2 + 100^2\end{aligned}$$

$$\begin{aligned}2r \frac{dr}{dt} &= 2x \frac{dx}{dt} + 0 \\ r \frac{dr}{dt} &= x \frac{dx}{dt} \\ \frac{dr}{dt} &= \frac{x \frac{dx}{dt}}{r} \\ \frac{dr}{dt} &= \frac{100\sqrt{3}(50)}{200} = 25\sqrt{3} \\ &\approx 43.3\end{aligned}$$

She must let out the string at a rate of $25\sqrt{3} \approx 43.3$ ft/min.

6.6 Differentials: Linear Approximation

Your Turn 1

$$y = 300x^{-2/3}, \quad x = 8, \quad dx = 0.05$$

$$\frac{dy}{dx} = \left(-\frac{2}{3}\right)(300)x^{-5/3} = -200x^{-5/3}$$

$$\begin{aligned}dy &= -200x^{-5/3}dx \\ &= (-200)(8^{-5/3})(0.05) \\ &= (-200)\left(\frac{1}{32}\right)\left(\frac{1}{20}\right) \\ &= -\frac{5}{16}\end{aligned}$$

Your Turn 2

Use the approximation formula for $f(x) = \sqrt{x}$ developed in Example 2:

$$f(x + \Delta x) \approx \sqrt{x} + \frac{1}{2\sqrt{x}} \Delta x$$

For this problem, $x = 100$ and $dx = -1$.

$$\begin{aligned}f(99) &= f(100 - 1) \\ &\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-1) \\ &= 10 - \frac{1}{20} \\ &= 9.95\end{aligned}$$

Your Turn 3

Use the approximation derived in Example 5:

$$dV = 4\pi r^2 dr$$

For this problem, $r = 1.25$ and $dr = \Delta r = \pm 0.025$.

$$\begin{aligned} dV &= 4\pi(1.25)^2(\pm 0.025) \\ &\approx 0.491 \\ &\approx 0.5 \end{aligned}$$

The maximum error in the volume is about 0.5 mm^3 .

6.6 Exercises

$$\begin{aligned} 1. \quad y &= 2x^3 - 5x; x = -2, \Delta x = 0.1 \\ dy &= (6x^2 - 5)dx \\ \Delta y &\approx (6x^2 - 5)\Delta x \approx [6(-2)^2 - 5](0.1) \approx 1.9 \end{aligned}$$

$$\begin{aligned} 2. \quad y &= 4x^3 - 3x; x = 3, \Delta x = 0.2 \\ dy &= (12x^2 - 3)dx \\ \Delta y &\approx (12x^2 - 3)\Delta x \approx [12(3)^2 - 3](0.2) \approx 21 \end{aligned}$$

$$\begin{aligned} 3. \quad y &= x^3 - 2x^2 + 3; x = 1, \Delta x = -0.1 \\ dy &= (3x^2 - 4x)dx \\ \Delta y &\approx (3x^2 - 4x)\Delta x \\ &= [3(1)^2 - 4(1)](-0.1) \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} 4. \quad y &= 2x^3 + x^2 - 4x; x = 2, \Delta x = -0.2 \\ dy &= (6x^2 + 2x - 4)dx \\ \Delta y &\approx (6x^2 + 2x - 4)\Delta x \\ &= [6(2)^2 + 2(2) - 4](-0.2) \\ &= (24 + 4 - 4)(-0.2) = -4.8 \end{aligned}$$

$$\begin{aligned} 5. \quad y &= \sqrt{3x + 2}; x = 4, \Delta x = 0.15 \\ dy &= 3\left(\frac{1}{2}(3x + 2)^{-1/2}\right)dx \\ \Delta y &\approx \frac{3}{2\sqrt{3x + 2}}\Delta x \approx \frac{3}{2(3.74)}(0.15) \approx 0.060 \end{aligned}$$

$$\begin{aligned} 6. \quad y &= \sqrt{4x - 1}; x = 5, \Delta x = 0.08 \\ dy &= \frac{1}{2}(4x - 1)^{-1/2}(4)dx \\ &= 2(4x - 1)^{-1/2}dx \\ \Delta y &\approx 2(4x - 1)^{-1/2}\Delta x \\ &= 2[4(5) - 1]^{-1/2}(0.08) \\ &= 2(19)^{-1/2}(0.08) \\ &= \frac{2(0.08)}{(19)^{-1/2}} = 0.037 \end{aligned}$$

$$\begin{aligned} 7. \quad y &= \frac{2x - 5}{x + 1}; x = 2, \Delta x = -0.03 \\ dy &= \frac{(x + 1)(2) - (2x - 5)(1)}{(x + 1)^2}dx \\ &= \frac{7}{(x + 1)^2}dx \\ \Delta y &\approx \frac{7}{(x + 1)^2}\Delta x \\ &= \frac{7}{(2 + 1)^2}(-0.03) \\ &= -0.023 \end{aligned}$$

$$\begin{aligned} 8. \quad y &= \frac{6x - 3}{2x + 1}; x = 3, \Delta x = -0.04 \\ dy &= \frac{6(2x + 1) - 2(6x - 3)}{(2x + 1)^2}dx \\ &= \frac{12}{(2x + 1)^2}dx \\ \Delta y &\approx \frac{12}{(2x + 1)^2}\Delta x \\ &= \frac{12}{[2(3) + 1]^2}(-0.04) \\ &= \frac{-0.48}{49} = -0.010 \end{aligned}$$

$$\begin{aligned} 9. \quad &\sqrt{145} \\ \text{We know } \sqrt{144} &= 12, \text{ so } f(x) = \sqrt{x}, x = 144, \\ &dx = 1. \\ \frac{dy}{dx} &= \frac{1}{2}x^{-1/2} \\ dy &= \frac{1}{2\sqrt{x}}dx \\ dy &= \frac{1}{2\sqrt{144}}(1) = \frac{1}{24} \end{aligned}$$

$$\begin{aligned}\sqrt{145} &\approx f(x) + dy = 12 + \frac{1}{24} \\ &\approx 12.0417\end{aligned}$$

By calculator, $\sqrt{145} \approx 12.0416$.

The difference is $|12.0417 - 12.0416| = 0.0001$.

10. $\sqrt{23}$

We know $\sqrt{25} = 5$, $f(x) = \sqrt{x}$, $x = 25$, and $dx = -2$.

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$dy = \frac{1}{2\sqrt{x}}dx = \frac{1}{2\sqrt{25}}(-2) = -\frac{1}{5} = -0.2.$$

$$\sqrt{23} \approx f(x) + dy = 5 - 0.2 = 4.8$$

By calculator, $\sqrt{23} \approx 4.7958$.

The difference is $|4.8 - 4.7958| = 0.0042$.

11. $\sqrt{0.99}$

We know $\sqrt{1} = 1$, so $f(x) = \sqrt{x}$, $x = 1$, $dx = -0.01$.

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$dy = \frac{1}{2\sqrt{x}}dx$$

$$dy = \frac{1}{2\sqrt{1}}(-0.01) = -0.005$$

$$\begin{aligned}\sqrt{0.99} &\approx f(x) + dy = 1 - 0.005 \\ &= 0.995\end{aligned}$$

By calculator, $\sqrt{0.99} \approx 0.9950$.

The difference is $|0.995 - 0.9950| = 0$.

12. $\sqrt{17.02}$

We know $\sqrt{16} = 4$, $f(x) = \sqrt{x}$, $x = 16$, and $dx = 1.02$.

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$dy = \frac{1}{2\sqrt{x}}dx = \frac{1}{2\sqrt{16}}(1.02)$$

$$= \frac{1}{8}(1.02) = 0.1275$$

$$\begin{aligned}\sqrt{17.02} &\approx f(x) + dy \\ &= 4 + 0.1275 = 4.1275\end{aligned}$$

By calculator, $\sqrt{17.02} \approx 4.1255$.

The difference is $|4.1275 - 4.1255| = 0.0020$.

13. $e^{0.01}$

We know $e^0 = 1$, so $f(x) = e^x$, $x = 0$, $dx = 0.01$.

$$\frac{dy}{dx} = e^x$$

$$dy = e^x dx$$

$$dy = e^0(0.01) = 0.01$$

$$e^{0.01} \approx f(x) + dy = 1 + 0.01 = 1.01$$

By calculator, $e^{0.01} \approx 1.0101$.

The difference is $|1.01 - 1.0101| = 0.0001$.

14. $e^{-0.002}$

We know $e^0 = 1$, $f(x) = e^x$, $x = 0$, and $dx = -0.002$.

$$\frac{dy}{dx} = e^x$$

$$dy = e^x dx = e^0(-0.002) = -0.002$$

$$e^{-0.002} \approx f(x) + dy = 1 - 0.002 = 0.998$$

By calculator, $e^{-0.002} \approx 0.9980$.

The difference is $|0.9980 - 0.998| = 0$.

15. $\ln 1.05$

We know $\ln 1 = 0$, so $f(x) = \ln x$, $x = 1$, $dx = 0.05$.

$$\frac{dy}{dx} = \frac{1}{x}$$

$$dy = \frac{1}{x}dx$$

$$dy = \frac{1}{1}(0.05) = 0.05$$

$$\ln 1.05 \approx f(x) + dy = 0 + 0.05 = 0.05$$

By calculator, $\ln 1.05 \approx 0.0488$.

The difference is $|0.05 - 0.0488| = 0.0012$.

16. $\ln 0.98$

We know $\ln 1 = 0$, $f(x) = \ln x$, $x = 1$, and $dx = -0.02$.

$$\frac{dy}{dx} = \frac{1}{x}$$

$$dy = \frac{1}{x} dx = \frac{1}{1}(-0.02) = -0.02$$

$$\ln 0.98 \approx f(x) + dy = 0 - 0.02 = -0.02$$

By calculator, $\ln 0.98 \approx -0.0202$.

The difference is $|-0.02 - (-0.0202)| = 0.0002$.

17. Let D = the demand in thousands of pounds;
 x = the price in dollars.

$$D(q) = -3q^3 - 2q^2 + 1500$$

(a) $q = 2, \Delta q = 0.10$

$$dD = (-9q^2 - 4q)dq$$

$$\Delta D \approx (-9q^2 - 4q) \Delta q$$

$$\approx [-9(4) - 4(2)](0.10)$$

$$\approx -4.4 \text{ thousand pounds}$$

(b) $q = 6, \Delta q = 0.15$

$$\Delta D \approx [-9(36) - 4(6)](0.15)$$

$$\approx -52.2 \text{ thousand pounds}$$

18. $A(x) = 0.04x^3 + 0.1x^2 + 0.5x + 6$

(a) $x = 3, \Delta x = 1$

$$dA = (0.12x^2 + 0.2x + 0.5)dx$$

$$= (0.12x^2 + 0.2x + 0.5) \Delta x$$

$$= [(0.12)(3)^2 + (0.2)(3) + (0.5)](1)$$

$$= 2.18$$

The change in average cost is about \$2.18.

(b) $x = 5, \Delta x = 1$

$$dA = [(0.12)(5)^2 + (0.2)(5) + (0.5)](1)$$

$$= 4.5$$

The change in average cost is about \$4.50.

19. $R(x) = 12,000 \ln(0.01x + 1)$

$$x = 100, \Delta x = 1$$

$$dR = \frac{12,000}{0.01x + 1}(0.01)dx$$

$$\Delta R \approx \frac{120}{0.01x + 1} \Delta x$$

$$\approx \frac{120}{0.01(100) + 1}(1)$$

$$\approx \$60$$

20. $P = R - C$

$$= 12,000 \ln(0.01x + 1) - (150 + 75x)$$

$$x = 100, \Delta x = 1$$

$$dP = \frac{12,000}{0.01x + 1}(0.01)dx - 75dx$$

$$\Delta P \approx \frac{12,000}{0.01x + 1}(0.01) \Delta x - 75 \Delta x$$

$$\approx \frac{12,000}{0.01(100) + 1}(0.01)(1) - 75(1)$$

$$\approx 60 - 75$$

$$\approx -15$$

The change in profit is a loss of about \$15.

21. If a cube is given a coating 0.1 in. thick, each edge increases in length by twice that amount, or 0.2 in. because there is a face at both ends of the edge.

$$V = x^3, x = 4, \Delta x = 0.2$$

$$dV = 3x^2 dx$$

$$\Delta V \approx 3x^2 \Delta x$$

$$= 3(4^2)(0.2)$$

$$= 9.6$$

For 1000 cubes $9.6(1000) = 9600 \text{ in.}^3$ of coating should be ordered.

22. Let x = the number of beach balls;

$$V = \text{the volume of } x \text{ beach balls.}$$

Then $\frac{dV}{dr} \approx$ the volume of material in beach balls since they are hollow.

$$V = \frac{4}{3} \pi r^3 x$$

$$r = 6 \text{ in.}, x = 5000, \Delta r = 0.03 \text{ in.}$$

$$dV = \frac{4}{3} \pi (3r^2 x + r^3) \Delta r$$

$$= \frac{4}{3} \pi (3 \cdot 36 \cdot 5000 + 216)(0.03)$$

$$= 21,608.64 \pi$$

$21,608 \pi \text{ in.}^3$ of material would be needed.

$$23. \text{ (a) } A(x) = y = 0.003631x^3 - 0.03746x^2 + 0.1012x + 0.009$$

$$\text{Let } x = 1, dx = 0.2.$$

$$\frac{dy}{dx} = 0.010893x^2 - 0.07492x + 0.1012$$

$$dy = (0.010893x^2 - 0.07492x + 0.1012)dx$$

$$\begin{aligned}\Delta y &\approx (0.010893x^2 - 0.07492x + 0.1012)\Delta x \\ &\approx (0.010893 \cdot 1^2 - 0.07492 \cdot 1 + 0.1012) \cdot 0.2 \\ &\approx 0.007435\end{aligned}$$

The alcohol concentration increases by about 0.74 percent.

(b)

$$\begin{aligned}\Delta y &\approx (0.010893 \cdot 3^2 - 0.07492 \cdot 3 + 0.1012) \cdot 0.2 \\ &\approx -0.005105\end{aligned}$$

The alcohol concentration decreases by about 0.51 percent.

$$24. \quad C = \frac{5x}{9 + x^2}$$

$$\begin{aligned}dC &= \frac{5(9 + x^2) - 2x(5x)}{(9 + x^2)^2} dx \\ &= \frac{45 + 5x^2 - 10x^2}{(9 + x^2)^2} dx \\ &= \frac{45 - 5x^2}{(9 + x^2)^2} dx \\ &\approx \frac{45 - 5x^2}{(9 + x^2)^2} \Delta x\end{aligned}$$

$$\text{(a) } x = 1, \Delta x = 0.5$$

$$\begin{aligned}dC &\approx \frac{45 - 5(1)^2}{(9 + 1)^2} (0.5) \\ &= \frac{40}{100} (0.5) \\ &= 0.2\end{aligned}$$

$$\text{(b) } x = 2, \Delta x = 0.25$$

$$\begin{aligned}dC &\approx \frac{45 - 5(2)^2}{(9 + 4)^2} (0.25) \\ &= 0.037\end{aligned}$$

$$25. \quad P(x) = \frac{25x}{8 + x^2}$$

$$\begin{aligned}dP &= \frac{(8 + x^2)(25) - 25x(2x)}{(8 + x^2)^2} dx \\ &= \frac{(8 + x^2)(25) - 25x(2x)}{(8 + x^2)^2} \Delta x\end{aligned}$$

$$\text{(a) } x = 2, \Delta x = 0.5$$

$$\begin{aligned}dP &= \frac{[(8 + 4)(25) - (25)(2)(4)](0.5)}{(8 + 4)^2} \\ &= 0.347 \text{ million}\end{aligned}$$

$$\text{(b) } x = 3, \Delta x = 0.25$$

$$\begin{aligned}dP &= \frac{[(8 + 9)(25) - 25(3)(6)](0.25)}{(8 + 9)^2} \\ &\approx -0.022 \text{ million}\end{aligned}$$

$$26. \quad A = \pi r^2, r = 1.7 \text{ mm}, \Delta r = -0.1 \text{ mm}$$

$$dA = 2\pi r dr$$

$$\Delta A \approx 2\pi r \Delta r$$

$$= 2\pi(1.7)(-0.1)$$

$$= -0.34\pi \text{ mm}^2$$

$$27. \quad r \text{ changes from 14 mm to 16 mm, so } \Delta r = 2.$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = \frac{4}{3}(3)\pi r^2 dr$$

$$\Delta V \approx 4\pi r^2 \Delta r$$

$$= 4\pi(14)^2(2)$$

$$= 1568\pi \text{ mm}^3$$

$$28. \quad A = \pi r^2, r = 1.2 \text{ mi}, \Delta r = 0.2 \text{ mi}$$

$$dA = 2\pi r dr$$

$$\Delta A \approx 2\pi r \Delta r$$

$$= 2\pi(1.2)(0.2)$$

$$= 0.48\pi \text{ mi}^2$$

$$29. \quad r \text{ increases from 20 mm to 22 mm, so } \Delta r = 2.$$

$$A = \pi r^2$$

$$dA = 2\pi r dr$$

$$\Delta A \approx 2\pi r \Delta r$$

$$= 2\pi(20)(2)$$

$$= 80\pi \text{ mm}^2$$

30. $A(p) = \frac{1.181p}{94.359 - p}$

- (a) Since values for p must be non-negative and the denominator can't be zero, a sensible domain would be from 0 to about 94.

$$\begin{aligned} \text{(b)} \quad dA &= \frac{(94.359 - p)(1.181) - 1.181p(-1)}{(94.359 - p)^2} dp \\ &= \frac{111.437979 - 1.181p + 1.181p}{(94.359 - p)^2} dp \\ &= \frac{111.437979}{(94.359 - p)^2} dp \end{aligned}$$

We are given $p = 60$ and $dp = 65 - 60 = 5$.

$$dA \approx \frac{111.437979}{(94.359 - 60)^2} (5) \approx 0.472$$

It will take about 0.47 years.

The actual value is

$$A(65) - A(60) \approx 2.615 - 2.062 = 0.553$$

or about 0.55 years.

31. $W(t) = -3.5 + 197.5e^{-0.01394(t-108.4)}$

$$\begin{aligned} \text{(a)} \quad dW &= 197.5e^{-0.01394(t-108.4)} (-1)e^{-0.01394(t-108.4)} (-0.01394) dt \\ &= 2.75315e^{-0.01394(t-108.4)} e^{-0.01394(t-108.4)} dt \end{aligned}$$

We are given $t = 80$ and $dt = 90 - 80 = 10$.

$$dW \approx 9.258$$

The pig will gain about 9.3 kg.

- (b) The actual weight gain is calculated as

$$\begin{aligned} W(90) - W(80) &\approx 50.736 - 41.202 \\ &= 9.534 \end{aligned}$$

or about 9.5 kg.

$$32. \quad V = \frac{4}{3}\pi r^3, \quad r = 4 \text{ cm}, \quad \Delta r = 0.2 \text{ cm}$$

$$dV = 4\pi r^2 dr$$

$$\begin{aligned}\Delta V &\approx 4\pi r^2 \Delta r \\ &= 4\pi(4)^2(0.2) \\ &= 12.8\pi \text{ cm}^3\end{aligned}$$

$$33. \quad r = 3 \text{ cm}, \quad \Delta r = -0.2 \text{ cm}$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr$$

$$\begin{aligned}\Delta V &\approx 4\pi r^2 \Delta r \\ &= 4\pi(9)(-0.2) \\ &= -7.2\pi \text{ cm}^3\end{aligned}$$

$$34. \quad V = x^3, \quad V = 27, \quad x = 3, \quad \Delta V = 0.1$$

$$dV = 3x^2 dx$$

$$\Delta V \approx 3x^2 \Delta x$$

$$\begin{aligned}\Delta x &\approx \frac{\Delta V}{3x^2} \\ &\approx \frac{0.1}{3 \cdot 3^2} \\ &\approx 0.0037 \text{ mm}\end{aligned}$$

$$35. \quad V = \frac{1}{3}\pi r^2 h, \quad h = 13, \quad dh = 0.2$$

$$\begin{aligned}V &= \frac{1}{3}\pi \left(\frac{h}{15}\right)^2 h \\ &= \frac{\pi}{775} h^3\end{aligned}$$

$$\begin{aligned}dV &= \frac{\pi}{775} \cdot 3h^2 dh \\ &= \frac{\pi}{225} h^2 dh\end{aligned}$$

$$\begin{aligned}\Delta V &\approx \frac{\pi}{225} h^2 \Delta h \\ &\approx \frac{\pi}{225} (13^2)(0.2) \\ &\approx 0.472 \text{ cm}^3\end{aligned}$$

$$36. \quad A = x^2; \quad x = 3.45, \quad \Delta x = \pm 0.002$$

$$dA = 2x dx$$

$$\begin{aligned}\Delta A &\approx 2x \Delta x = 2(3.45)(\pm 0.002) \\ &= \pm 0.0138 \text{ in.}^2\end{aligned}$$

$$37. \quad A = x^2; \quad x = 4, \quad dA = 0.01$$

$$dA = 2x dx$$

$$\Delta A \approx 2x \Delta x$$

$$\Delta x \approx \frac{\Delta A}{2x} \approx \frac{0.01}{2(4)} \approx 0.00125 \text{ cm}$$

$$38. \quad r = 4.87 \text{ in.}, \quad \Delta r = \pm 0.040$$

$$A = \pi r^2$$

$$dA = 2\pi r dr$$

$$\begin{aligned}\Delta A &\approx 2\pi r \Delta r = 2\pi(4.87)(\pm 0.040) \\ &= \pm 1.224 \text{ in.}^2\end{aligned}$$

$$39. \quad V = \frac{4}{3}\pi r^3; \quad r = 5.81, \quad \Delta r = \pm 0.003$$

$$dV = \frac{4}{3}\pi(3r^2)dr$$

$$\begin{aligned}\Delta V &\approx \frac{4}{3}\pi(3r^2)\Delta r \\ &= 4\pi(5.81)^2(\pm 0.003) \\ &= \pm 0.405\pi \approx \pm 1.273 \text{ in.}^3\end{aligned}$$

$$40. \quad V = x^3; \quad x = 5, \quad dV = 0.3$$

$$dV = 3x^2 dx$$

$$\Delta V \approx 3x^2 \Delta x$$

$$\begin{aligned}\Delta x &\approx \frac{\Delta V}{3x^2} \\ &\approx \frac{0.3}{3(5^2)} \\ &\approx 0.004 \text{ ft}\end{aligned}$$

$$41. \quad h = 7.284 \text{ in.}, \quad r = 1.09 \pm 0.007 \text{ in.}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{2}{3}\pi r h dr$$

$$\begin{aligned}\Delta V &\approx \frac{2}{3}\pi r h \Delta r \\ &= \frac{2}{3}\pi(1.09)(7.284)(0.007) \\ &= \pm 0.116 \text{ in.}^3\end{aligned}$$

Chapter 6 Review Exercises

1. False: The absolute maximum might occur at the endpoint of the interval of interest.

2. True

3. False: It could have either. For example $f(x) = 1/(1 - x^2)$ has an absolute minimum of 1 on $(-1, 1)$.

4. True

5. True

6. True

7. True

8. True

9. True

10. True

11. $f(x) = -x^3 + 6x^2 + 1; [-1, 6]$
 $f'(x) = -3x^2 + 12x = 0$ when $x = 0, 4$.

$$f(-1) = 8$$

$$f(0) = 1$$

$$f(4) = 33$$

$$f(6) = 1$$

Absolute maximum of 33 at 4; absolute minimum of 1 at 0 and 6.

12. $f(x) = 4x^3 - 9x^2 - 3; [-1, 2]$
 $f'(x) = 12x^2 - 18x = 0$ when $x = 0, \frac{3}{2}$.

$$f(-1) = -16$$

$$f(0) = -3$$

$$f\left(\frac{3}{2}\right) = -9.75$$

$$f(2) = -7$$

Absolute maximum of -3 at 0; absolute minimum of -16 at -1 .

13. $f(x) = x^3 + 2x^2 - 15x + 3; [-4, 2]$
 $f'(x) = 3x^2 + 4x - 15 = 0$ when
 $(3x - 5)(x + 3) = 0$

$$x = \frac{5}{3} \quad \text{or} \quad x = -3.$$

$$f(-4) = 31$$

$$f(-3) = 39$$

$$f\left(\frac{5}{3}\right) = -\frac{319}{27}$$

$$f(2) = -11$$

Absolute maximum of 39 at -3 ; absolute minimum of $-\frac{319}{27}$ at $\frac{5}{3}$.

14. $f(x) = -2x^3 - 2x^2 + 2x - 1; [-3, 1]$
 $f'(x) = -6x^2 - 4x + 2$
 $f'(x) = 0$ when

$$3x^2 + 2x - 1 = 0$$

$$(3x - 1)(x + 1) = 0$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -1.$$

$$f(-3) = 29$$

$$f(-1) = -3$$

$$f\left(\frac{1}{3}\right) = -\frac{17}{27}$$

$$f(1) = -3$$

Absolute maximum of 29 at -3 ; absolute minimum of -3 at -1 and 1 .

17. (a) $f(x) = \frac{2 \ln x}{x^2}; [1, 4]$

$$f'(x) = \frac{x^2 \left(\frac{2}{x}\right) - (2 \ln x)(2x)}{x^4}$$

$$= \frac{2x - 4x \ln x}{x^4}$$

$$= \frac{2 - 4 \ln x}{x^3}$$

$$f'(x) = 0 \quad \text{when}$$

$$2 - 4 \ln x = 0$$

$$2 = 4 \ln x$$

$$0.5 = \ln x$$

$$e^{0.5} = x$$

$$x \approx 1.6487.$$

x	$f(x)$
1	0
$e^{0.5}$	0.36788
4	0.17329

Maximum is 0.37; minimum is 0.

(b) [2, 5]

Note that the critical number of f is not in the domain, so we only test the endpoints.

x	$f(x)$
2	0.34657
5	0.12876

Maximum is 0.35, minimum is 0.13.

18. $f(x) = \frac{e^{2x}}{x^2}$

First find the critical numbers.

$$\begin{aligned} f'(x) &= \frac{2e^{2x}x^2 - e^{2x}(2x)}{(x^2)^2} \\ &= \frac{2e^{2x}(x^2 - x)}{x^4} \end{aligned}$$

$f'(x) = 0$ when $x^2 - x = (x)(x - 1) = 0$, that is, at $x = 1$. (Both the derivative and the original function are undefined at $x = 0$.)

(a) The function f is continuous on the interval $[1/2, 2]$ and the critical number lies in this interval, so we evaluate the function at three points:

x	$f(x)$
1/2	10.873
1	7.389
2	13.650

The absolute minimum is 7.39 and the absolute maximum is 13.65.

(b) The function f is continuous on the interval $[1, 3]$. The critical number coincides with an endpoint so we need to look at only two function values:

x	$f(x)$
1	7.389
3	44.825

The absolute minimum is 7.39 and the absolute maximum is 44.83.

21. $x^2 - 4y^2 = 3x^3y^4$

$$\frac{d}{dx}(x^2 - 4y^2) = \frac{d}{dx}(3x^3y^4)$$

$$2x - 8y \frac{dy}{dx} = 9x^2y^4 + 3x^3 \cdot 4y^3 \frac{dy}{dx}$$

$$(-8y - 3x^3 \cdot 4y^3) \frac{dy}{dx} = 9x^2y^4 - 2x$$

$$\frac{dy}{dx} = \frac{2x - 9x^2y^4}{8y + 12x^3y^3}$$

22. $x^2y^3 + 4xy = 2$

$$\frac{d}{dx}(x^2y^3 + 4xy) = \frac{d}{dx}(2)$$

$$2xy^3 + 3y^2 \left(\frac{dy}{dx} \right) x^2 + 4y + 4x \frac{dy}{dx} = 0$$

$$(3x^2y^2 + 4x) \frac{dy}{dx} = -2xy^3 - 4y$$

$$\frac{dy}{dx} = \frac{-2xy^2 - 4y}{3x^2y^2 + 4x}$$

23. $2\sqrt{y-1} = 9x^{2/3} + y$

$$\frac{d}{dx}[2(y-1)^{1/2}] = \frac{d}{dx}(9x^{2/3} + y)$$

$$2 \cdot \frac{1}{2} \cdot (y-1)^{-1/2} \frac{dy}{dx} = 6x^{-1/3} + \frac{dy}{dx}$$

$$[(y-1)^{-1/2} - 1] \frac{dy}{dx} = 6x^{-1/3}$$

$$\frac{1 - \sqrt{y-1}}{\sqrt{y-1}} \cdot \frac{dy}{dx} = \frac{6}{x^{1/3}}$$

$$\frac{dy}{dx} = \frac{6\sqrt{y-1}}{x^{1/3}(1 - \sqrt{y-1})}$$

24. $9\sqrt{x} + 4y^3 = 2\sqrt{y}$

$$\frac{d}{dx}(9\sqrt{x} + 4y^3) = \frac{d}{dx}(2\sqrt{y})$$

$$\frac{9}{2}x^{-1/2} + 12y^2 \frac{dy}{dx} = 2 \cdot \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx}$$

$$\frac{9}{2x^{1/2}} = \left(\frac{1}{y^{1/2}} - 12y^2 \right) \frac{dy}{dx}$$

$$\frac{9}{2x^{1/2}} = \frac{1 - 12y^{5/2}}{y^{1/2}} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{9y^{1/2}}{2x^{1/2}(1 - 12y^{5/2})}$$

$$= \frac{9\sqrt{y}}{2\sqrt{x}(1 - 12y^{5/2})}$$

$$25. \frac{6 + 5x}{2 - 3y} = \frac{1}{5x}$$

$$5x(6 + 5x) = 2 - 3y$$

$$30x + 25x^2 = 2 - 3y$$

$$\frac{d}{dx}(30x + 25x^2) = \frac{d}{dx}(2 - 3y)$$

$$30 + 50x = -3 \frac{dy}{dx}$$

$$-\frac{30 + 50x}{3} = \frac{dy}{dx}$$

$$26. \frac{x + 2y}{x - 3y} = y^{1/2}$$

$$x + 2y = y^{1/2}(x - 3y)$$

$$\frac{d}{dx}(x + 2y) = \frac{d}{dx}[y^{1/2}(x - 3y)]$$

$$1 + 2 \frac{dy}{dx} = y^{1/2} \left(1 - 3 \frac{dy}{dx} \right)$$

$$+ \frac{1}{2}(x - 3y)y^{-1/2} \frac{dy}{dx}$$

$$1 + 2 \frac{dy}{dx} = y^{1/2} - 3y^{1/2} \frac{dy}{dx} + \frac{1}{2}xy^{-1/2} \frac{dy}{dx}$$

$$- \frac{3}{2}y^{1/2} \frac{dy}{dx}$$

$$(2 + 3y^{1/2} - \frac{1}{2}xy^{-1/2} + \frac{3}{2}y^{1/2}) \frac{dy}{dx} = y^{1/2} - 1$$

$$\frac{2y^{1/2} \left(2 + \frac{9}{2}y^{1/2} - \frac{1}{2}xy^{-1/2} \right) \frac{dy}{dx}}{2y^{1/2}} = y^{1/2} - 1$$

$$\left(\frac{4y^{1/2} + 9y - x}{2y^{1/2}} \right) \frac{dy}{dx} = y^{1/2} - 1$$

$$\frac{dy}{dx} = \frac{2y - 2y^{1/2}}{4y^{1/2} + 9y - x}$$

$$27. \ln(xy + 1) = 2xy^3 + 4$$

$$\frac{d}{dx}[\ln(xy + 1)] = \frac{d}{dx}(2xy^3 + 4)$$

$$\frac{1}{xy + 1} \cdot \frac{d}{dx}(xy + 1) = 2y^3 + 2x \cdot 3y^2 \frac{dy}{dx} + \frac{d}{dx}(4)$$

$$\frac{1}{xy + 1} \left(y + x \frac{dy}{dx} + \frac{d}{dx}(1) \right) = 2y^3 + 6xy^2 \frac{dy}{dx}$$

$$\frac{y}{xy + 1} + \frac{x}{xy + 1} \cdot \frac{dy}{dx} = 2y^3 + 6xy^2 \frac{dy}{dx}$$

$$\left(\frac{x}{xy + 1} - 6xy^2 \right) \frac{dy}{dx} = 2y^3 - \frac{y}{xy + 1}$$

$$\frac{dy}{dx} = \frac{2y^3 - \frac{y}{xy + 1}}{\frac{x}{xy + 1} - 6xy^2}$$

$$= \frac{2y^3(xy + 1) - y}{x - 6xy^2(xy + 1)}$$

$$= \frac{2xy^4 + 2y^3 - y}{x - 6x^2y^3 - 6xy^2}$$

$$28. \ln(x + y) = 1 + x^2 + y^3$$

$$\frac{d}{dx}[\ln(x + y)] = \frac{d}{dx}(1 + x^2 + y^3)$$

$$\frac{1}{x + y} \cdot \frac{d}{dx}(x + y) = 2x + 3y^2 \cdot \frac{dy}{dx}$$

$$\frac{1}{x + y} \left(1 + \frac{dy}{dx} \right) = 2x + 3y^2 \cdot \frac{dy}{dx}$$

$$\left(\frac{1}{x + y} - 3y^2 \right) \frac{dy}{dx} = 2x - \frac{1}{x + y}$$

$$\frac{dy}{dx} = \frac{2x - \frac{1}{x + y}}{\frac{1}{x + y} - 3y^2}$$

$$= \frac{2x(x + y) - 1}{1 - 3y^2(x + y)}$$

$$= \frac{2x^2 + 2xy - 1}{1 - 3xy^2 - 3y^3}$$

$$= \frac{1 - 2x^2 - 2xy}{3xy^2 + 3y^3 - 1}$$

29. $\sqrt{2y} - 4xy = -22$, tangent line at $(3, 2)$.

$$\begin{aligned}\frac{d}{dx}(\sqrt{2y} - 4xy) &= \frac{d}{dx}(-22) \\ \frac{1}{2}(2)(2y)^{-1/2} \frac{dy}{dx} - \left(4y + 4x \frac{dy}{dx}\right) &= 0 \\ ((2y)^{-1/2} - 4x) \frac{dy}{dx} &= 4y \\ \frac{dy}{dx} &= \frac{4y}{\frac{1}{\sqrt{2y}} - 4x}\end{aligned}$$

To find the slope m of the tangent line, substitute 3 for x and 2 for y .

$$\begin{aligned}m &= \frac{4y}{\frac{1}{2\sqrt{2y}} - 4x} \\ &= \frac{4(2)}{\frac{1}{\sqrt{2(2)}} - 4(3)} \\ &= \frac{8}{\frac{1}{2} - 12} \\ &= \frac{16}{1 - 24} = -\frac{16}{23}\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ y - 2 &= -\frac{16}{23}(x - 3) \\ y - 2 - \frac{48}{23} &= -\frac{16}{23}x \\ y &= -\frac{16}{23}x + \frac{94}{23}.\end{aligned}$$

We can also write this equation as $16x + 23y = 94$.

30. $8y^3 - 4xy^2 = 20$, tangent line at $(-3, 1)$.

$$\begin{aligned}\frac{d}{dx}(8y^3 - 4xy^2) &= \frac{d}{dx}(20) \\ 24y^2 \frac{dy}{dx} - \left(8xy \frac{dy}{dx} + 4y^2\right) &= 0 \\ (24y^2 - 8xy) \frac{dy}{dx} &= 4y^2 \\ \frac{dy}{dx} &= \frac{4y^2}{24y^2 - 8xy} \\ &= \frac{y^2}{6y^2 - 2xy}\end{aligned}$$

To find the slope m of the tangent line, substitute -3 for x and 1 for y .

$$\begin{aligned}m &= \frac{y^2}{6y^2 - 2xy} \\ &= \frac{1^2}{6(1^2) - 2(-3)(1)} \\ &= \frac{1}{12}\end{aligned}$$

The equation of the tangent line is

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{12}(x - (-3)) \\ y - 1 - \frac{3}{12} &= \frac{1}{12}x \\ y &= \frac{1}{12}x + \frac{5}{4}.\end{aligned}$$

We can also write this as $1x - 12y = -15$.

33. $y = 8x^3 - 7x^2$, $\frac{dx}{dt} = 4$, $x = 2$

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}(8x^3 - 7x^2) \\ &= 24x^2 \frac{dx}{dt} - 14x \frac{dx}{dt} \\ &= 24(2)^2(4) - 14(2)(4) \\ &= 272\end{aligned}$$

34. $y = \frac{9 - 4x}{3 + 2x}$, $\frac{dx}{dt} = -1$, $x = -3$

$$\begin{aligned}\frac{dy}{dt} &= \frac{(-4)(3 + 2x) - (2)(9 - 4x)}{(3 + 2x)^2} \frac{dx}{dt} \\ &= \frac{-30}{(3 + 2x)^2} \frac{dx}{dt} \\ &= \frac{-30}{[3 + 2(-3)]^2} (-1) = \frac{30}{9} = \frac{10}{3}\end{aligned}$$

35. $y = \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$, $\frac{dx}{dt} = -4$, $x = 4$

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt} \left[\frac{1 + \sqrt{x}}{1 - \sqrt{x}} \right] \\ &= \frac{\left[(1 - \sqrt{x}) \left(\frac{1}{2} x^{-1/2} \frac{dx}{dt} \right) - 1(1 + \sqrt{x}) \left(-\frac{1}{2} \right) \left(x^{-1/2} \frac{dx}{dt} \right) \right]}{(1 - \sqrt{x})^2}\end{aligned}$$

$$= \frac{\begin{bmatrix} (1-2)\left(\frac{1}{2.2}\right)(-4) \\ -(1+2)\left(\frac{-1}{2.2}\right)(-4) \end{bmatrix}}{(1-2)^2}$$

$$= \frac{1-3}{1} = -2$$

$$36. \frac{x^2 + 5y}{x - 2y} = 2;$$

$$\frac{dx}{dt} = 1, x = 2, y = 0$$

$$x^2 + 5y = 2(x - 2y)$$

$$x^2 + 5y = 2x - 4y$$

$$9y = -x^2 + 2x$$

$$y = \frac{1}{9}(-x^2 + 2x)$$

$$y = -\frac{1}{9}x^2 + \frac{2}{9}x$$

$$\frac{dy}{dt} = \left(-\frac{2}{9}x + \frac{2}{9}\right)\frac{dx}{dt}$$

$$= \left[\left(-\frac{2}{9}\right)(2) + \frac{2}{9}\right](1)$$

$$= -\frac{4}{9} + \frac{2}{9} = -\frac{2}{9}$$

$$37. y = xe^{3x}; \frac{dx}{dt} = -2, x = 1$$

$$\frac{dy}{dt} = \frac{d}{dt}(xe^{3x})$$

$$= \frac{dx}{dt} \cdot e^{3x} + x \cdot \frac{d}{dt}(e^{3x})$$

$$= \frac{dx}{dt} \cdot e^{3x} + xe^{3x} \cdot 3 \frac{dx}{dt}$$

$$= (1 + 3x)e^{3x} \frac{dx}{dt}$$

$$= (1 + 3 \cdot 1)e^{3(1)}(-2) = -8e^3$$

$$38. y = \frac{1}{e^{x^2} + 1}; \frac{dx}{dt} = 3, x = 1$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{1}{e^{x^2} + 1}\right)$$

$$= \frac{-1}{(e^{x^2} + 1)^2} \cdot \frac{d}{dt}(e^{x^2} + 1)$$

$$= \frac{-1}{(e^{x^2} + 1)^2} \left[e^{x^2} \cdot \frac{d}{dt}(x^2) + \frac{d}{dt}(1) \right]$$

$$= \frac{-1}{(e^{x^2} + 1)^2} \left[e^{x^2} \cdot 2x \frac{dx}{dt} \right]$$

$$= \frac{-1}{(e + 1)^2} [e \cdot 2 \cdot 1 \cdot 3]$$

$$= \frac{-6e}{(e + 1)^2}$$

$$41. y = \frac{3x - 7}{2x + 1}; x = 2, \Delta x = 0.003$$

$$dy = \frac{(3)(2x + 1) - (2)(3x - 7)}{(2x + 1)^2} dx$$

$$dy = \frac{17}{(2x + 1)^2} dx$$

$$\Delta y \approx \frac{17}{(2x + 1)^2} \Delta x$$

$$= \frac{17}{(2[2] + 1)^2} (0.003)$$

$$= 0.00204$$

$$42. y = 8 - x^2 + x^3, x = -1, \Delta x = 0.02$$

$$dy = (-2x + 3x^2)dx$$

$$\Delta y \approx (-2x + 3x^2)\Delta x$$

$$= [-2(-1) + 3(-1)^2](0.02)$$

$$= 0.1$$

43. $-12x + x^3 + y + y^2 = 4$

$$\frac{dy}{dx}(-12x + x^3 + y + y^2) = \frac{d}{dx}(4)$$

$$-12 + 3x^2 + \frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$(1 + 2y)\frac{dy}{dx} = 12 - 3x^2$$

$$\frac{dy}{dx} = \frac{12 - 3x^2}{1 + 2y}$$

(a) If $\frac{dy}{dx} = 0$,

$$12 - 3x^2 = 0$$

$$12 = 3x^2$$

$$\pm 2 = x.$$

$x = 2$:

$$-24 + 8 + y + y^2 = 4$$

$$y + y^2 = 20$$

$$y^2 + y - 20 = 0$$

$$(y + 5)(y - 4) = 0$$

$$y = -5 \text{ or } y = 4$$

$(2, -5)$ and $(2, 4)$ are critical points.

$x = -2$:

$$24 - 8 + y + y^2 = 4$$

$$y + y^2 = -12$$

$$y^2 + y + 12 = 0$$

$$y = \frac{-1 \pm \sqrt{1^2 - 48}}{2}$$

This leads to imaginary roots.

$x = -2$ does not produce critical points.

(b)

x	y_1	y_2
1.9	-4.99	3.99
2	-5	4
2.1	-4.99	3.99

The point $(2, -5)$ is a relative minimum.

The point $(2, 4)$ is a relative maximum.

(c) There is no absolute maximum or minimum for x or y .

45. (a) $P(x) = -x^3 + 10x^2 - 12x$

$$P'(x) = -3x^2 + 20x - 12 = 0$$

$$3x^2 - 20x + 12 = 0$$

$$(3x - 2)(x - 6) = 0$$

$$3x - 2 = 0 \quad \text{or} \quad x - 6 = 0$$

$$x = \frac{2}{3} \quad \text{or} \quad x = 6$$

$$P''(x) = -6x + 20$$

$$P''\left(\frac{2}{3}\right) = 16,$$

which implies that $x = \frac{2}{3}$ is the location of the minimum.

$$P''(6) = -16,$$

which implies that $x = 6$ is the location of the maximum. Thus, 600 boxes will produce a maximum profit.

(b) Maximum profit $= P(6)$

$$= -(6)^3 + 10(6)^2 - 12(6) = 72$$

The maximum profit is \$720.

46. Let x = the length and width of a side of the base;

h = the height.

The volume is 32 m^3 ; the base is square and there is no top. Find the height, length, and width for minimum surface area.

$$\text{Volume} = x^2h$$

$$x^2h = 32$$

$$h = \frac{32}{x^2}$$

$$\text{Surface area} = x^2 + 4xh$$

$$A = x^2 + 4x\left(\frac{32}{x^2}\right)$$

$$= x^2 + 128x^{-1}$$

$$A' = 2x - 128x^{-2}$$

If $A' = 0$,

$$\frac{2x^3 - 128}{x^2} = 0$$

$$x^3 = 64$$

$$x = 4.$$

$$A''(x) = 2 + 2(128)x^{-3}$$

$$A''(4) = 6 > 0$$

The minimum is at $x = 4$, where

$$h = \frac{32}{4^2} = 2.$$

The dimensions are 2 m by 4 m by 4 m.

47. Volume of cylinder $= \pi r^2 h$

Surface area of cylinder open at one end

$$= 2\pi rh + \pi r^2.$$

$$V = \pi r^2 h = 27\pi$$

$$h = \frac{27\pi}{\pi r^2} = \frac{27}{r^2}$$

$$A = 2\pi r \left(\frac{27}{r^2} \right) + \pi r^2$$

$$= 54\pi r^{-1} + \pi r^2$$

$$A' = -54\pi r^{-2} + 2\pi r$$

If $A' = 0$,

$$2\pi r = \frac{54\pi}{r^2}$$

$$r^3 = 27$$

$$r = 3.$$

If $r = 3$,

$$A'' = 108\pi r^{-3} + 2\pi > 0,$$

so the value at $r = 3$ is a minimum.

For the maximum cost, the radius of the bottom should be 3 inches.

48. $V = \pi r^2 h = 40$, so $h = \frac{40}{\pi r^2}$.

$$A = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r^2 + 2\pi r \left(\frac{40}{\pi r^2} \right)$$

$$= 2\pi r^2 + \frac{80}{r}$$

$$\text{Cost} = C(r) = 4(2\pi r^2) + 3 \left(\frac{80}{r} \right)$$

$$= 8\pi r^2 + \frac{240}{r}$$

$$C'(r) = 16\pi r - \frac{240}{r^2}$$

$$16\pi r - \frac{240}{r^2} = 0$$

$$16\pi r^3 = 240$$

$$r^3 = \frac{15}{\pi}$$

$$r \approx 1.684$$

$$C''(r) = 16\pi + \frac{480}{r^3} > 0, \text{ so } r = 1.684$$

minimizes cost.

$$h = \frac{40}{\pi r^2} = \frac{40}{\pi(1.684)^2} = 4.490$$

The radius should be 1.684 in. and the height should be 4.490 in.

49. Here $k = 0.15$, $M = 20,000$, and $f = 12$.

We have

$$\begin{aligned} q &= \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(12)20,000}{0.15}} \\ &= \sqrt{3,200,000} \approx 1789 \end{aligned}$$

Ordering 1789 rolls each time minimizes annual cost.

50. $M = 180,000$ cases sold per year

$k = \$12$, cost to store 1 case for 1 yr

$f = \$20$, fixed cost for order

$x =$ the number of cases per order

$$\begin{aligned} x &= \sqrt{\frac{2fM}{k}} \\ &= \sqrt{\frac{2(20)(180,000)}{12}} \\ &= \sqrt{600,000} \\ &= 100\sqrt{60} \\ &\approx 775 \end{aligned}$$

The store should order 775 cases each time.

51. Use equation (3) from Section 6.3 with $k = 1$,

$M = 128,000$, and $f = 10$.

$$\begin{aligned} q &= \sqrt{\frac{2fM}{k}} = \sqrt{\frac{2(10)(128,000)}{1}} \\ &= \sqrt{2,560,000} = 1600 \end{aligned}$$

The number of lots that should be produced annually is

$$\frac{M}{q} = \frac{128,000}{1600} = 80.$$

52. Use equation (3) from Section 6.3 with $k = 2$, $M = 240,000$, and $f = 15$.

$$q = \frac{\sqrt{2fM}}{k} = \sqrt{\frac{2(15)(240,000)}{2}} \\ = \sqrt{3,600,000} \approx 1897.4$$

$T(1897) \approx 3794.7333$ and $T(1898) \approx 3794.7334$. Since $T(1897) < T(1898)$, then the number of batches per year should be

$$\frac{M}{q} = \frac{240,000}{1897} \approx 127.$$

53. $\ln q = D - 0.447 \ln p$

$$\frac{1}{q} \frac{dq}{dp} = -\frac{0.47}{p}$$

$$\frac{dp}{dq} = -0.47 \frac{q}{p}$$

$$E = -\frac{p}{q} \frac{dq}{dp}$$

$$= -\frac{p}{q} \left(-0.47 \frac{q}{p} \right) \\ = 0.47$$

E is less than 1, so the demand is inelastic.

54. $q = \frac{A}{p^k}$

$$\frac{dq}{dp} = -k \frac{A}{p^{k+1}}$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$= \left(-\frac{p}{\frac{A}{p^k}} \right) (-k) \left(\frac{A}{p^{k+1}} \right)$$

$$= \left(-\frac{p^{k+1}}{A} \right) \left(-k \frac{A}{p^{k+1}} \right) \\ = k$$

The demand is elastic when $k > 1$ and inelastic when $k < 1$.

55. $A = \pi r^2$; $\frac{dr}{dt} = 4$ ft/min, $r = 7$ ft

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(7)(4)$$

$$\frac{dA}{dt} = 56\pi$$

The rate of change of the area is 56π ft²/min.

56. $\frac{dx}{dt} = rx(N - x)$

$$= rxN - rx^2$$

$$\frac{d^2x}{dt^2} = rN \frac{dx}{dt} - 2rx \frac{dx}{dt}$$

$$= r \frac{dx}{dt} (N - 2x)$$

$$= r[rx(N - x)](N - 2x)$$

$$= r^2x(N - x)(N - 2x)$$

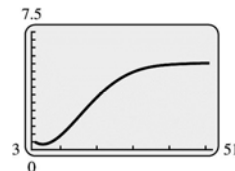
$$\frac{d^2x}{dt^2} = 0 \text{ when } x = 0, x = N, \text{ or } x = \frac{N}{2}.$$

On $\left(0, \frac{N}{2}\right)$, $\frac{d^2x}{dt^2} > 0$; therefore, the curve is concave upward.

On $\left(\frac{N}{2}, N\right)$, $\frac{d^2x}{dt^2} < 0$; therefore, the curve is concave downward.

Hence $x = \frac{N}{2}$ is a point of inflection.

57. (a)

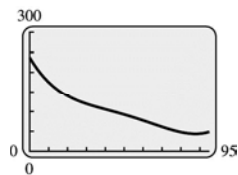


- (b) We use a graphing calculator to graph

$$M'(t) = -0.4321173 + 0.1129024t \\ - 0.0061518t^2 + 0.0001260t^3 \\ - 0.0000008925t^4$$

on $[3, 51]$ by $[0, 0.3]$. We find the maximum value of $M'(t)$ on this graph at about 15.41, or on about the 15th day.

58. (a)

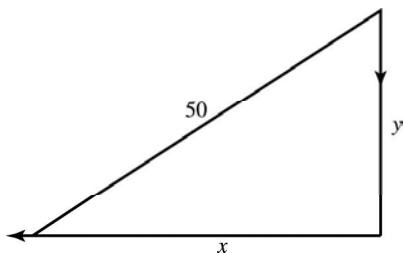


- (b) To find where the maximum and minimum numbers occur, use a graphing calculator to locate any extreme points on the graph. One critical number is formed at about 87.78.

t	$P(t)$
0	237.09
87.78	43.56
95	48.66

The maximum number of polygons is about 237 at birth. The minimum number is about 44.

59. Let x = the distance from the base of the ladder to the building;
 y = the height on the building at the top of the ladder.



$$\frac{dy}{dt} = -2$$

$$50^2 = x^2 + y^2$$

$$0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

$$\text{When } x = 30, y = \sqrt{2500 - (30)^2} = 40.$$

So

$$\frac{dx}{dt} = \frac{-40}{30}(-2) = \frac{80}{30} = \frac{8}{3}$$

The base of the ladder is slipping away from the building at a rate of $\frac{8}{3}$ ft/min.

$$60. \frac{dV}{dt} = 0.9 \text{ ft}^3/\text{min}$$

find $\frac{dr}{dt}$ when $r = 1.7$ ft

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi(3)r^2 \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt}$$

$$0.9 = 4\pi(1.7^2) \frac{dr}{dt}$$

$$\frac{0.9}{4\pi(1.7^2)} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{0.9}{11.56\pi} \approx 0.0248$$

The radius is changing at the rate of

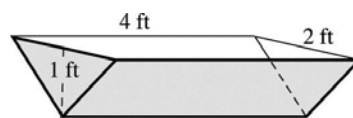
$$\frac{0.9}{11.56\pi} \approx 0.0248 \text{ ft/min.}$$

61. Let x = one-half of the width of the triangular cross section;
 h = the height of the water;
 V = the volume of the water.

$$\frac{dV}{dt} = 3.5 \text{ ft}^3/\text{min.}$$

Find $\frac{dV}{dt}$ when $h = \frac{1}{3}$.

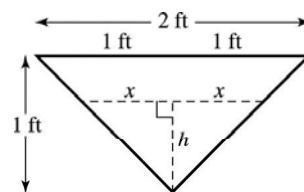
$$V = \left(\text{Area of triangular side} \right) (\text{length})$$



Area of triangular cross section

$$= \frac{1}{2}(\text{base})(\text{altitude})$$

$$= \frac{1}{2}(2x)(h) = xh$$



By similar triangles, $\frac{2x}{h} = \frac{2}{1}$, so $x = h$.

$$\begin{aligned} V &= (xh)(4) \\ &= h^2 \cdot 4 \\ &= 4h^2 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt} &= 8h \frac{dh}{dt} \\ \frac{1}{8h} \cdot \frac{dV}{dt} &= \frac{dh}{dt} \\ \frac{1}{8\left(\frac{1}{3}\right)}(3.5) &= \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{21}{16} = 1.3125 \end{aligned}$$

The depth of water is changing at the rate of 1.3125 ft/min.

62. $V = \frac{4}{3}\pi r^3, r = 4\text{ in.}, \Delta r = 0.02\text{ in.}$

$$\begin{aligned} dV &= 4\pi r^2 dr \\ \Delta V &\approx 4\pi r^2 \Delta r \\ &= 4\pi(4)^2(0.02) \\ &= 1.28\pi \text{ or about } 4.021 \end{aligned}$$

The volume of the coating is $1.28\pi \text{ in.}^3$ or about 4.021 in.^3 .

63. $A = s^2; s = 9.2, \Delta s = \pm 0.04$

$$\begin{aligned} ds &= 2s ds \\ \Delta A &\approx 2s \Delta s \\ &= 2(9.2)(\pm 0.04) \\ &= \pm 0.736 \text{ in.}^2 \end{aligned}$$

64. $V = l \cdot w \cdot h$
 $w = 4 + h$
 $l + g = 130; g = 2(w + h)$

$$\begin{aligned} l + 2(w + h) &= 130 \\ l + 2w + 2h &= 130 \end{aligned}$$

$$\begin{aligned} l &= 130 - 2w - 2h \\ &= 130 - 2(4 + h) - 2h \\ &= 122 - 4h \end{aligned}$$

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= (122 - 4h)(4 + h)h \\ &= 488h + 106h^2 - 4h^3 \\ \frac{dV}{dh} &= 488 + 212h - 12h^2 \end{aligned}$$

Set $\frac{dV}{dh} = 0$.

$$488 + 212h - 12h^2 = 0$$

$$12h^2 - 212h - 488 = 0$$

$$3h^2 - 53h - 122 = 0$$

$$\begin{aligned} h &= \frac{53 \pm \sqrt{2809 + 1464}}{6} \\ &\approx -2.06 \text{ or } h = 19.73 \end{aligned}$$

h can't be negative, so $h \approx 19.73$.

Thus,

$$\begin{aligned} l &\approx 122 - 4h \\ &\approx 43.1. \end{aligned}$$

The length that produces the maximum volume is about 43.1 inches.

65. We need to minimize y . Note that $x > 0$.

$$\frac{dy}{dx} = \frac{x}{8} - \frac{2}{x}$$

Set the derivative equal to 0.

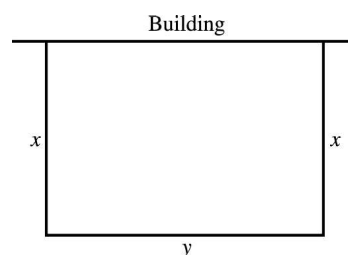
$$\begin{aligned} \frac{x}{8} - \frac{2}{x} &= 0 \\ \frac{x}{8} &= \frac{2}{x} \\ x^2 &= 16 \\ x &= 4 \end{aligned}$$

Since $\lim_{x \rightarrow 0} y = \infty$, $\lim_{x \rightarrow \infty} y = \infty$, and $x = 4$ is the only critical value in $(0, \infty)$, $x = 4$ produces a minimum value.

$$\begin{aligned} y &= \frac{4^2}{16} - 2 \ln 4 + \frac{1}{4} + 2 \ln 6 \\ &= 1.25 + 2(\ln 6 - \ln 4) \\ &= 1.25 + 2 \ln 1.5 \end{aligned}$$

The y coordinate of the Southern most point of the second boat's path is $1.25 + 2 \ln 1.5$.

66. Let x = width of play area;
 y = length of play area.



An equation describing the amount of fencing is

$$900 = 2x + y \text{ or } y = 900 - 2x.$$

Then $A = xy$ so

$$\begin{aligned} A(x) &= x(900 - 2x) \\ &= 900x - 2x^2. \end{aligned}$$

$$\text{If } A'(x) = 900 - 4x = 0, x = 225.$$

$$\text{Then } y = 900 - 2(225) = 450.$$

$A''(x) = -4 < 0$, so the area is maximized if the dimensions are 225 m by 450 m.

67. Distance on shore: $40 - x$ feet

Speed on shore: 5 feet per second

Distance in water: $\sqrt{x^2 + 40^2}$ feet

Speed in water: 3 feet for second

The total travel time t is

$$t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}.$$

$$\begin{aligned} t(x) &= \frac{40 - x}{5} + \frac{\sqrt{x^2 + 40^2}}{3} \\ &= 8 - \frac{x}{5} + \frac{\sqrt{x^2 + 1600}}{3} \\ t'(x) &= -\frac{1}{5} + \frac{1}{3} \cdot \frac{1}{2}(x^2 + 1600)^{-1/2}(2x) \\ &= -\frac{1}{5} + \frac{x}{3\sqrt{x^2 + 1600}} \end{aligned}$$

Minimize the travel time $t(x)$. If $t'(x) = 0$:

$$\begin{aligned} \frac{x}{3\sqrt{x^2 + 1600}} &= \frac{1}{5} \\ 5x &= 3\sqrt{x^2 + 1600} \\ \frac{5x}{3} &= \sqrt{x^2 + 1600} \\ \frac{25}{9}x^2 &= x^2 + 1600 \\ x^2 &= \frac{1600 \cdot 9}{16} \\ x &= \frac{40 \cdot 3}{4} = 30 \end{aligned}$$

(Discard the negative solution.)

To minimize the time, he should walk
 $40 - x = 40 - 30 = 10$ ft along the shore
 before paddling toward the desired destination.
 The minimum travel time is

$$\frac{40 - 30}{5} + \frac{\sqrt{30^2 + 40^2}}{3} \approx 18.67 \text{ seconds.}$$

68. Distance on shore: $25 - x$ feet

Speed on shore: 5 feet per second

Distance in water: $\sqrt{x^2 + 40^2}$ feet

Speed in water: 3 feet per second

The total travel time t is $t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}$.

$$\begin{aligned} t(x) &= \frac{25 - x}{5} + \frac{\sqrt{x^2 + 40^2}}{3} \\ &= 5 - \frac{x}{5} + \frac{\sqrt{x^2 + 1600}}{3} \\ t'(x) &= -\frac{1}{5} + \frac{1}{3} \cdot \frac{1}{2}(x^2 + 1600)^{-1/2}(2x) \\ &= -\frac{1}{5} + \frac{x}{3\sqrt{x^2 + 1600}} \end{aligned}$$

Minimize the travel time $t(x)$. If $t'(x) = 0$:

$$\begin{aligned} \frac{x}{3\sqrt{x^2 + 1600}} &= \frac{1}{5} \\ 5x &= 3\sqrt{x^2 + 1600} \\ \frac{5x}{3} &= \sqrt{x^2 + 1600} \\ \frac{25}{9}x^2 &= x^2 + 1600 \\ \frac{16}{9}x^2 &= 1600 \\ x^2 &= \frac{1600 \cdot 9}{16} \\ x &= \frac{40 \cdot 3}{4} = 30 \end{aligned}$$

(Discard the negative solution.)

$x = 30$ is impossible since the closest point on the shore to the desired destination is only 25 ft from where he is standing.

Check the end points.

x	$t(x)$
0	18.33
25	15.72

The time is minimized when $x = 25$.

$25 - x = 25 - 25 = 0$ ft, so the mathematician should start paddling where he is standing. The travel time is $t(25) = 15.72$ sec.

Extended Application: A Total Cost Model for a Training Program

$$1. \quad Z(m) = \frac{C_1}{m} + DtC_2 + DC_3 \left(\frac{m-1}{2} \right)$$

$$\begin{aligned} Z'(m) &= -\frac{C_1}{m^2} + 0 + \frac{DC_3}{2} \\ &= -\frac{C_1}{m^2} + \frac{DC_3}{2} \end{aligned}$$

$$2. \quad Z'(m) = 0 \text{ when}$$

$$\begin{aligned} \frac{DC_3}{2} &= \frac{C_1}{m^2} \\ m^2 &= \frac{2C_1}{DC_3} \\ m &= \sqrt{\frac{2C_1}{DC_3}} \end{aligned}$$

$$3. \quad D = 3, t = 12, C_1 = 15,000, C_3 = 900$$

$$m = \sqrt{\frac{2(15,000)}{3(900)}} = \sqrt{\frac{100}{9}} \approx 3.33$$

$$4. \quad 3 < 3.33 < 4$$

$$m^+ = 4 \text{ and } m^- = 3$$

$$5. \quad Z(m) = \frac{C_1 + mDtC_2}{m} + \frac{DC_3 \left[\frac{m(m-1)}{2} \right]}{m}$$

$$= \frac{C_1}{m} + DtC_2 + DC_3 \left(\frac{m-1}{2} \right)$$

$$C_1 = 15,000; D = 3, t = 12,$$

$$C_2 = 100; C_3 = 900$$

$$Z(m^+) = Z(4)$$

$$= \frac{15,000}{4} + 3(12)(100) + 3(900) \left(\frac{4-1}{2} \right)$$

$$= 3750 + 3600 + 4050$$

$$= \$11,400$$

$$Z(m^-) = Z(3)$$

$$= \frac{15,000}{3} + 3600 + 3 \frac{(900)(3-1)}{2}$$

$$= 5000 + 3600 + 2700$$

$$= \$11,300$$

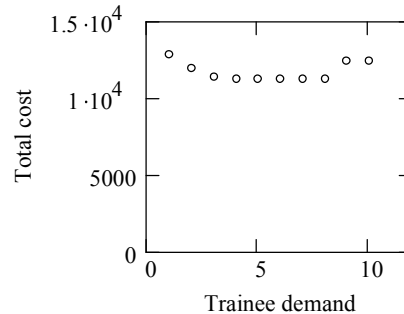
$$6. \quad \text{Since } Z(3) < Z(4), \text{ the optimal time interval is 3 months.}$$

$$N = mD = 3 \cdot 3 = 9$$

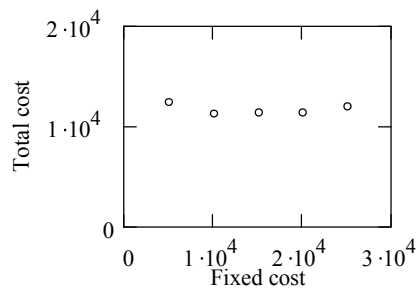
There should be 9 trainees per batch.

7. The graphs below show total cost per batch of trainees in dollars for four scenarios in which one of the values used Exercise 3 is allowed to vary.

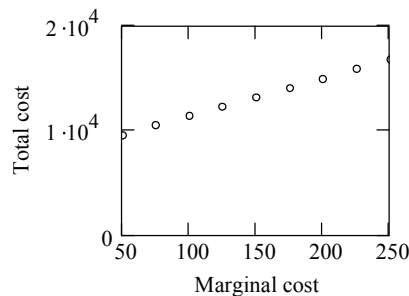
Demand D for trainees per month varies from 1 to 10, with other values as in Exercise 3:



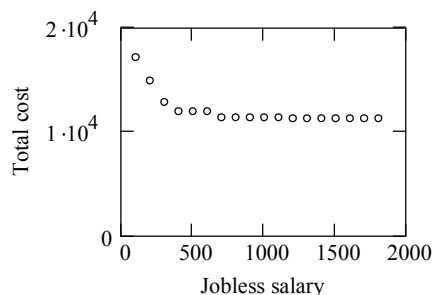
Fixed cost C_1 of training a batch varies from \$5000 to \$20,000, with other values as in Exercise 3:



Marginal cost per trainee per month C_2 varies from \$50 to \$250, with other values as in Exercise 3:



Salary for jobless trainee C_3 varies from \$100 to \$1800, with other values as in Exercise 3:



INTEGRATION

7.1 Antiderivatives

Your Turn 1

Find an antiderivative for the function $f(x) = 8x^7$.

Since the derivative of x^n is nx^{n-1} , the derivative of x^8 is $8x^7$. Thus x^8 is an antiderivative of $8x^7$. The general antiderivative is $x^8 + C$.

Your Turn 2

Find $\int \frac{1}{t^4} dt$.

Use the power rule with $n = -4$.

$$\begin{aligned}\int \frac{1}{t^4} dt &= \int t^{-4} dt \\ &= \frac{t^{-4+1}}{-4+1} + C \\ &= \frac{t^{-3}}{-3} + C \\ &= -\frac{1}{3t^3} + C\end{aligned}$$

Your Turn 3

Find $\int (6x^2 + 8x - 9) dx$.

Use the sum or difference rule and the constant multiple rule.

$$\begin{aligned}\int (6x^2 + 8x - 9) dx &= \int 6x^2 dx + \int 8x dx - \int 9 dx \\ &= 6 \int x^2 dx + 8 \int x dx - 9 \int dx\end{aligned}$$

Now use the power rule on each term.

$$\begin{aligned}6 \int x^2 dx + 8 \int x dx - 9 \int dx \\ &= 6 \left(\frac{x^3}{3} \right) + 8 \left(\frac{x^2}{2} \right) - 9x + C \\ &= 2x^3 + 4x^2 - 9x + C\end{aligned}$$

Your Turn 4

$$\begin{aligned}\int \frac{x^3 - 2}{\sqrt{x}} dx &= \int \left(\frac{x^3}{\sqrt{x}} - \frac{2}{\sqrt{x}} \right) \\ &= \int \frac{x^3}{x^{1/2}} dx - \int \frac{2}{x^{1/2}} dx \\ &= \int x^{5/2} dx - 2 \int x^{-1/2} dx \\ &= \frac{2}{7} x^{7/2} - 2 \left(\frac{2}{1} x^{1/2} \right) + C \\ &= \frac{2}{7} x^{7/2} - 4x^{1/2} + C\end{aligned}$$

Your Turn 5

$$\begin{aligned}\int \left(\frac{3}{x} + e^{-3x} \right) dx &= \int \frac{3}{x} dx + \int e^{-3x} dx \\ &= 3 \int \frac{1}{x} dx + \int e^{-3x} dx \\ &= 3 \ln |x| - \frac{1}{3} e^{-3x} + C\end{aligned}$$

Your Turn 6

Suppose an object is thrown down from the top of the 2717-ft tall Burj Khalifa with an initial velocity of -20 ft/sec. Find when it hits the ground and how fast it is traveling when it hits the ground.

In Example 11 (b) we derived the formulas

$$\begin{aligned}v(t) &= -32t - 20 \\ s(t) &= -16t^2 - 20t + 1100\end{aligned}$$

for the velocity v and distance above the ground s for an object thrown down from the Willis Tower. The only change required for the new problem is to change 1100 to the height of the Burj Khalifa, 2717, so we have

$$s(t) = -16t^2 - 20t + 2717.$$

To find when the object hits the ground we solve

$$s(t) = -16t^2 - 20t + 2717 = 0.$$

Use the quadratic formula.

$$t = \frac{20 \pm \sqrt{20^2 - 4(-16)(2717)}}{2(-16)}$$

$$t \approx -13.62 \quad \text{or} \quad t \approx 12.42$$

Only the positive root is relevant. To find the speed on impact, substitute $t = 12.42$ into the formula for v .

$$\begin{aligned} v(12.42) &= -32(12.42) - 20 \\ &\approx -417 \end{aligned}$$

The object hits the ground after 12.42 sec, traveling downward at 417 ft/sec.

Your Turn 7

$$f'(x) = 3x^{1/2} + 4$$

$$\begin{aligned} f(x) &= \int f'(x) dx \\ &= \int (3x^{1/2} + 4) dx \\ &= 3 \int x^{1/2} dx + 4 \int dx \\ &= 3 \left(\frac{2}{3} x^{3/2} \right) + 4x + C \\ &= 2x^{3/2} + 4x + C \end{aligned}$$

Since the graph of f is to go through the point $(1, -2)$,

$$f(1) = -2.$$

$$\begin{aligned} 2(1)^{3/2} + 4(1) + C &= -2 \\ 2 + 4 + C &= -2 \\ C &= -8 \end{aligned}$$

Thus,

$$f(x) = 2x^{3/2} + 4x - 8.$$

7.1 Exercises

1. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then there is a constant C such that $F(x) - G(x) = C$.

The two functions can differ only by a constant.

$$\begin{aligned} 5. \quad \int 6 dk &= 6 \int 1 dk \\ &= 6 \int k^0 dk \\ &= 6 \cdot \frac{1}{1} k^{0+1} + C \\ &= 6k + C \end{aligned}$$

$$\begin{aligned} 6. \quad \int 9 dy &= 9 \int 1 dy = 9 \int y^0 dy \\ &= \frac{9}{1} y^{0+1} + C \\ &= 9y + C \end{aligned}$$

$$\begin{aligned} 7. \quad \int (2z + 3) dz &= 2 \int z dz + 3 \int z^0 dz \\ &= 2 \cdot \frac{1}{1+1} z^{1+1} + 3 \cdot \frac{1}{0+1} z^{0+1} + C \\ &= z^2 + 3z + C \end{aligned}$$

$$\begin{aligned} 8. \quad \int (3x - 5) dx &= 3 \int x dx - 5 \int x^0 dx \\ &= 3 \cdot \frac{1}{2} x^2 - 5 \cdot \frac{1}{1} x + C \\ &= \frac{3x^2}{2} - 5x + C \end{aligned}$$

$$\begin{aligned} 9. \quad \int (6t^2 - 8t + 7) dt &= 6 \int t^2 dt - 8 \int t dt + 7 \int t^0 dt \\ &= \frac{6t^3}{3} - \frac{8t^2}{2} + 7t + C \\ &= 2t^3 - 4t^2 + 7t + C \end{aligned}$$

$$\begin{aligned} 10. \quad \int (5x^2 - 6x + 3) dx &= 5 \int x^2 dx - 6 \int x dx + 3 \int x^0 dx \\ &= \frac{5x^3}{3} - \frac{6x^2}{2} + 3x + C \\ &= \frac{5x^3}{3} - 3x^2 + 3x + C \end{aligned}$$

$$\begin{aligned} 11. \quad \int (4z^3 + 3z^2 + 2z - 6) dz &= 4 \int z^3 dz + 3 \int z^2 dz + 2 \int z dz \\ &\quad - 6 \int z^0 dz \\ &= \frac{4z^4}{4} + \frac{3z^3}{3} + \frac{2z^2}{2} - 6z + C \\ &= z^4 + z^3 + z^2 - 6z + C \end{aligned}$$

$$\begin{aligned}
 12. \quad & \int (16y^3 + 9y^2 - 6y + 3) dy \\
 &= 16 \int y^3 dy + 9 \int y^2 dy \\
 &\quad - 6 \int y dy + 3 \int dy \\
 &= \frac{16y^4}{4} + \frac{9y^3}{3} - \frac{6y^2}{2} + 3y + C \\
 &= 4y^4 + 3y^3 - 3y^2 + 3y + C
 \end{aligned}$$

$$\begin{aligned}
 13. \quad & \int (5\sqrt{z} + \sqrt{2}) dz = 5 \int z^{1/2} dz + \sqrt{2} \int dz \\
 &= \frac{5z^{3/2}}{\frac{3}{2}} + \sqrt{2}z + C \\
 &= 5\left(\frac{2}{3}\right)z^{3/2} + \sqrt{2}z + C \\
 &= \frac{10z^{3/2}}{3} + \sqrt{2}z + C
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \int (t^{1/4} + \pi^{1/4}) dt = \frac{t^{1/4+1}}{\frac{1}{4}+1} + \pi^{1/4}t + C \\
 &= \frac{t^{5/4}}{\frac{5}{4}} + \pi^{1/4}t + C \\
 &= \frac{4t^{5/4}}{5} + \pi^{1/4}t + C
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \int 5x(x^2 - 8) dx = \int (5x^3 - 40x) dx \\
 &= \frac{5x^4}{4} - \frac{40x^2}{2} + C \\
 &= \frac{5x^4}{4} - 20x^2 + C
 \end{aligned}$$

$$\begin{aligned}
 16. \quad & \int x^2(x^4 + 4x + 3) dx \\
 &= \int (x^6 + 4x^3 + 3x^2) dx \\
 &= \frac{x^7}{7} + \frac{4x^4}{4} + \frac{3x^3}{3} + C \\
 &= \frac{x^7}{7} + x^4 + x^3 + C
 \end{aligned}$$

$$\begin{aligned}
 17. \quad & \int (4\sqrt{v} - 3v^{3/2}) dv \\
 &= 4 \int v^{1/2} dv - 3 \int v^{3/2} dv \\
 &= \frac{4v^{3/2}}{\frac{3}{2}} - \frac{3v^{5/2}}{\frac{5}{2}} + C \\
 &= \frac{8v^{3/2}}{3} - \frac{6v^{5/2}}{5} + C
 \end{aligned}$$

$$\begin{aligned}
 18. \quad & \int (15x\sqrt{x} + 2\sqrt{x}) dx \\
 &= 15 \int x(x^{1/2}) dx + 2 \int x^{1/2} dx \\
 &= 15 \int x^{3/2} dx + 2 \int x^{1/2} dx \\
 &= \frac{15x^{5/2}}{\frac{5}{2}} + \frac{2x^{3/2}}{\frac{3}{2}} + C \\
 &= 15\left(\frac{2}{5}\right)x^{5/2} + 2\left(\frac{2}{3}\right)x^{3/2} + C \\
 &= 6x^{5/2} + \frac{4x^{3/2}}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & \int (10u^{3/2} - 14u^{5/2}) du \\
 &= 10 \int u^{3/2} du - 14 \int u^{5/2} du \\
 &= \frac{10u^{5/2}}{\frac{5}{2}} - \frac{14u^{7/2}}{\frac{7}{2}} + C \\
 &= 10\left(\frac{2}{5}\right)u^{5/2} - 14\left(\frac{2}{7}\right)u^{7/2} + C \\
 &= 4u^{5/2} - 4u^{7/2} + C
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & \int (56t^{5/2} + 18t^{7/2}) dt \\
 &= 56 \int t^{5/2} dt + 18 \int t^{7/2} dt \\
 &= \frac{56t^{7/2}}{\frac{7}{2}} + \frac{18t^{9/2}}{\frac{9}{2}} + C \\
 &= 16t^{7/2} + 4t^{9/2} + C
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \int \left(\frac{7}{z^2} \right) dz &= \int 7z^{-2} dz \\
 &= 7 \int z^{-2} dz \\
 &= 7 \left(\frac{z^{-2+1}}{-2+1} \right) + C \\
 &= \frac{7z^{-1}}{-1} + C \\
 &= -\frac{7}{z} + C
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \int \left(\frac{4}{x^3} \right) dx &= \int 4x^{-3} dx \\
 &= 4 \int x^{-3} dx \\
 &= \frac{4x^{-2}}{-2} + C \\
 &= -2x^{-2} + C \\
 &= -\frac{2}{x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \int \left(\frac{\pi^3}{y^3} - \frac{\sqrt{\pi}}{\sqrt{y}} \right) dy &= \int \pi^3 y^{-3} dy - \int \sqrt{\pi} y^{-1/2} dy \\
 &= \pi^3 \int y^{-3} dy - \sqrt{\pi} \int y^{-1/2} dy \\
 &= \pi^3 \left(\frac{y^{-2}}{-2} \right) - \sqrt{\pi} \left(\frac{y^{-1/2}}{-1/2} \right) + C \\
 &= -\frac{\pi^3}{2y^2} - 2\sqrt{\pi y} + C
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \int \left(\sqrt{u} + \frac{1}{u^2} \right) du &= \int u^{1/2} du + \int u^{-2} du \\
 &= \frac{u^{3/2}}{\frac{3}{2}} + \frac{u^{-1}}{-1} + C \\
 &= \frac{2u^{3/2}}{3} - \frac{1}{u} + C
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int (-9t^{-2.5} - 2t^{-1}) dt &= -9 \int t^{-2.5} dt - 2 \int t^{-1} dt \\
 &= \frac{-9t^{-1.5}}{-1.5} - 2 \int \frac{dt}{t} \\
 &= 6t^{-1.5} - 2 \ln |t| + C
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int (10x^{-3.5} + 4x^{-1}) dx &= 10 \int x^{-3.5} dx + 4 \int x^{-1} dx \\
 &= \frac{10x^{-2.5}}{-2.5} + 4 \ln |x| + C \\
 &= -4x^{-2.5} + 4 \ln |x| + C
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \int \frac{1}{3x^2} dx &= \int \frac{1}{3} x^{-2} dx \\
 &= \frac{1}{3} \int x^{-2} dx \\
 &= \frac{1}{3} \left(\frac{x^{-1}}{-1} \right) + C \\
 &= -\frac{1}{3} x^{-1} + C \\
 &= -\frac{1}{3x} + C
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \int \frac{2}{3x^4} dx &= \int \frac{2}{3} x^{-4} dx \\
 &= \frac{2}{3} \int x^{-4} dx \\
 &= \frac{2}{3} \left(\frac{x^{-3}}{-3} \right) + C \\
 &= -\frac{2}{9} x^{-3} + C \\
 &= -\frac{2}{9x^3} + C
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \int 3e^{-0.2x} dx &= 3 \int e^{-0.2x} dx \\
 &= 3 \left(\frac{1}{-0.2} \right) e^{-0.2x} + C \\
 &= \frac{3(e^{-0.2x})}{-0.2} + C \\
 &= -15e^{-0.2x} + C
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \int -4e^{0.2v} dv &= -4 \int e^{0.2v} dv \\
 &= (-4) \frac{1}{0.2} e^{0.2v} + C \\
 &= -20e^{0.2v} + C
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \int \left(-\frac{3}{x} + 4e^{-0.4x} + e^{0.1} \right) dx \\
 &= -3 \int \frac{dx}{x} + 4 \int e^{-0.4x} dx + e^{0.1} \int dx \\
 &= -3 \ln |x| + \frac{4e^{-0.4x}}{-0.4} + e^{0.1}x + C \\
 &= -3 \ln |x| - 10e^{-0.4x} + e^{0.1}x + C
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \int \left(\frac{9}{x} - 3e^{-0.4x} \right) dx \\
 &= \int \frac{9}{x} dx - 3 \int e^{-0.4x} dx \\
 &= 9 \ln |x| - 3 \left(-\frac{1}{0.4} \right) e^{-0.4x} + C \\
 &= 9 \ln |x| + \frac{15e^{-0.4x}}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \int \left(\frac{1+2t^3}{4t} \right) dt &= \int \left(\frac{1}{4t} + \frac{t^2}{2} \right) dt \\
 &= \frac{1}{4} \int \frac{1}{t} dt + \frac{1}{2} \int t^2 dt \\
 &= \frac{1}{4} \ln |t| + \frac{1}{2} \left(\frac{t^3}{3} \right) + C \\
 &= \frac{1}{4} \ln |t| + \frac{t^3}{6} + C
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \int \left(\frac{2y^{1/2} - 3y^2}{6y} \right) dy \\
 &= \int \frac{2y^{1/2}}{6y} dy - \int \frac{3y^2}{6y} dy \\
 &= \frac{1}{3} \int y^{-1/2} dy - \frac{1}{2} \int y dy \\
 &= \frac{1}{3} \left(\frac{y^{1/2}}{\frac{1}{2}} \right) - \frac{y^2}{4} + C \\
 &= \frac{2y^{1/2}}{3} - \frac{y^2}{4} + C
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \int (e^{2u} + 4u) du &= \frac{e^{2u}}{2} + \frac{4u^2}{2} + C \\
 &= \frac{e^{2u}}{2} + 2u^2 + C
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \int (v^2 - e^{3v}) dv &= \int v^2 dv - \int e^{3v} dv \\
 &= \frac{v^3}{3} - \frac{e^{3v}}{3} + C \\
 &= \frac{v^3 - e^{3v}}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 37. \quad \int (x+1)^2 dx &= \int (x^2 + 2x + 1) dx \\
 &= \frac{x^3}{3} + \frac{2x^2}{2} + x + C \\
 &= \frac{x^3}{3} + x^2 + x + C
 \end{aligned}$$

$$\begin{aligned}
 38. \quad \int (2y-1)^2 dy &= \int (4y^2 - 4y + 1) dy \\
 &= \frac{4y^3}{3} - \frac{4y^2}{2} + y + C \\
 &= \frac{4y^3}{3} - 2y^2 + y + C
 \end{aligned}$$

$$\begin{aligned}
 39. \quad \int \frac{\sqrt{x}+1}{\sqrt[3]{x}} dx &= \int \left(\frac{\sqrt{x}}{\sqrt[3]{x}} + \frac{1}{\sqrt[3]{x}} \right) dx \\
 &= \int \left(x^{(1/2-1/3)} + x^{-1/3} \right) dx \\
 &= \int x^{1/6} dx + \int x^{-1/3} dx \\
 &= \frac{x^{7/6}}{\frac{7}{6}} + \frac{x^{2/3}}{\frac{2}{3}} + C \\
 &= \frac{6x^{7/6}}{7} + \frac{3x^{2/3}}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 40. \quad \int \frac{1-2\sqrt[3]{z}}{\sqrt[3]{z}} dz &= \int \left(\frac{1}{\sqrt[3]{z}} + \frac{2\sqrt[3]{z}}{\sqrt[3]{z}} \right) dz \\
 &= \int (z^{-1/3} - 2) dz \\
 &= \frac{z^{2/3}}{\frac{2}{3}} - 2z + C \\
 &= \frac{3z^{2/3}}{2} - 2z + C
 \end{aligned}$$

$$41. \int 10^x dx = \frac{10^x}{\ln 10} + C$$

$$42. \int 3^{2x} dx = \frac{3^{2x}}{2(\ln 3)} + C$$

43. Find $f(x)$ such that $f'(x) = x^{2/3}$, and $(1, \frac{3}{5})$ is on the curve.

$$\int x^{2/3} dx = \frac{x^{5/3}}{\frac{5}{3}} + C$$

$$f(x) = \frac{3x^{5/3}}{5} + C$$

Since $(1, \frac{3}{5})$ is on the curve,

$$f(1) = \frac{3}{5}.$$

$$f(1) = \frac{3(1)^{5/3}}{5} + C = \frac{3}{5}$$

$$\frac{3}{5} + C = \frac{3}{5}$$

$$C = 0.$$

Thus,

$$f(x) = \frac{3x^{5/3}}{5}.$$

44. Find $f(x)$ such that $f'(x) = 6x^2 - 4x + 3$, and $(0, 1)$ is on the curve.

$$\begin{aligned} f(x) &= \int (6x^2 - 4x + 3) dx \\ &= \frac{6x^3}{3} - \frac{4x^2}{2} + 3x + C \\ &= 2x^3 - 2x^2 + 3x + C \end{aligned}$$

Since $(0, 1)$ is on the curve, then $f(0) = 1$.

$$\begin{aligned} f(0) &= 2(0)^3 - 2(0)^2 + 3(0) + C = 1 \\ C &= 1 \end{aligned}$$

Thus,

$$f(x) = 2x^3 - 2x^2 + 3x + 1.$$

45. $C'(x) = 4x - 5$; fixed cost is \$8.

$$C(x) = \int (4x - 5) dx$$

$$= \frac{4x^2}{2} - 5x + k$$

$$= 2x^2 - 5x + k$$

$$C(0) = 2(0)^2 - 5(0) + k = k$$

$$\text{Since } C(0) = 8, k = 8.$$

Thus,

$$C(x) = 2x^2 - 5x + 8.$$

46. $C'(x) = 0.2x^2 + 5x$; fixed cost is \$10.

$$C(x) = \int (0.2x^2 + 5x) dx$$

$$= \frac{0.2x^3}{3} + \frac{5x^2}{2} + k$$

$$C(0) = \frac{0.2(0)^3}{3} + \frac{5(0)^2}{2} + k = k$$

$$\text{Since } C(0) = 10, k = 10.$$

Thus,

$$C(x) = \frac{0.2x^3}{3} + \frac{5x^2}{2} + 10.$$

47. $C'(x) = 0.03e^{0.01x}$; fixed cost \$8.

$$C(x) = \int 0.03e^{0.01x} dx$$

$$= 0.03 \int e^{0.01x} dx$$

$$= 0.03 \left(\frac{1}{0.01} e^{0.01x} \right) + k$$

$$= 3e^{0.01x} + k$$

$$C(0) = 3e^{0.01(0)} + k = 3(1) + k$$

$$= 3 + k$$

$$\text{Since } C(0) = 8, 3 + k = 8, \text{ and } k = 5.$$

Thus,

$$C(x) = 3e^{0.01x} + 5.$$

48. $C'(x) = x^{1/2}$, 16 units cost \$45.

$$C(x) = \int x^{1/2} dx = \frac{x^{3/2}}{\frac{3}{2}} + k = \frac{2}{3}x^{3/2} + k$$

$$C(16) = \frac{2}{3}(16)^{3/2} + k = \frac{2}{3}(64) + k = \frac{128}{3} + k$$

$$\text{Since } C(16) = 45,$$

$$\frac{128}{3} + k = 45$$

$$k = \frac{7}{3}.$$

Thus,

$$C(x) = \frac{2}{3}x^{3/2} + \frac{7}{3}.$$

49. $C'(x) = x^{2/3} + 2$; 8 units cost \$58.

$$C(x) = \int (x^{2/3} + 2) dx$$

$$= \frac{3x^{5/3}}{5} + 2x + k$$

$$C(8) = \frac{3(8)^{5/3}}{5} + 2(8) + k$$

$$= \frac{3(32)}{5} + 16 + k$$

$$\text{Since } C(8) = 58,$$

$$58 - 16 - \frac{96}{5} = k$$

$$\frac{114}{5} = k.$$

Thus,

$$C(x) = \frac{3x^{5/3}}{5} + 2x + \frac{114}{5}.$$

50. $C'(x) = x + \frac{1}{x^2}$; 2 units cost \$5.50, so

$$C(2) = 5.50.$$

$$C(x) = \int \left(x + \frac{1}{x^2} \right) dx$$

$$= \int (x + x^{-2}) dx$$

$$= \frac{x^2}{2} + \frac{x^{-1}}{-1} + k$$

$$C(x) = \frac{x^2}{2} - \frac{1}{x} + k$$

$$C(2) = \frac{(2)^2}{2} - \frac{1}{2} + k$$

$$= 2 - \frac{1}{2} + k$$

$$\text{Since } C(2) = 5.50,$$

$$5.50 - 1.5 = k$$

$$4 = k.$$

Thus,

$$C(x) = \frac{x^2}{2} - \frac{1}{x} + 4.$$

51. $C'(x) = 5x - \frac{1}{x}$; 10 units cost \$94.20, so

$$C(10) = 94.20.$$

$$C(x) = \int \left(5x - \frac{1}{x} \right) dx = \frac{5x^2}{2} - \ln|x| + k$$

$$C(10) = \frac{5(10)^2}{2} - \ln(10) + k$$

$$= 250 - 2.30 + k.$$

$$\text{Since } C(10) = 94.20,$$

$$94.20 = 247.70 + k$$

$$-153.50 = k.$$

$$\text{Thus, } C(x) = \frac{5x^2}{2} - \ln|x| - 153.50.$$

52. $C'(x) = 1.2^x(\ln 1.2)$; 2 units cost \$9.44

(Hint: Recall that $a^x = e^{x \ln a}$.)

$$C(x) = \int 1.2^x(\ln 1.2) dx$$

$$= \ln 1.2 \int 1.2^x dx$$

$$= \ln 1.2 \int e^{x \ln 1.2} dx$$

$$= \ln 1.2 \left(\frac{1}{\ln 1.2} e^{x \ln 1.2} \right) + k$$

$$= e^{x \ln 1.2} + k$$

$$C(2) = e^{2 \ln 1.2} + k = 1.44 + k$$

$$\text{Since } C(2) = 9.44,$$

$$1.44 + k = 9.44$$

$$k = 8.$$

$$\text{Thus, } C(x) = e^{x \ln 1.2} + 8 = 1.2^x + 8.$$

53. $R'(x) = 175 - 0.02x - 0.03x^2$

$$R = \int (175 - 0.02x - 0.03x^2) dx$$

$$= 175x - 0.01x^2 - 0.01x^3 + C.$$

If $x = 0$, then $R = 0$ (no items sold means no revenue), and

$$0 = 175(0) - 0.01(0)^2 - 0.01(0)^3 + C$$

$$0 = C.$$

Thus, $R = 175x - 0.01x^2 - 0.01x^3$ gives the revenue function. Now, recall that $R = xp$, where p is the demand function. Then

$$175x - 0.01x^2 - 0.01x^3 = xp$$

$$175 - 0.01x - 0.01x^2 = p,$$

the demand function.

54. $R'(x) = 50 - 5x^{2/3}$

$$R = \int (50 - 5x^{2/3}) dx$$

$$= 50x - 3x^{5/3} + C$$

If $x = 0$, then $R = 0$ (no items sold means no revenue), and

$$0 = 50(0) - 3(0)^{5/3} + C$$

$$0 = C$$

Thus, $R = 50x - 3x^{5/3}$ gives the revenue function. Now, recall that $R = xp$, where p is demand function. Then

$$50x - 3x^{5/3} = xp$$

$$50 - 3x^{2/3} = p,$$

which gives the demand function.

55. $R'(x) = 500 - 0.15\sqrt{x}$

$$R = \int (500 - 0.15\sqrt{x}) dx$$

$$= 500x - 0.1x^{3/2} + C.$$

If $x = 0$, $R = 0$ (no items sold means no revenue), and

$$0 = 500(0) - 0.1(0)^{3/2} + C$$

$$0 = C.$$

Thus, $R = 500x - 0.1x^{3/2}$ gives the revenue function. Now, recall that $R = xp$, where p is the demand function. Then

$$500x - 0.1x^{3/2} = xp$$

$$500 - 0.1\sqrt{x} = p, \text{ the demand function.}$$

56. $R'(x) = 600 - 5e^{0.0002x}$

$$R = \int (600 - 5e^{0.0002x}) dx$$

$$= 600x - 25,000e^{0.0002x} + C.$$

If $x = 0$, then $R = 0$ (no items sold means no revenue), and

$$0 = 600(0) - 25,000e^{0.0002(0)} + C$$

$$0 = -25,000 + C$$

$$0 = 25,000.$$

$$\text{Thus, } R = 600x - 25,000e^{0.0002x} + 25,000$$

$$= 600x - 25,000(1 - e^{0.0002x})$$

gives the revenue function. Now, recall that $R = xp$, where p is the demand function.

Then

$$600x + 25,000(1 - e^{0.0002x}) = xp$$

$$600 + \frac{25,000}{x}(1 - e^{0.0002x}) = p,$$

Which gives the demand function.

57. $f'(t) = 7.50t - 16.8$

(a) $f(t) = \int (7.50t - 16.8) dt$

$$= \frac{7.50}{2}t^2 - 16.8t + C$$

$$= 3.75t^2 - 16.8t + C$$

In 2005 ($t = 5$), $f(t) = 9.8$ and

$$9.8 = 3.75(5)^2 - 16.8(5) + C$$

$$9.8 = 9.75 + C$$

$$0.05 = C$$

Thus, $f(t) = 3.75t^2 - 16.8t + 0.05$.

(b) In 2009, $t = 9$ and

$$f(9) = 3.75(9)^2 - 16.8(9) + 0.05 = 152.6$$

The function derived in (a) predicts 152.6 billion monthly text messages, quite close to the actual value of 152.7 billion messages.

58. $P'(x) = \sqrt{x} + \frac{1}{2}$; profit is -1 when 0 hamburgers are sold.

$$P(x) = \int \left(\sqrt{x} + \frac{1}{2} \right) dx$$

$$= \frac{2x^{3/2}}{3} + \frac{x}{2} + k$$

$$P(0) = \frac{2(0)^{3/2}}{3} + \frac{0}{2} + k$$

$$\text{Since } P(0) = -1, k = -1.$$

$$P(x) = \frac{2}{3}x^{3/2} + \frac{x}{2} - 1$$

59. (a) $P'(x) = 50x^3 + 30x^2$; profit is -40 when no cheese is sold.

$$\begin{aligned} P(x) &= \int (50x^3 + 30x^2) dx \\ &= \frac{25x^4}{2} + 10x^3 + k \end{aligned}$$

$$P(0) = \frac{25(0)^4}{2} + 10(0)^3 + k$$

Since

$$P(0) = -40,$$

$$-40 = k.$$

Thus,

$$P(x) = \frac{25x^4}{2} + 10x^3 - 40.$$

$$(b) \quad P(2) = \frac{25(2)^4}{2} + 10(2)^3 - 40 = 240$$

The profit from selling 200 lbs of Brie cheese is \$240.

60. (a) $f'(t) = 0.01e^{-0.01t}$

$$\begin{aligned} f(t) &= \int 0.01e^{-0.01t} dt \\ &= -\frac{0.01e^{-0.01t}}{0.01} + k \\ &= -e^{-0.01t} + k \end{aligned}$$

(b)

$$f(0) = -e^{-0.01(0)} + k = -e^0 + k = -1 + k$$

Since $f(0) = 0$,

$$0 = -1 + k$$

$$k = 1.$$

$$f(t) = -e^{-0.01t} + 1$$

$$f(10) = -e^{-0.01(10)} + 1$$

$$= -e^{-0.1} + 1$$

$$= -0.905 + 1$$

$$= 0.095$$

0.095 unit is excreted in 10 min.

$$\begin{aligned} 61. \quad \int \frac{g(x)}{x} dx &= \int \frac{a - bx}{x} dx \\ &= \int \left(\frac{a}{x} - b \right) dx \\ &= a \int \frac{dx}{x} - b \int dx \\ &= a \ln|x| - bx + C \end{aligned}$$

Since x represents a positive quantity, the absolute value sign can be dropped.

$$\int \frac{g(x)}{x} dx = a \ln x - bx + C$$

$$\begin{aligned} 62. \quad (a) \quad c(t) &= (c_0 - C)e^{-kAt/V} + M \\ c'(t) &= (c_0 - C) \left(\frac{-kA}{V} \right) e^{-kAt/V} \\ &= \frac{-kA}{V} (c_0 - C) e^{-kAt/V} \end{aligned}$$

(b) According to (1) and (2),

$$\begin{aligned} c'(t) &= \frac{kA}{V} \left[C - (c_0 - C)e^{-kAt/V} + C \right] \\ &= \frac{kA}{V} (C - c_0) e^{-kAt/V}. \end{aligned}$$

This is also what we get for $c'(t)$ by differentiating Equation (2).

$$63. \quad N'(t) = Ae^{kt}$$

$$(a) \quad N(t) = \frac{A}{k} e^{kt} + C$$

$$A = 50, N(t) = 300 \text{ when } t = 0.$$

$$N(0) = \frac{50}{k} e^0 + C = 300$$

$$N'(5) = 250$$

Therefore,

$$N'(5) = 50e^{5k} = 250$$

$$e^{5k} = 5$$

$$5k = \ln 5$$

$$k = \frac{\ln 5}{5}.$$

$$N(0) = \frac{50}{\frac{\ln 5}{5}} + C = 300$$

$$\frac{250}{\ln 5} + C = 300$$

$$C = 300 - \frac{250}{\ln 5} \approx 144.67$$

$$\begin{aligned}
 N(t) &= \frac{50}{\frac{\ln 5}{5}} e^{(\ln 5/5)t} + 144.67 \\
 &= 155.3337e^{0.321888t} + 144.67 \\
 &\approx 155.3e^{0.3219} + 144.7
 \end{aligned}$$

$$(b) \quad N(12) = 155.3337e^{0.321888(12)} + 144.67 \approx 7537$$

There are 7537 cells present after 12 days.

$$\begin{aligned}
 64. \quad V'(t) &= -kP(t) \\
 P(t) &= P_0 e^{-mt} \\
 V'(t) &= -kP_0 e^{-mt} \\
 V(t) &= \frac{k}{m} P_0 e^{-mt} + C \\
 V(0) &= \frac{k}{m} P_0 e^0 + C \\
 V_0 - \frac{k}{m} P_0 &= C
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 V(t) &= \frac{k}{m} P_0 e^{-mt} + V_0 - \frac{k}{m} P_0 \\
 &= \frac{kP_0}{m} e^{-mt} + V_0 - \frac{kP_0}{m}.
 \end{aligned}$$

$$65. \quad B'(t) = 0.06048t^2 - 1.292t + 15.86$$

$$\begin{aligned}
 (a) \quad B(t) &= \int (0.06048t^2 - 1.292t + 15.86) dt \\
 &= \frac{0.06048}{3} t^3 - \frac{1.292}{2} t^2 \\
 &\quad + 15.86t + C \\
 &= 0.02016t^3 - 0.6460t^2 \\
 &\quad + 15.86t + C
 \end{aligned}$$

In 1970, when $t = 0$, 839,700 or about 839.7 thousand degrees were conferred, so $B(0) = 839.7$ and thus $C = 839.7$ and the formula for B is

$$B(t) = 0.02016t^3 - 0.6460t^2 + 15.86t + 839.7.$$

(b) To project the number of bachelor's degrees conferred in 2015 we set t equal to 45 and evaluate $B(45)$.

$$\begin{aligned}
 B(45) &= 0.02016(45)^3 - 0.6460(45)^2 \\
 &\quad + 15.86(45) + 839.7 \\
 &\approx 2082
 \end{aligned}$$

The formula predicts that 2082 thousand or about 2,082,000 bachelor's degrees will be conferred in 2015.

$$66. \quad D'(t) = 29.25e^{0.03572t}$$

$$\begin{aligned}
 (a) \quad D(t) &= \int 29.25e^{0.03572t} dt \\
 &= \frac{29.25}{0.03572} e^{0.03572t} + C \\
 &= 818.9e^{0.03572t} + C
 \end{aligned}$$

In 1980, when $t = 0$, $D = 700$, so

$$\begin{aligned}
 818.9e^{0.03572(0)} + C &= 700 \\
 818.9 + C &= 700 \\
 C &= -118.9
 \end{aligned}$$

Thus

$$D(t) = 818.9e^{0.03572t} - 118.9.$$

(b) To project the number of dentistry degrees conferred in 2015 we set t equal to 35 and evaluate $D(35)$.

$$\begin{aligned}
 D(35) &= 818.9e^{0.03572(35)} - 118.9 \\
 &\approx 2740
 \end{aligned}$$

The formula predicts that 2740 dentistry degrees will be conferred in 2015.

$$67. \quad a(t) = 5t^2 + 4$$

$$\begin{aligned}
 v(t) &= \int (5t^2 + 4) dt \\
 &= \frac{5t^3}{3} + 4t + C \\
 v(0) &= \frac{5(0)^3}{3} + 4(0) + C
 \end{aligned}$$

Since $v(0) = 6$, $C = 6$.

$$v(t) = \frac{5t^3}{3} + 4t + 6$$

68. $v(t) = 9t^2 - 3\sqrt{t}$

$$s = \int v(t) dt$$

$$= \int (9t^2 - 3\sqrt{t}) dt$$

$$= 3t^3 - 2t^{3/2} + C$$

$$s = 3t^3 - 2t^{3/2} + C$$

Since $s(1) = 8$,

$$8 = 3(1)^3 - 2(1)^{3/2} + C$$

$$8 = 1 + C$$

$$7 = C.$$

Thus,

$$s(t) = 3t^3 - 2t^{3/2} + 7.$$

69. $a(t) = -32$

$$v(t) = \int -32 dt = -32t + C_1$$

$$v(0) = -32(0) + C_1$$

Since $v(0) = 0$, $C_1 = 0$.

$$v(t) = -32t$$

$$s(t) = \int -32t dt$$

$$= \frac{-32t^2}{2} + C_2$$

$$= -16t^2 + C_2$$

At $t = 0$, the plane is at 6400 ft.

That is, $s(0) = 6400$.

$$s(0) = -16(0)^2 + C_2$$

$$6400 = 0 + C_2$$

$$C_2 = 6400$$

$$s(t) = -16t^2 + 6400$$

When the object hits the ground, $s(t) = 0$.

$$-16t^2 + 6400 = 0$$

$$-16t^2 = -6400$$

$$t^2 = 400$$

$$t = \pm 20$$

Discard -20 since time must be positive.

The object hits the ground in 20 sec.

70. $a(t) = 18t + 8$

$$v(t) = \int (18t + 8) dt$$

$$= 9t^2 + 8t + C_1$$

$$v(1) = 9(1)^2 + 8(1) + C_1 = 17 + C_1$$

Since $v(1) = 15$, $C_1 = -2$.

$$v(t) = 9t^2 + 8t - 2$$

$$s(t) = \int (9t^2 + 8t - 2) dt$$

$$= 3t^3 + 4t^2 - 2t + C_2$$

$$s(1) = 3(1)^3 + 4(1)^2 - 2(1) + C_2$$

$$= 5 + C_2$$

Since $s(1) = 19$, $C_2 = 14$.

Thus,

$$s(t) = 3t^3 + 4t^2 - 2t + 14.$$

71. $a(t) = \frac{15}{2}\sqrt{t} = 3e^{-t}$

$$v(t) = \int \left(\frac{15}{2}\sqrt{t} + 3e^{-t} \right) dt$$

$$= \int \left(\frac{15}{2}t^{1/2} + 3e^{-t} \right) dt$$

$$= \frac{15}{2} \left(\frac{t^{3/2}}{\frac{3}{2}} \right) + 3 \left(\frac{1}{-1} e^{-t} \right) + C_1$$

$$= 5t^{3/2} - 3e^{-t} + C_1$$

$$v(0) = 5(0)^{3/2} - 3e^{-0} + C_1 = -3 + C_1$$

Since $v(0) = -3$, $C_1 = 0$.

$$v(t) = 5t^{3/2} - 3e^{-t}$$

$$s(t) = \int (5t^{3/2} - 3e^{-t}) dt$$

$$= 5 \left(\frac{t^{5/2}}{\frac{5}{2}} \right) - 3 \left(-\frac{1}{1} e^{-t} \right) + C_2$$

$$= 2t^{5/2} + 3e^{-t} + C_2$$

$$s(0) = 2(0)^{5/2} + 3e^{-0} + C_2 = 3 + C_2$$

Since $s(0) = 4$, $C_2 = 1$.

Thus,

$$s(t) = 2t^{5/2} + 3e^{-t} + 1.$$

72. The acceleration of gravity is a constant with value -32 ft/sec^2 ; that is,

$$a(t) = -32.$$

First find $v(t)$ by integrating $a(t)$:

$$v(t) = \int (-32)dt = -32t + k.$$

When $t = 0$, $v(t) = v_0$:

$$v_0 = -32(0) + k$$

$$v_0 = k$$

and

$$v(t) = -32t + v_0.$$

Now integrate $v(t)$ to find $h(t)$.

$$h(t) = \int (-32t + v_0)dt = -16t^2 + v_0t + C$$

Since $h(t) = h_0$ when $t = 0$, we can substitute these values into the equation for $h(t)$ to get

$$C = h_0 \text{ and}$$

$$h(t) = -16t^2 + v_0t + h_0.$$

Recall that g is a constant with value -32 ft/sec^2 , so $\frac{1}{2}g$ has value -16 ft/sec^2 , and

$$h(t) = \frac{1}{2}gt^2 + v_0t + h_0.$$

73. First find $v(t)$ by integrating $a(t)$:

$$v(t) = \int (-32)dt = -32t + k.$$

When $t = 5$, $v(t) = 0$:

$$0 = -32(5) + k$$

$$160 = k$$

and

$$v(t) = -32t + 160.$$

Now integrate $v(t)$ to find $h(t)$.

$$h(t) = \int (-32t + 160)dt = -16t^2 + 160t + C$$

Since $h(t) = 412$ when $t = 5$, we can substitute these values into the equation for $h(t)$ to get

$$C = 12 \text{ and}$$

$$h(t) = -16t^2 + 160t + 12.$$

Therefore, from the equation given in Exercise 72, the initial velocity v_0 is 160 ft/sec and the initial height of the rocket h_0 is 12 ft.

74. (a) First find $v(t)$ by integrating $a(t)$:

$$v(t) = \int (-32)dt = -32t + k.$$

When $t = 0$, $v(t) = v_0$:

$$v_0 = -32(0) + k$$

$$v_0 = k$$

and $v(t) = -32t + v_0$.

Now integrate $v(t)$ to find $s(t)$.

$$\begin{aligned} s(t) &= \int (-32t + v_0)dt \\ &= -16t^2 + v_0t + C \end{aligned}$$

Since $s(t) = 0$ when $t = 0$, we can substitute these values into the equation for $s(t)$ to get $C = 0$ and

$$s(t) = -16t^2 + v_0t.$$

- (b) When $t = 14$, $s(t) = 0$, so

$$0 = -16(14)^2 + v_0(14)$$

$$v_0 = 224$$

The velocity was 224 feet per second at time $t = 0$.

- (c) $v_0t = 224(14) = 3136$

The distance the rocket would travel horizontally would be 3136 feet.

7.2 Substitution

Your Turn 1

Find $\int 8x(4x^2 + 8)^6 dx$.

Let $u = 4x^2 + 8$.

Then $du = 8x dx$.

Now substitute.

$$\begin{aligned}
 \int 8x(4x^2 + 8)^6 dx &= \int (4x^2 + 8)^6 (8x dx) \\
 &= \int u^6 du \\
 &= \frac{1}{7} u^7 + C
 \end{aligned}$$

Now replace u with $4x^2 + 8$.

$$\begin{aligned}
 \int 8x(4x^2 + 8)^6 dx &= \frac{1}{7} u^7 + C \\
 &= \frac{1}{7} (4x^2 + 8)^7 + C
 \end{aligned}$$

Your Turn 2

Find $\int x^3 \sqrt{3x^4 + 10} dx$.

Let $u = 3x^4 + 10$.

Then $du = 12x^3 dx$.

Multiply the integral by $\frac{12}{12}$ to introduce the factor of 12 needed for du , and then substitute.

$$\begin{aligned}
 \int x^3 \sqrt{3x^4 + 10} dx &= \frac{1}{12} \int 12x^3 \sqrt{3x^4 + 10} dx \\
 &= \frac{1}{12} \int \sqrt{3x^4 + 10} (12x^3 dx) \\
 &= \frac{1}{12} \int u^{1/2} du \\
 &= \frac{1}{12} \left(\frac{2}{3} u^{3/2} + C \right) \\
 &= \frac{1}{18} u^{3/2} + C
 \end{aligned}$$

(Note that $(1/12) C$ is just a different constant, which we can also call C .) Now replace u with $3x^4 + 10$.

$$\begin{aligned}
 \int x^3 \sqrt{3x^4 + 10} dx &= \frac{1}{18} u^{3/2} + C \\
 &= \frac{1}{18} (3x^4 + 10)^{3/2} + C
 \end{aligned}$$

Your Turn 3

Find $\int \frac{x+1}{(4x^2+8x)^3} dx$.

Let $u = 4x^2 + 8x$.

Then $du = (8x + 8)dx = 8(x + 1)dx$

Multiply the integral by $\frac{8}{8}$ to introduce the factor of 8 needed for du , and then substitute.

$$\begin{aligned}
 \int \frac{x+1}{(4x^2+8x)^3} dx &= \frac{1}{8} \int \frac{8(x+1)}{(4x^2+8x)^3} dx \\
 &= \frac{1}{8} \int \frac{1}{u^3} du \\
 &= \frac{1}{8} \int u^{-3} du \\
 &= \frac{1}{8} \left(-\frac{1}{2} u^{-2} + C \right) \\
 &= -\frac{1}{16u^2} + C
 \end{aligned}$$

Now replace u with $4x^2 + 8x$.

$$\begin{aligned}
 \int \frac{x+1}{(4x^2+8x)^3} dx &= -\frac{1}{16u^2} + C \\
 &= -\frac{1}{16(4x^2+8x)^2}
 \end{aligned}$$

Your Turn 4

Find $\int \frac{x+3}{x^2+6x} dx$.

Let $u = x^2 + 6x$.

Then $du = 2x + 6 dx = 2(x + 3) dx$

Multiply the integral by $\frac{2}{2}$ to introduce the factor of 2 needed for du , and then substitute.

$$\begin{aligned}
 \int \frac{x+3}{x^2+6x} dx &= \frac{1}{2} \int \frac{2(x+3)}{x^2+6x} dx \\
 &= \frac{1}{2} \int \frac{1}{u} du \\
 &= \frac{1}{2} (\ln|u| + C) \\
 &= \frac{1}{2} \ln|u| + C
 \end{aligned}$$

Now replace u with $x^2 + 6x$.

$$\begin{aligned}
 \int \frac{x+3}{x^2+6x} dx &= \frac{1}{2} \ln|u| + C \\
 &= \frac{1}{2} \ln|x^2 + 6x| + C
 \end{aligned}$$

Your Turn 5

Find $\int x^3 e^{x^4} dx$.

Let $u = x^4$.

Then $du = 4x^3 dx$.

Multiply the integral by $\frac{1}{4}$ to introduce the factor of 4 needed for du , and then substitute.

$$\begin{aligned}\int x^3 e^{x^4} dx &= \frac{1}{4} \int 4x^3 e^{x^4} dx \\ &= \frac{1}{4} \int e^{x^4} (4x^3 dx) \\ &= \frac{1}{4} \int e^u du \\ &= \frac{1}{4} e^u + C\end{aligned}$$

Now replace u with x^4 .

$$\int x^3 e^{x^4} dx = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C$$

Your Turn 6

Find $\int x\sqrt{3+x} dx$.

Let $u = 3 + x$.

Then $du = dx$ and $x = u - 3$.

Now substitute.

$$\begin{aligned}\int x\sqrt{3+x} dx &= \int (u-3)\sqrt{u} du \\ &= \int u\sqrt{u} du - 3 \int \sqrt{u} du \\ &= \int u^{3/2} du - 3 \int u^{1/2} du \\ &= \frac{2}{5} u^{5/2} - 3 \left(\frac{2}{3} u^{3/2} \right) + C \\ &= \frac{2}{5} u^{5/2} - 2u^{3/2} + C\end{aligned}$$

Now replace u with $3 + x$.

$$\begin{aligned}\int x\sqrt{3+x} dx &= \frac{2}{5} u^{5/2} - 2u^{3/2} + C \\ &= \frac{2}{5} (3+x)^{5/2} - 2(3+x)^{3/2} + C\end{aligned}$$

7.2 Exercises

2. (a) $\int (3x^2 - 5)^4 2x dx$

Let $u = 3x^2 - 5$; then $du = 6x dx$.

(b) $\int \sqrt{1-x} dx$

Let $u = 1 - x$; then $du = -dx$.

(c) $\int \frac{x^2}{2x^3 + 1} dx$

Let $u = 2x^3 + 1$; then $du = 6x^2 dx$.

(d) $\int 4x^3 e^{x^4} dx$

Let $u = x^4$; then $du = 4x^3 dx$.

3. $\int 4(2x+3)^4 dx = 2 \int 2(2x+3)^4 dx$

Let $u = 2x + 3$, so that $du = 2 dx$.

$$\begin{aligned}&= 2 \int u^4 du \\ &= \frac{2 \cdot u^5}{5} + C \\ &= \frac{2(2x+3)^5}{5} + C\end{aligned}$$

4. $\int (-4t+1)^3 dt = -\frac{1}{4} \int -4(-4t+1)^3 dt$

Let $u = -4t + 1$, so that $du = -4dt$.

$$\begin{aligned}&= -\frac{1}{4} \int u^3 du \\ &= -\frac{1}{4} \cdot \frac{u^4}{4} + C \\ &= \frac{-u^4}{16} + C \\ &= \frac{-(-4t+1)^4}{16} + C\end{aligned}$$

5. $\int \frac{2 dm}{(2m+1)^3} = \int 2(2m+1)^{-3} dm$

Let $u = 2m + 1$, so that $du = 2 dm$.

$$\begin{aligned}&= \int u^{-3} du \\ &= \frac{u^{-2}}{-2} + C = \frac{-(2m+1)^{-2}}{2} + C\end{aligned}$$

$$6. \int \frac{3 du}{\sqrt{3u-5}} = \int 3(3u-5)^{-1/2} du$$

Let $w = 3u - 5$, so that $dw = 3 du$.

$$\begin{aligned} &= \int w^{-1/2} dw \\ &= \frac{w^{1/2}}{\frac{1}{2}} + C \\ &= 2w^{1/2} + C \\ &= 2(3u-5)^{1/2} + C \end{aligned}$$

$$7. \int \frac{2x+2}{(x^2+2x-4)^4} dx \\ = \int (2x+2)(x^2+2x-4)^{-4} dx$$

Let $w = x^2 + 2x - 4$, so that
 $dw = (2x+2)dx$.

$$\begin{aligned} &= \int w^{-4} dw \\ &= \frac{w^{-3}}{-3} + C \\ &= -\frac{(x^2+2x-4)^{-3}}{3} + C \\ &= -\frac{1}{3(x^2+2x-4)^3} + C \end{aligned}$$

$$8. \int \frac{6x^2 dx}{(2x^3+7)^{3/2}} \\ = \int 6x^2(2x^3+7)^{-3/2} dx$$

Let $u = 2x^3 + 7$, so that $du = 6x^2 dx$.

$$\begin{aligned} &= \int u^{-3/2} du \\ &= \frac{u^{-1/2}}{-\frac{1}{2}} + C \\ &= -2u^{-1/2} + C \\ &= \frac{-2}{u^{1/2}} + C \\ &= \frac{-2}{(2x^3+7)^{1/2}} + C \end{aligned}$$

$$9. \int z\sqrt{4z^2-5} dz = \int z(4z^2-5)^{1/2} dz \\ = \frac{1}{8} \int 8z(4z^2-5)^{1/2} dz$$

Let $u = 4z^2 - 5$, so that $du = 8z dz$.

$$\begin{aligned} &= \frac{1}{8} \int u^{1/2} du \\ &= \frac{1}{8} \cdot \frac{u^{3/2}}{\frac{3}{2}} + C \\ &= \frac{1}{8} \cdot \left(\frac{2}{3}\right) u^{3/2} + C \\ &= \frac{(4z^2-5)^{3/2}}{12} + C \end{aligned}$$

$$10. \int r\sqrt{5r^2+2} dr = \int r(5r^2+2)^{1/2} dr \\ = \frac{1}{10} \int 10r(5r^2+2)^{1/2} dr$$

Let $u = 5r^2 + 2$, so that $du = 10r dr$.

$$\begin{aligned} &= \frac{1}{10} \int u^{1/2} du \\ &= \frac{1}{10} \cdot \frac{u^{3/2}}{\frac{3}{2}} + C \\ &= \frac{u^{3/2}}{15} + C \\ &= \frac{(5r^2+2)^{3/2}}{15} + C \end{aligned}$$

$$11. \int 3x^2 e^{2x^3} dx = \frac{1}{2} \int 2 \cdot 3x^2 e^{2x^3} dx$$

Let $u = 2x^3$, so that $du = 6x^2 dx$.

$$\begin{aligned} &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \frac{e^{2x^3}}{2} + C \end{aligned}$$

12. $\int r e^{-r^2} dr$

Let $u = -r^2$, so that $du = -2r dr$.

$$\begin{aligned}\int r e^{-r^2} dr &= -\frac{1}{2} \int -2r e^{-r^2} dr \\ &= -\frac{1}{2} \int e^u du \\ &= \frac{-e^u}{2} + C = \frac{-e^{-r^2}}{2} + C\end{aligned}$$

13. $\int (1-t)e^{2t-t^2} dt$

$$= \frac{1}{2} \int 2(1-t)e^{2t-t^2} dt$$

Let $u = 2t - t^2$, so that $du = (2 - 2t)dt$.

$$\begin{aligned}&= \frac{1}{2} \int e^u du \\ &= \frac{e^u}{2} + C = \frac{e^{2t-t^2}}{2} + C\end{aligned}$$

14. $\int (x^2 - 1)e^{x^3 - 3x} dx$

Let $u = x^3 - 3x$, so that

$$du = (3x^2 - 3)dx = 3(x^2 - 1)dx.$$

$$\begin{aligned}\int (x^2 - 1)e^{x^3 - 3x} dx &= \frac{1}{3} \int 3(x^2 - 1)e^{x^3 - 3x} dx \\ &= \frac{1}{3} \int e^u du = \frac{e^u}{3} + C \\ &= \frac{e^{x^3 - 3x}}{3} + C\end{aligned}$$

15. $\int \frac{e^{1/z}}{z^2} dz = -\int e^{1/z} \cdot \frac{-1}{z^2} dz$

Let $u = \frac{1}{z}$, so that $du = \frac{-1}{z^2} dz$.

$$\begin{aligned}\int \frac{e^{1/z}}{z^2} dz &= -\int e^u du \\ &= -e^u + C \\ &= -e^{1/z} + C\end{aligned}$$

16. $\int \frac{e^{\sqrt{y}}}{2\sqrt{y}} dy = \int \frac{e^{y^{1/2}}}{2y^{1/2}} dy$
 $= \int \frac{1}{2} y^{-1/2} e^{y^{1/2}} dy$

Let $u = y^{1/2}$, so that $du = \frac{1}{2} y^{-1/2} dy$.

$$\begin{aligned}&= \int e^u du = e^u + C \\ &= e^{y^{1/2}} + C = e^{\sqrt{y}} + C\end{aligned}$$

17. $\int \frac{t}{t^2 + 2} dt$

Let $t^2 + 2 = u$, so that $2t dt = du$.

$$\begin{aligned}&= \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln|u| + C \\ &= \frac{\ln(t^2 + 2)}{2} + C\end{aligned}$$

18. $\int \frac{-4x}{x^2 + 3} dx$

Let $u = x^2 + 3$, so that $du = 2x dx$.

$$\begin{aligned}\int \frac{-4x}{x^2 + 3} dx &= -2 \int \frac{2x dx}{x^2 + 3} = -2 \int \frac{du}{u} \\ &= -2 \ln|u| + C \\ &= -2 \ln(x^2 + 3) + C\end{aligned}$$

19. $\int \frac{x^3 + 2x}{x^4 + 4x^2 + 7} dx$

Let $u = x^4 + 4x^2 + 7$.

Then $du = (4x^3 + 8x)dx = 4(x^3 + 2x)dx$.

$$\begin{aligned}\int \frac{x^3 + 2x}{x^4 + 4x^2 + 7} dx &= \frac{1}{4} \int \frac{(4x^3 + 2x)}{x^4 + 4x^2 + 7} dx \\ &= \frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} \ln|u| + C \\ &= \frac{1}{4} \ln(x^4 + 4x^2 + 7) + C\end{aligned}$$

Since $x^4 + 4x^2 + 7 > 0$ for all x , we can write

this answer as $\frac{1}{4} \ln(x^4 + 4x^2 + 7) + C$.

20. $\int \frac{t^2 + 2}{t^3 + 6t + 3} dt$

Let $u = t^3 + 6t + 3$, so that $du = (3t^2 + 6)dt$.

$$\begin{aligned}\int \frac{t^2 + 2}{t^3 + 6t + 3} dt &= \frac{1}{3} \int \frac{(3t^2 + 6)dt}{t^3 + 6t + 3} \\ &= \frac{1}{3} \int \frac{du}{u} \\ &= \frac{1}{3} \ln|u| + C \\ &= \frac{1}{3} \ln|t^3 + 6t + 3| + C\end{aligned}$$

21. $\int \frac{2x + 1}{(x^2 + x)^3} dx$

$$= \int (2x + 1)(x^2 + x)^{-3} dx$$

Let $u = x^2 + x$, so that $du = (2x + 1)dx$.

$$\begin{aligned}&= \int u^{-3} du = \frac{u^{-2}}{-2} + C \\ &= \frac{-1}{2u^2} + C = \frac{-1}{2(x^2 + x)^2} + C\end{aligned}$$

22. $\int \frac{y^2 + y}{(2y^3 + 3y^2 + 1)^{2/3}} dy$

$$= \int (y^2 + y)(2y^3 + 3y^2 + 1)^{-2/3} dy$$

Let $u = 2y^3 + 3y^2 + 1$, so that

$$du = (6y^2 + 6y) dy.$$

$$\begin{aligned}&= \frac{1}{6} \int 6(y^2 + y)(2y^3 + 3y^2 + 1)^{-2/3} dy \\ &= \frac{1}{6} \int u^{-2/3} du \\ &= \frac{1}{6} \cdot \frac{u^{1/3}}{\frac{1}{3}} + C \\ &= \frac{u^{1/3}}{2} + C \\ &= \frac{(2y^3 + 3y^2 + 1)^{1/3}}{2} + C\end{aligned}$$

23. $\int p(p + 1)^5 dp$

Let $u = p + 1$, so that $du = dp$; also,
 $p = u - 1$.

$$\begin{aligned}&= \int (u - 1)u^5 du \\ &= \int (u^6 - u^5) du \\ &= \frac{u^7}{7} - \frac{u^6}{6} + C \\ &= \frac{(p + 1)^7}{7} - \frac{(p + 1)^6}{6} + C\end{aligned}$$

24. $\int 4r\sqrt{8 - r} dr$

$$= \int 4r(8 - r)^{1/2} dr$$

Let $u = 8 - r$, so that $du = -dr$; also,
 $r = 8 - u$.

$$\begin{aligned}&= -4 \int -r(8 - r)^{1/2} dr \\ &= -4 \int (8 - u)u^{1/2} du \\ &= -4 \int (8u^{1/2} - u^{3/2}) du \\ &= -4 \left(\frac{8u^{3/2}}{\frac{3}{2}} - \frac{u^{5/2}}{\frac{5}{2}} \right) + C \\ &= \frac{8(8 - r)^{5/2}}{5} - \frac{64(8 - r)^{3/2}}{3} + C\end{aligned}$$

25. $\int \frac{u}{\sqrt{u - 1}} du$

$$= \int u(u - 1)^{-1/2} du$$

Let $w = u - 1$, so that $dw = du$ and
 $u = w + 1$.

$$\begin{aligned}&= \int (w + 1)w^{-1/2} dw \\ &= \int (w^{1/2} + w^{-1/2}) dw \\ &= \frac{w^{3/2}}{\frac{3}{2}} + \frac{w^{1/2}}{\frac{1}{2}} + C \\ &= \frac{2(u - 1)^{3/2}}{3} + 2(u - 1)^{1/2} + C\end{aligned}$$

26. $\int \frac{2x}{(x + 5)^6} dx$

$$= \int 2x(x + 5)^{-6} dx = 2 \int x(x + 5)^{-6} dx$$

Let $u = x + 5$, so that

$$du = dx; \text{ also, } u - 5 = x.$$

$$\begin{aligned} &= 2 \int (u - 5)u^{-6} du = 2 \int (u^{-5} - 5u^{-6}) du \\ &= 2 \left(\frac{u^{-4}}{-4} \right) - 10 \left(\frac{u^{-5}}{-5} \right) + C \\ &= -\frac{u^{-4}}{2} + 2u^{-5} + C \\ &= \frac{-1}{2(x+5)^4} + \frac{2}{(x+5)^5} + C \end{aligned}$$

$$\begin{aligned} 27. \quad & \int \left(\sqrt{x^2 + 12x} \right) (x + 6) dx \\ &= \int (x^2 + 12x)^{1/2} (x + 6) dx \end{aligned}$$

$$\text{Let } x^2 + 12x = u, \text{ so that}$$

$$\begin{aligned} (2x + 12) dx &= du \\ 2(x + 6) dx &= du. \\ &= \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \left(\frac{2}{3} \right) u^{3/2} + C \\ &= \frac{(x^2 + 12x)^{3/2}}{3} + C \end{aligned}$$

$$\begin{aligned} 28. \quad & \int \left(\sqrt{x^2 - 6x} \right) (x - 3) dx \\ &= \int (x^2 - 6x)^{1/2} (x - 3) dx \end{aligned}$$

$$\text{Let } u = x^2 - 6x, \text{ so that}$$

$$\begin{aligned} du &= (2x - 6) dx = 2(x - 3) dx. \\ &= \frac{1}{2} \int (x^2 - 6x)^{1/2} 2(x - 3) dx \\ &= \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \left(\frac{u^{3/2}}{\frac{3}{2}} \right) + C \\ &= \frac{u^{3/2}}{3} + C = \frac{(x^2 - 6x)^{3/2}}{3} + C \end{aligned}$$

$$29. \quad \int \frac{3(1 + 3 \ln x)^2}{x} dx$$

$$\text{Let } u = 1 + 3 \ln x, \text{ so that } du = \frac{3}{x} dx.$$

$$\begin{aligned} &= \frac{1}{3} \int \frac{3(1 + 3 \ln x)^2}{x} dx \\ &= \frac{1}{3} \int u^2 du \\ &= \frac{1}{3} \cdot \frac{u^3}{3} + C \\ &= \frac{(1 + 3 \ln x)^3}{9} + C \end{aligned}$$

$$30. \quad \int \frac{\sqrt{2 + \ln x}}{x} dx$$

$$\text{Let } u = 2 + \ln x, \text{ so that } du = \frac{1}{x} dx.$$

$$\begin{aligned} \int \frac{\sqrt{2 + \ln x}}{x} dx &= \int \sqrt{u} du \\ &= \int u^{1/2} du \\ &= \frac{u^{3/2}}{3/2} + C \\ &= \frac{2}{3} u^{3/2} + C \\ &= \frac{2}{3} (2 + \ln x)^{3/2} + C \end{aligned}$$

$$31. \quad \int \frac{e^{2x}}{e^{2x} + 5} dx$$

$$\text{Let } u = e^{2x} + 5, \text{ so that } du = 2e^{2x} dx.$$

$$\begin{aligned} &= \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln |e^{2x} + 5| + C \\ &= \frac{1}{2} \ln (e^{2x} + 5) + C \end{aligned}$$

$$32. \quad \int \frac{1}{x(\ln x)} dx$$

$$\text{Let } u = \ln x, \text{ so that}$$

$$du = \frac{1}{x} dx.$$

$$\begin{aligned} \int \frac{1}{x(\ln x)} dx &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |\ln x| + C \end{aligned}$$

33. $\int \frac{\log x}{x} dx$

Let $u = \log x$, so that $du = \frac{1}{(\ln 10)x} dx$.

$$\begin{aligned}\int \frac{\log x}{x} dx &= (\ln 10) \int \frac{\log x}{(\ln 10)x} dx = (\ln 10) \int u du \\ &= (\ln 10) \left(\frac{u^2}{2} \right) + C \\ &= \frac{(\ln 10)(\log x)^2}{2} + C\end{aligned}$$

34. $\int \frac{[\log_2(5x+1)]^2}{5x+1} dx$

Let $u = \log_2(5x+1)$, so that

$$\begin{aligned}du &= \frac{5}{(\ln 2)(5x+1)} dx \\ &= \frac{\ln 2}{5} \int \frac{5[\log_2(5x+1)]^2}{(\ln 2)(5x+1)} dx \\ &= \frac{\ln 2}{5} \int u^2 du \\ &= \frac{\ln 2}{5} \left(\frac{u^3}{3} \right) + C \\ &= \frac{(\ln 2)[\log_2(5x+1)]^3}{15} + C\end{aligned}$$

35. $\int x 8^{3x^2+1} dx$

Let $u = 3x^2 + 1$, so that $du = 6x dx$.

$$\begin{aligned}&= \frac{1}{6} \int 6x \cdot 8^{3x^2+1} dx \\ &= \frac{1}{6} \int 8^u du \\ &= \frac{1}{6} \left(\frac{8^u}{\ln 8} \right) + C \\ &= \frac{8^{3x^2+1}}{6 \ln 8} + C\end{aligned}$$

36. $\int \frac{10^{5\sqrt{x}+2}}{\sqrt{x}} dx$

Let $u = 5\sqrt{x} + 2$, so that $du = \frac{5}{2\sqrt{x}} dx$.

$$\begin{aligned}&= \frac{2}{5} \int \frac{5 \cdot 10^{5\sqrt{x}+2}}{2\sqrt{x}} dx \\ &= \frac{2}{5} \int 10^u du \\ &= \frac{2}{5} \cdot \frac{10^u}{\ln 10} + C = \frac{2 \cdot 10^{5\sqrt{x}+2}}{5 \ln 10} + C\end{aligned}$$

39. (a) $R'(x) = 4x(x^2 + 27,000)^{-2/3}$

$$\begin{aligned}R(x) &= \int 4x(x^2 + 27,000)^{-2/3} dx \\ &= 2 \int 2x(x^2 + 27,000)^{-2/3} dx\end{aligned}$$

Let $u = x^2 + 27,000$, so that $du = 2x dx$.

$$\begin{aligned}R &= 2 \int u^{-2/3} du \\ &= 2 \cdot 3u^{1/3} + C \\ &= 6(x^2 + 27,000)^{1/3} + C \\ R(125) &= 6(125^2 + 27,000)^{1/3} + C\end{aligned}$$

Since $R(125) = 29.591$,

$$\begin{aligned}6(125^2 + 27,000)^{1/3} + C &= 29.591 \\ C &= -180\end{aligned}$$

Thus,

$$R(x) = 6(x^2 + 27,000)^{1/3} - 180.$$

(b) $R(x) = 6(x^2 + 27,000)^{1/3} - 180 \geq 40$

$$6(x^2 + 27,000)^{1/3} \geq 220$$

$$(x^2 + 27,000)^{1/3} \geq 36.6667$$

$$x^2 + 27,000 \geq 49,296.43$$

$$x^2 \geq 22,296.43$$

$$x \geq 149.4$$

For a revenue of at least \$40,000, 150 players must be sold.

40. (a) $D'(t) = 90(t+6)\sqrt{t^2+12t}$

$$= 90(t+6)(t^2+12t)^{1/2}$$

$$D(t) = \int 90(t+6)(t^2+12t)^{1/2} dt$$

$$= 45 \int (2t+12)(t^2+12t)^{1/2} dt$$

Let $u = t^2 + 12t$, so that $du = (2t + 12) dt$.

$$\begin{aligned}
 D &= 45 \int u^{1/2} du \\
 &= 45 \cdot \frac{2u^{3/2}}{3} + C \\
 &= 30u^{3/2} + C \\
 D(t) &= 30(t^2 + 12t)^{3/2} + C
 \end{aligned}$$

Since $D(4) = 16,260$,

$$\begin{aligned}
 30(4^2 + 12 \cdot 4)^{3/2} + C &= 16,260 \\
 15,360 + C &= 16,260 \\
 C &= 900
 \end{aligned}$$

Thus,

$$D(t) = 30(t^2 + 12t)^{3/2} + 900.$$

$$\begin{aligned}
 \text{(b)} \quad D(t) &= 30(t^2 + 12t)^{3/2} + 900 = 40,000 \\
 30(t^2 + 12t)^{3/2} &= 39,100 \\
 (t^2 + 12t)^{3/2} &= 1303.333 \\
 t^2 + 12t &= 119.317 \\
 t^2 + 12t - 119.317 &= 0 \\
 x &= \frac{-12 \pm \sqrt{12^2 - 4(1)(-119.317)}}{2(1)} \\
 x &\approx \frac{-12 \pm 24.93}{2(1)} \\
 x &\approx 6.465 \text{ or } x \approx -18.464
 \end{aligned}$$

7 years must pass.

$$41. \quad C'(x) = \frac{60x}{5x^2 + e}$$

(a) Let $u = 5x^2 + e$, so that $du = 10x \, dx$.

$$\begin{aligned}
 C(x) &= \int C'(x) \, dx \\
 &= \int \frac{60x}{5x^2 + e} \, dx \\
 &= 6 \int \frac{du}{u} = 6 \ln|u| + C \\
 &= 6 \ln(5x^2 + e) + C
 \end{aligned}$$

Since $C(0) = 10$, $C = 4$.

Therefore,

$$\begin{aligned}
 C(x) &= 6 \ln|5x^2 + e| + 4 \\
 &= 6 \ln(5x^2 + e) + 4.
 \end{aligned}$$

$$\text{(b)} \quad C(5) = 6 \ln(5 \cdot 5^2 + e) + 4 \approx 33.099$$

Since this represents \$33,099 dollars which is greater than \$20,000, a new source of investment income should be sought.

$$42. \quad \text{(a)} \quad P'(x) = xe^{-x^2}$$

Let $-x^2 = u$, so that $-2x \, dx = du$, or $x \, dx = -\frac{1}{2} du$.

$$\begin{aligned}
 P(x) &= \int xe^{-x^2} \, dx \\
 &= -\frac{1}{2} \int e^u \, du = -\frac{e^u}{2} + C \\
 &= -\frac{e^{-x^2}}{2} + C \\
 P(3) &= -\frac{e^{-9}}{2} + C
 \end{aligned}$$

Since $10,000 = 0.01$ million and $P(3) = 0.01$,

$$\begin{aligned}
 -\frac{e^{-9}}{2} + C &= 0.01 \\
 C &= 0.01 + \frac{e^{-9}}{2} \\
 &= 0.01006 \approx 0.01. \\
 P(x) &= \frac{-e^{-x^2}}{2} + 0.01
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \lim_{x \rightarrow \infty} (x) &= \lim_{x \rightarrow \infty} \left(\frac{-e^{-x^2}}{2} + 0.01 \right) \\
 &= \lim_{x \rightarrow \infty} \left(-\frac{1}{2e^{x^2}} + 0.01 \right) = 0.01
 \end{aligned}$$

Since profit is expressed in millions of dollars, the profit approaches $0.01(1,000,000) = \$10,000$.

43. $f'(t) = 4.0674 \cdot 10^{-4} t(t - 1970)^{0.4}$

- (a) Let $u = t - 1970$. To get the t outside the parentheses in terms of u , solve $u = t - 1970$ for t to get $t = u + 1970$. Then $dt = du$ and we can substitute as follows.

$$\begin{aligned} f(t) &= \int f'(t) dt = \int 4.0674 \cdot 10^{-4} t(t - 1970)^{0.4} dt \\ &= \int 4.0674 \cdot 10^{-4} (u + 1970)(u)^{0.4} du \\ &= 4.0674 \cdot 10^{-4} \int (u + 1970)(u)^{0.4} du \\ &= 4.0674 \cdot 10^{-4} \int (u^{1.4} + 1970u^{0.4}) du \\ &= 4.0674 \cdot 10^{-4} \left(\frac{u^{2.4}}{2.4} + \frac{1970u^{1.4}}{1.4} \right) + C \\ &= 4.0674 \cdot 10^{-4} \left[\frac{(t - 1970)^{2.4}}{2.4} + \frac{1970(t - 1970)^{1.4}}{1.4} \right] + C \end{aligned}$$

Since $f(1970) = 61.298$, $C = 61.298$.

Therefore, $f(t) = 4.0674 \cdot 10^{-4} \left[\frac{(t - 1970)^{2.4}}{2.4} + \frac{1970(t - 1970)^{1.4}}{1.4} \right] + 61.298$.

(b) $f(2015) = 4.0674 \cdot 10^{-4} \left[\frac{(2015 - 1970)^{2.4}}{2.4} + \frac{1970(2015 - 1970)^{1.4}}{1.4} \right] + 61.298 \approx 180.9$.

In the year 2015, there will be about 181,000 local transit vehicles.

44. $f'(t) = 0.001483t(t - 1980)^{0.75}$

- (a) Let $u = t - 1980$. To get the t outside the parentheses in terms of u , solve $u = t - 1980$ for t to get $t = u + 1980$. Then $dt = du$ and we can substitute as follows.

$$\begin{aligned} f(t) &= \int f'(t) dt = \int 0.001483t(t - 1980)^{0.75} dt \\ &= \int 0.001483(u + 1980)(u)^{0.75} du \\ &= 0.001483 \int (u + 1980)(u)^{0.75} du \\ &= 0.001483 \int (u^{1.75} + 1980u^{0.75}) du \\ &= 0.001483 \left(\frac{u^{2.75}}{2.75} + \frac{1980u^{1.75}}{1.75} \right) + C \\ &= 0.001483 \left[\frac{(t - 1980)^{2.75}}{2.75} + \frac{1980(t - 1980)^{1.75}}{1.75} \right] + C \end{aligned}$$

Since $f(1980) = 262.951$, $C = 262.951$.

Therefore,

$$f(t) = 0.001483 \left[\frac{(t - 1980)^{2.75}}{2.75} + \frac{1980(t - 1980)^{1.75}}{1.75} \right] + 262.951.$$

$$(b) \quad f(2015) = 0.001483 \left[\frac{(2015 - 1980)^{2.75}}{2.75} + \frac{1980(2015 - 1980)^{1.75}}{1.75} \right] + 262.951 \approx 1117.517$$

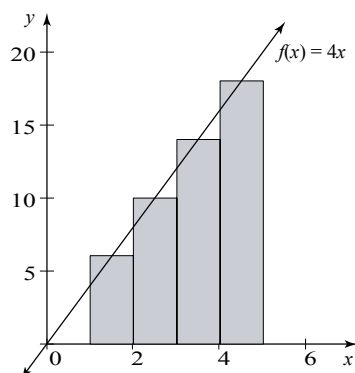
In the year 2012, there will be about 1,118,000,000 outpatient visits.

7.3 Area and the Definite Integral

Your Turn 1

Approximate $\int_1^5 4x \, dx$ using four rectangles.

Find the area of the shaded region:



Build a table giving the heights of the rectangles, which are the values of $f(x) = 4x$ at the midpoint of each interval.

i	x_i	$f(x_i)$
1	1.5	6.0
2	2.5	10.0
3	3.5	14.0
4	4.5	18.0

For each interval, $\Delta x = 1$. The sum of the areas of the rectangles is

$$\begin{aligned} \sum_{i=1}^4 f(x_i) \Delta x_i &= f(1.5)\Delta x + f(2.5)\Delta x + f(3.5)\Delta x + f(4.5)\Delta x \\ &= 1(6) + 1(10) + 1(14) + 1(18) \\ &= 48. \end{aligned}$$

Thus our approximation to the integral is 48. In this case the approximation is exact.

Your Turn 2

A driver has the following velocities at various times:

Time (hr)	0	0.5	1	1.5	2
Velocity (mph)	0	50	56	40	48

Approximate the total distance traveled during the 2-hour period.

Using left endpoints:

$$\begin{aligned}\text{distance} &= 0(0.5) + 50(0.5) + 56(0.5) + 40(0.5) \\ &= 73 \text{ miles}\end{aligned}$$

Using right endpoints:

$$\begin{aligned}\text{distance} &= 50(0.5) + 56(0.5) + 40(0.5) + 48(0.5) \\ &= 97 \text{ miles}\end{aligned}$$

Averaging these two estimates:

$$\text{distance} = \frac{73 + 97}{2} = 85 \text{ miles}$$

7.3 Exercises

$$2. \int_0^4 (x^2 + 3)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n (x_i^2 + 3)\Delta x,$$

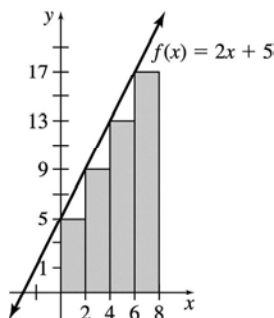
where $\Delta x = \frac{4-0}{n} = \frac{4}{n}$ and x_i is any value of x in the i th interval.

$$3. f(x) = 2x + 5, x_1 = 0, x_2 = 2, x_3 = 4, x_4 = 6, \text{ and } \Delta x = 2$$

$$(a) \sum_{i=1}^4 f(x_i)\Delta x$$

$$\begin{aligned}&= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= f(0)(2) + f(2)(2) + f(4)(2) + f(6)(2) \\ &= [2(0) + 5](2) + [2(2) + 5](2) \\ &\quad + [2(4) + 5](2) + [2(6) + 5](2) \\ &= 10 + 9(2) + 13(2) + 17(2) \\ &= 88\end{aligned}$$

(b)



The sum of these rectangles approximates

$$\int_0^8 (2x + 5) dx.$$

$$4. f(x) = \frac{1}{x} \text{ and } x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2, \text{ and } \Delta x = \frac{1}{2}$$

$$(a) \sum_{i=1}^4 f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$f(x_1) = f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}} = 2$$

$$f(x_2) = f(1) = \frac{1}{1} = 1$$

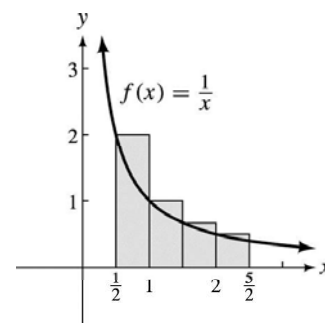
$$f(x_3) = f\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$f(x_4) = f(2) = \frac{1}{2}$$

Thus,

$$\begin{aligned}\sum_{i=1}^4 f(x_i)\Delta x &= (2)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{12 + 6 + 4 + 3}{12} = \frac{25}{12}.\end{aligned}$$

(b)



The sum approximates the integral $\int_{1/2}^{5/2} \frac{1}{x} dx$.

$$5. f(x) = 2x + 5 \text{ from } x = 2 \text{ to } x = 4$$

For $n = 4$ rectangles:

$$\Delta x = \frac{4 - 2}{4} = 0.5$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	2	9
2	2.5	10
3	3	11
4	3.5	12

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 9(0.5) + 10(0.5) + 11(0.5) + 12(0.5) \\
 &= 21
 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	2.5	10
2	3	11
3	3.5	12
4	4	13

$$A = 10(0.5) + 11(0.5) + 12(0.5) + 13(0.5) = 23$$

$$(c) \text{ Average} = \frac{21 + 23}{2} = \frac{44}{2} = 22$$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	2.25	9.5
2	2.75	10.5
3	3.25	11.5
4	3.75	12.5

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x = 9.5(0.5) + 10.5(0.5) \\
 &\quad + 11.5(0.5) + 12.5(0.5) = 22
 \end{aligned}$$

6. $f(x) = 3x + 2$ from $x = 1$ to $x = 3$ For $n = 4$ rectangles:

$$\Delta x = \frac{3 - 1}{4} = 0.5$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	1	5
2	1.5	6.5
3	2	8
4	2.5	9.5

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 5(0.5) + 6.5(0.5) + 8(0.5) + 9.5(0.5) = 14.5
 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	1.5	6.5
2	2	8
3	2.5	9.5
4	3	11

$$\begin{aligned}
 A &= 6.5(0.5) + 8(0.5) + 9.5(0.5) + 11(0.5) \\
 &= 17.5
 \end{aligned}$$

$$(c) \text{ Average} = \frac{14.5 + 17.5}{2} = \frac{32}{2} = 16$$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	1.25	5.75
2	1.75	7.25
3	2.25	8.75
4	2.75	10.25

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 5.75(0.5) + 7.25(0.5) + 8.75(0.5) + 10.25(0.5) \\
 &= 16
 \end{aligned}$$

7. $f(x) = -x^2 + 4$ from $x = -2$ to $x = 2$ For $n = 4$ rectangles:

$$\Delta x = \frac{2 - (-2)}{4} = 1$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	-2	$-(-2)^2 + 4 = 0$
2	-1	$-(-1)^2 + 4 = 3$
3	0	$-(0)^2 + 4 = 4$
4	1	$-(1)^2 + 4 = 3$

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= (0)(1) + (3)(1) + (4)(1) + (3)(1) \\
 &= 10
 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	-1	3
2	0	4
3	1	3
4	2	0

$$\text{Area} = 1(3) + 1(4) + 1(3) + 1(0) = 10$$

$$(c) \text{ Average} = \frac{10 + 10}{2} = 10$$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	$-\frac{3}{2}$	$\frac{7}{4}$
2	$-\frac{1}{2}$	$\frac{15}{4}$
3	$\frac{1}{2}$	$\frac{15}{4}$
4	$\frac{3}{2}$	$\frac{7}{4}$

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= \frac{7}{4}(1) + \frac{15}{4}(1) + \frac{15}{4}(1) + \frac{7}{4}(1) \\
 &= 11
 \end{aligned}$$

8. $f(x) = x^2$ from $x = 1$ to 5For $n = 4$ rectangles:

$$\Delta x = \frac{5-1}{4} = 1$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	1	1
2	2	4
3	3	9
4	4	16

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 1(1) + 4(1) + 9(1) + 16(1) \\
 &= 30
 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	2	4
2	3	9
3	4	16
4	5	25

$$A = 4(1) + 9(1) + 16(1) + 25(1) = 54$$

$$(c) \text{ Average} = \frac{30 + 54}{2} = \frac{84}{2} = 42$$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	$\frac{3}{2}$	$\frac{9}{4}$
2	$\frac{5}{2}$	$\frac{25}{4}$
3	$\frac{7}{2}$	$\frac{49}{4}$
4	$\frac{9}{2}$	$\frac{81}{4}$

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= \frac{9}{4}(1) + \frac{25}{4}(1) + \frac{49}{4}(1) + \frac{81}{4}(1) \\
 &= 41
 \end{aligned}$$

9. $f(x) = e^x + 1$ from $x = -2$ to $x = 2$ For $n = 4$ rectangles:

$$\Delta x = \frac{2 - (-2)}{4} = 1$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	-2	$e^{-2} + 1$
2	-1	$e^{-1} + 1$
3	0	$e^0 + 1 = 2$
4	1	$e^1 + 1$

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x = \sum_{i=1}^4 f(x_i)(1) = \sum_{i=1}^4 f(x_i) \\
 &= (e^{-2} + 1) + (e^{-1} + 1) + 2 + e^1 + 1 \\
 &\approx 8.2215 \approx 8.22
 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	-1	$e^{-1} + 1$
2	0	2
3	1	$e + 1$
4	2	$e^2 + 1$

$$\begin{aligned}
 \text{Area} &= 1(e^{-1} + 1) + 1(2) + 1(e + 1) + 1(e^2 + 1) \\
 &\approx 15.4752 \approx 15.48
 \end{aligned}$$

$$\begin{aligned}
 (c) \text{ Average} &= \frac{8.2215 + 15.4752}{2} \\
 &= 11.84835 \\
 &\approx 11.85
 \end{aligned}$$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	$-\frac{3}{2}$	$e^{-3/2} + 1$
2	$-\frac{1}{2}$	$e^{-1/2} + 1$
3	$\frac{1}{2}$	$e^{1/2} + 1$
4	$\frac{3}{2}$	$e^{3/2} + 1$

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= (e^{-3/2} + 1)(1) + (e^{-1/2} + 1)(1) \\
 &\quad + (e^{1/2} + 1)(1) + (e^{3/2} + 1)(1) \\
 &\approx 10.9601 \approx 10.96
 \end{aligned}$$

10. $f(x) = e^x - 1$ from $x = 0$ to $x = 4$ For $n = 4$ rectangles:

$$\Delta x = \frac{4 - 0}{4} = 1$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	0	0
2	1	1.718
3	2	6.389
4	3	19.086

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 0(1) + 1.718(1) + 6.389(1) + 19.086(1) \\
 &\approx 27.19
 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	1	1.718
2	2	6.389
3	3	19.086
4	4	53.598

$$\begin{aligned}
 A &= 1.718(1) + 6.389(1) + 19.086(1) + 53.598(1) \\
 &\approx 80.79
 \end{aligned}$$

$$(c) \text{ Average} = \frac{27.19 + 80.79}{2} = 53.99$$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	$\frac{1}{2}$	0.649
2	$\frac{3}{2}$	3.482
3	$\frac{5}{2}$	11.182
4	$\frac{7}{2}$	32.115

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 0.649(1) + 3.482(1) + 11.182(1) + 32.115(1) \\
 &\approx 47.43
 \end{aligned}$$

11. $f(x) = \frac{2}{x}$ from $x = 1$ to $x = 9$ For $n = 4$ rectangles:

$$\Delta x = \frac{9 - 1}{4} = 2$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	1	$\frac{2}{1} = 2$
2	3	$\frac{2}{3}$
3	5	$\frac{2}{5} = 0.4$
4	7	$\frac{2}{7}$

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= (2)(2) + \frac{2}{3}(2) + (0.4)(2) + \left(\frac{2}{7}\right)(2) \\
 &\approx 6.7048 \approx 6.70
 \end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	3	$\frac{2}{3}$
2	5	$\frac{2}{5}$
3	7	$\frac{2}{7}$
4	9	$\frac{2}{9}$

$$\begin{aligned}\text{Area} &= 2\left(\frac{2}{3}\right) + 2\left(\frac{2}{5}\right) + 2\left(\frac{2}{7}\right) + 2\left(\frac{2}{9}\right) \\ &= \frac{4}{3} + \frac{4}{5} + \frac{4}{7} + \frac{4}{9} \approx 3.1492 \approx 3.15\end{aligned}$$

(c) Average = $\frac{6.7 + 3.15}{2} = 4.93$

(d) Using the midpoints:

i	x_i	$f(x_i)$
1	2	1
2	4	$\frac{1}{2}$
3	6	$\frac{1}{3}$
4	8	$\frac{1}{4}$

$$\begin{aligned}A &= \sum_{i=1}^4 f(x_i)\Delta x \\ &= 1(2) + \frac{1}{2}(2) + \frac{1}{3}(2) + \frac{1}{4}(2) \\ &\approx 4.1667 \approx 4.17\end{aligned}$$

12. $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 3$

For $n = 4$ rectangles:

$$\Delta x = \frac{3 - 1}{4} = 0.5$$

(a) Using the left endpoints:

i	x_i	$f(x_i)$
1	1	1
2	1.5	0.6667
3	2	0.5
4	2.5	0.4

$$\begin{aligned}A &= \sum_{i=1}^4 f(x_i)\Delta x \\ &= 1(0.5) + 0.6667(0.5) + 0.5(0.5) + 0.4(0.5) \\ &= 1.283\end{aligned}$$

(b) Using the right endpoints:

i	x_i	$f(x_i)$
1	1.5	0.6667
2	2	0.5
3	2.5	0.4
4	3	0.3333

$$\begin{aligned}A &= 0.6667(0.5) + 0.5(0.5) \\ &\quad + 0.4(0.5) + 0.3333(0.5) = 0.95\end{aligned}$$

(c) Average = $\frac{1.283 + 0.95}{2}$
 $= \frac{2.233}{2} = 1.117$

(d) Using the midpoints:

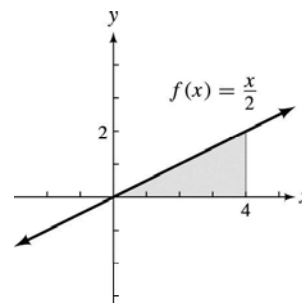
i	x_i	$f(x_i)$
1	1.25	0.8
2	1.75	0.5714
3	2.25	0.4444
4	2.75	0.3636

$$\begin{aligned}A &= \sum_{i=1}^4 f(x_i)\Delta x \\ &= 0.8(0.5) + 0.5714(0.5) + 0.4444(0.5) \\ &\quad + 0.3636(0.5) \\ &= 1.090\end{aligned}$$

13. (a) Width = $\frac{4 - 0}{4} = 1$; $f(x) = \frac{x}{2}$

$$\begin{aligned}\text{Area} &= 1 \cdot f\left(\frac{1}{2}\right) + 1 \cdot f\left(\frac{3}{2}\right) \\ &\quad + 1 \cdot f\left(\frac{5}{2}\right) + 1 \cdot f\left(\frac{7}{2}\right) \\ &= \frac{1}{4} + \frac{3}{4} + \frac{5}{4} + \frac{7}{4} = \frac{16}{4} = 4\end{aligned}$$

(b)



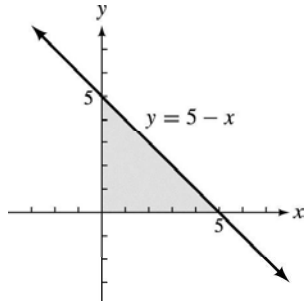
$$\begin{aligned}\int_0^4 f(x)dx &= \int_0^4 \frac{x}{2}dx = \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(4)(2) = 4\end{aligned}$$

14. (a) Width = $\frac{5 - 0}{5} = 1$; $f(x) = 5 - x$

$$\begin{aligned}\text{Area} &= f\left(\frac{1}{2}\right)(1) + f\left(\frac{3}{2}\right)(1) + f\left(\frac{5}{2}\right)(1) \\ &\quad + f\left(\frac{7}{2}\right)(1) + f\left(\frac{9}{2}\right)(1) \\ &= 4.5 + 3.5 + 2.5 + 1.5 + 0.5 = 12.5\end{aligned}$$

(b) $\int_0^5 (5 - x) dx$

Graph $y = 5 - x$.



$\int_0^5 (5 - x) dx$ is the area of a triangle with base $= 5 - 0 = 5$ and altitude $= 5$.

$$\begin{aligned}\text{Area} &= \frac{1}{2}(\text{altitude})(\text{base}) \\ &= \frac{1}{2}(5)(5) = 12.5\end{aligned}$$

15. (a) Area of triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height}$.

The base is 4; the height is 2.

$$\int_0^4 f(x) dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$$

- (b) The larger triangle has an area of $\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$. The smaller triangle has an area of $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. The sum is $\frac{9}{2} + \frac{1}{2} = \frac{10}{2} = 5$.

16. (a) $\int_0^2 f(x) dx$ is the area of a rectangle with width $x = 2$ and length $y = 4$. The rectangle has area $2 \cdot 4 = 8$. $\int_2^6 f(x) dx$ is the area of one-fourth of a circle that has radius 4. The area is $\frac{1}{4} \pi r^2 = \frac{1}{4} \pi (4)^2 = 4\pi$.

Therefore, $\int_0^6 f(x) dx = 8 + 4\pi$.

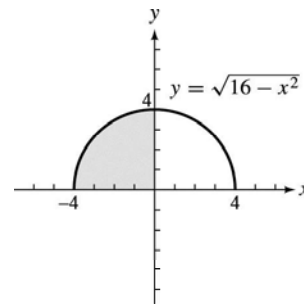
- (b) $\int_0^2 f(x) dx$ is the area of one-fourth of a circle that has radius 2. The area is $\frac{1}{4} \pi r^2 = \frac{1}{4} \pi (2)^2 = \pi$.

$\int_2^6 f(x) dx$ is the area of a triangle with base 4 and height 2. The triangle has area $\frac{1}{2} \cdot 4 \cdot 2 = 4$.

Therefore, $\int_0^6 f(x) dx = 4 + \pi$.

17. $\int_{-4}^0 \sqrt{16 - x^2} dx$

Graph $y = \sqrt{16 - x^2}$.

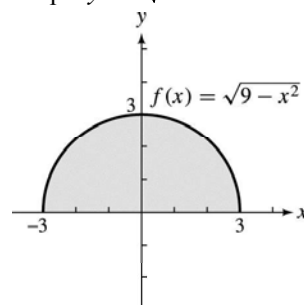


$\int_{-4}^0 \sqrt{16 - x^2} dx$ is the area of the portion of the circle in the second quadrant, which is one-fourth of a circle. The circle has radius 4.

$$\text{Area} = \frac{1}{4} \pi r^2 = \frac{1}{4} \pi (4)^2 = 4\pi$$

18. $\int_{-3}^3 \sqrt{9 - x^2} dx$

Graph $y = \sqrt{9 - x^2}$.

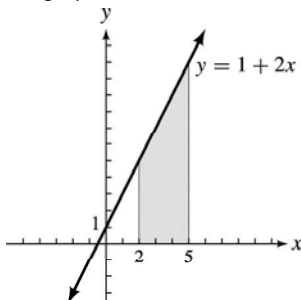


$\int_{-3}^3 \sqrt{9 - x^2} dx$ is the area of a semicircle with radius 3 centered at the origin.

$$\text{Area} = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi (3)^2 = \frac{9}{2} \pi$$

19. $\int_2^5 (1 + 2x) dx$

Graph $y = 1 + 2x$.



$\int_2^5 (1 + 2x) dx$ is the area of the trapezoid with $B = 11$, $b = 5$, and $h = 3$. The formula for the area is

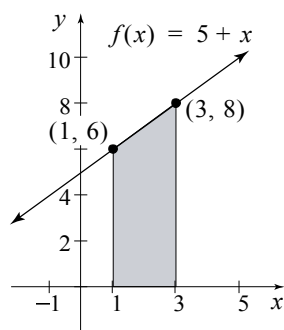
$$A = \frac{1}{2}(B + b)h,$$

so we have

$$A = \frac{1}{2}(11 + 5)(3) = 24.$$

20. $\int_1^3 (5 + x) dx$

Graph $y = 5 + x$.



$\int_1^3 (5 + x) dx$ is the area of a trapezoid with bases of length 6 and 8 and height of length 2.

$$\begin{aligned} \text{Area} &= \frac{1}{2}(\text{height})(\text{base}_1 + \text{base}_2) \\ &= \frac{1}{2}(2)(6 + 8) = 14 \end{aligned}$$

21. (a) With $n = 10$, $\Delta x = \frac{1-0}{10} = 0.1$, and $x_1 = 0 + 0.1 = 0.1$, use the command $\text{seq}(X^2, X, 0.1, 1, 0.1) \rightarrow L1$. The resulting screen is:

```
seq(X^2,X,.1,1,.1)
→L1
{.01 .04 .09 .1...
```

- (b) Since $\sum_{i=1}^n f(x_i)\Delta x = \Delta x \left(\sum_{i=1}^n f(x_i) \right)$, use the command $0.1 * \text{sum}(L1)$ to approximate $\int_0^1 x^2 dx$. The resulting screen is:

```
.1*sum(L1)
.385
```

$$\int_0^1 x^2 dx \approx 0.385$$

- (c) With $n = 100$, $\Delta x = \frac{1-0}{100} = 0.01$ and $x_1 = 0 + 0.01 = 0.01$, use the command $\text{seq}(X^2, X, 0.01, 1, 0.01) \rightarrow L1$. The resulting screen is:

```
seq(X^2,X,.01,1,.01)
→L1
{1E-4 4E-4 9E-4...
```

- Use the command $0.01 * \text{sum}(L1)$ to approximate $\int_0^1 x^2 dx$. The resulting screen is:

```
.01*sum(L1)
.33835
```

$$\int_0^1 x^2 dx \approx 0.33835$$

- (d) With $n = 500$, $\Delta x = \frac{1-0}{500} = 0.002$, and $x_1 = 0 + 0.002 = 0.002$, use the command $\text{seq}(X^2, X, 0.002, 1, 0.002) \rightarrow L1$. The resulting screen is:

```
seq(X^2,X,.002,1,
.002)→L1
{4E-6 1.6E-5 3...
```

Use the command $0.002*\text{sum}(L1)$ to approximate $\int_0^1 x^2 dx$. The resulting screen is:

```
.002*sum(L1)
.334334
```

$$\int_0^1 x^2 dx \approx 0.334334$$

- (e) As n gets larger the approximation for $\int_0^1 x^2 dx$ seems to be approaching 0.333333 or $\frac{1}{3}$. We estimate $\int_0^1 x^2 dx = \frac{1}{3}$.

22. (a) With $n = 10$, $\Delta x = \frac{1-0}{10} = 0.1$, and $x_1 = 0 + 0.1 = 0.1$, use the command $\text{seq}(X^3, X, 0.1, 0.1, 0.1) \rightarrow L1$. The resulting screen is:

```
seq(X^3,X,.1,1,.
1)→L1
{.001 .008 .027...
```

- (b) Since $\sum_{i=1}^n f(x_i)\Delta x = \Delta x \left(\sum_{i=1}^n f(x_i) \right)$, use the command $0.1*\text{sum}(L1)$ to approximate $\int_0^1 x^3 dx$. The resulting screen is:

```
.1*sum(L1)
.3025
```

$$\int_0^1 x^3 dx \approx 0.3025$$

- (c) With $n = 100$, $\Delta x = \frac{1-0}{100} = 0.01$, and $x_1 = 0 + 0.01 = 0.01$, use the command $\text{seq}(X^3, X, 0.01, 0.1, 0.01) \rightarrow L1$. The resulting screen is:

```
seq(X^3,X,.01,1,
.01)→L1
{1E-6 8E-6 2.7E...
```

Use the command $0.01*\text{sum}(L1)$ to approximate $\int_0^1 x^3 dx$. The resulting screen is:

```
.01*sum(L1)
.255025
```

$$\int_0^1 x^3 dx \approx 0.255025$$

- (d) With $n = 500$, $\Delta x = \frac{1-0}{500} = 0.002$, and $x_1 = 0 + 0.002 = 0.002$, use the command $\text{seq}(X^3, X, 0.002, 1, 0.002) \rightarrow L1$. The resulting screen is:

```
seq(X^3,X,.002,1
,.002)→L1
{8E-9 6.4E-8 2...
```

Use the command $0.002*\text{sum}(L1)$ to approximate $\int_0^1 x^3 dx$. The resulting screen is:

```
.002*sum(L1)
.251001
```

$$\int_0^1 x^3 dx \approx 0.251001$$

- (e) As n gets larger the approximation for $\int_0^1 x^3 dx$ seems to be approaching 0.25 or $\frac{1}{4}$. We estimate $\int_0^1 x^3 dx = \frac{1}{4}$.

For Exercises 24–34, reading on the graphs and answers may vary.

24. Left endpoints:

Read values of the function from the graph for every 2 hours from midnight to 10 P.M. These values give the heights of 12 rectangles. The width of each rectangle is $\Delta x = 2$. We estimate the area under the curve as

$$\begin{aligned}
 A &= \sum_{i=1}^{12} f(x_i) \Delta x \\
 &= 3.0(2) + 3.2(2) + 3.5(2) + 4.2(2) \\
 &\quad + 5.2(2) + 6.2(2) + 8.0(2) + 11.0(2) \\
 &\quad + 11.8(2) + 10.0(2) + 6.0(2) + 4.4(2) \\
 &= 153.0
 \end{aligned}$$

Right endpoints:

Read values of the function from the graph for every 2 hours from 2 A.M. to midnight. Now we estimate the area under the curve as

$$\begin{aligned}
 A &= \sum_{i=1}^{12} f(x_i) \Delta x \\
 &= 3.2(2) + 3.5(2) + 4.2(2) + 5.2(2) \\
 &\quad + 6.2(2) + 8.0(2) + 11.0(2) + 11.8(2) \\
 &\quad + 10.0(2) + 6.0(2) + 4.4(2) + 3.8(2) \\
 &= 154.6
 \end{aligned}$$

Average:

$$\begin{aligned}
 \frac{153.0 + 154.6}{2} &= \frac{307.6}{2} \\
 &= 153.8 \text{ million kilowatt-hours}
 \end{aligned}$$

The area under the curve represents the total electricity usage. We estimate this usage as about 153.8 million kilowatt hours.

25. Left endpoints:

Read values of the function on the graph every three years from 1997 to 2006. These values give us the heights of four rectangles. The width of each rectangle is $\Delta x = 3$. We estimate the area under the curve as follows:

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 34(3) + 57(3) + 115(3) + 264(3) \\
 &= 1410
 \end{aligned}$$

Right endpoints:

Read values of the function on the graph every three years from 2000 to 2009. We estimate the area under the curve as follows:

$$\begin{aligned}
 A &= \sum_{i=1}^4 f(x_i) \Delta x \\
 &= 57(3) + 115(3) + 264(3) + 697(3) \\
 &= 3399
 \end{aligned}$$

$$\text{Average: } \frac{1410 + 3399}{2} = 2404.5 \text{ trillion BTUs}$$

We estimate the total wind energy consumption over the 12-year period from 1997 to 2009 as 2404.5 trillion BTUs.

26. Left endpoints:

Read values of the function from the graph for every minute from 0 minutes to 19 minutes. These values give the height of 20 rectangles. The width of the each rectangle is $\Delta x = 1$. We estimate the area under the curve as

$$\begin{aligned}
 &\sum_{i=1}^{20} f(x_i) \Delta x \\
 &= 0(1) + 2.4(1) + 2.9(1) + 3.1(1) + 3.2(1) \\
 &\quad + 3.3(1) + 3.3(1) + 3.3(1) + 3.3(1) + 3.3(1) \\
 &\quad + 3.3(1) + 1.1(1) + 0.7(1) + 0.6(1) + 0.5(1) \\
 &\quad + 0.4(1) + 0.3(1) + 0.3(1) + 0.2(1) + 0.2(1) \\
 &\approx 35.7.
 \end{aligned}$$

Right endpoints:

Read values of the function from the graph for every minute from 1 minutes to 20 minutes.

Now we estimate the area under the curve as

$$\begin{aligned}
 &\sum_{i=1}^{20} f(x_i) \Delta x \\
 &= 2.4(1) + 2.9(1) + 3.1(1) + 3.2(1) + 3.3(1) \\
 &\quad + 3.3(1) + 3.3(1) + 3.3(1) + 3.3(1) + 3.3(1) \\
 &\quad + 1.1(1) + 0.7(1) + 0.6(1) + 0.5(1) + 0.4(1) \\
 &\quad + 0.3(1) + 0.3(1) + 0.2(1) + 0.2(1) + 0.2(1) \\
 &\approx 35.9.
 \end{aligned}$$

$$\text{Average: } \frac{35.7 + 35.9}{2} = 35.8 \text{ liters}$$

The area under the curve represents the total volume of oxygen inhaled. We estimate this volume as about 35.8 liters.

27. First read approximate data values from the graph. These readings are just estimates, and you may get different answers if your estimated readings differ from these. Month 1 represents mid-February.

Cows		Pigs	
Month	Cases	Month	Cases
1	3000	1	2000
2	165,000	2	62,000
3	267,000	3	68,000
4	54,000	4	3000
5	44,000	5	1000
6	21,000	6	9000
7	16,500	7	1000
8	11,500	8	0
9	1000	9	0

(a) Left endpoints:

Add up the values corresponding to months 1 through 8 in the Cows table. The total is 582,000 cases.

Right endpoints:

Add up the values corresponding to months 2 through 9. The total is 580,000 cases.

The average of these two values is 581,000 cases.

(b) Left endpoints:

Add up the values corresponding to months 1 through 8 in the Pigs table. The total is 146,000 cases.

Right endpoints:

Add up the values corresponding to months 2 through 9. The total is 144,000 cases.

The average of these two values is 145,000 cases.

28. Left endpoints:

Read values of the function on the graph every two years from 2000 to 2006. These values give us the heights of four rectangles. The width of each rectangle is $\Delta x = 2$. We estimate the area under the graph as follows:

$$A = \sum_{i=1}^4 f(x_i)\Delta x = 3331(2) + 3650(2) + 3701(2) + 3793(2) = 28,950$$

Right endpoints:

Read values of the function on the graph every two years from 2002 to 2008. We estimate the area under the graph as follows.

$$A = \sum_{i=1}^4 f(x_i)\Delta x = 3650(2) + 3701(2) + 3793(2) + 3113(2) = 28,514$$

$$\text{Average: } \frac{28,950 + 28,514}{2} = 28,732 \text{ accidents}$$

The area under the graph represents the number of fatal automobile accidents in California from 2000 to 2008. We estimate this number as 28,732 fatal accidents.

29. Read the value of the function for every 5 sec from $x = 2.5$ to $x = 12.5$. These are the midpoints of rectangle with width $\Delta x = 5$. Then read the function for $x = 17$, which is the midpoint of a rectangle with width $\Delta x = 4$.

$$\begin{aligned} \sum_{i=1}^4 f(x_i)\Delta x &\approx 36(5) + 63(5) + 84(5) + 95(4) \approx 1295 \\ \frac{1295}{3600}(5280) &\approx 1900 \end{aligned}$$

The Porsche 928 traveled about 1900 ft.

30. Read the value for the speed every 5 sec from $x = 2.5$ to $x = 22.5$. These are the midpoints of rectangles with width $\Delta x = 5$. Then read the speed for $x = 26.5$, which is the midpoint of a rectangle with width $\Delta x = 3$.

$$\begin{aligned} \sum_{i=1}^6 f(x_i)\Delta x &\approx 28(5) + 54(5) + 72(5) + 82(5) + 92(5) + 98(3) = 1934 \\ \frac{1934}{3600}(5280) &\approx 2800 \end{aligned}$$

The BMW 733i traveled about 2800 ft.

31. Left endpoints:

Read values of the function from the table for every number of seconds from 2.0 to 19.3. These values give the heights of 10 rectangles. The width of each rectangle varies. We estimate the area under the curve as

$$\begin{aligned}
& \sum_{i=1}^{10} f(x_i) \Delta x \\
&= 30(2.9 - 2.0) + 40(4.1 - 2.9) + 50(5.3 - 4.1) + 60(6.9 - 5.3) + 70(8.7 - 6.9) + 80(10.7 - 8.7) \\
&\quad + 90(13.2 - 10.7) + 100(16.1 - 13.2) + 110(19.3 - 16.1) + 120(23.4 - 19.3) \\
&= 1876 \\
&\quad \frac{5280}{3600}(1876) \approx 2751
\end{aligned}$$

Right endpoints:

Read values of the function from the table for every number of seconds from 2.0 to 23.4. These values give the heights of 11 rectangles. The width of each rectangle varies. We estimate the area under the curve as

$$\begin{aligned}
& \sum_{i=1}^{11} f(x_i) \Delta x \\
&= 30(2.0 - 0) + 40(2.9 - 2.0) + 50(4.1 - 2.9) + 60(5.3 - 4.1) + 70(6.9 - 5.3) + 80(8.7 - 6.9) \\
&\quad + 90(10.7 - 8.7) + 100(13.2 - 10.7) + 110(16.1 - 13.2) + 120(19.3 - 16.1) + 130(23.4 - 19.3) \\
&= 2150 \\
&\quad \frac{5280}{3600}(2150) \approx 3153
\end{aligned}$$

$$\text{Average: } \frac{2751 + 3153}{2} = \frac{5904}{2} = 2952 \text{ ft}$$

The distance traveled by the Mercedes-Benz S550 is about 2952 ft.

32. Left endpoints:

Read values of the function from the table for every number of seconds from 2.4 to 19.2. These values give the heights of 8 rectangles. The width of each rectangle varies. We estimate the area under the curve as

$$\begin{aligned}
& \sum_{i=1}^8 f(x_i) \Delta x \\
&= 30(3.5 - 2.4) + 40(5.1 - 3.5) + 50(6.9 - 5.1) + 60(8.9 - 6.9) + 70(11.2 - 8.9) \\
&\quad + 80(14.9 - 11.2) + 90(19.2 - 14.9) + 100(24.4 - 19.2) \\
&= 1671 \\
&\quad \frac{5280}{3600}(1671) \approx 2451
\end{aligned}$$

Right endpoints:

Read values of the function from the table for every number of seconds from 2.4 to 24.4. These values give the heights of 9 rectangles. The width of each rectangle varies. We estimate the area under the curve as

$$\begin{aligned}
& \sum_{i=1}^9 f(x_i) \Delta x \\
&= 30(2.4 - 0) + 40(3.5 - 2.4) + 50(5.1 - 3.5) + 60(6.9 - 5.1) + 70(8.9 - 6.9) \\
&\quad + 80(11.2 - 8.9) + 90(14.9 - 11.2) + 100(19.2 - 14.9) + 110(24.4 - 19.2) \\
&= 1963 \\
&\quad \frac{5280}{3600}(1963) \approx 2879
\end{aligned}$$

$$\text{Average: } \frac{2451 + 2879}{2} = \frac{5330}{2} = 2665$$

The distance traveled by the Chevrolet Malibu Maxx SS is about 2665 ft.

33. (a) Read values of the function on the plain glass graph every 2 hr from 6 to 6. These are at midpoints of the widths $\Delta x = 2$ and represent the heights of the rectangles.

$$f(x_i)\Delta x = 132(2) + 215(2) + 150(2) + 44(2) + 34(2) + 26(2) + 12(2) \approx 1226$$

The total heat gain was about 1230 BTUs per square foot.

- (b) Read values on the ShadeScreen graph every 2 hr from 6 to 6.

$$\sum f(x_i)\Delta x = 38(2) + 25(2) + 16(2) + 12(2) + 10(2) + 10(2) + 5(2) \approx 232$$

The total heat gain was about 230 BTUs per square foot.

34. (a) Read the value for a plain glass window facing south for every 2 hr from 6 to 6. These are the heights, at the midpoints, of rectangles with width $\Delta x = 2$.

$$\sum = 10(2) + 30(2) + 80(2) + 107(2) + 79(2) + 29(2) + 10(2) \approx 690$$

The heat gain is about 690 BTUs per square foot.

- (b) Read the value for a window with Shadescreen facing south for every 2 hr from 6 to 6. These are the heights, at the midpoints, of rectangles with width $\Delta x = 2$.

$$\sum = 4(2) + 10(2) + 20(2) + 22(2) + 20(2) + 10(2) + 4(2) \approx 180$$

The heat gain is about 180 BTUs per square foot.

35. (a) Then area of a trapezoid is

$$A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(6)(1 + 2) = 9.$$

Car A has traveled 9 ft.

- (b) Car A is furthest ahead of car B at 2 sec. Notice that from $t = 0$ to $t = 2$, $v(t)$ is larger for car A than for car B. For $t > 2$, $v(t)$ is larger for car B than for car A.

- (c) As seen in part (a), car A drove 9 ft in 2 sec. The distance of car B can be calculated as follows:

$$\begin{aligned} \frac{2 - 0}{4} &= \frac{1}{2} = \text{width} \\ \text{Distance} &= \frac{1}{2} \cdot v(0.25) + \frac{1}{2}v(0.75) + \frac{1}{2}v(1.25) + \frac{1}{2}v(1.75) \\ &= \frac{1}{2}(0.2) + \frac{1}{2}(1) + \frac{1}{2}(2.6) + \frac{1}{2}(5) \\ &= 4.4 \\ 9 - 4.4 &= 4.6 \end{aligned}$$

The furthest car A can get ahead of car B is about 4.6 ft.

- (d) At $t = 3$, car A travels $\frac{1}{2}(6)(2 + 3) = 15$ ft and car B travels approximately 13 ft.

At $t = 3.5$, car A travels $\frac{1}{2}(6)(2.5 + 3.5) = 18$ ft and car B travels approximately 18.25 ft. Therefore, car B catches up with car A between 3 and 3.5 sec.

36. Using the left endpoints:

$$\begin{aligned}\text{Distance} &= v_0(1) + v_1(1) + v_2(1) + v_3(1) \\ &= 0 + 8 + 13 + 17 = 38 \text{ ft}\end{aligned}$$

Using the right endpoints:

$$\begin{aligned}\text{Distance} &= v_1(1) + v_2(1) + v_3(1) + v_4(1) \\ &= 8 + 13 + 17 + 18 = 56 \text{ ft}\end{aligned}$$

37. Using the left endpoints:

$$\begin{aligned}\text{Distance} &= v_0(1) + v_1(1) + v_2(1) \\ &= 10 + 6.5 + 6 = 22.5 \text{ ft}\end{aligned}$$

Using the right endpoints:

$$\begin{aligned}\text{Distance} &= v_1(1) + v_2(1) + v_3(1) \\ &= 6.5 + 6 + 5.5 = 18 \text{ ft}\end{aligned}$$

38. (a) Using the left endpoints:

$$\begin{aligned}\text{Distance} &= \sum_{i=1}^n f(x_i) \Delta x_i \\ &= 0(1.84) + 12.9(1.96) + 23.8(2.58) \\ &\quad + 26.3(0.85) + 26.3(1.73) + 26.0(0.87) \\ &= 0 + 25.284 + 61.404 + 22.355 \\ &\quad + 45.499 + 22.62 \\ &= 177.162\end{aligned}$$

Since we multiplied the units of seconds by miles per hour, we need to divide by 3600 (the number of seconds in an hour) to get a distance in miles.

$$\frac{177.162}{3600} \approx 0.0492$$

The estimate of the distance is 0.0492 miles.

- (b) Using the right endpoints:

$$\begin{aligned}\text{Distance} &= \sum_{i=1}^n f(x_i) \Delta x_i \\ &= 12.9(1.84) + 23.8(1.96) + 26.3(2.58) \\ &\quad + 26.3(0.85) + 26.0(1.73) + 25.7(0.87) \\ &= 23.736 + 46.648 + 67.854 + 22.355 \\ &\quad + 44.98 + 22.359 \\ &= 227.932\end{aligned}$$

Divide by 3600 (the number of seconds in an hour) to get a distance in miles.

$$\frac{227.932}{3600} \approx 0.0633$$

The estimate of the distance is 0.0633 miles.

$$(c) \frac{100}{1609} \approx 0.0622$$

Johnson actually ran 0.0622 miles. The answer to part b is closer.

39. (a) Read values from the graph for every hour from 1 A.M. through 11 P.M. The values give the heights of 23 rectangles. The width of each rectangle is $\Delta x = 1$. We estimate the area under the curve as

$$\begin{aligned}A &= \sum_{i=1}^{23} f(x_i) \Delta x \\ &= 500(1) + 550(1) + 800(1) + 1600(1) + 4000(1) + 7000(1) \\ &\quad + 7000(1) + 5900(1) + 4500(1) + 3500(1) + 3100(1) \\ &\quad + 3100(1) + 3500(1) + 3800(1) + 4100(1) + 4800(1) \\ &\quad + 4750(1) + 4000(1) + 2500(1) + 2250(1) + 1800(1) \\ &\quad + 1500(1) + 1050(1) \\ &= 75,600 \text{ vehicles}\end{aligned}$$

- (b) Read values from the graph for every hour from 1 A.M. through 11 P.M. The values give the heights of 23 rectangles. The width of each rectangle is $\Delta x = 1$. We estimate the area under the curve as

$$\begin{aligned}A &= \sum_{i=1}^{23} f(x_i) \Delta x \\ &= 500(1) + 400(1) + 400(1) + 700(1) + 1500(1) \\ &\quad + 3000(1) + 4100(1) + 3900(1) + 3200(1) + 3600(1) \\ &\quad + 4000(1) + 4000(1) + 4300(1) + 5200(1) \\ &\quad + 6000(1) + 6500(1) + 6400(1) + 6000(1) + 4700(1) \\ &\quad + 3100(1) + 2600(1) + 1900(1) + 1300(1) \\ &= 77,300 \text{ vehicles}\end{aligned}$$

7.4 The Fundamental Theorem of Calculus

Your Turn 1

Find $\int_1^3 3x^2 dx$.

The indefinite integral is $\int 3x^2 dx = x^3 + C$.

By the Fundamental Theorem,

$$\begin{aligned}\int_1^3 3x^2 dx &= x^3 \Big|_1^3 \\ &= 3^3 - 1^3 \\ &= 27 - 1 \\ &= 26\end{aligned}$$

Your Turn 2

Find $\int_3^5 (2x^3 - 3x + 4) dx$.

$$\begin{aligned}
 & \int_3^5 (2x^3 - 3x + 4) dx \\
 &= 2 \int_3^5 x^3 dx - 3 \int_3^5 x dx + 4 \int_3^5 dx \\
 &= \frac{2}{4} x^4 \Big|_3^5 - \frac{3}{2} x^2 \Big|_3^5 + 4x \Big|_3^5 \\
 &= \frac{1}{2} (5^4 - 3^4) - \frac{3}{2} (5^2 - 3^2) + 4(5 - 3) \\
 &= 256
 \end{aligned}$$

Your Turn 3

Find $\int_1^3 \frac{2}{y} dy$.

$$\begin{aligned}
 \int_1^3 \frac{2}{y} dy &= 2 \int_1^3 \frac{1}{y} dy \\
 &= 2 \ln |y| \Big|_1^3 \\
 &= 2(\ln 3 - \ln 1) \\
 &= 2 \ln 3 \text{ or } \ln 3^2 = \ln 9
 \end{aligned}$$

Your Turn 4

Evaluate $\int_0^4 2x\sqrt{16 - x^2} dx$.

Using Method 1:

Let $u = 16 - x^2$.

Then $du = -2x$.

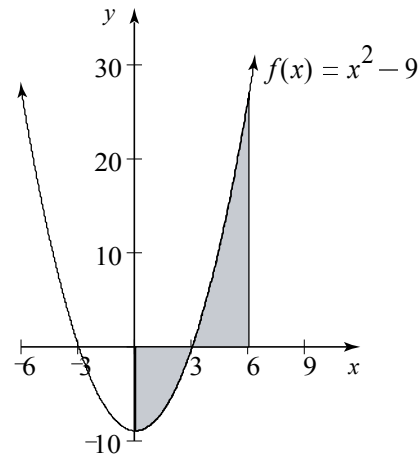
If $x = 4$, then $u = 16 - 4^2 = 0$.

If $x = 0$, then $u = 16 - 0^2 = 16$.

Now substitute.

Your Turn 5

Find the area between the graph of the function $f(x) = x^2 - 9$ and the x -axis from $x = 0$ to $x = 6$. Here is a graph of the function and the area to be found.



Since the curve is below the x -axis on the interval $(0, 3)$, the definite integral will count this area as negative. The total positive area is thus

$$\begin{aligned}
 & \left| \int_0^3 (x^2 - 9) dx \right| + \int_3^6 (x^2 - 9) dx \\
 &= \left| \left(\frac{1}{3} x^3 - 9x \right) \Big|_0^3 \right| + \left(\frac{1}{3} x^3 - 9x \right) \Big|_3^6 \\
 &= |9 - 27| + (27 - 54 - (9 - 27)) \\
 &= 18 + 18 = 36
 \end{aligned}$$

7.4 Exercises

$$\begin{aligned}
 1. \quad \int_{-2}^4 (-3) dp &= -3 \int_{-2}^4 dp = -3 \cdot p \Big|_{-2}^4 \\
 &= -3[4 - (-2)] \\
 &= -18
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \int_{-4}^1 \sqrt{2} dx &= \sqrt{2} \int_{-4}^1 dx = \sqrt{2} \cdot x \Big|_{-4}^1 \\
 &= \sqrt{2}[1 - (-4)] \\
 &= 5\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \int_{-1}^2 (5t - 3) dt &= 5 \int_{-1}^2 t dt - 3 \int_{-1}^2 dt \\
 &= \frac{5}{2} t^2 \Big|_{-1}^2 - 3t \Big|_{-1}^2 \\
 &= \frac{5}{2} [2^2 - (-1)^2] - 3[2 - (-1)] \\
 &= \frac{5}{2} (4 - 1) - 3(2 + 1)
 \end{aligned}$$

$$= \frac{15}{2} - 9$$

$$= \frac{15}{2} - \frac{18}{2} = -\frac{3}{2}$$

$$4. \quad \int_{-2}^2 (4z + 3) dz = 4 \int_{-2}^2 z dz + 3 \int_{-2}^2 dz$$

$$= 2z^2 \Big|_{-2}^2 + 3z \Big|_{-2}^2$$

$$= 2[2^2 - (-2)^2] + 3[2 - (-2)]$$

$$= 2(4 - 4) + 3(4) = 12$$

$$5. \quad \int_0^2 (5x^2 - 4x + 2) dx$$

$$= 5 \int_0^2 x^2 dx - 4 \int_0^2 x dx + 2 \int_0^2 dx$$

$$= \frac{5x^3}{3} \Big|_0^2 - 2x^2 \Big|_0^2 + 2x \Big|_0^2$$

$$= \frac{5}{3}(2^3 - 0^3) - 2(2^2 - 0^2) + 2(2 - 0)$$

$$= \frac{5}{3}(8) - 2(4) + 2(2) = \frac{40 - 24 + 12}{3}$$

$$= \frac{28}{3}$$

$$6. \quad \int_{-2}^3 (-x^2 - 3x + 5) dx$$

$$= - \int_{-2}^3 x^2 dx - 3 \int_{-2}^3 x dx + 5 \int_{-2}^3 dx$$

$$= -\frac{1}{3}x^3 \Big|_{-2}^3 - \frac{3}{2}x^2 \Big|_{-2}^3 + 5x \Big|_{-2}^3$$

$$= -\frac{1}{3}[3^3 - (-2)^3] - \frac{3}{2}[3^2 - (-2)^2] + 5[3 - (-2)]$$

$$= -\frac{1}{3}(27 + 8) - \frac{3}{2}(9 - 4) + 5(5)$$

$$= -\frac{35}{3} - \frac{15}{2} + 25 = \frac{35}{6}$$

$$7. \quad \int_0^2 3\sqrt{4u+1} du$$

Let $4u + 1 = x$, so that $4 du = dx$.

When $u = 0$, $x = 4(0) + 1 = 1$.

When $u = 2$, $x = 4(2) + 1 = 9$.

$$\int_0^2 3\sqrt{4u+1} du$$

$$= \frac{3}{4} \int_0^2 \sqrt{4u+1} (4 du)$$

$$= \frac{3}{4} \int_1^9 x^{1/2} dx$$

$$= \frac{3}{4} \cdot \frac{x^{3/2}}{3/2} \Big|_1^9$$

$$= \frac{3}{4} \cdot \frac{2}{3} (9^{3/2} - 1^{3/2})$$

$$= \frac{1}{2} (27 - 1) = \frac{26}{2} = 13$$

$$8. \quad \int_3^9 \sqrt{2r-2} dr = \int_3^9 (2r-2)^{1/2} dr$$

Let $u = 2r - 2$, so that $du = 2 dr$.

If $r = 9$, $u = 2 \cdot 9 - 2 = 16$.

If $r = 3$, $u = 2 \cdot 3 - 2 = 4$.

$$\int_3^9 (2r-2)^{1/2} dr = \frac{1}{2} \int_4^{16} u^{1/2} du$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} \Big|_4^{16}$$

$$= \frac{1}{3} \cdot u^{3/2} \Big|_4^{16}$$

$$= \frac{1}{3} (16^{3/2} - 4^{3/2})$$

$$= \frac{1}{3} (64 - 8) = \frac{56}{3}$$

$$9. \quad \int_0^4 2(t^{1/2} - t) dt = 2 \int_0^4 t^{1/2} dt - 2 \int_0^4 t dt$$

$$= 2 \cdot \frac{t^{3/2}}{3/2} \Big|_0^4 - 2 \cdot \frac{t^2}{2} \Big|_0^4$$

$$= \frac{4}{3} (4^{3/2} - 0^{3/2}) - (4^2 - 0^2)$$

$$= \frac{32}{3} - 16 = -\frac{16}{3}$$

$$\begin{aligned}
 10. \quad & \int_0^4 -(3x^{3/2} + x^{1/2}) dx \\
 &= -3 \int_0^4 x^{3/2} dx - \int_0^4 x^{1/2} dx \\
 &= -3 \left. \frac{x^{5/2}}{\frac{5}{2}} \right|_0^4 - \left. \frac{x^{3/2}}{\frac{3}{2}} \right|_0^4 \\
 &= -\frac{6}{5}(32) - \frac{2}{3}(8) \\
 &= -\frac{192}{5} - \frac{16}{3} = -\frac{656}{15}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & \int_1^4 (5y\sqrt{y} + 3\sqrt{y}) dy \\
 &= 5 \int_1^4 y^{3/2} dy + 3 \int_1^4 y^{1/2} dy \\
 &= 5 \left(\frac{y^{5/2}}{\frac{5}{2}} \right) \Big|_1^4 + 3 \left(\frac{y^{3/2}}{\frac{3}{2}} \right) \Big|_1^4 \\
 &= 2y^{5/2} \Big|_1^4 + 2y^{3/2} \Big|_1^4 \\
 &= 2(4^{5/2} - 1) + 2(4^{3/2} - 1) \\
 &= 2(32 - 1) + 2(8 - 1) \\
 &= 62 + 14 \\
 &= 76
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & \int_4^9 (4\sqrt{r} - 3r\sqrt{r}) dr \\
 &= 4 \int_4^9 r^{1/2} dr - 3 \int_4^9 r^{3/2} dr \\
 &= 4 \left. \frac{r^{3/2}}{\frac{3}{2}} \right|_4^9 - 3 \left. \frac{r^{5/2}}{\frac{5}{2}} \right|_4^9 \\
 &= \frac{8}{3} r^{3/2} \Big|_4^9 - \frac{6}{5} r^{5/2} \Big|_4^9 \\
 &= \frac{8}{3}(27 - 8) - \frac{6}{5}(243 - 32) \\
 &= \frac{8}{3} \cdot 19 - \frac{6}{5}(211) \\
 &= \frac{760}{15} - \frac{3798}{15} = -\frac{3038}{15}
 \end{aligned}$$

$$13. \quad \int_4^6 \frac{2}{(2x-7)^2} dx$$

Let $u = 2x - 7$, so that $du = 2dx$.

When $x = 6$, $u = 2 \cdot 6 - 7 = 5$.

When $x = 4$, $u = 2 \cdot 4 - 7 = 1$.

$$\begin{aligned}
 \int_4^6 \frac{2}{(2x-7)^2} dx &= \int_1^5 u^{-2} du \\
 &= \left. \frac{u^{-1}}{-1} \right|_1^5 \\
 &= -u^{-1} \Big|_1^5 \\
 &= -\left(\frac{1}{5} - 1\right) \\
 &= -\left(-\frac{4}{5}\right) \\
 &= \frac{4}{5}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \int_1^4 \frac{-3}{(2p+1)^2} dp \\
 &= -3 \int_1^4 (2p+1)^{-2} dp
 \end{aligned}$$

Let $u = 2p + 1$, so that $du = 2dp$.

If $p = 4$, $u = 2 \cdot 4 + 1 = 9$.

If $p = 1$, $u = 2 \cdot 1 + 1 = 3$.

$$\begin{aligned}
 & -3 \int_1^4 (2p+1)^{-2} dp \\
 &= -\frac{3}{2} \int_3^9 u^{-2} du \\
 &= -\frac{3}{2} \cdot \left. \frac{u^{-1}}{-1} \right|_3^9 \\
 &= \left. \frac{3}{2u} \right|_3^9 \\
 &= \frac{3}{18} - \frac{3}{6} = -\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
15. \quad & \int_1^5 (6n^{-2} - n^{-3}) dn \\
&= 6 \int_1^5 n^{-2} dn - \int_1^5 n^{-3} dn \\
&= 6 \cdot \left. \frac{n^{-1}}{-1} \right|_1^5 - \left. \frac{n^{-2}}{-2} \right|_1^5 \\
&= \left. \frac{-6}{n} \right|_1^5 + \left. \frac{1}{2n^2} \right|_1^5 \\
&= \frac{-6}{5} - \left(\frac{-6}{1} \right) + \left[\frac{1}{2(25)} - \frac{1}{2(1)} \right] \\
&= \frac{-6}{5} - \frac{6}{1} + \frac{1}{50} - \frac{1}{2} \\
&= \frac{108}{25}
\end{aligned}$$

$$\begin{aligned}
16. \quad & \int_2^3 (3x^{-3} - 5x^{-4}) dx \\
&= 3 \int_2^3 x^{-3} dx - 5 \int_2^3 x^{-4} dx \\
&= 3 \cdot \left. \frac{x^{-2}}{-2} \right|_2^3 - 5 \left. \frac{x^{-3}}{-3} \right|_2^3 \\
&= -\frac{3}{2} \cdot \left. \frac{1}{x^2} \right|_2^3 + \frac{5}{3} \cdot \left. \frac{1}{x^3} \right|_2^3 \\
&= -\frac{3}{2} \left(\frac{1}{9} - \frac{1}{4} \right) + \frac{5}{3} \left(\frac{1}{27} - \frac{1}{8} \right) \\
&= -\frac{1}{6} + \frac{3}{8} + \frac{5}{81} - \frac{5}{24} \\
&= \frac{5}{81}
\end{aligned}$$

$$\begin{aligned}
17. \quad & \int_{-3}^{-2} \left(2e^{-0.1y} + \frac{3}{y} \right) dy \\
&= 2 \int_{-3}^{-2} e^{-0.1y} dy + \int_{-3}^{-2} \frac{3}{y} dy \\
&= 2 \cdot \left. \frac{e^{-0.1y}}{-0.1} \right|_{-3}^{-2} + 3 \ln |y| \Big|_{-3}^{-2} \\
&= -20e^{-0.1y} \Big|_{-3}^{-2} + 3 \ln |y| \Big|_{-3}^{-2} \\
&= 20e^{0.3} - 20e^{0.2} + 3 \ln 2 - 3 \ln 3 \\
&\approx 1.353
\end{aligned}$$

$$\begin{aligned}
18. \quad & \int_{-2}^{-1} \left(\frac{-2}{t} + 3e^{0.3t} \right) dt \\
&= -2 \int_{-2}^{-1} \frac{1}{t} dt + \frac{3}{0.3} \int_{-2}^{-1} 0.3e^{0.3t} dt \\
&= -2 \ln |t| \Big|_{-2}^{-1} + \frac{3}{0.3} e^{0.3t} \Big|_{-2}^{-1} \\
&= 2 \ln 2 + 10e^{-0.3} - 10e^{-0.6} \\
&\approx 3.306
\end{aligned}$$

$$\begin{aligned}
19. \quad & \int_1^2 \left(e^{4u} - \frac{1}{(u+1)^2} \right) du \\
&= \int_1^2 e^{4u} du - \int_1^2 \frac{1}{(u+1)^2} du \\
&= \left. \frac{e^{4u}}{4} \right|_1^2 - \left. \frac{-1}{u+1} \right|_1^2 \\
&= \frac{e^8}{4} - \frac{e^4}{4} + \frac{1}{2+1} - \frac{1}{1+1} \\
&= \frac{e^8}{4} - \frac{e^4}{4} - \frac{1}{6} \\
&\approx 731.4
\end{aligned}$$

$$\begin{aligned}
20. \quad & \int_{0.5}^1 (p^3 - e^{4p}) dp = \int_{0.5}^1 p^3 dp - \int_{0.5}^1 e^{4p} dp \\
&= \left. \frac{p^4}{4} \right|_{0.5}^1 - \left. \frac{e^{4p}}{4} \right|_{0.5}^1 \\
&= \frac{1}{4} - \frac{1}{64} - \left(\frac{e^4}{4} - \frac{e^2}{4} \right) \\
&= \frac{15}{64} - \frac{e^4}{4} + \frac{e^2}{4} \\
&\approx -11.57
\end{aligned}$$

$$\begin{aligned}
21. \quad & \int_{-1}^0 y(2y^2 - 3)^5 dy \\
&\text{Let } u = 2y^3 - 3, \text{ so that} \\
&\quad du = 4y dy \text{ and } \frac{1}{4} du = y dy. \\
&\text{When } y = -1, u = 2(-1)^2 - 3 = -1. \\
&\text{When } y = 0, u = 2(0)^2 - 3 = -3.
\end{aligned}$$

$$\begin{aligned}
 \frac{1}{4} \int_{-1}^{-3} u^5 du &= \frac{1}{4} \cdot \frac{u^6}{6} \Big|_{-1}^{-3} \\
 &= \frac{1}{24} u^6 \Big|_{-1}^{-3} \\
 &= \frac{1}{24} (-3)^6 - \frac{1}{24} (-1)^6 \\
 &= \frac{729}{24} - \frac{1}{24} \\
 &= \frac{728}{24} = \frac{91}{3}
 \end{aligned}$$

$$22. \int_0^3 m^2(4m^3 + 2)^3 dm$$

Let $u = 4m^3 + 2$, so that

$$du = 12m^2 dm \text{ and } \frac{1}{12} du = m^2 dm.$$

Also, when $m = 3$,

$$u = 4(3^3) + 2 = 110,$$

and when $m = 0$,

$$u = 4(0^3) + 2 = 2.$$

$$\begin{aligned}
 \frac{1}{12} \int_2^{110} u^3 du &= \frac{1}{12} \cdot \frac{u^4}{4} \Big|_2^{110} = \frac{1}{48} u^4 \Big|_2^{110} \\
 &= \frac{146,410,000}{48} - \frac{16}{48} \\
 &= \frac{146,409,984}{48} \\
 &= \frac{9,150,624}{3} \\
 &\approx 3,050,208
 \end{aligned}$$

$$23. \int_1^{64} \frac{\sqrt{z} - 2}{\sqrt[3]{z}} dz$$

$$\begin{aligned}
 &= \int_1^{64} \left(\frac{z^{1/2}}{z^{1/3}} - 2z^{-1/3} \right) dz \\
 &= \int_1^{64} z^{1/6} dz - 2 \int_1^{64} z^{-1/3} dz \\
 &= \frac{z^{7/6}}{7/6} \Big|_1^{64} - 2 \frac{z^{2/3}}{2/3} \Big|_1^{64} \\
 &= \frac{6z^{7/6}}{7} \Big|_1^{64} - 3z^{2/3} \Big|_1^{64}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{6(64)^{7/6}}{7} - \frac{6(1)^{7/6}}{7} \\
 &\quad - 3(64^{2/3} - 1^{2/3}) \\
 &= \frac{6(128)}{7} - \frac{6}{7} - 3(16 - 1) \\
 &= \frac{768 - 6 - 315}{7} = \frac{447}{7} \approx 63.86
 \end{aligned}$$

$$24. \int_1^8 \frac{3 - y^{1/3}}{y^{2/3}} dy$$

$$\begin{aligned}
 &= \int_1^8 (3y^{-2/3} - y^{-1/3}) dy \\
 &= \int_1^8 3y^{-2/3} dy - \int_1^8 y^{-1/3} dy \\
 &= \frac{3y^{1/3}}{\frac{1}{3}} \Big|_1^8 - \frac{y^{2/3}}{\frac{2}{3}} \Big|_1^8 \\
 &= 9y^{1/3} \Big|_1^8 - \frac{3y^{2/3}}{2} \Big|_1^8 \\
 &= 9(2 - 1) - \frac{3}{2}(4 - 1) = 9 - \frac{9}{2} = \frac{9}{2}
 \end{aligned}$$

$$25. \int_1^2 \frac{\ln x}{x} dx$$

Let $u = \ln x$, so that

$$du = \frac{1}{x} dx.$$

When $x = 1$, $u = \ln 1 = 0$.

When $x = 2$, $u = \ln 2$.

$$\begin{aligned}
 \int_0^{\ln 2} u du &= \frac{u^2}{2} \Big|_0^{\ln 2} \\
 &= \frac{(\ln 2)^2}{2} - 0 \\
 &= \frac{(\ln 2)^2}{2} \\
 &\approx 0.2402
 \end{aligned}$$

$$26. \int_1^3 \frac{\sqrt{\ln x}}{x} dx$$

Let $u = \ln x$, so that

$$du = \frac{1}{x} dx.$$

When $x = 3$, $u = \ln 3$, and

when $x = 1$, $u = \ln 1 = 0$.

$$\begin{aligned}
 \int_0^{\ln 3} \sqrt{u} \, du &= \int_0^{\ln 3} u^{1/2} \, du \\
 &= \left. \frac{u^{3/2}}{\frac{3}{2}} \right|_0^{\ln 3} \\
 &= \frac{2}{3} u^{3/2} \Big|_0^{\ln 3} \\
 &= \frac{2}{3} (\ln 3)^{3/2} - \frac{2}{3} (0)^{3/2} \\
 &= \frac{2}{3} (\ln 3)^{3/2} \\
 &\approx 0.7677
 \end{aligned}$$

$$27. \int_0^8 x^{1/3} \sqrt{x^{4/3} + 9} \, dx$$

Let $u = x^{4/3} + 9$, so that

$$du = \frac{4}{3} x^{1/3} dx \text{ and } \frac{3}{4} du = x^{1/3} dx.$$

When $x = 0$, $u = 0^{4/3} + 9 = 9$.

When $x = 8$, $u = 8^{4/3} + 9 = 25$.

$$\begin{aligned}
 \frac{3}{4} \int_9^{25} \sqrt{u} \, du &= \frac{3}{4} \int_9^{25} u^{1/2} \, du \\
 &= \frac{3}{4} \cdot \left. \frac{u^{3/2}}{\frac{3}{2}} \right|_9^{25} \\
 &= \frac{1}{2} u^{3/2} \Big|_9^{25} \\
 &= \frac{1}{2} (25)^{3/2} - \frac{1}{2} (9)^{3/2} \\
 &= \frac{125}{2} - \frac{27}{2} = 49
 \end{aligned}$$

$$28. \int_1^2 \frac{3}{x(1 + \ln x)} \, dx$$

Let $u = 1 + \ln x$, so that

$$du = \frac{1}{x} dx.$$

When $x = 2$, $u = 1 + \ln 2$, and

when $x = 1$, $u = 1 + \ln 1 = 1$.

$$\begin{aligned}
 \int_1^{1+\ln 2} \frac{3}{u} \, du &= 3 \ln |u| \Big|_1^{1+\ln 2} \\
 &= 3 \ln (1 + \ln 2) - 3 \ln 1 \\
 &= 3 \ln (1 + \ln 2) \approx 1.580
 \end{aligned}$$

$$29. \int_0^1 \frac{e^{2t}}{(3 + e^{2t})^2} \, dt$$

Let $u = 3 + e^{2t}$, so that $du = 2e^{2t} \, dt$.

When $x = 1$, $u = 3 + e^{2 \cdot 1} = 3 + e^2$.

When $x = 0$, $u = 3 + e^{2 \cdot 0} = 4$.

$$\begin{aligned}
 \int_0^1 \frac{e^{2t}}{(3 + e^{2t})^2} \, dt &= \frac{1}{2} \int_4^{3+e^2} u^{-2} \, du \\
 &= \frac{1}{2} \cdot \left. \frac{u^{-1}}{-1} \right|_4^{3+e^2} = \left. \frac{-1}{2u} \right|_4^{3+e^2} \\
 &= \frac{1}{8} - \frac{1}{2(3 + e^2)} \\
 &\approx 0.07687
 \end{aligned}$$

$$30. \int_0^1 \frac{e^{2z}}{\sqrt{1 + e^{2z}}} \, dz$$

Let $u = 1 + e^{2z}$, so that

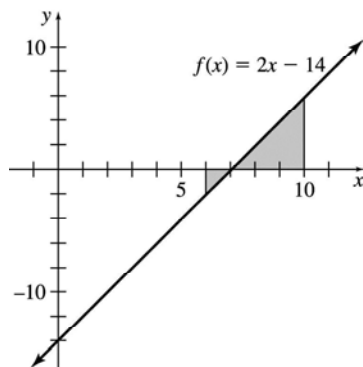
$$du = 2e^{2z} \, dz \text{ and } \frac{1}{2} du = e^{2z} \, dz.$$

When $z = 1$, $u = 1 + e^2$, and

when $z = 0$, $u = 1 + e^0 = 2$.

$$\begin{aligned}
 \frac{1}{2} \int_2^{1+e^2} \frac{1}{\sqrt{u}} \, du &= \frac{1}{2} \int_2^{1+e^2} u^{-1/2} \, du \\
 &= \frac{1}{2} \cdot \left. \frac{u^{1/2}}{\frac{1}{2}} \right|_2^{1+e^2} \\
 &= u^{1/2} \Big|_2^{1+e^2} \\
 &= \sqrt{1 + e^2} - \sqrt{2} \\
 &\approx 1.482
 \end{aligned}$$

31. $f(x) = 2x - 14; [6, 10]$



The graph crosses the x -axis at

$$0 = 2x - 14$$

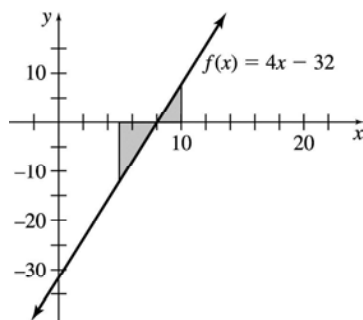
$$2x = 14$$

$$x = 7.$$

This location is in the interval. The area of the region is

$$\begin{aligned} & \left| \int_6^7 (2x - 14) dx \right| + \int_7^{10} (2x - 14) dx \\ &= |(x^2 - 14x)|_6^7 + (x^2 - 14x)|_7^{10} \\ &= |(7^2 - 98) - (6^2 - 84)| \\ &\quad + (10^2 - 140) - (7^2 - 98) \\ &= |-1| + (-40) - (-49) \\ &= 10. \end{aligned}$$

32. $f(x) = 4x - 32; [5, 10]$



The graph crosses the x -axis at

$$0 = 4x - 32$$

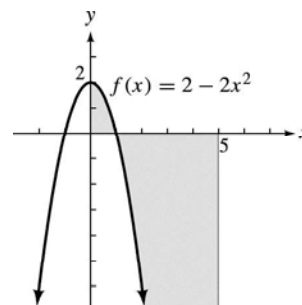
$$4x = 32$$

$$x = 8.$$

This location is in the interval. The area of the region is

$$\begin{aligned} & \left| \int_5^8 (4x - 32) dx \right| + \int_8^{10} (4x - 32) dx \\ &= |(2x^2 - 32x)|_5^8 + (2x^2 - 32x)|_8^{10} \\ &= |(128 - 256) - (50 - 160)| \\ &\quad + (200 - 320) - (128 - 256) \\ &= |-18| + (-120) + 128 \\ &= 26. \end{aligned}$$

33. $f(x) = 2 - 2x^2; [0, 5]$



Find the points where the graph crosses the x -axis by solving $2 - 2x^2 = 0$.

$$2 - 2x^2 = 0$$

$$2x^2 = 2$$

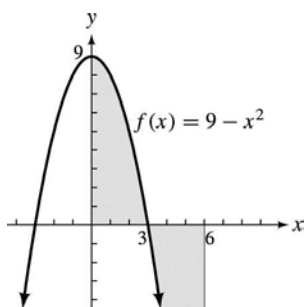
$$x^2 = 1$$

$$x = \pm 1.$$

The only solution in the interval $[0, 5]$ is 1. The total area is

$$\begin{aligned} & \int_0^1 (2 - 2x^2) dx + \left| \int_1^5 (2 - 2x^2) dx \right| \\ &= \left(2x - \frac{2x^3}{3} \right) \Big|_0^1 + \left| \left(2x - \frac{2x^3}{3} \right) \Big|_1^5 \right| \\ &= 2 - \frac{2}{3} + \left| 10 - \frac{2(5^3)}{3} - 2 + \frac{2}{3} \right| \\ &= \frac{4}{3} + \left| \frac{-224}{3} \right| \\ &= \frac{228}{3} \\ &= 76. \end{aligned}$$

34. $f(x) = 9 - x^2$; $[0, 6]$



The graph crosses the x -axis at

$$0 = 9 - x^2$$

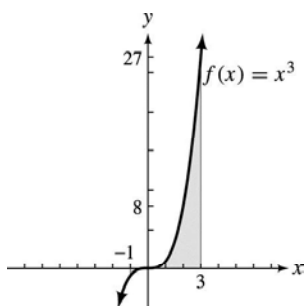
$$x^2 = 9$$

$$x = \pm 3.$$

In the interval, the graph crosses at $x = 3$. The area of the region is

$$\begin{aligned} & \int_0^3 (9 - x^2) dx + \left| \int_3^6 (9 - x^2) dx \right| \\ &= \left(9x - \frac{x^3}{3} \right) \Big|_0^3 + \left| \left(9x - \frac{x^3}{3} \right) \Big|_3^6 \right| \\ &= (27 - 9) + |(54 - 72) - (27 - 9)| \\ &= 18 + |-36| \\ &= 18 + 36 \\ &= 54. \end{aligned}$$

35. $f(x) = x^3$; $[-1, 3]$



The solution

$$x^3 = 0$$

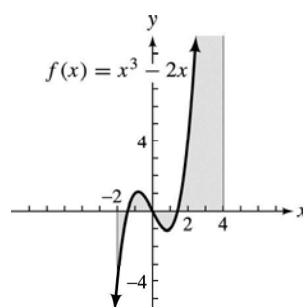
$$x = 0$$

indicates that the graph crosses the x -axis at 0 in the given interval $[-1, 3]$.

The total area is

$$\begin{aligned} & \left| \int_{-1}^0 x^3 dx \right| + \int_0^3 x^3 dx \\ &= \left| \frac{x^4}{4} \Big|_{-1}^0 \right| + \left| \frac{x^4}{4} \Big|_0^3 \right| \\ &= \left| \left(0 - \frac{1}{4} \right) \right| + \left(\frac{3^4}{4} - 0 \right) \\ &= \frac{1}{4} + \frac{81}{4} = \frac{82}{4} \\ &= \frac{41}{2}. \end{aligned}$$

36. $f(x) = x^3 - 2x$; $[-2, 4]$



The graph crosses the x -axis at

$$0 = x^3 - 2x$$

$$= x(x^2 - 2)$$

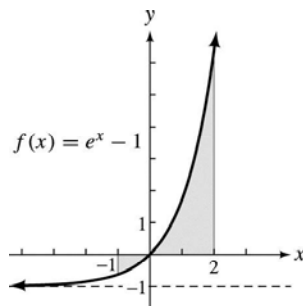
$$x = 0, x = \sqrt{2}, \text{ and } x = -\sqrt{2}.$$

These location are all in the interval.

The area of the region is

$$\begin{aligned} & \left| \int_{-2}^{-\sqrt{2}} (x^3 - 2x) dx \right| + \left| \int_{-\sqrt{2}}^0 (x^3 - 2x) dx \right| \\ &+ \left| \int_0^{\sqrt{2}} (x^3 - 2x) dx \right| + \left| \int_{\sqrt{2}}^4 (x^3 - 2x) dx \right| \\ &= \left| \left(\frac{x^4}{4} - x^2 \right) \Big|_{-2}^{-\sqrt{2}} \right| + \left| \left(\frac{x^4}{4} - x^2 \right) \Big|_{-\sqrt{2}}^0 \right| \\ &+ \left| \left(\frac{x^4}{4} - x^2 \right) \Big|_0^{\sqrt{2}} \right| + \left| \left(\frac{x^4}{4} - x^2 \right) \Big|_{\sqrt{2}}^4 \right| \\ &= |(1 - 2) - (4 - 4)| + |0 - (1 - 2)| \\ &+ |(1 - 2) - 0 + (64 - 16) - 1 - 2| \\ &= |-1| + |1| + |-1| + |49| \\ &= 1 + 1 + 1 + 49 \\ &= 52. \end{aligned}$$

37. $f(x) = e^x - 1; [-1, 2]$



Solve

$$e^x - 1 = 0.$$

$$e^x = 1$$

$$x \ln e = \ln 1$$

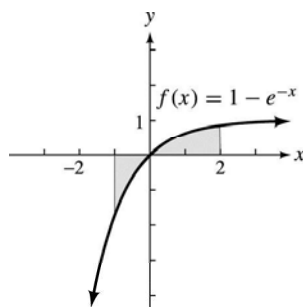
$$x = 0$$

The graph crosses the x -axis at 0 in the given interval $[-1, 2]$.

The total area is

$$\begin{aligned} & \left| \int_{-1}^0 (e^x - 1) dx \right| + \int_0^2 (e^x - 1) dx \\ &= \left| (e^x - x) \Big|_{-1}^0 + (e^x - x) \Big|_0^2 \right| \\ &= |(1 - 0) - (e^{-1} + 1)| + (e^2 - 2) - (1 - 0) \\ &= |1 - e^{-1} - 1| + e^2 - 2 - 1 \\ &= \frac{1}{e} + e^2 - 3 \\ &\approx 4.757. \end{aligned}$$

38. $f(x) = 1 - e^{-x}; [-1, 2]$



The graph crosses the x -axis at

$$0 = 1 - e^{-x}$$

$$e^{-x} = 1$$

$$-x \ln e = \ln 1$$

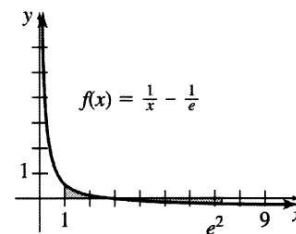
$$-x = 0$$

$$x = 0.$$

The area of the region is

$$\begin{aligned} & \left| \int_{-1}^0 (1 - e^{-x}) dx \right| + \int_0^2 (1 - e^{-x}) dx \\ &= \left| (x + e^{-x}) \Big|_{-1}^0 + (x + e^{-x}) \Big|_0^2 \right| \\ &= |(1) - (-1 + e^1)| + (2 + e^{-2}) - (e^0) \\ &= |2 - e| + 2 + e^{-2} - 1 \\ &= e - 2 + 1 + e^{-2} \\ &= e - 1 + e^{-2} \\ &\approx 1.854. \end{aligned}$$

39. $f(x) = \frac{1}{x} - \frac{1}{e}; [1, e^2]$



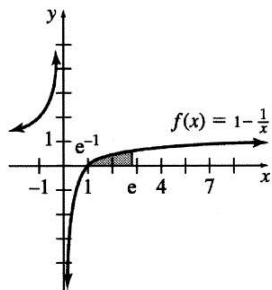
The graph crosses the x -axis at

$$\begin{aligned} 0 &= \frac{1}{x} - \frac{1}{e} \\ \frac{1}{x} &= \frac{1}{e} \\ x &= e. \end{aligned}$$

This location is in the interval. The area of the region is

$$\begin{aligned} & \int_1^e \left(\frac{1}{x} - \frac{1}{e} \right) dx + \left| \int_e^{e^2} \left(\frac{1}{x} - \frac{1}{e} \right) dx \right| \\ &= \left| \ln|x| - \frac{x}{e} \Big|_1^e + \left| \left(\ln|x| - \frac{x}{e} \right) \Big|_e^{e^2} \right| \\ &= 0 - \left(-\frac{1}{e} \right) + |(2 - e) - 0| \\ &= \frac{1}{e} + |2 - e| \\ &= e - 2 + \frac{1}{e}. \end{aligned}$$

40. $f(x) = 1 - \frac{1}{x}; [e^{-1}, e]$



The graph crosses the x -axis at

$$\begin{aligned} 0 &= 1 - \frac{1}{x} \\ \frac{1}{x} &= 1 \\ x &= 1. \end{aligned}$$

This location is in the interval. The area of the region is

$$\begin{aligned} &\left| \int_{e^{-1}}^1 \left(1 - \frac{1}{x}\right) dx \right| + \left| \int_1^e \left(1 - \frac{1}{x}\right) dx \right| \\ &= \left| (x - \ln|x|) \Big|_{e^{-1}}^1 \right| + (x - \ln|x|) \Big|_1^e \\ &= |(1 - 0) - [e^{-1} - (-1)]| + (e - 1) - (1 - 0) \\ &= \left| 1 - \left(\frac{1}{e} + 1\right) \right| + (e - 1) - 1 \\ &= e - 2 + \frac{1}{e}. \end{aligned}$$

41. $y = 4 - x^2; [0, 3]$

From the graph, we see that the total area is

$$\begin{aligned} &\int_0^2 (4 - x^2) dx + \left| \int_2^3 (4 - x^2) dx \right| \\ &= \left(4x - \frac{x^3}{3} \right) \Big|_0^2 + \left| \left(4x - \frac{x^3}{3} \right) \Big|_2^3 \right| \\ &= \left[\left(8 - \frac{8}{3} \right) - 0 \right] \\ &\quad + \left| \left[(12 - 9) - \left(8 - \frac{8}{3} \right) \right] \right| \\ &= \frac{16}{3} + \left| 3 - \frac{16}{3} \right| \\ &= \frac{16}{3} + \frac{7}{3} \\ &= \frac{23}{3} \end{aligned}$$

42. $f(x) = x^2 - 2x; [-1, 2]$

From the graph, we see that the total area is

$$\begin{aligned} &\int_{-1}^0 (x^2 - 2x) dx + \left| \int_0^2 (x^2 - 2x) dx \right| \\ &= \left(\frac{x^3}{3} - x^2 \right) \Big|_{-1}^0 + \left| \left(\frac{x^3}{3} - x^2 \right) \Big|_0^2 \right| \\ &= -\left(-\frac{1}{3} - 1 \right) + \left| \frac{8}{3} - 4 \right| \\ &= \frac{4}{3} + \left| -\frac{4}{3} \right| = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}. \end{aligned}$$

43. $y = e^x - e; [0, 2]$

From the graph, we see that total area is

$$\begin{aligned} &\left| \int_0^1 (e^x - e) dx \right| + \int_1^2 (e^x - e) dx \\ &= \left| (e^x - xe) \Big|_0^1 \right| + (e^x - xe) \Big|_1^2 \\ &= |(e^1 - e) - (e^0 + 0)| + (e^2 - 2e) - (e^1 - e) \\ &= |-1| + e^2 - 2e \\ &= 1 + e^2 - 2e \approx 2.952. \end{aligned}$$

44. $y = \frac{\ln x}{x}; \left[\frac{1}{e}, e \right]$

From the graph, we can see that the total area is

$$\begin{aligned} &\left| \int_{1/e}^1 \left(\frac{\ln x}{x} \right) dx \right| + \int_1^e \left(\frac{\ln x}{x} \right) dx \\ &= \left| \frac{(\ln x)^2}{2} \Big|_{1/e}^1 \right| + \frac{(\ln x)^2}{2} \Big|_1^e \\ &= \left| 0 - \frac{1}{2} \right| + \frac{1}{2} - 0 \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

45. $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$

46. The equation for Exercise 45 is

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

- (a) If
- b
- is replaced by
- d
- , we get

$$\int_a^c f(x) dx = \int_a^d f(x) dx + \int_d^c f(x) dx.$$

This is a correct statement.

- (b) If
- b
- is replaced by
- g
- , we get

$$\int_a^c f(x) dx = \int_a^g f(x) dx + \int_g^c f(x) dx.$$

This is a correct statement.

$$\begin{aligned} 47. \quad \int_0^{16} f(x) dx &= \int_0^2 f(x) dx + \int_2^5 f(x) dx \\ &\quad + \int_5^8 f(x) dx + \int_8^{16} f(x) dx \\ &= \frac{1}{2} \cdot 2(1+3) + \frac{\pi(3^2)}{4} \\ &\quad - \frac{\pi(3^2)}{4} - \frac{1}{2}(3)(8) \\ &= 4 + \frac{9}{4}\pi - \frac{9}{4}\pi - 12 = -8 \end{aligned}$$

$$48. \quad \text{Prove: } \int_a^b k f(x) dx = k \int_a^b f(x) dx.$$

Assume $F(x)$ is an antiderivative of $f(x)$.

Then $kF(x)$ is an antiderivative of $kf(x)$.

$$\begin{aligned} \int_a^b k f(x) dx &= kF(x) \Big|_a^b \\ &= kF(b) - kF(a) \\ &= k[F(b) - F(a)] \\ &= k \int_a^b f(x) dx \end{aligned}$$

$$\begin{aligned} 49. \quad \text{Prove: } \int_a^b f(x) dx \\ &= \int_a^c f(x) dx + \int_c^b f(x) dx. \end{aligned}$$

Let $F(x)$ be an antiderivative of $f(x)$.

$$\begin{aligned} \int_a^c f(x) dx + \int_c^b f(x) dx \\ &= F(x) \Big|_a^c + F(x) \Big|_c^b \\ &= [F(c) - F(a)] + [F(b) - F(c)] \\ &= F(c) - F(a) + F(b) - F(c) \\ &= F(b) - F(a) \\ &= \int_a^b f(x) dx \end{aligned}$$

$$50. \quad \text{Prove: } \int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Assume $F(x)$ is an antiderivative of $f(x)$.

$$\begin{aligned} \int_a^b f(x) dx &= [F(x)]_a^b \\ &= F(b) - F(a) \\ &= (-1)[F(a) - F(b)] \\ &= -1 \int_b^a f(x) dx \\ &= - \int_b^a f(x) dx \end{aligned}$$

$$\begin{aligned} 51. \quad \int_{-1}^4 f(x) dx \\ &= \int_{-1}^0 (2x+3) dx + \int_0^4 \left(-\frac{x}{4} - 3\right) dx \\ &= (x^2 + 3x) \Big|_{-1}^0 + \left(-\frac{x^2}{8} - 3x\right) \Big|_0^4 \\ &= -(1-3) + (-2-12) \\ &= 2-14 \\ &= -12 \end{aligned}$$

$$52. \quad \int_0^1 e^{x^2} dx = 1.46265; \int_0^2 e^{x^2} dx = 16.45263$$

- (a) Since
- e^{x^2}
- is symmetric about the
- y
- axis,

$$\begin{aligned} \int_{-1}^1 e^{x^2} dx &= 2 \int_0^1 e^{x^2} dx \\ &= 2(1.46265) \\ &= 2.92530. \end{aligned}$$

$$\begin{aligned} (b) \quad \int_1^2 e^{x^2} dx &= \int_0^2 e^{x^2} dx - \int_0^1 e^{x^2} dx \\ &= 16.45263 - 1.46265 \\ &= 14.98998 \end{aligned}$$

53. (a) $g(t) = t^4$ and $c = 1$, use substitution.

$$\begin{aligned} f(x) &= \int_c^x g(t) dt \\ &= \int_1^x t^4 dt \\ &= \left. \frac{t^5}{5} \right|_1^x \\ &= \frac{x^5}{5} - \frac{(1)^5}{5} \\ &= \frac{x^5}{5} - \frac{1}{5} \end{aligned}$$

(b) $f'(x) = \frac{d}{dx}(f(x))$

$$\begin{aligned} &= \frac{d}{dx} \left(\frac{x^5}{5} - \frac{1}{5} \right) \\ &= \frac{1}{5} \cdot \frac{d}{dx}(x^5) - \frac{d}{dx} \left(\frac{1}{5} \right) \\ &= \frac{1}{5} \cdot 5x^4 - 0 = x^4 \end{aligned}$$

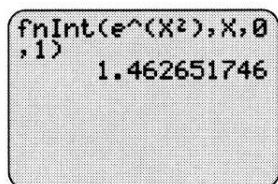
Since $g(t) = t^4$, then $g(x) = x^4$ and we see $f'(x) = g(x)$.

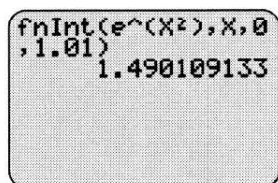
- (c) Let $g(t) = e^{t^2}$ and $c = 0$, then

$$\begin{aligned} f(x) &= \int_0^x e^{t^2} dt. \\ f(1) &= \int_0^1 e^{t^2} dt \text{ and } f(1.01) = \int_0^{1.01} e^{t^2} dt. \end{aligned}$$

Use the fnInt command in the Math menu of your calculator to find $\int_0^1 e^{x^2} dx$ and

$\int_0^{1.01} e^{x^2} dx$. The resulting screens are:





$$f(1) \approx 1.46265$$

$$f(1.01) \approx 1.49011$$

Use $\frac{f(1+h) - f(1)}{h}$ to approximate

$f'(1)$ with $h = 0.01$

$$\begin{aligned} \frac{f(1+h) - f(1)}{h} &= \frac{f(1.01) - f(1)}{0.01} \\ &\approx \frac{1.49011 - 1.46265}{0.01} \\ &= 2.746 \end{aligned}$$

So $f'(1) \approx 2.746$, and $g(1) = e^{1^2} = e \approx 2.718$.

54. (a) $\int_{-5}^5 x(x^2 + 3)^7 dx$

Let $u = x^2 + 3$, so that $du = 2x dx$.

When $x = 5$, $u = 5^2 + 3 = 28$.

When $x = -5$, $u = (-5)^2 + 3 = 28$.

$$\begin{aligned} \int_{-5}^5 x(x^2 + 3)^7 dx &= \frac{1}{2} \int_{28}^{28} u^7 du \\ &= 0 \end{aligned}$$

55. $P'(t) = (3t + 3)(t^2 + 2t + 2)^{1/3}$

(a) $\int_0^3 3(t + 1)(t^2 + 2t + 2)^{1/3} dt$

Let $u = t^2 + 2t + 2$, so that

$du = (2t + 2) dt$ and $\frac{1}{2} du = (t + 1) dt$.

When $t = 0$, $u = 0^2 + 2 \cdot 0 + 2 = 2$.

When $t = 3$, $u = 3^2 + 2 \cdot 3 + 2 = 17$.

$$\begin{aligned} \frac{3}{2} \int_2^{17} u^{1/3} du &= \frac{3}{2} \cdot \frac{u^{4/3}}{\frac{4}{3}} \Big|_2^{17} \\ &= \frac{9}{8} u^{4/3} \Big|_2^{17} \\ &= \frac{9}{8} (17)^{4/3} - \frac{9}{8} (2)^{4/3} \\ &\approx 46.341 \end{aligned}$$

Total profits for the first 3 yr were

$$\frac{9000}{8} (17^{4/3} - 2^{4/3}) \approx \$46,341.$$

$$(b) \int_3^4 3(t+1)(t^2+2t+2)^{1/3} dt$$

Let $u = t^2 + 2t + 2$, so that
 $du = (2t + 2) dt = 2(t + 1) dt$ and
 $\frac{3}{2} du = 3(t + 1) dt$.

When $t = 3$, $u = 3^2 + 2 \cdot 3 + 2 = 17$.

When $t = 4$, $u = 4^2 + 2 \cdot 4 + 2 = 26$.

$$\begin{aligned} \frac{3}{2} \int_{17}^{26} u^{1/3} du &= \frac{9}{8} u^{4/3} \Big|_{17}^{26} \\ &= \frac{9}{8} (26)^{4/3} - \frac{9}{8} (17)^{4/3} \approx 37,477 \end{aligned}$$

Profit in the fourth year was

$$\frac{9000}{8} (26^{4/3} - 17^{4/3}) \approx \$37,477.$$

$$\begin{aligned} (c) \lim_{x \rightarrow \infty} P'(t) &= \lim_{x \rightarrow \infty} (3t + 3)(t^2 + 2t + 2)^{1/3} \\ &= \infty \end{aligned}$$

The annual profit is slowly increasing without bound.

56. $H'(x) = 20 - 2x$ is the rate of change of the number of hours it takes a worker to produce the x th item.

- (a) The total number of hours required to produce the first 5 item is

$$\begin{aligned} \int_0^5 (20 - 2x) dx &= (20x - x^2) \Big|_0^5 \\ &= 100 - 25 = 75. \end{aligned}$$

It would take 75 hr to produce 5 items.

- (b) The total number of hours required to produce the first 10 items is

$$\begin{aligned} \int_0^{10} (20 - 2x) dx &= (20x - x^2) \Big|_0^{10} \\ &= (200 - 100) - (0) \\ &= 100. \end{aligned}$$

It would take 100 hr to produce the first 10 items.

$$57. P'(t) = 140t^{5/2}$$

$$\begin{aligned} \int_0^4 140t^{5/2} dt &= 140 \cdot \frac{t^{7/2}}{\frac{7}{2}} \Big|_0^4 \\ &= 40t^{7/2} \Big|_0^4 \\ &= 5120 \end{aligned}$$

Since 5120 is above the total level of acceptable pollution (4850), the factory cannot operate for 4 years without killing all the fish in the lake.

58. The tanker is leaking oil at a rate in barrels per hour of

$$L'(t) = \frac{80 \ln(t+1)}{t+1}.$$

$$(a) \int_0^{24} \frac{80 \ln(t+1)}{t+1} dt$$

Let $u = \ln(t+1)$, so that $du = \frac{1}{t+1} dt$.

When $t = 24$, $u = \ln 25$.

When $t = 0$, $u = \ln 1 = 0$.

$$\begin{aligned} 80 \int_0^{\ln 25} u du &= 80 \frac{u^2}{2} \Big|_0^{\ln 25} \\ &= 40u^2 \Big|_0^{\ln 25} \\ &= 40(\ln 25)^2 - 40(0)^2 \\ &\approx 414 \end{aligned}$$

About 414 barrels will leak on the first day.

$$(b) \int_{24}^{48} \frac{80 \ln(t+1)}{t+1} dt$$

Let $u = \ln(t+1)$, so that the limits of integration with respect to u are $\ln 25$ and $\ln 49$.

$$\begin{aligned} 80 \int_{\ln 25}^{\ln 49} u du &= 40u^2 \Big|_{\ln 25}^{\ln 49} \\ &= 40(\ln 49)^2 - 40(\ln 25)^2 \\ &\approx 191 \end{aligned}$$

About 191 barrels will leak on the second day.

$$(c) \lim_{t \rightarrow \infty} L'(t) = \lim_{t \rightarrow \infty} \frac{80 \ln(t+1)}{t+1} = 0$$

The number of barrels of oil leaking per day is decreasing to 0.

59. Growth rate is $0.6 + \frac{4}{(t+1)^3}$ ft/yr.

(a) Total growth in the second year is

$$\begin{aligned} & \int_1^2 \left[0.6 + \frac{4}{(t+1)^3} \right] dt \\ &= \left[0.6t + \frac{4}{-2(t+1)^2} \right] \bigg|_1^2 \\ &= \left[0.6(2) - \frac{2}{(2+1)^2} \right] \\ &\quad - \left[0.6(1) - \frac{2}{(1+1)^2} \right] \\ &= \frac{44}{45} - \frac{1}{10} \\ &\approx 0.8778 \text{ ft.} \end{aligned}$$

(b) Total growth in the third year is

$$\begin{aligned} & \int_2^3 \left[0.6 + \frac{4}{(t+1)^3} \right] dt \\ &= \left[0.6t + \frac{4}{-2(t+1)^2} \right] \bigg|_2^3 \\ &= \left[0.6(3) - \frac{2}{(3+1)^2} \right] \\ &\quad - \left[0.6(2) - \frac{2}{(2+1)^2} \right] \\ &= \frac{67}{40} - \frac{44}{45} \\ &\approx 0.6972 \text{ ft.} \end{aligned}$$

60. Total growth after 3.5 days is

$$\begin{aligned} \int_0^{3.5} R'(x) dx &= \int_0^{3.5} 150e^{0.2x} dx \\ &= 150 \cdot \frac{e^{0.2x}}{0.2} \bigg|_0^{3.5} \\ &= 750e^{0.2x} \bigg|_0^{3.5} \\ &= 750e^{0.7} - 750e^0 \\ &\approx 760.3. \end{aligned}$$

$$61. \quad R'(t) = \frac{5}{t+1} + \frac{2}{\sqrt{t+1}}$$

(a) Total reaction from $t = 1$ to $t = 12$ is

$$\begin{aligned} & \int_1^{12} \left(\frac{5}{t+1} + \frac{2}{\sqrt{t+1}} \right) dt \\ &= \left[5 \ln(t+1) + 4\sqrt{t+1} \right] \bigg|_1^{12} \\ &= (5 \ln 13 + 4\sqrt{13}) - (5 \ln 2 + 4\sqrt{2}) \\ &\approx 18.12. \end{aligned}$$

(b) Total reaction from $t = 12$ to $t = 24$ is

$$\begin{aligned} & \int_{12}^{24} \left(\frac{5}{t+1} + \frac{2}{\sqrt{t+1}} \right) dt \\ &= \left[5 \ln(t+1) + 4\sqrt{t+1} \right] \bigg|_{12}^{24} \\ &= (5 \ln 25 + 4\sqrt{25}) - (5 \ln 13 + 4\sqrt{13}) \\ &\approx 8.847. \end{aligned}$$

$$\begin{aligned} 62. \quad F(T) &= \int_0^T f(x) dx \\ &= \int_0^T kb^x dx \\ &= \int_0^T ke^{(\ln b)x} dx \\ &= k \int_0^T e^{(\ln b)x} dx \\ &= \frac{k}{\ln b} \cdot e^{(\ln b)x} \bigg|_0^T \\ &= \frac{k}{\ln b} [e^{(\ln b)T} - 1] \\ &= \frac{k}{\ln b} [b^T - 1] \end{aligned}$$

$$63. \quad (b) \quad \int_0^{60} n(x) dx$$

$$(c) \quad \int_5^{10} \sqrt{5x+1} dx$$

Let $u = 5x + 1$. Then $du = 5 dx$.

When $x = 5$, $u = 26$; when $x = 10$, $u = 51$.

$$\begin{aligned}
 & \frac{1}{5} \int_{26}^{51} u^{1/2} du \\
 &= \frac{1}{5} \cdot \frac{u^{3/2}}{\frac{3}{2}} \bigg|_{26}^{51} \\
 &= \frac{2}{15} u^{3/2} \bigg|_{26}^{51} \\
 &= \frac{2}{15} (51^{3/2} - 26^{3/2}) \\
 &\approx 30.89 \text{ million}
 \end{aligned}$$

64. $w'(t) = (3t + 2)^{1/3}$

$$\begin{aligned}
 w(t) &= \int_0^3 (3t + 2)^{1/3} dt = \frac{1}{3} \cdot \frac{(3t + 2)^{4/3}}{\frac{4}{3}} \bigg|_0^3 \\
 &= \frac{(3t + 2)^{4/3}}{4} \bigg|_0^3 \\
 &= \frac{1}{4} (11^{4/3} - 2^{4/3}) \\
 &\approx 5.486 \text{ mg}
 \end{aligned}$$

65. $v = k(R^2 - r^2)$

$$\begin{aligned}
 \text{(a)} \quad Q(R) &= \int_0^R 2\pi vr \, dr \\
 &= \int_0^R 2\pi k(R^2 - r^2)r \, dr \\
 &= 2\pi k \int_0^R (R^2 r - r^3) \, dr \\
 &= 2\pi k \left(\frac{R^2 r^2}{2} - \frac{r^4}{4} \right) \bigg|_0^R \\
 &= 2\pi k \left(\frac{R^4}{2} - \frac{R^4}{4} \right) \\
 &= 2\pi k \left(\frac{R^4}{4} \right) \\
 &= \frac{\pi k R^4}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad Q(0.4) &= \frac{\pi k (0.4)^4}{2} \\
 &= 0.04k \text{ mm/min}
 \end{aligned}$$

66. $\int_3^9 (0.1762x^2 - 3.986x + 22.68) dx$

$$\begin{aligned}
 &= \left(\frac{0.1762}{3} x^3 - \frac{3.986}{2} x^2 + 22.68x \right) \bigg|_3^9 \\
 &= 85.5036 - 51.6888 = 33.8148
 \end{aligned}$$

The total increase in the length of a ram's horn during the period is about 33.8 cm.

67. $E(t) = 753t^{-0.1321}$

- (a) Since t is the age of the beagle in years, to convert the formula to days, let $T = 365t$, or $t = \frac{T}{365}$.

$$\begin{aligned}
 E(T) &= 753 \left(\frac{T}{365} \right)^{-0.1321} \\
 &\approx 1642T^{-0.1321}
 \end{aligned}$$

Now, replace T with t .

$$E(t) = 1642t^{-0.1321}$$

- (b) The beagle's age in days after one year is 365 days and after 3 years she is 1095 days old.

$$\begin{aligned}
 &\int_{365}^{1095} 1642t^{-0.1321} dt \\
 &= 1642 \frac{1}{0.8679} t^{0.8679} \bigg|_{365}^{1095} \\
 &\approx 1892(1,095^{0.8679} - 365^{0.8679}) \\
 &\approx 505,155
 \end{aligned}$$

The beagle's total energy requirements are about 505,000 kJ/W^{0.67}, where W represents weight.

68. $\int_0^{100} 0.85e^{0.0133x} dx$

Let $u = 0.0133x$, so that $du = 0.0133dx$, or $dx = \frac{1}{0.0133} du$

When $x = 100$, $u = 1.33$.

When $x = 0$, $u = 0$.

$$\begin{aligned}
 \int_0^{100} 0.85e^{0.0133x} dx &= \frac{0.85}{0.0133} e^u \bigg|_0^{1.33} \\
 &= \frac{0.85}{0.0133} (e^{1.33} - e^0) \\
 &\approx 177.736
 \end{aligned}$$

The total mass of the column is about 178 g.

69. (a) $f(x) = 40.2 + 3.50x - 0.897x^2$

$$\begin{aligned} \int_0^9 (40.2 + 3.50x - 0.897x^2) dx \\ = (40.2x + 1.74x^2 - 0.299x^3) \Big|_0^9 \\ \approx 286 \end{aligned}$$

The integral represents the population aged 0 to 90, which is about 286 million.

(b) $\int_{4.5}^{6.5} (40.2 + 3.50x - 0.897x^2) dx$

$$\begin{aligned} = (40.2x + 1.74x^2 - 0.299x^3) \Big|_{4.5}^{6.5} \\ \approx 64 \end{aligned}$$

The number of baby boomers is about 64 million.

70. $\int_{2.5}^5 (0.0353x^3 - 0.541x^2 + 3.78x + 4.29) dx$

$$\begin{aligned} = (0.008825x^4 - 0.18033x^3 + 1.8900x^2 \\ + 4.2900x) \Big|_{2.5}^5 \\ \approx 0.32 \end{aligned}$$

About 32% of families have incomes between \$25,000 and \$50,000.

71. $c'(t) = ke^{rt}$

(a) $c'(t) = 1.2e^{0.04t}$

(b) The amount of oil that the company will sell in the next ten years is given by the integral

$$\int_0^{10} 1.2e^{0.04t} dt.$$

(c) $\int_0^{10} 1.2e^{0.04t} dx = \frac{1.2e^{0.04t}}{0.04} \Big|_0^{10}$

$$\begin{aligned} &= 30e^{0.04t} \Big|_0^{10} \\ &= 30e^{0.4} - 30 \\ &\approx 14.75 \end{aligned}$$

This represents about 14.75 billion barrels of oil.

(d) $\int_0^T 1.2e^{0.04t} dt = 30e^{0.04t} \Big|_0^T$

$$= 30e^{0.04T} - 30$$

Solve

$$20 = 30e^{0.04T} - 30.$$

$$50 = 30e^{0.04T}$$

$$\frac{5}{3} = e^{0.04T}$$

$$\ln \frac{5}{3} = 0.04T \ln e$$

$$T = \frac{\ln \frac{5}{3}}{0.04}$$

$$\approx 12.8$$

The oil will last about 12.8 years.

(e) $\int_0^T 1.2e^{0.02t} dt = 60e^{0.02t} \Big|_0^T$

$$= 60e^{0.02T} - 60$$

Solve

$$20 = 60e^{0.02T} - 60.$$

$$80 = 60e^{0.02T}$$

$$\frac{4}{3} = e^{0.02T}$$

$$\ln \frac{4}{3} = 0.02T \ln e$$

$$T = \frac{\ln \frac{4}{3}}{0.02} \approx 14.4$$

The oil will last about 14.4 years.

72. $C'(t) = 1.2e^{0.04t}$

$$\begin{aligned} C(T) &= \int_0^T 1.2e^{0.04t} \\ &= \frac{1.2}{0.04} e^{0.04t} \Big|_0^T \\ &= 30(e^{0.04T} - e^0) \\ &= 30(e^{0.04T} - 1) \end{aligned}$$

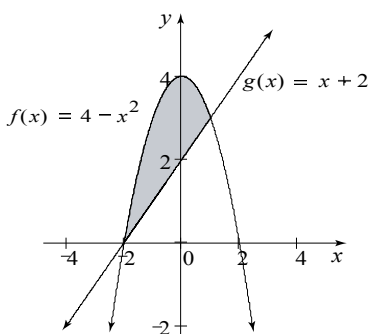
In 5 yr,

$$\begin{aligned} C(5) &= 30(e^{0.04(5)} - 1) \\ &= 30(e^{0.2} - 1) \\ &\approx 6.64 \text{ billion barrels.} \end{aligned}$$

7.5 The Area Between Two Curves

Your Turn 1

Find the area bounded by $f(x) = 4 - x^2$, $g(x) = x + 2$, $x = -2$, and $x = 1$. A sketch such as the one below shows that the two graphs intersect at the points $(-2, 0)$ and $(1, 3)$.



Over the interval $[-2, 1]$, $f(x) \geq g(x)$, so the area will be given by $\int_{-2}^1 [f(x) - g(x)] dx$.

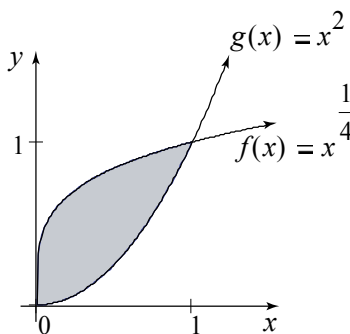
$$\begin{aligned} \int_{-2}^1 [f(x) - g(x)] dx &= \int_{-2}^1 [4 - x^2 - (x + 2)] dx \\ &= \int_{-2}^1 (2 - x - x^2) dx \\ &= \left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_{-2}^1 \\ &= \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - 2 + \frac{8}{3} \right) \\ &= \frac{9}{2} \end{aligned}$$

Your Turn 2

Find the area between the curves $y = x^{1/4}$ and $y = x^2$. First find where these two curves intersect by setting the two righthand sides equal.

$$\begin{aligned} x^{1/4} &= x^2 \\ x &= x^8 \\ x^8 - x &= 0 \\ x(x^7 - 1) &= 0 \\ x &= 0 \text{ or } x^7 - 1 = 0 \\ x &= 0 \text{ or } x = 1. \end{aligned}$$

The corresponding y values are 0 and 1, respectively, so the curves intersect at $(0, 0)$ and $(1, 1)$, as shown in the graph below. Here $f(x) = x^{1/4}$ and $g(x) = x^2$.



Over the interval $[0, 1]$, $f(x) \geq g(x)$, so the area is given by the integral $\int_0^1 [f(x) - g(x)] dx$.

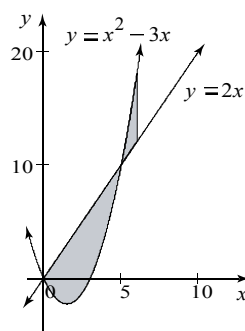
$$\begin{aligned} \int_0^1 [f(x) - g(x)] dx &= \int_0^1 (x^{1/4} - x^2) dx \\ &= \left(\frac{4}{5}x^{5/2} - \frac{1}{3}x^3 \right) \Big|_0^1 \\ &= \left(\frac{4}{5} - \frac{1}{3} \right) - (0 - 0) = \frac{7}{15} \end{aligned}$$

Your Turn 3

Find the area enclosed by $y = x^2 - 3x$ and $y = 2x$ on $[0, 6]$. First find where the two graphs intersect.

$$\begin{aligned} x^2 - 3x &= 2x \\ x^2 - 5x &= 0 \\ x(x - 5) &= 0 \\ x &= 0 \text{ or } x = 5. \end{aligned}$$

The intersection points are $(0, 0)$ and $(5, 10)$, so we will need to use two integrals. On $(0, 5)$, $2x$ is the larger function and on $(5, 6)$, $x^2 - 3x$ is the larger function, as illustrated in the following graph.



$$\begin{aligned}
 \text{Area} &= \int_0^5 [2x - (x^2 - 3x)] dx + \int_5^6 [(x^2 - 3x) - 2x] dx \\
 &= \int_0^5 (5x - x^2) dx + \int_5^6 (x^2 - 5x) dx \\
 &= \left(\frac{5}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^5 + \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 \right) \Big|_5^6 \\
 &= \left(\frac{125}{2} - \frac{125}{3} - 0 \right) + \left(\frac{216}{3} - \frac{180}{2} - \frac{125}{3} + \frac{125}{2} \right) \\
 &= \frac{71}{3}
 \end{aligned}$$

Your Turn 4

Find the consumers' surplus and the producers' surplus for oat bran when the price in dollars per ton is $D(q) = 600 - e^{q/3}$ when the demand is q tons, and the price in dollars per ton is $S(q) = e^{q/3} - 100$ when the demand is q tons.

First find the equilibrium quantity.

$$\begin{aligned}
 e^{q/3} - 100 &= 600 - e^{q/3} \\
 2e^{q/3} &= 700 \\
 e^{q/3} &= 350 \\
 \frac{q}{3} &= \ln 350 \\
 q &= 3 \ln 350 \\
 q &\approx 17.57380
 \end{aligned}$$

The equilibrium price is

$$\begin{aligned}
 S(17.57380) &= e^{17.57380/3} - 100 \\
 &\approx 250.00
 \end{aligned}$$

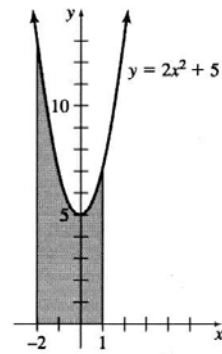
The consumers' surplus is given by the following integral:

$$\begin{aligned}
 &\int_0^{17.57380} (600 - e^{q/3} - 250) dq \\
 &= \left(350q - 3e^{q/3} \right) \Big|_0^{17.57380} \\
 &= (350(17.57380) - 3e^{17.57380/3}) - (0 - 3) \\
 &\approx 5103.83
 \end{aligned}$$

The consumers' surplus is \$5103.83. As in Example 5, the producers' surplus has the same value, \$5103.83.

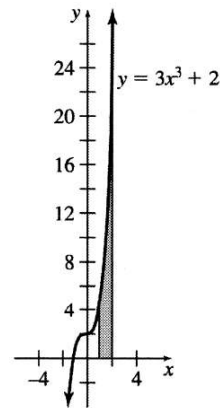
7.5 Exercises

1. $x = -2, x = 1, y = 2x^2 + 5, y = 0$



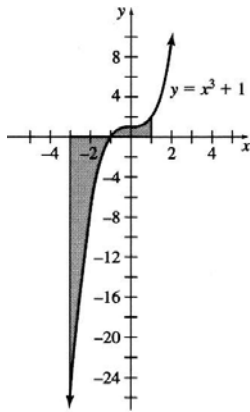
$$\begin{aligned}
 \int_{-2}^1 [(2x^2 + 5) - 0] dx &= \left(\frac{2x^3}{3} + 5x \right) \Big|_{-2}^1 \\
 &= \left(\frac{2}{3} + 5 \right) - \left(-\frac{16}{3} - 10 \right) \\
 &= 21
 \end{aligned}$$

2. $x = 1, x = 2, y = 3x^3 + 2, y = 0$



$$\begin{aligned}
 \int_1^2 [(3x^3 + 2) - 0] dx &= \left(\frac{3x^4}{4} + 2x \right) \Big|_1^2 \\
 &= (12 + 4) - \left(\frac{3}{4} + 2 \right) \\
 &= \frac{53}{4}
 \end{aligned}$$

3. $x = -3, x = 1, y = x^3 + 1, y = 0$



To find the points of intersection of the graphs, substitute for y .

$$\begin{aligned}x^3 + 1 &= 0 \\x^3 &= -1 \\x &= -1\end{aligned}$$

The region is composed of two separate regions because $y = x^3 + 1$ intersects $y = 0$ at $x = -1$.

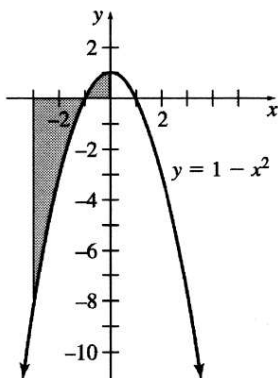
Let $f(x) = x^3 + 1, g(x) = 0$.

In the interval $[-3, -1], g(x) \geq f(x)$.

In the interval $[-1, 1], f(x) \geq g(x)$.

$$\begin{aligned}&\int_{-3}^{-1} [0 - (x^3 + 1)] dx + \int_{-1}^1 [(x^3 + 1) - 0] dx \\&= \left(\frac{-x^4}{4} - x \right) \Big|_{-3}^{-1} + \left(\frac{x^4}{4} + x \right) \Big|_{-1}^1 \\&= \left(-\frac{1}{4} + 1 \right) - \left(-\frac{81}{4} + 3 \right) + \left(\frac{1}{4} + 1 \right) - \left(\frac{1}{4} - 1 \right) \\&= 20\end{aligned}$$

4. $x = -3, x = 0, y = 1 - x^2, y = 0$



To find the points of intersection of the graphs in $[-3, 0]$, substitute for y .

$$\begin{aligned}1 - x^2 &= 0 \\x^2 &= 1 \\x &= -1 \text{ or } x = 1\end{aligned}$$

The region is composed of two separate regions because $y = 1 - x^2$ intersects $y = 0$ at $x = -1$.

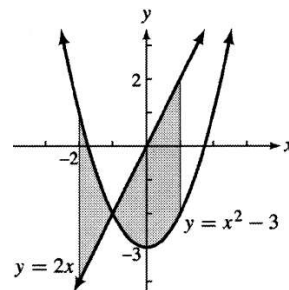
Let $f(x) = 1 - x^2, g(x) = 0$.

In the interval $[-3, -1], g(x) \geq f(x)$.

In the interval $[-1, 0], f(x) \geq g(x)$.

$$\begin{aligned}&\int_{-3}^{-1} [0 - (1 - x^2)] dx + \int_{-1}^0 [(1 - x^2) - 0] dx \\&= \left(-x + \frac{x^3}{3} \right) \Big|_{-3}^{-1} + \left(x - \frac{x^3}{3} \right) \Big|_{-1}^0 \\&= \left(1 - \frac{1}{3} \right) - (3 - 9) + 0 - \left(-1 + \frac{1}{3} \right) \\&= \frac{22}{3}\end{aligned}$$

5. $x = -2, x = 1, y = 2x, y = x^2 - 3$



Find the points of intersection of the graphs of $y = 2x$ and $y = x^2 - 3$ by substituting for y .

$$\begin{aligned}2x &= x^2 - 3 \\0 &= x^2 - 2x - 3 \\0 &= (x - 3)(x + 1)\end{aligned}$$

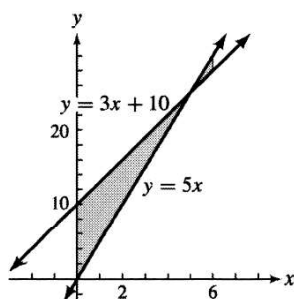
The only intersection in $[-2, 1]$ is at $x = -1$.

In the interval $[-2, -1], (x^2 - 3) \geq 2x$.

In the interval $[-1, 1], 2x \geq (x^2 - 3)$.

$$\begin{aligned}
& \int_{-2}^{-1} [(x^2 - 3) - (2x)] dx + \int_{-1}^1 [(2x) - (x^2 - 3)] dx \\
&= \int_{-2}^{-1} (x^2 - 3 - 2x) dx + \int_{-1}^1 (2x - x^2 + 3) dx \\
&= \left(\frac{x^3}{3} - 3x - x^2 \right) \Big|_{-2}^{-1} + \left(x^2 - \frac{x^3}{3} + 3x \right) \Big|_{-1}^1 \\
&= -\frac{1}{3} + 3 - 1 - \left(-\frac{8}{3} + 6 - 4 \right) + 1 - \frac{1}{3} + 3 \\
&\quad - \left(1 + \frac{1}{3} - 3 \right) \\
&= \frac{5}{3} + 6 = \frac{23}{3}
\end{aligned}$$

6. $x = 0, x = 6, y = 5x, y = 3x + 10$



To find the intersection of $y = 5x$ and $y = 3x + 10$, substitute for y .

$$\begin{aligned}
5x &= 3x + 10 \\
2x &= 10 \\
x &= 5
\end{aligned}$$

If $x = 5, y = 5(5) = 25$.

The region is composed of two separate regions because $y = 5x$ and $y = 3x + 10$ intersect at $x = 5$ (that is, $(5, 25)$).

Let $f(x) = 3x + 10, g(x) = 5x$.

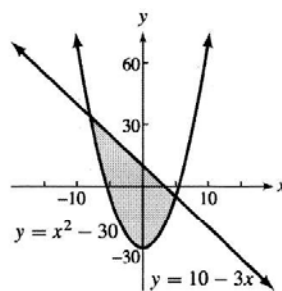
In the interval $[0, 5], f(x) \geq g(x)$.

In the interval $[5, 6], g(x) \geq f(x)$.

$$\begin{aligned}
& \int_0^5 (3x + 10 - 5x) dx \\
& \quad + \int_5^6 [5x - (3x + 10)] dx \\
&= \int_0^5 (-2x + 10) dx + \int_5^6 (2x - 10) dx \Big|_5^6
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{-2x^2}{2} + 10x \right) \Big|_0^5 + \left(\frac{2x^2}{2} - 10x \right) \Big|_5^6 \\
&= (-x^2 + 10x) \Big|_0^5 + (x^2 - 10x) \Big|_5^6 \\
&= -25 + 50 + (36 - 60) - (25 - 50) \\
&= 26
\end{aligned}$$

7. $y = x^2 - 30$
 $y = 10 - 3x$



Find the points of intersection.

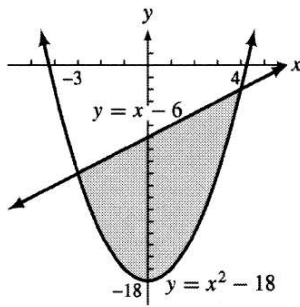
$$\begin{aligned}
x^2 - 30 &= 10 - 3x \\
x^2 + 3x - 40 &= 0 \\
(x + 8)(x - 5) &= 0 \\
x &= -8 \text{ or } x = 5
\end{aligned}$$

Let $f(x) = 10 - 3x$ and $g(x) = x^2 - 30$.

The area between the curves is given by

$$\begin{aligned}
& \int_{-8}^5 [f(x) - g(x)] dx \\
&= \int_{-8}^5 [(10 - 3x) - (x^2 - 30)] dx \\
&= \int_{-8}^5 (-x^2 - 3x + 40) dx \\
&= \left(\frac{-x^3}{3} - \frac{3x^2}{2} + 40x \right) \Big|_{-8}^5 \\
&= \frac{-5^3}{3} - \frac{3(5)^2}{2} + 40(5) \\
&\quad - \left[\frac{-(-8)^3}{3} - \frac{3(-8)^2}{2} + 40(-8) \right] \\
&= \frac{-125}{3} - \frac{75}{2} + 200 - \frac{512}{3} + \frac{192}{2} + 320 \\
&\approx 366.1667.
\end{aligned}$$

8. $y = x^2 - 18, y = x - 6$



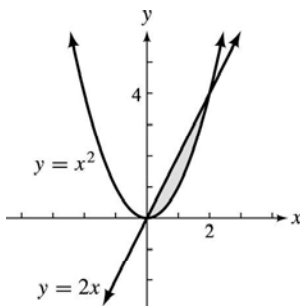
Find the intersection points.

$$\begin{aligned}x^2 - 18 &= x - 6 \\x^2 - x - 12 &= 0 \\(x - 4)(x + 3) &= 0\end{aligned}$$

The curves intersect at $x = -3$ and $x = 4$.

$$\begin{aligned}\int_{-3}^4 [(x - 6) - (x^2 - 18)] dx \\&= \int_{-3}^4 (x - 6 - x^2 + 18) dx \\&= \int_{-3}^4 (-x^2 + x + 12) dx \\&= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 12x \right) \bigg|_{-3}^4 \\&= \left(-\frac{64}{3} + 8 + 48 \right) - \left(9 + \frac{9}{2} - 36 \right) \\&= \left(-\frac{64}{3} + 56 \right) - \left(-27 + \frac{9}{2} \right) \\&= \frac{-64}{3} + 83 - \frac{9}{2} \\&= \frac{343}{6} \\&\approx 57.167\end{aligned}$$

9. $y = x^2, y = 2x$



Find the points of intersection.

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x - 2) = 0$$

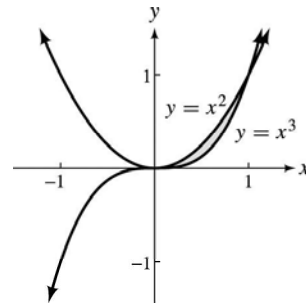
$$x = 0 \text{ or } x = 2$$

Let $f(x) = 2x$ and $g(x) = x^2$.

The area between the curves is given by

$$\begin{aligned}\int_0^2 [f(x) - g(x)] dx &= \int_0^2 (2x - x^2) dx \\&= \left(\frac{2x^2}{2} - \frac{x^3}{3} \right) \bigg|_0^2 \\&= 4 - \frac{8}{3} = \frac{4}{3}.\end{aligned}$$

10. $y = x^2, y = x^3$



Find the intersection points.

$$x^2 = x^3$$

$$x^2 - x^3 = 0$$

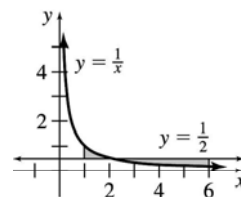
$$x^2(1 - x) = 0$$

The curves intersect at $x = 0$ and $x = 1$.

In the interval $[0, 1]$, $x^2 > x^3$.

$$\begin{aligned}\int_0^1 (x^2 - x^3) dx &= \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_0^1 \\&= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}\end{aligned}$$

11. $x = 1, x = 6, y = \frac{1}{x}, y = \frac{1}{2}$



To find the points of intersection of the graphs, substitute for y .

$$\begin{aligned}\frac{1}{x} &= \frac{1}{2} \\ x &= 2\end{aligned}$$

The region is composed of two separate regions because $y = \frac{1}{x}$ intersects $y = \frac{1}{2}$ at $x = 2$.

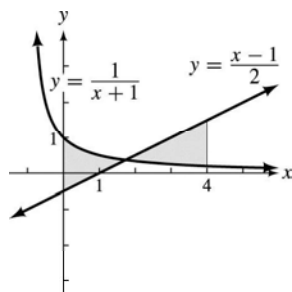
Let $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{2}$.

In the interval $[1, 2]$, $f(x) \geq g(x)$.

In the interval $[2, 6]$, $g(x) \geq f(x)$.

$$\begin{aligned}& \int_1^2 \left(\frac{1}{x} - \frac{1}{2} \right) dx + \int_2^6 \left(\frac{1}{2} - \frac{1}{x} \right) dx \\&= \left(\ln|x| - \frac{x}{2} \right) \Big|_1^2 + \left(\frac{x}{2} - \ln|x| \right) \Big|_2^6 \\&= (\ln 2 - 1) - \left(0 - \frac{1}{2} \right) + (3 - \ln 6) - (1 - \ln 2) \\&= 2 \ln 2 - \ln 6 + \frac{3}{2} \approx 1.095\end{aligned}$$

12. $x = 0, x = 4, y = \frac{1}{x+1}, y = \frac{x-1}{2}$



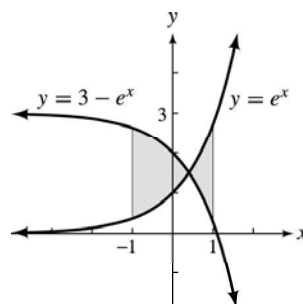
Find the intersection points.

$$\begin{aligned}\frac{1}{x+1} &= \frac{x-1}{2} \\ x^2 - 1 &= 2 \\ x^2 - 3 &= 0\end{aligned}$$

In the interval $[0, 4]$, the only intersection point is at $x = \sqrt{3}$.

$$\begin{aligned}& \int_0^{\sqrt{3}} \left(\frac{1}{x+1} - \frac{x-1}{2} \right) dx + \int_{\sqrt{3}}^4 \left(\frac{x-1}{2} - \frac{1}{x+1} \right) dx \\&= \left(\ln|x+1| - \frac{x^2}{4} + \frac{x}{2} \right) \Big|_0^{\sqrt{3}} \\&\quad + \left(\frac{x^2}{4} - \frac{x}{2} - \ln|x+1| \right) \Big|_{\sqrt{3}}^4 \\&= \ln(\sqrt{3}+1) - \frac{3}{4} + \frac{\sqrt{3}}{2} \\&\quad + \left\{ (4-2-\ln 5) - \left[\frac{3}{4} - \frac{\sqrt{3}}{2} - \ln(\sqrt{3}+1) \right] \right\} \\&= \ln(\sqrt{3}+1) - \frac{3}{4} + \frac{\sqrt{3}}{2} + 2 \\&\quad - \ln 5 - \frac{3}{4} + \frac{\sqrt{3}}{2} + \ln(\sqrt{3}+1) \\&= \ln(\sqrt{3}+1) + \ln(\sqrt{3}+1) - \ln 5 + \frac{1}{2} + \sqrt{3} \\&= \ln \frac{(\sqrt{3}+1)^2}{5} + \frac{1}{2} + \sqrt{3} \approx 2.633\end{aligned}$$

13. $x = -1, x = 1, y = e^x, y = 3 - e^x$



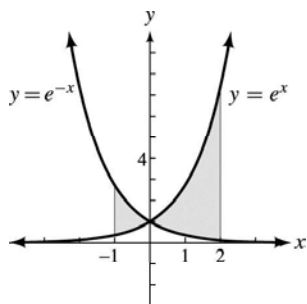
To find the point of intersection, set $e^x = 3 - e^x$ and solve for x .

$$\begin{aligned}e^x &= 3 - e^x \\ 2e^x &= 3 \\ e^x &= \frac{3}{2} \\ \ln e^x &= \ln \frac{3}{2} \\ x \ln e &= \ln \frac{3}{2} \\ x &= \ln \frac{3}{2}\end{aligned}$$

The area of the region between the curves from $x = -1$ to $x = 1$ is

$$\begin{aligned}
& \int_{-1}^{\ln 3/2} [(3 - e^x) - e^x] dx \\
& + \int_{\ln 3/2}^1 [e^x - (3 - e^x)] dx \\
& = \int_{-1}^{\ln 3/2} (3 - 2e^x) dx + \int_{\ln 3/2}^1 (2e^x - 3) dx \\
& = (3x - 2e^x) \Big|_{-1}^{\ln 3/2} + (2e^x - 3x) \Big|_{\ln 3/2}^1 \\
& = \left[\left(3 \ln \frac{3}{2} - 2e^{\ln 3/2} \right) - [3(-1) - 2e^{-1}] \right] \\
& + \left[2e^1 - 3(1) - \left(2e^{\ln 3/2} - 3 \ln \frac{3}{2} \right) \right] \\
& = \left[\left(3 \ln \frac{3}{2} - 3 \right) - \left(-3 - \frac{2}{e} \right) \right] \\
& + \left[2e - 3 - \left(3 - 3 \ln \frac{3}{2} \right) \right] \\
& = 6 \ln \frac{3}{2} + \frac{2}{e} + 2e - 6 \approx 2.605.
\end{aligned}$$

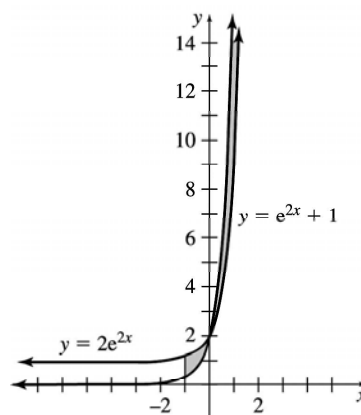
14. $x = -1, x = 2, y = e^{-x}, y = e^x$



The total area between the curves from $x = -1$ to $x = 2$ is

$$\begin{aligned}
& \int_{-1}^0 (e^{-x} - e^x) dx + \int_0^2 (e^x - e^{-x}) dx \\
& = (-e^{-x} - e^x) \Big|_{-1}^0 + (e^x + e^{-x}) \Big|_0^2 \\
& = [(-1 - 1) - (-e - e^{-1})] \\
& + [(e^2 + e^{-2}) - (1 + 1)] \\
& = e^2 + e^{-2} + e + e^{-1} - 4 \approx 6.611.
\end{aligned}$$

15. $x = -1, x = 2, y = 2e^{2x}, y = e^{2x} + 1$



To find the points of intersection of the graphs, substitute for y .

$$\begin{aligned}
2e^{2x} &= e^{2x} + 1 \\
e^{2x} &= 1 \\
2x &= 0 \\
x &= 0
\end{aligned}$$

The region is composed of two separate regions because $y = 2e^{2x}$ intersects $y = e^{2x} + 1$ at $x = 0$.

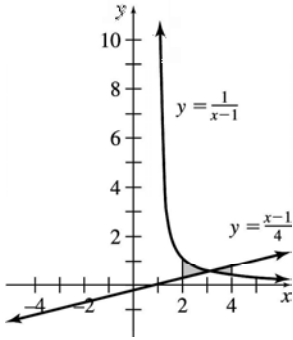
Let $f(x) = 2e^{2x}$, $g(x) = e^{2x} + 1$.

In the interval $[-1, 0]$, $g(x) \geq f(x)$.

In the interval $[0, 2]$, $f(x) \geq g(x)$.

$$\begin{aligned}
& \int_{-1}^0 (e^{2x} + 1 - 2e^{2x}) dx \\
& + \int_0^2 [2e^{2x} - (e^{2x} + 1)] dx \\
& = \left(-\frac{e^{2x}}{2} + x \right) \Big|_{-1}^0 + \left(\frac{e^{2x}}{2} - x \right) \Big|_0^2 \\
& = \left(-\frac{1}{2} + 0 \right) - \left(-\frac{e^{-2}}{2} - 1 \right) \\
& + \left(\frac{e^4}{2} - 2 \right) - \left(\frac{1}{2} - 0 \right) \\
& = \frac{e^{-2} + e^4}{2} - 2 \approx 25.37
\end{aligned}$$

16. $x = 2, x = 4, y = \frac{x-1}{4}, y = \frac{1}{x-1}$



To find the points of intersection of the graphs in $[2, 4]$, substitute for y .

$$\begin{aligned}\frac{x-1}{4} &= \frac{1}{x-1} \\ (x-1)(x-1) &= 4 \\ x^2 - 2x + 1 &= 4 \\ x^2 - 2x - 3 &= 0 \\ x &= -1 \text{ or } x = 3\end{aligned}$$

The region is composed of two separate regions because $y = \frac{x-1}{4}$ intersects $y = \frac{1}{x-1}$ at $x = 3$.

Let $f(x) = \frac{x-1}{4}, g(x) = \frac{1}{x-1}$.

In the interval $[2, 3], g(x) \geq f(x)$.

In the interval $[3, 4], f(x) \geq g(x)$.

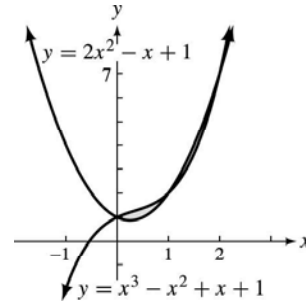
$$\begin{aligned}&\int_2^3 \left(\frac{1}{x-1} - \frac{x-1}{4} \right) dx + \int_3^4 \left(\frac{x-1}{4} - \frac{1}{x-1} \right) dx \\&= \left[\ln|x-1| - \frac{x(x-2)}{8} \right] \Big|_2^3 \\&\quad + \left[\frac{x(x-2)}{8} - \ln|x-1| \right] \Big|_3^4 \\&= \left(\ln 2 - \frac{3}{8} \right) - 0 + (1 - \ln 3) - \left(\frac{3}{8} - \ln 2 \right) \\&= 2 \ln 2 - \ln 3 + \frac{1}{4} \approx 0.5377\end{aligned}$$

17. $y = x^3 - x^2 + x + 1, y = 2x^2 - x + 1$

Find the points of intersection.

$$\begin{aligned}x^3 - x^2 + x + 1 &= 2x^2 - x + 1 \\ x^3 - 3x^2 + 2x &= 0 \\ x(x^2 - 3x + 2) &= 0 \\ x(x-2)(x-1) &= 0\end{aligned}$$

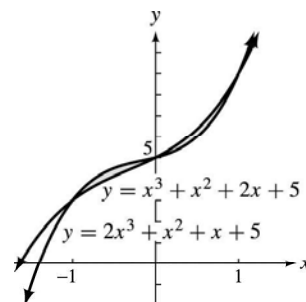
The points of intersection are at $x = 0$, $x = 1$, and $x = 2$.



Area between the curves is

$$\begin{aligned}&\int_0^1 [(x^3 - x^2 + x + 1) - (2x^2 - x + 1)] dx \\&\quad + \int_1^2 [(2x^2 - x + 1) - (x^3 - x^2 + x + 1)] dx \\&= \int_0^1 (x^3 - 3x^2 + 2x) dx + \int_1^2 (-x^3 + 3x^2 - 2x) dx \\&= \left[\frac{x^4}{4} - x^3 + x^2 \right] \Big|_0^1 + \left[-\frac{x^4}{4} + x^3 - x^2 \right] \Big|_1^2 \\&= \left[\left(\frac{1}{4} - 1 + 1 \right) - (0) \right] \\&\quad + \left[(-4 + 8 - 4) - \left(-\frac{1}{4} + 1 - 1 \right) \right] \\&= \frac{1}{4} + \frac{1}{4} \\&= \frac{1}{2}.\end{aligned}$$

18. $y = 2x^3 + x^2 + x + 5,$
 $y = x^3 + x^2 + 2x + 5$



To find the points of intersection, substitute for y .

$$\begin{aligned} 2x^3 + x^2 + x + 5 &= x^3 + x^2 + 2x + 5 \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \end{aligned}$$

The points of intersection are at $x = 0$,
 $x = -1$, and $x = 1$.

The area of the region between the curves is

$$\begin{aligned} &\int_{-1}^0 [(2x^3 + x^2 + x + 5) - (x^3 + x^2 + 2x + 5)] dx \\ &+ \int_0^1 [(x^3 + x^2 + 2x + 5) - (2x^3 + x^2 + x + 5)] dx \\ &= \int_{-1}^0 (x^3 - x) dx + \int_0^1 (-x^3 + x) dx \\ &= \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_{-1}^0 + \left(-\frac{x^4}{4} + \frac{x^2}{2} \right) \Big|_0^1 \\ &= \left[0 - \left(\frac{1}{4} - \frac{1}{2} \right) \right] + \left[\left(-\frac{1}{4} + \frac{1}{2} \right) - 0 \right] \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

19. $y = x^4 + \ln(x + 10)$,
 $y = x^3 + \ln(x + 10)$

Find the points of intersection.

$$\begin{aligned} x^4 + \ln(x + 10) &= x^3 + \ln(x + 10) \\ x^4 - x^3 &= 0 \\ x^3(x - 1) &= 0 \\ x &= 0 \text{ or } x = 1 \end{aligned}$$

The points of intersection are at $x = 0$ and
 $x = 1$.

The area between the curves is

$$\begin{aligned} &\int_0^1 [(x^3 + \ln(x + 10)) - (x^4 + \ln(x + 10))] dx \\ &= \int_0^1 (x^3 - x^4) dx \\ &= \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1 \\ &= \left(\frac{1}{4} - \frac{1}{5} \right) - (0) = \frac{1}{20}. \end{aligned}$$

20. $y = x^5 - 2 \ln(x + 5)$,
 $y = x^3 - 2 \ln(x + 5)$

To find the points of intersection, substitute for y .

$$\begin{aligned} x^5 - 2 \ln(x + 5) &= x^3 - 2 \ln(x + 5) \\ x^5 - x^3 &= 0 \\ x^3(x^2 - 1) &= 0 \end{aligned}$$

The points of intersection are at $x = 0$ and
 $x = 1$ and $x = -1$.

In the interval $[-1, 0]$,

$$x^5 - 2 \ln(x + 5) > x^3 - 2 \ln(x + 5).$$

In the interval $[0, 1]$,

$$x^5 - 2 \ln(x + 5) < x^3 - 2 \ln(x + 5).$$

The area between the curves is

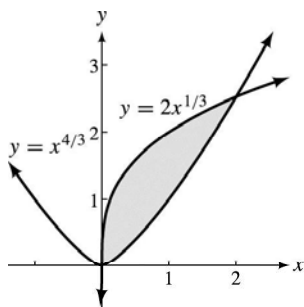
$$\begin{aligned} &\int_{-1}^0 [(x^5 - 2 \ln(x + 5)) - (x^3 - 2 \ln(x + 5))] dx \\ &+ \int_0^1 [(x^3 - 2 \ln(x + 5)) - (x^5 - 2 \ln(x + 5))] dx \\ &= \int_{-1}^0 (x^5 - x^3) dx + \int_0^1 (x^3 - x^5) dx \\ &= \left(\frac{x^6}{6} - \frac{x^4}{4} \right) \Big|_{-1}^0 + \left(-\frac{x^4}{4} + \frac{x^6}{6} \right) \Big|_0^1 \\ &= \left[0 - \left(\frac{1}{6} - \frac{1}{4} \right) \right] + \left[\left(-\frac{1}{4} + \frac{1}{6} \right) - 0 \right] \\ &= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}. \end{aligned}$$

21. $y = x^{4/3}$, $y = 2x^{1/3}$

Find the points of intersection.

$$\begin{aligned} x^{4/3} &= 2x^{1/3} \\ x^{4/3} - 2x^{1/3} &= 0 \\ x^{1/3}(x - 2) &= 0 \\ x &= 0 \text{ or } x = 2 \end{aligned}$$

The points of intersection are at $x = 0$ and
 $x = 2$.



The area between the curves is

$$\begin{aligned}
 \int_0^2 (2x^{1/3} - x^{4/3}) dx &= 2 \left. \frac{x^{4/3}}{\frac{4}{3}} - \frac{x^{7/3}}{\frac{7}{3}} \right|_0^2 \\
 &= \frac{3}{2} x^{4/3} - \frac{3}{7} x^{7/3} \Big|_0^2 \\
 &= \left[\frac{3}{2} (2)^{4/3} - \frac{3}{7} (2)^{7/3} \right] - 0 \\
 &= \frac{3(2^{4/3})}{2} - \frac{3(2^{7/3})}{7} \\
 &\approx 1.62.
 \end{aligned}$$

22. $y = \sqrt{x}, y = x\sqrt{x}$

To find the points of intersection, substitute for y .

$$\begin{aligned}
 \sqrt{x} &= x\sqrt{x} \\
 x\sqrt{x} - \sqrt{x} &= 0 \\
 \sqrt{x}(x - 1) &= 0
 \end{aligned}$$

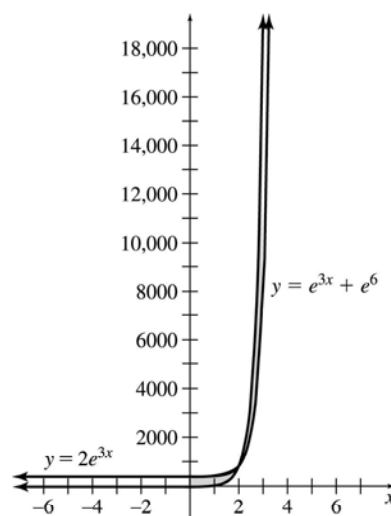
The points of intersection are at $x = 0$ and $x = 1$.

In $[0, 1]$, $\sqrt{x} > x\sqrt{x}$.

The area between the curves is

$$\begin{aligned}
 \int_0^1 (\sqrt{x} - x\sqrt{x}) dx &= \int_0^1 (x^{1/2} - x^{3/2}) dx \\
 &= \left(\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} \right) \Big|_0^1 \\
 &= \left(\frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} \right) \Big|_0^1 \\
 &= \left[\frac{2}{3} (1) - \frac{2}{5} (1) \right] - 0 \\
 &= \frac{4}{15}.
 \end{aligned}$$

23. $x = 0, x = 3, y = 2e^{3x}, y = e^{3x} + e^6$



To find the points of intersection of the graphs, substitute for y .

$$\begin{aligned}
 2e^{3x} &= e^{3x} + e^6 \\
 e^{3x} &= e^6 \\
 3x &= 6 \\
 x &= 2
 \end{aligned}$$

The region is composed of two separate regions because $y = 2e^{3x}$ intersects $y = e^{3x} + e^6$ at $x = 2$.

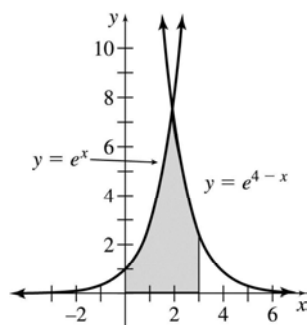
Let $f(x) = 2e^{3x}, g(x) = e^{3x} + e^6$.

In the interval $[0, 2]$, $g(x) \geq f(x)$.

In the interval $[2, 3]$, $f(x) \geq g(x)$.

$$\begin{aligned}
 \int_0^2 (e^{3x} + e^6 - 2e^{3x}) dx + \int_2^3 [2e^{3x} - (e^{3x} + e^6)] dx \\
 &= \left(-\frac{e^{3x}}{3} + e^6 x \right) \Big|_0^2 + \left(\frac{e^{3x}}{3} - e^6 x \right) \Big|_2^3 \\
 &= \left(-\frac{e^6}{3} + 2e^6 \right) - \left(-\frac{1}{3} + 0 \right) \\
 &\quad + \left(\frac{e^9}{3} - 3e^6 \right) - \left(\frac{e^6}{3} - 2e^6 \right) \\
 &= \frac{e^9 + e^6 + 1}{3} \\
 &\approx 2836
 \end{aligned}$$

24. $x = 0, x = 3, y = e^x, y = e^{4-x}$



To find the points of intersection of the graphs, substitute for y .

$$\begin{aligned} e^x &= e^{4-x} \\ x &= 4 - x \\ x &= 2 \end{aligned}$$

The region is composed of two separate regions because $y = e^x$ intersects $y = e^{4-x}$ at $x = 2$.

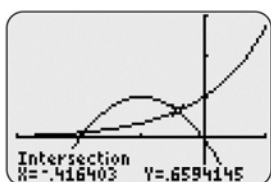
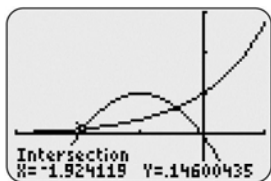
Let $f(x) = e^x, g(x) = e^{4-x}$.

In the interval $[0, 2], g(x) \geq f(x)$.

In the interval $[2, 3], f(x) \geq g(x)$.

$$\begin{aligned} &\int_0^2 (e^{4-x} - e^x) dx + \int_2^3 (e^x - e^{4-x}) dx \\ &= (-e^{4-x} - e^x) \Big|_0^2 + (e^x + e^{4-x}) \Big|_2^3 \\ &= (-e^2 - e^2) - (-e^4 - 1) \\ &\quad + (e^3 + e) - (e^2 + e^2) \\ &= e^4 + e^3 - 4e^2 + e + 1 \\ &\approx 48.85 \end{aligned}$$

25. Graph $y_1 = e^x$ and $y_2 = -x^2 - 2x$ on your graphing calculator. Use the intersect command to find the two intersection points. The resulting screens are:



These screens show that $e^x = -x^2 - 2x$ when $x \approx -1.9241$ and $x \approx -0.4164$.

In the interval $[-1.9241, -0.4164]$,

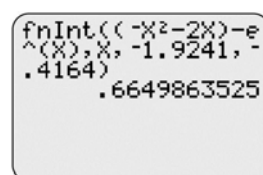
$$e^x < -x^2 - 2x.$$

The area between the curves is given by

$$\int_{-1.9241}^{-0.4164} [(-x^2 - 2x) - e^x] dx.$$

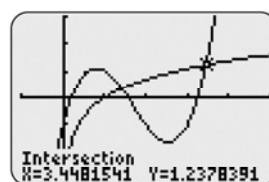
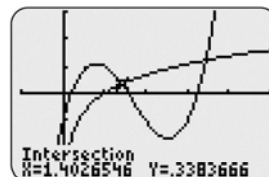
Use the fnInt command to approximate this definite integral.

The resulting screen is:



The last screen shows that the area is approximately 0.6650.

26. Graph $y_1 = \ln x$ and $y_2 = x^3 - 5x^2 + 6x - 1$ on your graphing calculator. Use the intersect command to find the two intersection points. The resulting screens are:



These screens show that $\ln x = x^3 - 5x^2 + 6x - 1$ when $x \approx 1.4027$ and $x \approx 3.4482$.

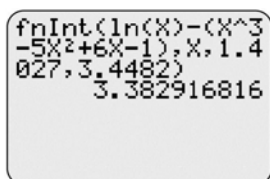
In the interval $[1.4027, 3.4482]$,

$$\ln x > x^3 - 5x^2 + 6x - 1.$$

The area between the curves is given by

$$\int_{1.4027}^{3.4482} [\ln x - (x^3 - 5x^2 + 6x - 1)] dx.$$

Use the fnInt command to approximate this definite integral. The resulting screen is:



The last screen shows that the area is approximately 3.3829.

27. (a) It is profitable to use the machine until $S'(x) = C'(x)$.

$$150 - x^2 = x^2 + \frac{11}{4}x$$

$$2x^2 + \frac{11}{4}x - 150 = 0$$

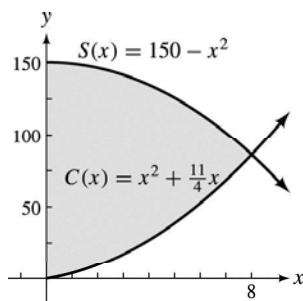
$$8x^2 + 11x - 600 = 0$$

$$x = \frac{-11 \pm \sqrt{121 - 4(8)(-600)}}{16}$$

$$= \frac{-11 \pm 139}{16}$$

$$x = 8 \text{ or } x = -9.375$$

It will be profitable to use this machine for 8 years. Reject the negative solution.



- (b) Since $150 - x^2 > x^2 + \frac{11}{4}x$, in the interval $[0, 8]$, the net total saving in the first year are

$$\int_0^8 \left[(150 - x^2) - \left(x^2 + \frac{11}{4}x \right) \right] dx$$

$$= \int_0^8 \left(-2x^2 - \frac{11}{4}x + 150 \right) dx$$

$$= \left(-\frac{2x^3}{3} - \frac{11x^2}{8} + 150x \right) \Big|_0^8$$

$$= -\frac{2}{3} - \frac{11}{8} + 150 \approx \$148.$$

- (c) The net total savings over the entire period of use are

$$\int_0^8 \left[(150 - x^2) - \left(x^2 + \frac{11}{4}x \right) \right] dx$$

$$= \left(-\frac{2x^3}{3} - \frac{11x^2}{8} + 150x \right) \Big|_0^8$$

$$= \frac{-2(8^3)}{3} - \frac{11(8^2)}{8} + 150(8)$$

$$= \frac{-1024}{3} - \frac{704}{8} + 1200 \approx \$771.$$

28. (a) $S'(x) = -x^2 + 4x + 8$, $C'(x) = \frac{3}{25}x^2$

$$S'(x) = C'(x)$$

$$-x^2 + 4x + 8 = \frac{3}{25}x^2$$

$$-25x^2 + 100x + 200 = 3x^2$$

$$0 = 28x^2 - 100x - 200$$

$$0 = 7x^2 - 25x - 50$$

$$0 = (7x + 10)(x - 5)$$

$$x = -\frac{10}{7} \text{ or } x = 5$$

Since time would not be negative, 5 is the only solution. It will pay to use the device for 5 yr.

- (b) The total savings over 5 yr is given by

$$\int_0^5 (-x^2 + 4x + 8) dx$$

$$= \left(-\frac{x^3}{3} + 2x^2 + 8x \right) \Big|_0^5$$

$$= \frac{-125}{3} + 90 = 48.33.$$

The total cost over 5 yr is given by

$$\int_0^5 \frac{3}{25}x^2 dx = \frac{x^3}{25} \Big|_0^5 = 5.$$

$$\begin{aligned} \text{Net savings} &= \$48.33 \text{ million} - \$5 \text{ million} \\ &= \$43.33 \text{ million} \end{aligned}$$

29. (a) $E'(x) = e^{0.1x}$ and $I'(x) = 98.8 - e^{0.1x}$

To find the point of intersection, where profit will be maximized, set the functions equal to each other and solve for x .

$$\begin{aligned}
 e^{0.1x} &= 98.8 - e^{0.1x} \\
 2e^{0.1x} &= 98.8 \\
 e^{0.1x} &= 49.4 \\
 0.1x &= \ln 49.4 \\
 x &= \frac{\ln 49.4}{0.1} \approx 39
 \end{aligned}$$

The optimum number of days for the job to last is 39.

- (b) The total income for 39 days is

$$\begin{aligned}
 \int_0^{39} (98.8 - e^{0.1x}) dx &= \left(98.8x - \frac{e^{0.1x}}{0.1} \right) \Big|_0^{39} \\
 &= \left(98.8x - 10e^{0.1x} \right) \Big|_0^{39} \\
 &= [98.8(39) - 10e^{3.9}] - (0 - 10) \\
 &= \$3369.18.
 \end{aligned}$$

- (c) The total expenditure for 39 days is

$$\begin{aligned}
 \int_0^{39} e^{0.1x} dx &= \frac{e^{0.1x}}{0.1} \Big|_0^{39} \\
 &= 10e^{0.1x} \Big|_0^{39} \\
 &= 10e^{3.9} - 10 \\
 &= \$484.02.
 \end{aligned}$$

- (d) Profit = Income - Expense
 $= 3369.18 - 484.02 = \$2885.16$

30. (a) $R'(t) = 104 - 0.4e^{t/2}$; $C'(t) = 0.3e^{t/2}$

It will no longer be profitable when $C'(t) > R'(t)$. Find t when $C'(t) > R'(t)$.

$$\begin{aligned}
 0.3e^{t/2} &> 104 - 0.4e^{t/2} \\
 0.7e^{t/2} &> 104 \\
 e^{t/2} &> \frac{104}{0.7} \\
 \ln e^{t/2} &> \ln \left(\frac{104}{0.7} \right) \\
 t &> 2 \ln \left(\frac{104}{0.7} \right) \\
 t &> 10
 \end{aligned}$$

It will no longer be profitable to use process after 10 yr.

- (b) The total net saving is

$$\begin{aligned}
 \int_0^{10} [(104 - 0.4e^{t/2}) - 0.3e^{t/2}] dt &= \int_0^{10} (104 - 0.7e^{t/2}) dt \\
 &= \left(104t - \frac{0.7e^{t/2}}{1/2} \right) \Big|_0^{10} \\
 &= (104t - 1.4e^{t/2}) \Big|_0^{10} \\
 &= [(104t - 1.4e^{t/2}) - (0 - 1.4)] \\
 &= 1041.4 - 1.4e^5 \\
 &\approx 834.
 \end{aligned}$$

The net total saving will be \$834,000.

31. $S(q) = q^{5/2} + 2q^{3/2} + 50$; $q = 16$ is the equilibrium quantity.

Producers surplus = $\int_0^{q_0} [p_0 - S(q)] dq$,
 where p_0 is the equilibrium price and q_0 is equilibrium supply.

$$\begin{aligned}
 p_0 &= S(16) = (16)^{5/2} + 2(16)^{3/2} + 50 \\
 &= 1202
 \end{aligned}$$

Therefore, the producers' surplus is

$$\begin{aligned}
 \int_0^{16} [1202 - (q^{5/2} + 2q^{3/2} + 50)] dq &= \int_0^{16} (1152 - q^{5/2} - 2q^{3/2}) dq \\
 &= \left(1152q - \frac{2}{7}q^{7/2} - \frac{4}{5}q^{5/2} \right) \Big|_0^{16} \\
 &= 1152(16) - \frac{2}{7}(16)^{7/2} - \frac{4}{5}(16)^{5/2} \\
 &= 18,432 - \frac{32,768}{7} - \frac{4096}{5} \\
 &= 12,931.66.
 \end{aligned}$$

The producers' surplus is 12,931.66.

32. $S(q) = 100 + 3q^{3/2} + q^{5/2}$; equilibrium quantity is $q = 9$.

$$\begin{aligned}\text{Producers' surplus} &= \int_0^{q_0} [p_0 - S(q)] dq \\ p_0 &= S(9) = 424\end{aligned}$$

$$\begin{aligned}&\int_0^9 [424 - (100 + 3q^{3/2} + q^{5/2})] dq \\&= \int_0^9 (324 - 3q^{3/2} - q^{5/2}) dq \\&= \left(324q - \frac{6}{5}q^{5/2} - \frac{2}{7}q^{7/2} \right) \Big|_0^9 \\&= \left[324(9) - \frac{6}{5}(9)^{5/2} - \frac{2}{7}(9)^{7/2} \right] - 0 \\&= 2916 - \frac{1458}{5} - \frac{4374}{7} \\&= 1999.54\end{aligned}$$

The producers' surplus is 1999.54.

33. $D(q) = \frac{200}{(3q+1)^2}$; $q = 3$ is the equilibrium quantity.

$$\begin{aligned}\text{Consumers' surplus} &= \int_0^{q_0} [D(q) - p_0] dq \\ p_0 &= D(3) = 2\end{aligned}$$

Therefore, the consumers' surplus is

$$\begin{aligned}&\int_0^3 \left[\frac{200}{(3q+1)^2} - 2 \right] dq \\&= \int_0^3 \frac{200}{(3q+1)^2} dq - \int_0^3 2 dq.\end{aligned}$$

Let $u = 3q + 1$, so that

$$du = 3 dq \text{ and } \frac{1}{3} du = dq.$$

$$\begin{aligned}&\int_0^3 \frac{200}{(3q+1)^2} dq - \int_0^3 2 dq \\&= \frac{1}{3} \int_1^{10} \frac{200}{u^2} du - \int_0^3 2 dq \\&= \frac{200}{3} \int_1^{10} u^{-2} du - \int_0^3 2 dq\end{aligned}$$

$$\begin{aligned}&= \frac{200}{3} \cdot \frac{u^{-1}}{-1} \Big|_1^{10} - 2q \Big|_0^3 \\&= -\frac{200}{3u} \Big|_1^{10} - 6 \\&= -\frac{200}{30} + \frac{200}{3} - 6 \\&= 54\end{aligned}$$

34. $D(q) = \frac{32,000}{(2q+8)^3}$; $q = 6$ is the equilibrium quantity.

$$\begin{aligned}\text{Consumers' surplus} &= \int_0^{q_0} [D(q) - p_0] dq \\ p_0 &= D(6) = \frac{32,000}{20^3} = 4\end{aligned}$$

Therefore, the consumers' surplus is

$$\begin{aligned}&\int_0^6 \left[\frac{32,000}{(2q+8)^3} - 4 \right] dq \\&= \int_0^6 \frac{32,000}{(2q+8)^3} dq - \int_0^6 4 dq.\end{aligned}$$

Let $u = 2q + 8$, so that

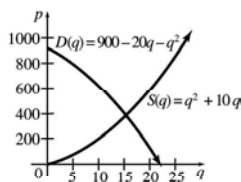
$$du = 2 dq \text{ and } \frac{1}{2} du = dq.$$

$$\begin{aligned}&\int_0^6 \frac{32,000}{(2q+8)^3} dq - \int_0^6 4 dq \\&= \frac{1}{2} \int_8^{20} \frac{32,000}{u^3} du - \int_0^6 4 dq \\&= 16,000 \int_8^{20} u^{-3} du - \int_0^6 4 dq \\&= 16,000 \cdot \frac{u^{-2}}{-2} \Big|_8^{20} - 4q \Big|_0^6 \\&= -\frac{8000}{u^2} \Big|_8^{20} - 24 \\&= -\frac{8000}{400} + \frac{8000}{64} - 24 \\&= 81\end{aligned}$$

35. $S(q) = q^2 + 10q$

$$D(q) = 900 - 20q - q^2$$

- (a) The graphs of the supply and demand functions are parabolas with vertices at $(-5, -25)$ and $(-10, 1900)$, respectively.



- (b) The graphs intersect at the point where the y -coordinates are equal.

$$q^2 + 10q = 900 - 20q - q^2$$

$$2q^2 + 30q - 900 = 0$$

$$q^2 + 15q - 450 = 0$$

$$(q + 30)(q - 15) = 0$$

$$q = -30 \text{ or } q = 15$$

Disregard the negative solution.

The supply and demand functions are in equilibrium when $q = 15$.

$$S(15) = 15^2 + 10(15) = 375$$

The point is $(15, 375)$.

- (c) Find the consumers' surplus.

$$\int_0^{q_0} [D(q) - p_0] dq$$

$$p_0 = D(15) = 375$$

$$\begin{aligned} & \int_0^{15} [(900 - 20q - q^2) - 375] dq \\ &= \int_0^{15} (525 - 20q - q^2) dq \\ &= \left[525q - 10q^2 - \frac{1}{3}q^3 \right]_0^{15} \\ &= \left[525(15) - 10(15)^2 - \frac{1}{3}(15)^3 \right] - 0 = 4500 \end{aligned}$$

The consumer's surplus is \$4500.

- (d) Find the producers' surplus.

$$\int_0^{q_0} [p_0 - S(q)] dq$$

$$p_0 = S(15) = 375$$

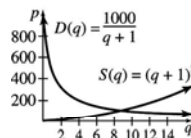
$$\begin{aligned} & \int_0^{15} [375 - (q^2 + 10q)] dq \\ &= \int_0^{15} (375 - q^2 - 10q) dq \\ &= \left[375q - \frac{1}{3}q^3 - 5q^2 \right]_0^{15} \\ &= \left[375(15) - \frac{1}{3}(15)^3 - 5(15)^2 \right] - 0 \\ &= 3375 \end{aligned}$$

The producer's surplus is \$3375.

36. $S(q) = (q + 1)^2$

$$D(q) = \frac{1000}{q + 1}$$

- (a) The graph of the supply function is a parabola with vertex at $(-1, 0)$. The graph of the demand function is the graph of a rational function with vertical asymptote of $x = -1$ and horizontal asymptote of $y = 0$.



- (b) Find the equilibrium point by setting the two functions equal.

$$(q + 1)^2 = \frac{1000}{q + 1}$$

$$(q + 1)^3 = 1000$$

$$q^3 + 3q^2 + 3q + 1 = 1000$$

$$q^3 + 3q^2 + 3q - 999 = 0$$

$$(q - 9)(q^2 + 12q + 111) = 0$$

Since $q^2 + 12q + 111$ has no real roots, $q = 9$ is the only root. At the equilibrium point where the supply and demand are both 9 items, the price is

$$S(9) = (9 + 1)^2 = 100.$$

The equilibrium point is $(9, 100)$.

- (c) The consumers' surplus is given by

$$\begin{aligned}
 & \int_0^9 \left(\frac{1000}{q+1} - 100 \right) dq \\
 &= (1000 \ln |q+1| - 100q) \Big|_0^9 \\
 &= 1000 \ln(9+1) - 100(9) - 0 \\
 &\approx 1402.59 \\
 &\text{Here the consumers' surplus is 1402.59.}
 \end{aligned}$$

- (d) The producers' surplus is given by

$$\begin{aligned}
 & \int_0^9 [100 - (q+1)^2] dq \\
 &= \int_0^9 (99 - q^2 - 2q) dq \\
 &= \left(99q - \frac{1}{3}q^3 - q^2 \right) \Big|_0^9 \\
 &= 99(9) - \frac{1}{3}(9)^3 - (9)^2 - 0 \\
 &= 567 \\
 &\text{Here the producers' surplus is 567.}
 \end{aligned}$$

37. (a)
- $S(q) = q^2 + 10q$
- ;
- $S(q) = 264$
- is the price the government set.

$$\begin{aligned}
 264 &= q^2 + 10q \\
 0 &= q^2 + 10q - 264 \\
 0 &= (q-12)(q+22) \\
 q &= 12 \quad \text{or} \quad q = -22
 \end{aligned}$$

Only 12 is a meaningful solution here. Thus, 12 units of oil will be produced.

- (b) The consumers' surplus is given by

$$\begin{aligned}
 & \int_0^{12} (900 - 20q - q^2 - 264) dq \\
 &= \int_0^{12} (636 - 20q - q^2) dq \\
 &= \left(636q - 10q^2 - \frac{1}{3}q^3 \right) \Big|_0^{12} \\
 &= 636(12) - 10(12)^2 - \frac{1}{3}(12)^3 - 0 \\
 &= 5616
 \end{aligned}$$

Here the consumer's surplus is \$5616. In this case, the consumers' surplus is $5616 - 4500 = \$1116$ larger.

- (c) The producers' surplus is given by

$$\begin{aligned}
 & \int_0^{12} [264 - (q^2 + 10q)] dq \\
 &= \int_0^{12} (264 - q^2 - 10q) dq \\
 &= \left(264q - \frac{1}{3}q^3 - 5q^2 \right) \Big|_0^{12} \\
 &= 264(12) - \frac{1}{3}(12)^3 - 5(12)^2 - 0 \\
 &= 1872
 \end{aligned}$$

Here the producers' surplus is \$1872. In this case, the producers' surplus is $3375 - 1872 = \$1503$ smaller.

- (d) For the equilibrium price, the total consumers' and producers' surplus is

$$4500 + 3375 = \$7875$$

For the government price, the total consumers' and producers' surplus is

$$5616 + 1872 = \$7488.$$

The difference is

$$7875 - 7488 = \$387.$$

39. (a) The pollution level in the lake is changing at the rate
- $f(t) - g(t)$
- at any time
- t
- . We find the amount of pollution by integrating.

$$\begin{aligned}
 & \int_0^{12} [f(t) - g(t)] dt \\
 &= \int_0^{12} [10(1 - e^{-0.5t}) - 0.4t] dt \\
 &= \left(10t - 10 \cdot \frac{1}{-0.5} e^{-0.5t} - 0.4 \cdot \frac{1}{2} t^2 \right) \Big|_0^{12} \\
 &= (20e^{-0.5t} + 10t - 0.2t^2) \Big|_0^{12} \\
 &= [20e^{-0.5(12)} + 10(12) - 0.2(12)^2] \\
 &\quad - [20e^{-0.5(0)} + 10(0) - 0.2(0)^2] \\
 &= (20e^{-6} + 91.2) - (20) \\
 &= 20e^{-6} + 71.2 \approx 71.25
 \end{aligned}$$

After 12 hours, there are about 71.25 gallons.

- (b) The graphs of the functions intersect at about 25.00. So the rate that pollution enters the lake equals the rate the pollution is removed at about 25 hours.

$$\begin{aligned}
 \text{(c)} \quad & \int_0^{25} [f(t) - g(t)] dt \\
 &= (20e^{-0.5t} + 10t - 0.2t^2) \Big|_0^{25} \\
 &= [20e^{-0.5(25)} + 10(25) - 0.2(25)^2] - 20 \\
 &= 20e^{-12.5} + 105 \approx 105
 \end{aligned}$$

After 25 hours, there are about 105 gallons.

- (d) For $t > 25$, $g(t) > f(t)$, and pollution is being removed at the rate $g(t) - f(t)$. So, we want to solve for c , where

$$\int_0^c [f(t) - g(t)] dt = 0.$$

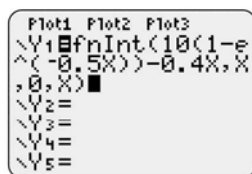
Alternatively, we could solve for c in

$$\int_{25}^c [g(t) - f(t)] dt = 105.$$

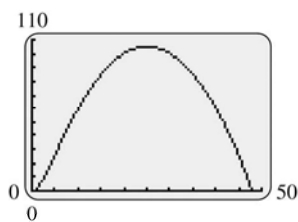
One way to do this with a graphing calculator is to graph the function

$$y = \int_0^x [f(t) - g(t)] dt$$

and determine the values of x for which $y = 0$. The first window shows how the function can be defined.



A suitable window for the graph is $[0, 50]$ by $[0, 110]$.



Use the calculator's features to approximate where the graph intersects the x -axis. These are at 0 and about 47.91. Therefore, the pollution will be removed from the lake after about 47.91 hours.

40. (a) The pollution level in the lake is changing at the rate $f(t) - g(t)$ at any time t . We find the amount of pollution by integrating.

$$\begin{aligned}
 & \int_0^{12} [f(t) - g(t)] dt \\
 &= \int_0^{12} [15(1 - e^{-0.05t}) - 0.3t] dt \\
 &= \left(15t - 15 \frac{1}{-0.05} e^{-0.05t} - 0.3 \frac{1}{2} t^2 \right) \Big|_0^{12} \\
 &= (300e^{-0.05t} + 15t - 0.15t^2) \Big|_0^{12} \\
 &= [300e^{-0.05(12)} + 15(12) - 0.15(12)^2] \\
 &\quad - [300e^{-0.05(0)} + 15(0) - 0.15(0)^2] \\
 &= (300e^{-0.6} + 158.4) - (300) \\
 &= 300e^{-0.6} - 141.6 \approx 23.04
 \end{aligned}$$

After 12 hours, there are about 23.04 gallons.

- (b) The graphs of the functions intersect at about 44.63. So the rate that pollution enters the lake equals the rate the pollution is removed at about 44.63 hours.

$$\begin{aligned}
 \text{(c)} \quad & \int_0^{44.63} [f(t) - g(t)] dt \\
 &= (300e^{-0.05t} + 15t - 0.15t^2) \Big|_0^{44.63} \\
 &= [300e^{-0.05(44.63)} + 15(44.63) \\
 &\quad - 0.15(44.63)^2] - 300 \\
 &= (300e^{-2.2315} + 370.674465) - 300 \\
 &= 300e^{-2.2315} + 70.674465 \approx 102.88
 \end{aligned}$$

After 44.63 hours, there are about 102.88 gallons.

- (d) For $t > 44.63$, $g(t) > f(t)$, and pollution is being removed at the rate $g(t) - f(t)$. So we want to solve for c , where

$$\int_0^c [f(t) - g(t)] dt = 0$$

(Alternatively, we could solve for c in

$$\int_{44.63}^c [g(t) - f(t)] dt = 102.88.)$$

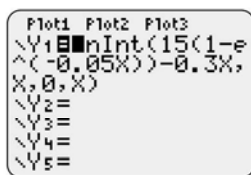
One way to do this with a graphing calculator is to graph the function

$$y = \int_0^x [f(t) - g(t)] dt$$

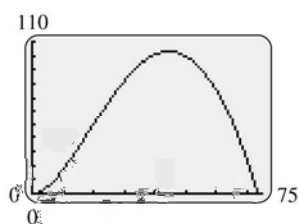
and determine the values of x for which

$$y = 0.$$

The first window shows how the function can be defined.



A suitable window for the graph is $[0, 75]$ by $[0, 110]$.



Use the calculator's features to approximate where the graph intersects the x -axis. These are at 0 and about 73.47. Therefore, the pollution will be removed from the lake after about 73.47 hours.

41. $I(x) = 0.9x^2 + 0.1x$

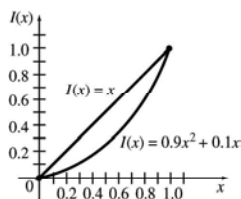
(a) $I(0.1) = 0.9(0.1)^2 + 0.1(0.1)$
 $= 0.019$

The lower 10% of income producers earn 1.9% of total income of the population.

(b) $I(0.4) = 0.9(0.4)^2 + 0.1(0.4) = 0.184$

The lower 40% of income producers earn 18.4% of total income of the population.

- (c) The graph of $I(x) = x$ is a straight line through the points $(0, 0)$ and $(1, 1)$. The graph of $I(x) = 0.9x^2 + 0.1x$ is a parabola with vertex $\left(-\frac{1}{18}, -\frac{1}{360}\right)$. Restrict the domain to $0 \leq x \leq 1$.



- (d) To find the points of intersection, solve

$$x = 0.9x^2 + 0.1x.$$

$$0.9x^2 - 0.9x = 0$$

$$0.9x(x - 1) = 0$$

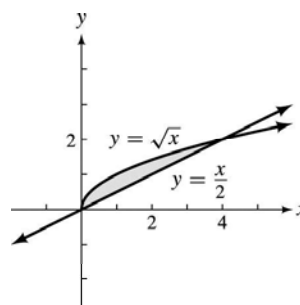
$$x = 0 \text{ or } x = 1$$

The area between the curves is given by

$$\begin{aligned} \int_0^1 [x - (0.9x^2 + 0.1x)] dx \\ &= \int_0^1 (0.9x - 0.9x^2) dx \\ &= \left(\frac{0.9x^2}{2} - \frac{0.9x^3}{3} \right) \bigg|_0^1 \\ &= \frac{0.9}{2} - \frac{0.9}{3} = 0.15. \end{aligned}$$

- (e) Income is distributed less equally in 2008 than in 1968.

42. $y = \sqrt{x}, y = \frac{x}{2}$



To find the points of intersection, substitute for y .

$$\sqrt{x} = \frac{x}{2}$$

$$\frac{x}{2} - \sqrt{x} = 0$$

$$x - 2\sqrt{x} = 0$$

$$\sqrt{x}(\sqrt{x} - 2) = 0$$

$$x = 0 \text{ or } x = 4$$

$$\begin{aligned} \text{Area} &= \int_0^4 \left(\sqrt{x} - \frac{x}{2} \right) dx \\ &= \int_0^4 \left(x^{1/2} - \frac{x}{2} \right) dx \\ &= \left(\frac{2}{3}x^{3/2} - \frac{1}{4}x^2 \right) \bigg|_0^4 \\ &= \frac{16}{3} - 4 = \frac{4}{3} \end{aligned}$$

7.6 Numerical Integration

Your Turn 1

Use the trapezoidal rule with $n = 4$ to approximate

$$\int_1^3 \sqrt{x^2 + 3} \, dx.$$

Here $f(x) = \sqrt{x^2 + 3}$, $a = 1$, $b = 3$, and $n = 4$. The subintervals have length $(3 - 1)/4 = 1/2$. The following table summarizes the information required.

i	x_i	$f(x_i)$
0	1	$f(1) = 2$
1	3/2	$f(3/2) \approx 2.29129$
2	2	$f(2) \approx 2.64575$
3	5/2	$f(5/2) \approx 3.04138$
4	3	$f(3) \approx 3.46410$

The trapezoidal rule gives

$$\begin{aligned} \int_1^3 \sqrt{x^2 + 3} \, dx &\approx \frac{3-1}{2} \left[\frac{1}{2}(2) + 2.29129 + 2.64575 \right. \\ &\quad \left. + 3.04138 + \frac{1}{2}(3.46410) \right] \\ &\approx 5.3552. \end{aligned}$$

Your Turn 2

Use Simpson's rule with $n = 4$ to approximate

$$\int_1^3 \sqrt{x^2 + 3} \, dx.$$

Here $f(x) = \sqrt{x^2 + 3}$, $a = 1$, $b = 3$, and $n = 4$. The subintervals have length $(3 - 1)/4 = 1/2$. The following table summarizes the information required; it is the same as the table used in Your Turn 1.

i	x_i	$f(x_i)$
0	1	$f(1) = 2$
1	3/2	$f(3/2) \approx 2.29129$
2	2	$f(2) \approx 2.64575$
3	5/2	$f(5/2) \approx 3.04138$
4	3	$f(3) \approx 3.46410$

For Simpson's rule, the factor in front is $(b - a)/3n = (3 - 1)/12 = 1/6$. Simpson's rule thus gives

$$\begin{aligned} \int_1^3 \sqrt{x^2 + 3} \, dx &\approx \frac{1}{6} [2 + 4(2.29129) + 2(2.64575) + 4(3.04138) + 3.46410] \\ &\approx 5.3477. \end{aligned}$$

7.6 Exercises

1. $\int_0^2 (3x^2 + 2) \, dx$

$$n = 4, b = 2, a = 0, f(x) = 3x^2 + 2$$

i	x_i	$f(x_i)$
0	0	2
1	$\frac{1}{2}$	2.75
2	1	5
3	$\frac{3}{2}$	8.75
4	2	14

(a) Trapezoidal rule:

$$\begin{aligned} \int_0^2 (3x^2 + 2) \, dx &\approx \frac{2-0}{4} \left[\frac{1}{2}(2) + 2.75 + 5 + 8.75 + \frac{1}{2}(14) \right] \\ &= 0.5(24.5) \\ &= 12.25 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned} \int_0^2 (3x^2 + 2) \, dx &\approx \frac{2-0}{3(4)} [2 + 4(2.75) + 2(5) + 4(8.75) + 14] \\ &= \frac{2}{12}(72) \\ &= 12 \end{aligned}$$

(c) Exact value:

$$\begin{aligned} \int_0^2 (3x^2 + 2) \, dx &= (x^3 + 2x) \Big|_0^2 \\ &= (8 + 4) - 0 \\ &= 12 \end{aligned}$$

2. $\int_0^2 (2x^2 + 1) dx$

$n = 4, b = 2, a = 0, f(x) = 2x^2 + 1$

i	x_i	$f(x_i)$
0	0	1
1	$\frac{1}{2}$	1.5
2	1	3
3	$\frac{3}{2}$	5.5
4	2	9

(a) Trapezoidal rule:

$$\begin{aligned} \int_0^2 (2x^2 + 1) dx &\approx \frac{2-0}{4} \left[\frac{1}{2}(1) + 1.5 + 3 + 5.5 + \frac{1}{2}(9) \right] \\ &= 0.5(15) = 7.5 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned} \int_0^2 (2x^2 + 1) dx &\approx \frac{2-0}{3(4)} [1 + 4(1.5) + 2(3) + 4(5.5) + 9] \\ &= \frac{2}{12}(44) \approx 7.333 \end{aligned}$$

(c) Exact value:

$$\begin{aligned} \int_0^2 (2x^2 + 1) dx &= \left(\frac{2x^3}{3} + x \right) \Big|_0^2 \\ &= \left(\frac{16}{3} + 2 \right) - 0 \\ &= \frac{22}{3} \approx 7.333 \end{aligned}$$

3. $\int_{-1}^3 \frac{3}{5-x} dx$

$n = 4, b = 3, a = -1, f(x) = \frac{3}{5-x}$

i	x_i	$f(x_i)$
0	-1	0.5
1	0	0.6
2	1	0.75
3	2	1
4	3	1.5

(a) Trapezoidal rule:

$$\begin{aligned} \int_{-1}^3 \frac{3}{5-x} dx &\approx \frac{3-(-1)}{4} \left[\frac{1}{2}(0.5) + 0.6 + 0.75 + 1 + \frac{1}{2}(1.5) \right] \\ &= 1(3.35) \\ &= 3.35 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned} \int_{-1}^3 \frac{3}{5-x} dx &\approx \frac{3-(-1)}{3(4)} [0.5 + 4(0.6) + 2(0.75) + 4(1) + 1.5] \\ &= \frac{1}{3} \left(\frac{99}{10} \right) \\ &= \frac{33}{10} \approx 3.3 \end{aligned}$$

(c) Exact value:

$$\begin{aligned} \int_{-1}^3 \frac{3}{5-x} dx &= -3 \ln |5-x| \Big|_{-1}^3 \\ &= -3(\ln |2| - \ln |6|) \\ &= 3 \ln 3 \approx 3.296 \end{aligned}$$

4. $\int_1^5 \frac{6}{2x+1} dx$

$n = 4, b = 5, a = 1, f(x) = \frac{6}{2x+1}$

i	x_i	$f(x_i)$
0	1	2
1	2	$\frac{6}{5}$
2	3	$\frac{6}{7}$
3	4	$\frac{2}{3}$
4	5	$\frac{6}{11}$

(a) Trapezoidal rule:

$$\begin{aligned} \int_1^5 \frac{6}{2x+1} dx &\approx \frac{5-1}{4} \left[\frac{1}{2}(2) + \frac{6}{5} + \frac{6}{7} + \frac{2}{3} + \frac{1}{2} \left(\frac{6}{11} \right) \right] \\ &= 1 \left(1 + \frac{6}{5} + \frac{6}{7} + \frac{2}{3} + \frac{3}{11} \right) \\ &\approx 3.997 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
& \int_1^5 \frac{6}{2x+1} dx \\
& \approx \frac{5-1}{3(4)} \left[2 + 4\left(\frac{6}{5}\right) + 2\left(\frac{6}{7}\right) + 4\left(\frac{6}{9}\right) + \left(\frac{6}{11}\right) \right] \\
& = \frac{1}{3} \left(2 + \frac{24}{5} + \frac{12}{7} + \frac{8}{3} + \frac{6}{11} \right) \\
& \approx 3.909
\end{aligned}$$

(c) Exact value:

$$\begin{aligned}
\int_1^5 \frac{6}{2x+1} dx &= 3 \ln |2x+1| \Big|_1^5 \\
&= 3(\ln |11| - \ln |3|) \\
&= 3 \ln \frac{11}{3} \approx 3.898
\end{aligned}$$

5. $\int_{-1}^2 (2x^3 + 1) dx$

$$n = 4, b = 2, a = -1, f(x) = 2x^3 + 1$$

i	x_i	$f(x)$
0	-1	-1
1	$-\frac{1}{4}$	$\frac{31}{32}$
2	$\frac{1}{2}$	$\frac{5}{4}$
3	$\frac{5}{4}$	$\frac{157}{32}$
4	2	17

(a) Trapezoidal rule:

$$\begin{aligned}
& \int_{-1}^2 (2x^3 + 1) dx \\
& \approx \frac{2 - (-1)}{4} \left[\frac{1}{2}(-1) + \frac{31}{32} + \frac{5}{4} \right. \\
& \quad \left. + \frac{157}{32} + \frac{1}{2}(17) \right] \\
& = 0.75(15.125) \\
& \approx 11.34
\end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
& \int_{-1}^2 (2x^3 + 1) dx \\
& \approx \frac{2 - (-1)}{3(4)} \left[-1 + 4\left(\frac{31}{32}\right) + 2\left(\frac{5}{4}\right) + 4\left(\frac{157}{32}\right) + 17 \right] \\
& = \frac{1}{4}(42) = 10.5
\end{aligned}$$

(c) Exact value:

$$\begin{aligned}
& \int_{-1}^2 (2x^3 + 1) dx \\
& = \left(\frac{x^4}{2} + x \right) \Big|_{-1}^2 \\
& = (8 + 2) - \left(\frac{1}{2} - 1 \right) \\
& = \frac{21}{2} \\
& = 10.5
\end{aligned}$$

6. $\int_0^3 (2x^3 + 1) dx$

$$n = 4, b = 3, a = 0, f(x) = 2x^3 + 1$$

i	x_i	$f(x)$
0	0	1
1	$\frac{3}{4}$	$\frac{59}{32}$
2	$\frac{3}{2}$	$\frac{31}{4}$
3	$\frac{9}{4}$	$\frac{761}{32}$
4	3	55

(a) Trapezoidal rule:

$$\begin{aligned}
& \int_0^3 (2x^3 + 1) dx \\
& \approx \frac{3 - 0}{4} \left[\frac{1}{2}(1) + \frac{59}{32} + \frac{31}{4} + \frac{761}{32} + \frac{1}{2}(55) \right] \\
& = \frac{3}{4} \left(\frac{491}{8} \right) \\
& \approx 46.03
\end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
& \int_0^3 (2x^3 + 1) dx \\
& = \frac{3 - 0}{3(4)} \left[1 + 4\left(\frac{59}{32}\right) + 2\left(\frac{31}{4}\right) \right. \\
& \quad \left. + 4\left(\frac{761}{32}\right) + 55 \right] \\
& = \frac{1}{4}(174) \\
& = 43.5
\end{aligned}$$

(c) Exact value:

$$\begin{aligned}
 \int_0^3 (2x^3 + 1) dx &= \left(\frac{x^4}{2} + x \right) \Big|_0^3 \\
 &= \left(\frac{81}{2} + 3 \right) - 0 \\
 &= \frac{87}{2} \\
 &= 43.5
 \end{aligned}$$

7. $\int_1^5 \frac{1}{x^2} dx$

$$n = 4, b = 5, a = 1, f(x) = \frac{1}{x^2}$$

i	x_i	$f(x_i)$
0	1	1
1	2	0.25
2	3	0.1111
3	4	0.0625
4	5	0.04

(a) Trapezoidal rule:

$$\begin{aligned}
 \int_1^5 \frac{1}{x^2} dx &\approx \frac{5-1}{4} \left[\frac{1}{2}(1) + 0.25 + 0.1111 \right. \\
 &\quad \left. + 0.0625 + \frac{1}{2}(0.04) \right] \\
 &\approx 0.9436
 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
 \int_1^5 \frac{1}{x^2} dx &\approx \frac{5-1}{12} [1 + 4(0.25) + 2(0.1111) \\
 &\quad + 4(0.0625) + 0.04] \\
 &\approx 0.8374
 \end{aligned}$$

(c) Exact value:

$$\begin{aligned}
 \int_1^5 x^{-2} dx &= -x^{-1} \Big|_1^5 \\
 &= -\frac{1}{5} + 1 \\
 &= \frac{4}{5} = 0.8
 \end{aligned}$$

8. $\int_2^4 \frac{1}{x^3} dx$

$$n = 4, b = 4, a = 2, f(x) = \frac{1}{x^3}$$

i	x_i	$f(x_i)$
0	2	0.125
1	2.5	0.064
2	3	0.03703
3	3.5	0.02332
4	4	0.015625

(a) Trapezoidal rule:

$$\begin{aligned}
 \int_2^4 \frac{dx}{x^3} &\approx \frac{4-2}{4} \left[\frac{1}{2}(0.125) + 0.064 + 0.03703 \right. \\
 &\quad \left. + 0.02332 + \frac{1}{2}(0.015625) \right] \\
 &\approx \frac{1}{2}(0.19466) \approx 0.0973
 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
 \int_2^4 \frac{dx}{x^3} &\approx \frac{4-2}{3(4)} [0.125 + 4(0.064) + 2(0.03703) \\
 &\quad + 4(0.02332) + 0.015625] \\
 &\approx \frac{1}{6}(0.056397) \\
 &\approx 0.0940
 \end{aligned}$$

(c) Exact value:

$$\begin{aligned}
 \int_2^4 \frac{dx}{x^3} &= \int_2^4 x^{-3} dx = \frac{x^{-2}}{-2} \Big|_2^4 = \frac{-1}{2x^2} \Big|_2^4 \\
 &= \frac{-1}{32} + \frac{1}{8} = \frac{3}{32} = 0.09375
 \end{aligned}$$

9. $\int_0^1 4xe^{-x^2} dx$

$$n = 4, b = 1, a = 0, f(x) = 4xe^{-x^2}$$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{1}{4}$	$e^{-1/16}$
2	$\frac{1}{2}$	$2e^{-1/4}$
3	$\frac{3}{4}$	$3e^{-9/16}$
4	1	$4e^{-1}$

(a) Trapezoidal rule:

$$\begin{aligned}
 & \int_0^1 4xe^{-x^2} dx \\
 & \approx \frac{1-0}{4} \left[\frac{1}{2}(0) + e^{-1/16} + 2e^{-1/4} \right. \\
 & \quad \left. + 3e^{-9/16} + \frac{1}{2}(4e^{-1}) \right] \\
 & = \frac{1}{4}(e^{-1/16} + 2e^{-1/4} + 3e^{-9/16} + 2e^{-1}) \\
 & \approx 1.236
 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
 & \int_0^1 4xe^{-x^2} dx \\
 & \approx \frac{1-0}{3(4)} [0 + 4(e^{-1/16}) + 2(2e^{-1/4}) \\
 & \quad + 4(3e^{-9/16}) + 4e^{-1}] \\
 & = \frac{1}{12}(4e^{-1/16} + 4e^{-1/4} + 12e^{-9/16} + 4e^{-1}) \\
 & \approx 1.265
 \end{aligned}$$

(c) Exact value:

$$\begin{aligned}
 \int_0^1 4xe^{-x^2} dx & = -2e^{-x^2} \Big|_0^1 \\
 & = (-2e^{-1}) - (-2) \\
 & = 2 - 2e^{-1} \approx 1.264
 \end{aligned}$$

10. $\int_0^4 x\sqrt{2x^2 + 1} dx$

$$n = 4, b = 4, a = 0, f(x) = x\sqrt{2x^2 + 1}$$

i	x_i	$f(x_i)$
0	0	0
1	1	$\sqrt{3}$
2	2	6
3	3	$3\sqrt{19}$
4	4	$4\sqrt{33}$

(a) Trapezoidal rule:

$$\begin{aligned}
 & \int_0^4 x\sqrt{2x^2 + 1} dx \\
 & \approx \frac{4-0}{4} \left[\frac{1}{2}(0) + \sqrt{3} + 6 + 3\sqrt{19} + \frac{1}{2}(4\sqrt{33}) \right] \\
 & = 1(\sqrt{3} + 6 + 3\sqrt{19} + 2\sqrt{33}) \\
 & \approx 32.30
 \end{aligned}$$

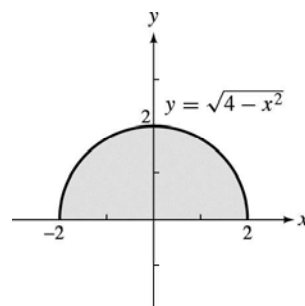
(b) Simpson's rule:

$$\begin{aligned}
 & \int_0^4 x\sqrt{2x^2 + 1} dx \\
 & \approx \frac{4-0}{3(4)} [0 + 4(\sqrt{3}) + 2(6) \\
 & \quad + 4(3\sqrt{19}) + 4\sqrt{33}] \\
 & = \frac{1}{3}(4\sqrt{3} + 12 + 12\sqrt{19} + 4\sqrt{33}) \\
 & \approx 31.40
 \end{aligned}$$

(c) Exact value:

$$\begin{aligned}
 \int_0^4 x\sqrt{2x^2 + 1} dx & = \frac{(2x^2 + 1)^{3/2}}{6} \Big|_0^4 \\
 & = \frac{33^{3/2} - 1}{6} \approx 31.43
 \end{aligned}$$

11. $y = \sqrt{4 - x^2}$



$$n = 8, b = 2, a = -2, f(x) = \sqrt{4 - x^2}$$

i	x_i	y
0	-2.0	0
1	-1.5	1.32289
2	-1.0	1.73205
3	-0.5	1.93649
4	0	2
5	0.5	1.93649
6	1.0	1.73205
7	1.5	1.32289
8	2.0	0

(a) Trapezoidal rule:

$$\begin{aligned}
 & \int_{-2}^2 \sqrt{4 - x^2} dx \\
 & \approx \frac{2 - (-2)}{8} \\
 & \quad \cdot \left[\frac{1}{2}(0) + 1.32289 + 1.73205 + \cdots + \frac{1}{2}(0) \right] \\
 & \approx 5.991
 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
 & \int_{-2}^2 \sqrt{4-x^2} \, dx \\
 & \approx \frac{2 - (-2)}{3(8)} \\
 & \cdot [0 + 4(1.32289) + 2(1.73205) + 4(1.93649) + 2(2) \\
 & + 4(1.93649) + 2(1.73205) + 4(1.32289) + 0] \\
 & \approx 6.167
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) Area of semicircle} &= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 \\
 &\approx 6.283
 \end{aligned}$$

Simpson's rule is more accurate.

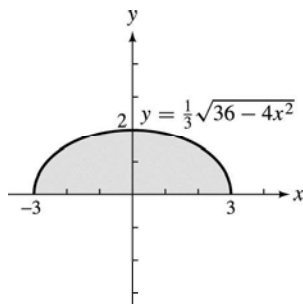
$$12. \quad 4x^2 + 9y^2 = 36$$

$$y^2 = \frac{36 - 4x^2}{9}$$

$$y = \pm \frac{1}{3}\sqrt{36 - 4x^2}$$

An equation of the semiellipse is

$$y = \frac{1}{3}\sqrt{36 - 4x^2}.$$



$$n = 12, b = -3, a = 3$$

i	x_i	y
0	-3	0
1	-2.5	1.1055
2	-2	1.4907
3	-1.5	1.7321
4	-1	1.8856
5	-0.5	1.9720
6	0	2
7	0.5	1.9720
8	1	1.8856
9	1.5	1.7321
10	2	1.4907
11	2.5	1.1055
12	3	0

(a) Trapezoidal rule:

$$\begin{aligned}
 \frac{A}{2} &= \frac{6}{12} \left[\frac{1}{2}(0) + 1.1055 + 1.4907 + 1.7321 \right. \\
 &\quad + 1.8856 + 1.972 + 2 + 1.972 + 1.8856 \\
 &\quad \left. + 1.7321 + 1.4907 + 1.1055 + 2(0) \right] \\
 &\approx 9.186
 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
 \frac{A}{2} &= \frac{6}{3(12)} [0 + 4(1.1055) + 2(1.4907) \\
 &\quad + 4(1.7321) + 2(1.8856) + 4(1.972) \\
 &\quad + 2(2) + 4(1.972) + 2(1.8856) + 4(1.7321) \\
 &\quad + 2(1.4907) + 4(1.1055) + 0] \\
 &= \frac{1}{6}(55.982) \\
 &\approx 9.330
 \end{aligned}$$

(c) The trapezoidal rule gives the area of the region as 9.1859. Simpson's rule gives the area of the region as 9.3304. The actual area is $3\pi \approx 9.4248$. Simpson's rule is a better approximation.

13. Since $f(x) > 0$ and $f''(x) > 0$ for all x between a and b , we know the graph of $f(x)$ on the interval from a to b is concave upward. Thus, the trapezoid that approximates the area will have an area greater than the actual area. Thus,

$$T > \int_a^b f(x) \, dx.$$

The correct choice is (b).

14. (a) $f(x) = x^2; [0, 3]$

$$T > \int_a^b f(x) \, dx$$

By looking at the graph of $y = x^2$ and dividing the area between 0 and 3 into an even number of trapezoids, you can see that each trapezoid has an area greater than the actual area [case (b)].

- (b) $f(x) = \sqrt{x}; [0, 9]$

$$T < \int_a^b f(x) \, dx$$

By looking at the graph of $y = \sqrt{x}$ and dividing area between 0 and 9 into an even number of trapezoids, you can see that each

trapezoid has an area less than the actual area [case (a)].

- (c) You can't say which is larger because some trapezoids are greater than the given area and some are less than the given area [case (c)].

$$15. \quad (a) \quad \int_0^1 x^4 dx = \left(\frac{1}{5} \right) x^5 \Big|_0^1 \\ = \frac{1}{5} \\ = 0.2$$

$$(b) \quad n = 4, b = 1, a = 0, f(x) = x^4$$

$$\int_0^1 x^4 dx \approx \frac{1-0}{4} \left[\frac{1}{2}(0) + \frac{1}{256} + \frac{1}{16} + \frac{81}{256} + \frac{1}{2}(1) \right] \\ = \frac{1}{4} \left(\frac{226}{256} \right) \\ \approx 0.220703$$

$$n = 8, b = 1, a = 0, f(x) = x^4$$

$$\int_0^1 x^4 dx \approx \frac{1-0}{8} \left[\frac{1}{2}(0) + \frac{1}{4096} + \frac{1}{256} + \frac{81}{4096} \right. \\ \left. + \frac{1}{16} + \frac{625}{4096} + \frac{81}{256} + \frac{2401}{4096} + \frac{1}{2}(1) \right] \\ = \frac{1}{8} \left(\frac{6724}{4096} \right) \\ \approx 0.20520$$

$$n = 16, b = 1, a = 0, f(x) = x^4$$

$$\int_0^1 x^4 dx \approx \frac{1-0}{16} \left[\frac{1}{2}(0) + \frac{1}{65,536} + \frac{1}{4096} \right. \\ \left. + \frac{81}{65,536} + \frac{1}{256} + \frac{625}{65,536} \right. \\ \left. + \frac{81}{4096} + \frac{2401}{65,536} + \frac{1}{16} \right. \\ \left. + \frac{6561}{65,536} + \frac{625}{4096} + \frac{14,641}{65,536} \right. \\ \left. + \frac{81}{256} + \frac{28,561}{65,536} + \frac{2401}{4096} \right. \\ \left. + \frac{50,625}{65,536} + \frac{1}{2}(1) \right] \\ \approx \frac{1}{16} \left(\frac{211,080}{65,536} \right) \\ \approx 0.201302$$

$$n = 32, b = 1, a = 0, f(x) = x^4$$

$$\int_0^1 x^4 dx \\ \approx \frac{1-0}{32} \left[\frac{1}{2}(0) + \frac{1}{1,048,576} + \frac{1}{65,536} \right. \\ \left. + \frac{81}{1,048,576} + \frac{1}{4096} + \frac{625}{1,048,576} \right. \\ \left. + \frac{81}{65,536} + \frac{2401}{1,048,576} + \frac{1}{256} + \frac{6561}{1,048,576} \right. \\ \left. + \frac{625}{65,536} + \frac{14,641}{1,048,576} + \frac{81}{4096} + \frac{28,561}{1,048,576} \right. \\ \left. + \frac{2401}{65,536} + \frac{50,625}{1,048,576} + \frac{1}{16} + \frac{83,521}{1,048,576} \right. \\ \left. + \frac{6561}{65,536} + \frac{130,321}{1,048,576} + \frac{625}{4096} + \frac{194,481}{1,048,576} \right. \\ \left. + \frac{14,641}{65,536} + \frac{279,841}{1,048,576} + \frac{81}{256} + \frac{390,625}{1,048,576} \right. \\ \left. + \frac{28,561}{65,536} + \frac{531,441}{1,048,576} + \frac{2401}{4096} + \frac{707,281}{1,048,576} \right. \\ \left. + \frac{50,625}{65,536} + \frac{923,521}{1,048,576} + \frac{1}{2}(1) \right] \\ \approx \frac{1}{32} \left(\frac{6,721,808}{1,048,576} \right) \approx 0.200325$$

To find error for each value of n , subtract as indicated.

$$n = 4: (0.220703 - 0.2) = 0.020703$$

$$n = 8: (0.205200 - 0.2) = 0.005200$$

$$n = 16: (0.201302 - 0.2) = 0.001302$$

$$n = 32: (0.200325 - 0.2) = 0.000325$$

$$(c) \quad p = 1$$

$$4^1(0.020703) = 4(0.020703) \\ = 0.082812$$

$$8^1(0.005200) = 8(0.005200) \\ = 0.0416$$

Since these are not the same, try $p = 2$.

$$p = 2:$$

$$4^2(0.020703) = 16(0.020703) \\ = 0.331248$$

$$8^2(0.005200) = 64(0.005200) = 0.3328$$

$$16^2(0.001302) = 256(0.001302) \\ = 0.333312$$

$$\begin{aligned} 32^2(0.000325) &= 1024(0.000325) \\ &= 0.3328 \end{aligned}$$

Since these values are all approximately the same, the correct choice is $p = 2$.

16. As n changes from 4 to 8, for example, the error changes from 0.020703 to 0.005200.

$$\begin{aligned} 0.020703a &= 0.005200 \\ a &\approx \frac{1}{4} \end{aligned}$$

Similar results would be obtained using other values for n .

The error is multiplied by $\frac{1}{4}$.

$$\begin{aligned} 17. \quad (a) \quad \int_0^1 x^4 dx &= \frac{1}{5} x^5 \Big|_0^1 \\ &= \frac{1}{5} \\ &= 0.2 \end{aligned}$$

$$(b) \quad n = 4, b = 1, a = 0, f(x) = x^4$$

$$\begin{aligned} \int_0^1 x^4 dx &\approx \frac{1-0}{3(4)} \left[0 + 4 \left(\frac{1}{256} \right) + 2 \left(\frac{1}{16} \right) \right. \\ &\quad \left. + 4 \left(\frac{81}{256} \right) + 1 \right] \\ &= \frac{1}{12} \left(\frac{77}{32} \right) \\ &\approx 0.2005208 \end{aligned}$$

$$n = 8, b = 1, a = 0, f(x) = x^4$$

$$\begin{aligned} \int_0^1 x^4 dx &\approx \frac{1-0}{3(8)} \left[0 + 4 \left(\frac{1}{4096} \right) + 2 \left(\frac{1}{256} \right) \right. \\ &\quad \left. + 4 \left(\frac{81}{4096} \right) + 2 \left(\frac{1}{16} \right) + 4 \left(\frac{625}{4096} \right) \right. \\ &\quad \left. + 2 \left(\frac{18}{256} \right) + 4 \left(\frac{2401}{4096} \right) + 1 \right] \\ &= \frac{1}{24} \left(\frac{4916}{1024} \right) \\ &\approx 0.2000326 \end{aligned}$$

$$n = 16, b = 1, a = 0, f(x) = x^4$$

$$\begin{aligned} \int_0^1 x^4 dx &\approx \frac{1-0}{3(16)} \left[0 + 4 \left(\frac{1}{65,536} \right) + 2 \left(\frac{1}{4096} \right) \right. \\ &\quad \left. + 4 \left(\frac{81}{65,536} \right) + 2 \left(\frac{1}{256} \right) + 4 \left(\frac{625}{65,536} \right) \right. \\ &\quad \left. + 2 \left(\frac{81}{4096} \right) + 4 \left(\frac{2401}{65,536} \right) + 2 \left(\frac{1}{16} \right) \right. \\ &\quad \left. + 4 \left(\frac{6561}{65,536} \right) + 2 \left(\frac{625}{4096} \right) + 4 \left(\frac{14,641}{65,536} \right) \right. \\ &\quad \left. + 2 \left(\frac{81}{256} \right) + 4 \left(\frac{28,561}{65,536} \right) + 2 \left(\frac{2401}{4096} \right) \right. \\ &\quad \left. + 4 \left(\frac{50,625}{65,536} + 1 \right) \right] \\ &= \frac{1}{48} \left(\frac{157,288}{16,384} \right) \approx 0.2000020 \end{aligned}$$

$$n = 32, b = 1, a = 0, f(x) = x^4$$

$$\begin{aligned} \int_0^1 x^4 dx &\approx \frac{1-0}{3(32)} \left[0 + 4 \left(\frac{1}{1,048,576} \right) + 2 \left(\frac{1}{65,536} \right) \right. \\ &\quad \left. + 4 \left(\frac{81}{1,048,576} \right) + 2 \left(\frac{1}{4096} \right) + 4 \left(\frac{625}{1,048,576} \right) \right. \\ &\quad \left. + 2 \left(\frac{625}{65,536} \right) + 4 \left(\frac{14,641}{1,048,576} \right) + 2 \left(\frac{81}{4096} \right) \right. \\ &\quad \left. + 4 \left(\frac{28,561}{1,048,576} \right) + 2 \left(\frac{2401}{65,536} \right) + 4 \left(\frac{50,625}{1,048,576} \right) \right. \\ &\quad \left. + 2 \left(\frac{1}{16} \right) + 4 \left(\frac{83,521}{1,048,576} \right) + 2 \left(\frac{6561}{65,536} \right) \right. \\ &\quad \left. + 4 \left(\frac{130,321}{1,048,576} \right) + 2 \left(\frac{625}{4096} \right) + 4 \left(\frac{194,481}{1,048,576} \right) \right. \\ &\quad \left. + 2 \left(\frac{14,641}{65,536} \right) + 4 \left(\frac{279,841}{1,048,576} \right) + 2 \left(\frac{81}{256} \right) \right. \\ &\quad \left. + 4 \left(\frac{390,625}{1,048,576} \right) + 2 \left(\frac{28,561}{65,536} \right) + 4 \left(\frac{531,441}{1,048,576} \right) \right. \\ &\quad \left. + 2 \left(\frac{2401}{4096} \right) + 4 \left(\frac{707,281}{1,048,576} \right) + 2 \left(\frac{50,625}{65,536} \right) \right. \\ &\quad \left. + 4 \left(\frac{923,521}{1,048,576} \right) + 1 \right] \\ &= \frac{1}{96} \left(\frac{50,033,168}{262,144} \right) \approx 0.2000001 \end{aligned}$$

To find error for each value of n , subtract as indicated.

$$n = 4: (0.2005208 - 0.2) = 0.0005208$$

$$n = 8: (0.2000326 - 0.2) = 0.0000326$$

$$n = 16: (0.2000020 - 0.2) = 0.0000020$$

$$n = 32: (0.2000001 - 0.2) = 0.0000001$$

(c) $p = 1$:

$$4^1(0.0005208) = 4(0.0005208) = 0.0020832$$

$$8^1(0.0000326) = 8(0.0000326) = 0.0002608$$

Try $p = 2$:

$$4^2(0.0005208) = 16(0.0005208) = 0.0083328$$

$$8^2(0.0000326) = 64(0.0000326) = 0.0020864$$

Try $p = 3$:

$$4^3(0.0005208) = 64(0.0005208) = 0.0333312$$

$$8^3(0.0000326) = 512(0.0000326) = 0.0166912$$

Try $p = 4$:

$$4^4(0.0005208) = 256(0.0005208) = 0.1333248$$

$$8^4(0.0000326) = 4096(0.0000326) = 0.1335296$$

$$16^4(0.0000020) = 65536(0.0000020) = 0.131072$$

$$32^4(0.0000001) = 1048576(0.0000001) = 0.1048576$$

These are the closest values we can get; thus,
 $p = 4$.

18. As n changes from 4 to 8, the error changes from 0.0005208 to 0.0000326.

$$0.0005208a = 0.0000326$$

$$a \approx \frac{1}{16}$$

Similar results would be obtained using other values for n .

The error is multiplied by $\frac{1}{16}$.

19. Midpoint rule:

$$n = 4, b = 5, a = 1, f(x) = \frac{1}{x^2}, \Delta x = 1$$

i	x_i	$f(x_i)$
0	$\frac{3}{2}$	$\frac{4}{9}$
2	$\frac{5}{2}$	$\frac{4}{25}$
3	$\frac{7}{2}$	$\frac{4}{49}$
4	$\frac{9}{2}$	$\frac{4}{81}$

$$\begin{aligned} \int_1^5 \frac{1}{x^2} dx &\approx \sum_{i=1}^4 f(x_i) \Delta x \\ &= \frac{4}{9}(1) + \frac{4}{25}(1) + \frac{4}{49}(1) + \frac{4}{81}(1) \\ &\approx 0.7355 \end{aligned}$$

Simpson's rule:

$$m = 8, b = 5, a = 1, f(x) = \frac{1}{x^2}$$

i	x_i	$f(x_i)$
0	1	1
1	$\frac{3}{2}$	$\frac{4}{9}$
2	2	$\frac{1}{4}$
3	$\frac{5}{2}$	$\frac{4}{25}$
4	3	$\frac{1}{9}$
5	$\frac{7}{2}$	$\frac{4}{49}$
6	4	$\frac{1}{16}$
7	$\frac{9}{2}$	$\frac{4}{81}$
8	5	$\frac{1}{25}$

$$\begin{aligned} \int_1^5 \frac{1}{x^2} dx &\approx \frac{5-1}{3(8)} \left[1 + 4\left(\frac{4}{9}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{4}{25}\right) \right. \\ &\quad \left. + 2\left(\frac{1}{9}\right) + 4\left(\frac{4}{49}\right) + 2\left(\frac{1}{16}\right) \right. \\ &\quad \left. + 4\left(\frac{4}{81}\right) + \frac{1}{25} \right] \\ &\approx \frac{1}{6}(4.82906) \\ &\approx 0.8048 \end{aligned}$$

From #7 part a, $T \approx 0.9436$, when $n = 4$.

To verify the formula evaluate $\frac{2M+T}{3}$.

$$\begin{aligned} \frac{2M+T}{3} &\approx \frac{2(0.7355) + 0.9436}{3} \\ &\approx 0.8048 \end{aligned}$$

20. Midpoint rule:

$$n = 4, b = 4, a = 2, f(x) = \frac{1}{x^3},$$

$$\Delta x = \frac{1}{2}$$

i	x_i	$f(x_i)$
1	$\frac{9}{4}$	$\frac{64}{729}$
2	$\frac{11}{4}$	$\frac{64}{1331}$
3	$\frac{13}{4}$	$\frac{1}{4}$
4	$\frac{15}{4}$	$\frac{64}{2197}$
4	$\frac{15}{4}$	$\frac{64}{3375}$

$$\begin{aligned} \int_2^4 \frac{1}{x^3} dx &\approx \sum_{i=1}^4 f(x_i) \Delta x \\ &= \frac{64}{729} \left(\frac{1}{2} \right) + \frac{64}{1331} \left(\frac{1}{2} \right) + \frac{64}{2197} \left(\frac{1}{2} \right) + \frac{64}{3375} \left(\frac{1}{2} \right) \\ &\approx 0.09198 \end{aligned}$$

Simpson's rule:

$$n = 8, b = 4, a = 2, f(x) = \frac{1}{x^3}$$

i	x_i	$f(x_i)$
0	2	$\frac{1}{8}$
1	$\frac{9}{4}$	$\frac{64}{729}$
2	$\frac{5}{2}$	$\frac{8}{125}$
3	$\frac{11}{4}$	$\frac{64}{1331}$
4	3	$\frac{1}{27}$
5	$\frac{13}{4}$	$\frac{64}{2197}$
6	$\frac{7}{2}$	$\frac{8}{343}$
7	$\frac{15}{4}$	$\frac{64}{3375}$
8	4	$\frac{1}{64}$

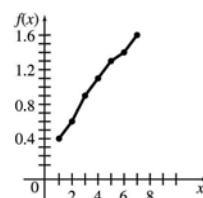
$$\begin{aligned} \int_2^4 \frac{1}{x^3} dx &\approx \frac{4-2}{3(8)} \left[\frac{1}{8} + 4 \left(\frac{64}{729} \right) + 2 \left(\frac{8}{125} \right) + 4 \left(\frac{64}{1331} \right) \right. \\ &\quad \left. + 2 \left(\frac{1}{27} \right) + 4 \left(\frac{64}{2197} \right) + 2 \left(\frac{8}{343} \right) \right. \\ &\quad \left. + 4 \left(\frac{64}{3375} \right) + \frac{1}{64} \right] \\ &\approx \frac{1}{12} (1.125223) \approx 0.09377 \end{aligned}$$

From #8 part a, $T \approx 0.0973$, when $n = 4$.

To verify the formula evaluate $\frac{2M+T}{3}$.

$$\begin{aligned} \frac{2M+T}{3} &\approx \frac{2(0.09198) + 0.0973}{3} \\ &\approx 0.09377 \end{aligned}$$

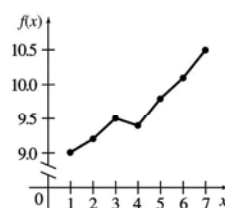
21. (a)



$$\begin{aligned} \text{(b)} \quad A &= \frac{7-1}{6} \left[\frac{1}{2} (0.4) + 0.6 + 0.9 + 1.1 \right. \\ &\quad \left. + 1.3 + 1.4 + \frac{1}{2} (1.6) \right] \\ &= 6.3 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad A &= \frac{7-1}{3(6)} [0.4 + 4(0.6) + 2(0.9) \\ &\quad + 4(1.1) + 2(1.3) + 4(1.4) + 1.6] \\ &\approx 6.27 \end{aligned}$$

22. (a)



$$\begin{aligned} \text{(b)} \quad A &= \frac{7-1}{6} \left[\frac{1}{2} (9) + 9.2 + 9.5 + 9.4 \right. \\ &\quad \left. + 9.8 + 10.1 + \frac{1}{2} (10.5) \right] \\ &= 57.75 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad A &= \frac{7-1}{3(6)}[9.0 + 4(9.2) + 2(9.5) \\
 &\quad + 4(9.4) + 2(9.8) + 4(10.1) + 10.5] \\
 &= \frac{1}{3}(172.9) \\
 &= 57.63
 \end{aligned}$$

$$23. \quad y = e^{-t^2} + \frac{1}{t+1}$$

The total reaction is

$$\int_1^9 \left(e^{-t^2} + \frac{1}{t+1} \right) dt.$$

$$n = 8, b = 9, a = 1, f(t) = e^{-t^2} + \frac{1}{t+1}$$

i	x_i	$f(x_i)$
0	1	0.8679
1	2	0.3516
2	3	0.2501
3	4	0.2000
4	5	0.1667
5	6	0.1429
6	7	0.1250
7	8	0.1111
8	9	0.1000

(a) Trapezoidal rule:

$$\begin{aligned}
 &\int_1^9 \left(e^{-t^2} + \frac{1}{t+1} \right) dt \\
 &\approx \frac{9-1}{8} \left[\frac{1}{2}(0.8679) + 0.3516 + 0.2501 \right. \\
 &\quad \left. + \cdots + \frac{1}{2}(0.1000) \right] \\
 &\approx 1.831
 \end{aligned}$$

(b) Simpson's rule:

$$\begin{aligned}
 &\int_1^9 \left(e^{-t^2} + \frac{1}{t+1} \right) dt \\
 &\approx \frac{9-1}{3(8)} [0.8679 + 4(0.3516) + 2(0.2501) \\
 &\quad + 4(0.2000) + 2(0.1667) + 4(0.1429) \\
 &\quad + 2(0.1250) + 4(0.1111) + 0.1000] \\
 &= \frac{1}{3}(5.2739) \\
 &\approx 1.758
 \end{aligned}$$

$$24. \quad y = \frac{2}{t+2} + e^{-t^2/2}$$

The total growth is

$$\int_1^7 \left(\frac{2}{t+2} + e^{-t^2/2} \right) dt.$$

$$n = 12, b = 7, a = 1, f(t) = \frac{2}{t+2} + e^{-t^2/2}$$

i	x_i	$f(x_i)$
0	1	1.2732
1	1.5	0.8961
2	2	0.6353
3	2.5	0.4884
4	3	0.4111
5	3.5	0.3658
6	4	0.3337
7	4.5	0.3077
8	5	0.2857
9	5.5	0.2667
10	6	0.2500
11	6.5	0.2353
12	7	0.2222

(a) Trapezoidal rule:

$$\begin{aligned}
 &\int_1^7 \left(\frac{2}{t+2} + e^{-t^2/2} \right) dt \\
 &= \frac{7-1}{12} \left[\frac{1}{2}(1.2732) + 0.8961 + 0.6353 \right. \\
 &\quad + 0.4884 + 0.4111 + 0.3658 + 0.3337 \\
 &\quad + 0.3077 + 0.2857 + 0.2667 + 0.2500 \\
 &\quad \left. + 0.2353 + \frac{1}{2}(0.2222) \right] \\
 &= \frac{1}{2}(5.2234) \approx 2.612
 \end{aligned}$$

(b) Simpson's rule

$$\begin{aligned}
 &\int_1^7 \left(\frac{2}{t+2} + e^{-t^2/2} \right) dt \\
 &= \frac{7-1}{3(12)} [1.2732 + 4(0.8961) + 2(0.6353) + 4(0.4884) \\
 &\quad + 2(0.4111) + 4(0.3658) + 2(0.3337) + 4(0.3077) \\
 &\quad + 2(0.2857) + 4(0.2667) + 2(0.2500) + 4(0.2353) \\
 &\quad + 0.2222] \\
 &= \frac{1}{6}(15.5670) \\
 &\approx 2.595
 \end{aligned}$$

25. Note that heights may differ depending on the readings of the graph. Thus, answers may vary.
 $n = 10$, $b = 20$, $a = 0$

i	x_i	$f(x_i)$
0	1	0
1	2	5
2	4	3
3	6	2
4	8	1.5
5	10	1.2
6	12	1
7	14	0.5
8	16	0.3
9	18	0.2
10	20	0.2

Area under curve for Formulation A

$$\begin{aligned}
 &= \frac{20 - 0}{10} \left[\frac{1}{2}(0) + 5 + 3 + 2 + 1.5 + 1.2 \right. \\
 &\quad \left. + 1 + 0.5 + 0.3 + 0.2 + \frac{1}{2}(0.2) \right] \\
 &= 2(14.8) \\
 &\approx 30 \text{ mcg(h)/ml}
 \end{aligned}$$

This represents the total amount of drug available to the patient for each ml of blood.

26. $n = 10$, $b = 20$, $a = 0$

i	x_i	y
0	0	0
1	2	2.0
2	4	2.9
3	6	3.0
4	8	2.5
5	10	2.0
6	12	1.75
7	14	1.0
8	16	0.75
9	18	0.50
10	20	0.25

$$\begin{aligned}
 A &= \frac{20 - 0}{10} \left[\frac{1}{2}(0) + 2 + 2.9 + 3 + 2.5 + 2 \right. \\
 &\quad \left. + 1.75 + 1.0 + 0.75 + 0.5 + \frac{1}{2}(0.25) \right] \\
 &= 33.05 \text{ (This answer may vary depending upon readings from the graph.)}
 \end{aligned}$$

The area under the curve, about 33 mcg(h)/ml, represents the total amount of drug available to the patient for each ml of blood.

27. As in Exercise 25, readings on the graph may vary, so answers may vary. The area both under the curve for Formulation A and above the minimum effective concentration line is on the interval $\left[\frac{1}{2}, 6\right]$.

Area under curve for Formulation A on $\left[\frac{1}{2}, 1\right]$, with $n = 1$

$$\begin{aligned}
 &= \frac{1 - \frac{1}{2}}{1} \left[\frac{1}{2}(2 + 6) \right] \\
 &= \frac{1}{2}(4) = 2
 \end{aligned}$$

Area under curve for Formulation A on $[1, 6]$, with $n = 5$

$$\begin{aligned}
 &= \frac{6 - 1}{5} \left[\frac{1}{2}(6) + 5 + 4 + 3 + 2.4 + \frac{1}{2}(2) \right] \\
 &= 18.4
 \end{aligned}$$

Area under minimum effective concentration line $\left[\frac{1}{2}, 6\right]$

$$= 5.5(2) = 11.0$$

Area under the curve for Formulation A and above minimum effective concentration line

$$\begin{aligned}
 &= 2 + 18.4 - 11.0 \\
 &\approx 9 \text{ mcg(h)/ml}
 \end{aligned}$$

This represents the total effective amount of drug available to the patient for each ml of blood.

28. The area both under the curve for Formulation B and above the minimum effective concentration line is on the interval $(2, 10)$.

$$n = 8, b = 10, a = 2$$

i	x_i	y
0	2	2.0
1	3	2.4
2	4	2.9
3	5	2.8
4	6	3.0
5	7	2.6
6	8	2.5
7	9	2.2
8	10	2.0

Let A_B = area under Formulation B curve between $t = 2$ and $t = 10$.

$$A_B = \frac{10-2}{8} \left[\frac{1}{2}(2) + 2.4 + 2.9 + 2.8 + 3 + 2.6 + 2.5 + 2.2 + \frac{1}{2}(2) \right]$$

$$A_B = 20.4$$

Let A_{ME} = area under minimum effective concentration curve between $t = 2$ and $t = 10$.

$$A_{ME} = (10 - 2)(2) = 16$$

So the area between A_B and A_{ME} between $t = 2$ and $t = 10$ is $20.4 - 16 = 4.4$.

This area, about 4.4 mcg(h)/ml, represents the total effective amount of the drug available to the patient for each ml of blood.

Notice that between $t = 0$ and $t = 12$, the graph for Formulation B is below the line.

Thus, no area exists under the curve for Formulation B and above the minimum effective concentration line in the intervals $(0, 2)$ and $(10, 12)$.

i	t_i	$f(t_i)$
0	0	0
1	1.75	6.89
2	35	7.57
3	52.5	7.78
4	70	7.77
5	87.5	7.65
6	105	7.46
7	122.5	7.23
8	140	6.97
9	157.5	6.70
10	175	6.42

Trapezoidal rule:

$$\begin{aligned} & \int_0^{175} 5.955 \left(\frac{t}{7} \right)^{0.233} e^{-0.027t/7} dt \\ & \approx \frac{175-0}{10} \left[\frac{1}{2}(0) + 6.89 + 7.57 + 7.78 + 7.77 \right. \\ & \quad \left. + 7.65 + 7.46 + 7.23 + 6.97 + 6.70 + \frac{1}{2}(6.42) \right] \\ & = 17.5(69.23) \\ & = 1211.525 \end{aligned}$$

The total milk consumed is about 1212 kg.

29. $y = b_0 w^{b_1} e^{-b_2 w}$

(a) if $t = 7w$ then $w = \frac{t}{7}$.

$$y = b_0 \left(\frac{t}{7} \right)^{b_1} e^{-b_2 t/7}$$

(b) Replacing the constants with the given values, we have

$$y = 5.955 \left(\frac{t}{7} \right)^{0.233} e^{-0.027t/7}$$

In 25 weeks, there are 175 days.

$$\int_0^{175} 5.955 \left(\frac{t}{7} \right)^{0.233} e^{-0.027t/7} dt$$

$$n = 10, b = 175, a = 0,$$

$$f(t) = 5.955 \left(\frac{t}{7} \right)^{0.233} e^{-0.027t/7}$$

Simpson's rule:

$$\begin{aligned} & \int_0^{175} 5.955 \left(\frac{t}{7} \right)^{0.233} e^{-0.027t/7} dt \\ & \approx \frac{175-0}{3(10)} [0 + 4(6.89) + 2(7.57) + 4(7.78) \\ & \quad + 2(7.77) + 4(7.65) + 2(7.46) + 4(7.23) \\ & \quad + 2(6.97) + 4(6.70) + 6.42] \end{aligned}$$

The total milk consumed is about 1231 kg.

(c) Replacing the constants with the given values, we have

$$y = 8.409 \left(\frac{t}{7} \right)^{0.143} e^{-0.037t/7}$$

In 25 weeks, there are 175 days.

$$\int_0^{175} 8.409 \left(\frac{t}{7} \right)^{0.143} e^{-0.037t/7} dt$$

$$n = 10, b = 175, a = 0,$$

$$f(t) = 8.409 \left(\frac{t}{7} \right)^{0.143} e^{-0.037t/7}$$

i	t_i	$f(t_i)$
0	0	0
1	17.5	8.74
2	35	8.80
3	52.5	8.50
4	70	8.07
5	87.5	7.60
6	105	7.11
7	122.5	6.63
8	140	6.16
9	157.5	5.71
10	175	5.28

Trapezoidal rule:

$$\begin{aligned}
 & \int_0^{175} 8.409 \left(\frac{t}{7} \right)^{0.143} e^{-0.037t/7} dt \\
 & \approx \frac{175 - 0}{10} \left[\frac{1}{2}(0) + 8.74 + 8.80 + 8.50 \right. \\
 & \quad + 8.07 + 7.60 + 7.11 + 6.63 \\
 & \quad \left. + 6.16 + 5.71 + \frac{1}{2}(5.28) \right] \\
 & = 17.5(69.96) \\
 & = 1224.30
 \end{aligned}$$

The total milk consumed is about 1224 kg.

Simpson's rule:

$$\begin{aligned}
 & \int_0^{175} 8.409 \left(\frac{t}{7} \right)^{0.143} e^{-0.037t/7} dt \\
 & \approx \frac{175 - 0}{3(10)} [0 + 4(8.74) + 2(8.80) + 4(8.50)] \\
 & \quad + 2(8.07) + 4(7.60) + 2(7.11) + 4(6.63) \\
 & \quad + 2(6.16) + 4(5.71) + 5.28] \\
 & = \frac{35}{6} (214.28) \\
 & = 1249.97
 \end{aligned}$$

The total milk consumed is about 1250 kg.

- 30.** First use the graph to approximate the nine mid-month measurements for both cows and pigs. You may get somewhat different estimates than those shown below, which will result in slightly different answers.

Cows		Pigs	
Month	Cases	Month	Cases
1	3000	1	2000
2	165,000	2	62,000
3	267,000	3	68,000
4	54,000	4	3000
5	44,000	5	1000
6	21,000	6	9000
7	16,500	7	1000
8	11,500	8	0
9	1000	9	0

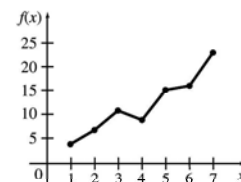
- (a)** Using Simpson's rule with $n = 9$ the computation is as follows:

$$\begin{aligned}
 \text{Total} &= \frac{8}{(3)(8)} \left[3000 + 4(165,000) + 2(267,000) \right. \\
 & \quad + 4(54,000) + 2(44,000) \\
 & \quad + 4(21,000) + 2(16,500) \\
 & \quad \left. + 4(11,500) + 1000 \right] \\
 & = 555,000 \text{ cases}
 \end{aligned}$$

- (b)** Using Simpson's rule with $n = 9$ the computation is as follows:

$$\begin{aligned}
 \text{Total} &= \frac{8}{(3)(8)} \left[2000 + 4(62,000) + 2(68,000) \right. \\
 & \quad + 4(3000) + 2(1000) \\
 & \quad + 4(9000) + 2(1000) \\
 & \quad \left. + 4(0) + 0 \right] \\
 & = 146,000 \text{ cases}
 \end{aligned}$$

- 31. (a)**

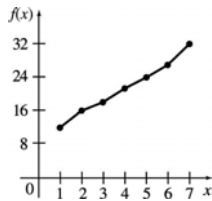


(b) $\frac{7-1}{6} \frac{1}{2}$

$$\begin{aligned}
 & \left[(4) + 7 + 11 + 9 + 15 + 16 + \frac{1}{2}(23) \right] \\
 & = 71.5
 \end{aligned}$$

(c) $\frac{7-1}{3(6)} [4 + 4(7) + 2(11) + 4(9) + 2(15) + 4(16) + 23] = 69.0$

32. (a)



$$(b) \quad A = \frac{7-1}{6} \left[\frac{1}{2}(12) + 16 + 18 + 21 + 24 + 27 + \frac{1}{2}(32) \right]$$

$$A = 1(128) = 128$$

$$(c) \quad A = \frac{7-1}{3(6)} [12 + 4(16) + 2(18) + 4(21) + 2(24) + 4(27) + 32]$$

$$A = 128$$

33. We need to evaluate

$$\int_{12}^{36} (105e^{0.01x} + 32) dx.$$

Using a calculator program for Simpson's rule with $n = 20$, we obtain 3979.242 as the value of this integral. This indicates that the total revenue between the twelfth and thirty-sixth months is about 3979.

34. We need to evaluate

$$\int_7^{182} 3.922t^{0.242} e^{-0.00357t} dt$$

Using a calculator program for Simpson's rule with $n = 20$, we obtain 1400.88 as the value of this integral. This indicates that the total amount of milk consumed by a calf from 7 to 182 days is about 1401 kg.

35. Use a calculator program for Simpson's rule with $n = 20$ to evaluate each of the integrals in this exercise.

$$(a) \quad \int_{-1}^1 \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx \approx 0.6827$$

The probability that a normal random variable is within 1 standard deviation of the mean is about 0.6827.

$$(b) \quad \int_{-2}^2 \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx \approx 0.9545$$

The probability that a normal random variable is within 2 standard deviation of the mean is about 0.9545.

$$(c) \quad \int_{-3}^3 \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx \approx 0.9973$$

The probability that a normal random variable is within 3 standard deviations of the mean is about 0.9973.

Chapter 7 Review Exercises

- True
- False: The statement is false for $n = -1$.
- False: For example, if $f(x) = 1$ the first expression is equal to $x^2/2 + C$ and the second is equal to $x^2 + C$.
- True
- True
- False: The derivative gives the instantaneous rate of change.
- False: If the function is positive over the interval of integration the definite integral gives the exact area.
- True
- True
- False: The definite integral may be positive, negative, or zero.
- True
- False: Sometimes true, but not in general.
- False: The trapezoidal rule allows any number of intervals.
- True
- $$\int (2x + 3) dx = \frac{2x^2}{2} + 3x + C$$

$$= x^2 + 3x + C$$
- $$\int (5x - 1) dx = \frac{5x^2}{2} - x + C$$

$$\begin{aligned}
 21. \quad \int (x^2 - 3x + 2) dx \\
 = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + C
 \end{aligned}$$

$$22. \quad \int (6 - x^2) dx = 6x - \frac{x^3}{3} + C$$

$$\begin{aligned}
 23. \quad \int 3\sqrt{x} dx &= 3 \int x^{1/2} dx \\
 &= \frac{3x^{3/2}}{\frac{3}{2}} + C \\
 &= 2x^{3/2} + C
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \int \frac{\sqrt{x}}{2} dx &= \int \frac{1}{2} x^{1/2} dx \\
 &= \frac{\frac{1}{2} x^{3/2}}{\frac{3}{2}} + C \\
 &= \frac{x^{3/2}}{3} + C
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int (x^{1/2} + 3x^{-2/3}) dx \\
 = \frac{x^{3/2}}{\frac{3}{2}} + \frac{3x^{1/3}}{\frac{1}{3}} + C \\
 = \frac{2x^{3/2}}{3} + 9x^{1/3} + C
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \int (2x^{4/3} + x^{-1/2}) dx \\
 = \frac{2x^{7/3}}{\frac{7}{3}} + \frac{x^{1/2}}{\frac{1}{2}} + C \\
 = \frac{6x^{7/3}}{7} + 2x^{1/2} + C
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \int \frac{-4}{x^3} dx &= \int -4x^{-3} dx \\
 &= \frac{-4x^{-2}}{-2} + C \\
 &= 2x^{-2} + C
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \int \frac{5}{x^4} dx &= \int 5x^{-4} dx \\
 &= \frac{5x^{-3}}{-3} + C \\
 &= -\frac{5}{3x^3} + C
 \end{aligned}$$

$$29. \quad \int -3e^{2x} dx = \frac{-3e^{2x}}{2} + C$$

$$30. \quad \int 5e^{-x} dx = -5e^{-x} + C$$

$$31. \quad \int xe^{3x^2} dx = \frac{1}{6} \int 6xe^{3x^2} dx$$

Let $u = 3x^2$, so that $du = 6x dx$.

$$\begin{aligned}
 &= \frac{1}{6} \int e^u du \\
 &= \frac{1}{6} e^u + C \\
 &= \frac{e^{3x^2}}{6} + C
 \end{aligned}$$

$$32. \quad \int 2xe^{x^2} dx = e^{x^2} + C$$

$$33. \quad \int \frac{3x}{x^2 - 1} dx = 3 \left(\frac{1}{2} \right) \int \frac{2x dx}{x^2 - 1}$$

Let $u = x^2 - 1$, so that $du = 2x dx$.

$$\begin{aligned}
 &= \frac{3}{2} \int \frac{du}{u} \\
 &= \frac{3}{2} \ln |u| + C \\
 &= \frac{3 \ln |x^2 - 1|}{2} + C
 \end{aligned}$$

$$34. \quad \int \frac{-x}{2 - x^2} dx = -\frac{1}{2} \int \frac{-2x dx}{2 - x^2}$$

Let $u = 2 - x^2$, so that
 $du = -2x dx$.

$$\begin{aligned}
 &= \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C \\
 &= \frac{1}{2} \ln |2 - x^2| + C
 \end{aligned}$$

$$35. \int \frac{x^2 dx}{(x^3 + 5)^4} = \frac{1}{3} \int \frac{3x^2 dx}{(x^3 + 5)^4}$$

Let $u = x^3 + 5$, so that

$$\begin{aligned} du &= 3x^2 dx. \\ &= \frac{1}{3} \int \frac{du}{u^4} \\ &= \frac{1}{3} \int u^{-4} du \\ &= \frac{1}{3} \left(\frac{u^{-3}}{-3} \right) + C \\ &= \frac{-(x^3 + 5)^{-3}}{9} + C \end{aligned}$$

$$36. \int (x^2 - 5x)^4 (2x - 5) dx$$

Let $u = x^2 - 5x$, so that

$$\begin{aligned} du &= (2x - 5) dx. \\ \int (x^2 - 5x)^4 (2x - 5) dx \\ &= \int u^4 du \\ &= \frac{u^5}{5} + C \\ &= \frac{(x^2 - 5x)^5}{5} + C \end{aligned}$$

$$37. \int \frac{x^3}{e^{3x^4}} dx = \int x^3 e^{-3x^4} dx$$

$$= -\frac{1}{12} \int -12x^3 e^{-3x^4} dx$$

Let $u = -3x^4$, so that $du = -12x^3 dx$.

$$\begin{aligned} &= -\frac{1}{12} \int e^u du \\ &= -\frac{1}{12} \int e^u + C \\ &= \frac{-e^{-3x^4}}{12} + C \end{aligned}$$

$$38. \int e^{3x^2+4} x dx$$

Let $u = 3x^2 + 4$ so that

$$du = 6x dx.$$

$$\begin{aligned} \int e^{3x^2+4} x dx &= \frac{1}{6} \int (6x) (e^{3x^2}) dx \\ &= \frac{1}{6} \int e^u du \\ &= \frac{1}{6} e^u + C = \frac{e^{3x^2+4}}{6} + C \end{aligned}$$

$$39. \int \frac{(3 \ln x + 2)^4}{x} dx$$

Let $u = 3 \ln x + 2$ so that

$$\begin{aligned} du &= \frac{3}{x} dx. \\ \int \frac{(3 \ln x + 2)^4}{x} dx &= \frac{1}{3} \int \frac{3(3 \ln x + 2)^4}{x} dx \\ &= \frac{1}{3} \int u^4 du \\ &= \frac{1}{3} \cdot \frac{u^5}{5} + C \\ &= \frac{(3 \ln x + 2)^5}{15} + C \end{aligned}$$

$$40. \int \frac{\sqrt{5 \ln x + 3}}{x} dx$$

Let $u = 5 \ln x + 3$ so that

$$\begin{aligned} du &= \frac{5}{x} dx. \\ \int \frac{\sqrt{5 \ln x + 3}}{x} dx &= \frac{1}{5} \int \frac{5\sqrt{5 \ln x + 3}}{x} dx \\ &= \frac{1}{5} \int u^{1/2} du \\ &= \frac{1}{5} \cdot \frac{2u^{3/2}}{3} + C \\ &= \frac{2(5 \ln x + 3)^{3/2}}{15} + C \end{aligned}$$

$$41. f(x) = 3x + 1, x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2, x_5 = 3$$

$$\begin{aligned} f(x_1) &= -2, f(x_2) = 1, f(x_3) = 4, \\ f(x_4) &= 7, f(x_5) = 10 \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^5 f(x_i) \\ &= f(1) + f(2) + f(3) + f(4) + f(5) \\ &= -2 + 1 + 4 + 7 + 10 \\ &= 20 \end{aligned}$$

42. (a) $\int_0^4 f(x) dx = 0$, since the area above the x -axis from 0 to 2 is identical to the area below the x -axis from 2 to 4.

- (b) $\int_0^4 f(x) dx$ can be computed by calculating the area of the rectangle and triangle that make up the region shown in graph.

$$\begin{aligned}\text{Area of rectangle} &= (\text{length})(\text{width}) \\ &= (3)(1) = 3\end{aligned}$$

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(1)(3) = \frac{3}{2}\end{aligned}$$

$$\int_0^4 f(x) dx = 3 + \frac{3}{2} = \frac{9}{2} = 4.5$$

43. $f(x) = 2x + 3$, from $x = 0$ to $x = 4$

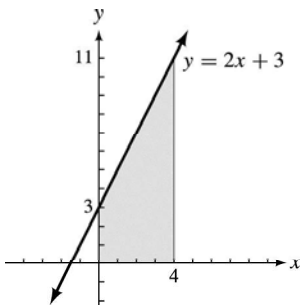
$$\Delta x = \frac{4 - 0}{4} = 1$$

i	x_i	$f(x_i)$
1	0	3
2	1	5
3	2	7
4	3	9

$$\begin{aligned}A &= \sum_{i=1}^4 f(x_i) \Delta x \\ &= 3(1) + 5(1) + 7(1) + 9(1) \\ &= 24\end{aligned}$$

44. $\int_0^4 (2x + 3) dx$

Graph $y = 2x + 3$.



$\int_0^4 (2x + 3) dx$ is the area of a trapezoid with $B = 11$, $b = 3$, $h = 4$. The formula for the area is

$$A = \frac{1}{2}(B + b)h.$$

$$A = \frac{1}{2}(11 + 3)(4) = 28,$$

so

$$\int_0^4 (2x + 3) dx = 28.$$

45. (a) Since $s(t)$ represents the odometer reading, the distance traveled between $t = 0$ and $t = T$ will be $s(T) - s(0)$.

- (b) $\int_0^T v(t) dt = s(T) - s(0)$ is equivalent to the Fundamental Theorem of Calculus with $a = 0$, and $b = T$ because $s(t)$ is an antiderivative of $v(t)$.

46. The Fundamental Theorem of Calculus states that

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a),$$

where f is continuous on $[a, b]$ and F is any antiderivative of f .

$$\begin{aligned}47. \int_1^2 (3x^2 + 5) dx &= \left(\frac{3x^3}{3} + 5x \right) \Big|_1^2 \\ &= (2^3 + 10) - (1 + 5) \\ &= 18 - 6 \\ &= 12\end{aligned}$$

48. $\int_1^6 (2x^2 + x) dx$

$$\begin{aligned}&= \left(\frac{2x^3}{3} + \frac{x^2}{2} \right) \Big|_1^6 \\ &= \left[\frac{2(6)^3}{3} + \frac{(6)^2}{2} \right] - \left[\frac{2(1)^3}{3} + \frac{(1)^2}{2} \right] \\ &= 144 + 18 - \frac{2}{3} - \frac{1}{2} \\ &= 162 - \frac{2}{3} - \frac{1}{2} = \frac{965}{6} \approx 160.83\end{aligned}$$

$$\begin{aligned}
 49. \quad \int_1^5 (3x^{-1} + x^{-3}) dx &= \left(3\ln|x| + \frac{x^{-2}}{-2} \right) \bigg|_1^5 \\
 &= \left(3\ln 5 - \frac{1}{50} \right) - \left(3\ln 1 - \frac{1}{2} \right) \\
 &= 3\ln 5 + \frac{12}{25} \approx 5.308
 \end{aligned}$$

$$\begin{aligned}
 50. \quad \int_1^3 (2x^{-1} + x^{-2}) dx &= \left(2\ln|x| + \frac{x^{-1}}{-1} \right) \bigg|_1^3 \\
 &= \left(2\ln 3 - \frac{1}{3} \right) - (2\ln 1 - 1) \\
 &= 2\ln 3 + \frac{2}{3} \approx 2.864
 \end{aligned}$$

$$51. \quad \int_0^1 x\sqrt{5x^2 + 4} dx$$

Let $u = 5x^2 + 4$, so that

$$du = 10x dx \text{ and } \frac{1}{10} du = x dx.$$

When $x = 0$, $u = 5(0^2) + 4 = 4$.

When $x = 1$, $u = 5(1^2) + 4 = 9$.

$$\begin{aligned}
 &= \frac{1}{10} \int_4^9 \sqrt{u} du = \frac{1}{10} \int_4^9 u^{1/2} du \\
 &= \frac{1}{10} \cdot \frac{u^{3/2}}{3/2} \bigg|_4^9 = \frac{1}{15} u^{3/2} \bigg|_4^9 \\
 &= \frac{1}{15} (9)^{3/2} - \frac{1}{15} (4)^{3/2} \\
 &= \frac{27}{15} - \frac{8}{15} \\
 &= \frac{19}{15}
 \end{aligned}$$

$$\begin{aligned}
 52. \quad \int_0^2 x^2(3x^3 + 1)^{1/3} dx &= \frac{(3x^3 + 1)^{4/3}}{12} \bigg|_0^2 \\
 &= \frac{25^{4/3}}{12} - \frac{1^{4/3}}{12} \\
 &= \frac{25^{4/3} - 1}{12} \approx 6.008
 \end{aligned}$$

$$\begin{aligned}
 53. \quad \int_0^2 3e^{-2x} dx &= \frac{-3e^{-2x}}{2} \bigg|_0^2 \\
 &= \frac{-3e^{-4}}{2} + \frac{3}{2} \\
 &= \frac{3(1 - e^{-4})}{2} \approx 1.473
 \end{aligned}$$

$$\begin{aligned}
 54. \quad \int_1^5 \frac{5}{2} e^{0.4x} dx &= \frac{5}{2} \cdot \frac{5}{2} \int_1^5 0.4e^{0.4x} dx \\
 &= \frac{5}{2} \cdot \frac{5}{2} \cdot e^{2x/5} \bigg|_1^5 \\
 &= \frac{25}{4} (e^2 - e^{0.4}) \\
 &= \frac{25(e^2 - e^{0.4})}{4} \approx 36.86
 \end{aligned}$$

$$55. \quad \int_0^{1/2} x\sqrt{1 - 16x^4} dx$$

Let $u = 4x^2$. Then $du = 8x dx$.

When $x = 0$, $u = 0$, and when $x = \frac{1}{2}$, $u = 1$.

Thus,

$$\int_0^{1/2} x\sqrt{1 - 16x^4} dx = \frac{1}{8} \int_0^1 \sqrt{1 - u^2} du.$$

Note that this integral represents the area of right upper quarter of a circle centered at the origin with a radius of 1.

$$\text{Area of circle} = \pi r^2 = \pi(1^2) = \pi$$

$$\int_0^1 \sqrt{1 - u^2} du = \frac{\pi}{4}$$

$$\frac{1}{8} \int_0^1 \sqrt{1 - u^2} du = \frac{1}{8} \cdot \frac{\pi}{4} = \frac{\pi}{32}$$

$$56. \quad \int_0^{\sqrt{2}} 4x\sqrt{4 - x^2} dx$$

Let $u = x^2$.

Then $du = 2x dx$.

When $x = 0$, $u = 0$.

When $x = \sqrt{2}$, $u = 2$.

Substitute:

$$\int_0^{\sqrt{2}} 4x\sqrt{4 - x^2} dx = 2 \int_0^2 \sqrt{4 - u^2} du$$

The integral on the right gives the area of a semicircle of radius 2 (the bottom half of a circle of radius 2 centered at the point $(0, 2)$).

This area is

$$\frac{1}{2}\pi(2)^2 = 2\pi. \text{ Thus}$$

$$\int_0^{\sqrt{2}} 4x\sqrt{4-x^2} dx = 2\pi.$$

$$57. \int_1^{e^5} \frac{\sqrt{25 - (\ln x)^2}}{x} dx$$

Let $u = \ln x$. Then $du = \frac{1}{x} dx$.

When $x = e^5$, $u = \ln(e^5) = 5$.

When $x = 1$, $u = \ln(1) = 0$.

Thus,

$$\int_1^{e^5} \frac{\sqrt{25 - (\ln x)^2}}{x} dx = \int_0^5 \sqrt{25 - u^2} du.$$

Note that this integral represents the area of a right upper quarter of a circle centered at the origin with a radius of 5.

$$\text{Area of circle} = \pi r^2 = \pi(5)^2 = 25\pi$$

$$\int_0^5 \sqrt{25 - u^2} du = \frac{25\pi}{4}$$

$$58. \int_1^{\sqrt{7}} 2x\sqrt{36 - (x^2 - 1)^2} dx$$

Let $u = x^2 - 1$. Then $du = 2x dx$.

When $x = \sqrt{7}$, $u = (\sqrt{7})^2 - 1 = 6$.

When $x = 1$, $u = (\sqrt{1})^2 - 1 = 0$.

Thus,

$$\int_1^{\sqrt{7}} 2x\sqrt{36 - (x^2 - 1)^2} dx = \int_0^6 \sqrt{36 - u^2} du.$$

Note that this integral represents the area of a right upper quarter of a circle centered at the origin with a radius of 6.

$$\text{Area of circle} = \pi r^2 = \pi(6)^2 = 36\pi$$

$$\int_0^6 \sqrt{36 - u^2} du = \frac{36\pi}{4} = 9\pi$$

$$59. f(x) = \sqrt{4x - 3}; [1, 3]$$

$$\begin{aligned} \text{Area} &= \int_1^3 \sqrt{4x - 3} dx \\ &= \int_1^3 (4x - 3)^{1/2} dx \\ &= \frac{2}{3} \cdot \frac{1}{4} \cdot (4x - 3)^{3/2} \Big|_1^3 \\ &= \frac{1}{6}(9)^{3/2} - \frac{1}{6}(1)^{3/2} \\ &= \frac{1}{6}(26) \\ &= \frac{13}{3} \end{aligned}$$

$$60. f(x) = (3x + 2)^6; [-2, 0]$$

$$\begin{aligned} \text{Area} &= \int_{-2}^0 (3x + 2)^6 dx \\ &= \frac{(3x + 2)^7}{21} \Big|_{-2}^0 \\ &= \frac{2^7}{21} - \frac{(-4)^7}{21} \\ &= \frac{5504}{7} \end{aligned}$$

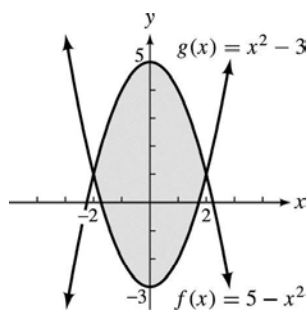
$$61. f(x) = xe^{x^2}; [0, 2]$$

$$\begin{aligned} \text{Area} &= \int_0^2 xe^{x^2} dx \\ &= \frac{e^{x^2}}{2} \Big|_0^2 \\ &= \frac{e^4}{2} - \frac{1}{2} \\ &= \frac{e^4 - 1}{2} \\ &\approx 26.80 \end{aligned}$$

$$62. f(x) = 1 + e^{-x}; [0, 4]$$

$$\begin{aligned} \int_0^4 (1 + e^{-x}) dx &= (x - e^{-x}) \Big|_0^4 \\ &= (4 - e^{-4}) - (0 - e^0) \\ &= 5 - e^{-4} \\ &\approx 4.982 \end{aligned}$$

63. $f(x) = 5 - x^2$, $g(x) = x^2 - 3$



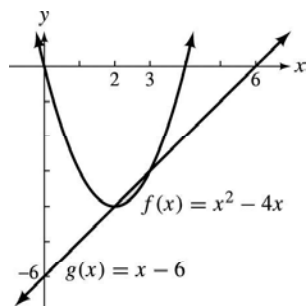
Points of intersection:

$$\begin{aligned} 5 - x^2 &= x^2 - 3 \\ 2x^2 - 8 &= 0 \\ 2(x^2 - 4) &= 0 \\ x &= \pm 2 \end{aligned}$$

Since $f(x) \geq g(x)$ in $[-2, 2]$, the area between the graphs is

$$\begin{aligned} \int_{-2}^2 [f(x) - g(x)] dx &= \int_{-2}^2 [(5 - x^2) - (x^2 - 3)] dx \\ &= \int_{-2}^2 (-2x^2 + 8) dx \\ &= \left(-\frac{2x^3}{3} + 8x \right) \bigg|_{-2}^2 \\ &= -\frac{2}{3}(8) + 16 + \frac{2}{3}(-8) - 8(-2) \\ &= -\frac{32}{3} + 32 = \frac{64}{3}. \end{aligned}$$

64. $f(x) = x^2 - 4x$; $g(x) = x - 6$



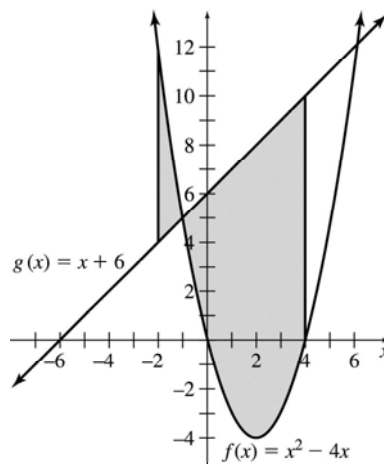
Find the points of intersection.

$$\begin{aligned} x^2 - 4x &= x - 6 \\ x^2 - 5x + 6 &= 0 \\ (x - 3)(x - 2) &= 0 \\ x &= 2 \text{ or } x = 3 \end{aligned}$$

Since $g(x) \geq f(x)$ in the interval $[2, 3]$, the area between the graphs is

$$\begin{aligned} \int_2^3 [g(x) - f(x)] dx &= \int_2^3 [(x - 6) - (x^2 - 4x)] dx \\ &= \int_2^3 (-x^2 + 5x - 6) dx \\ &= \left(-\frac{x^3}{3} + \frac{5x^2}{2} - 6x \right) \bigg|_2^3 \\ &= \frac{-27}{3} + \frac{5(9)}{2} - 6(3) - \left(\frac{-8}{3} - \frac{5(4)}{2} + 6(2) \right) \\ &= -\frac{19}{3} + \frac{25}{2} - 6 = \frac{1}{6}. \end{aligned}$$

65. $f(x) = x^2 - 4x$, $g(x) = x + 6$,
 $x = -2$, $x = 4$



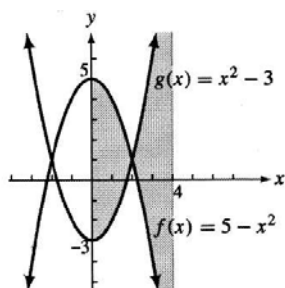
Points of intersection:

$$\begin{aligned} x^2 - 4x &= x + 6 \\ x^2 - 5x - 6 &= 0 \\ (x + 1)(x - 6) &= 0 \\ x &= -1 \text{ or } x = 6 \end{aligned}$$

Thus, the area is

$$\begin{aligned} \int_{-2}^{-1} [x^2 - 4x - (x + 6)] dx &+ \int_{-1}^4 [x + 6 - (x^2 - 4x)] dx \\ &= \left(\frac{x^3}{3} - \frac{5x^2}{2} - 6x \right) \bigg|_{-2}^{-1} + \left(-\frac{x^3}{3} + \frac{5x^2}{2} + 6x \right) \bigg|_{-1}^4 \\ &= \left(\frac{19}{6} + \frac{2}{3} \right) + \left(\frac{128}{3} + \frac{19}{6} \right) = \frac{149}{3} \end{aligned}$$

66. $f(x) = 5 - x^2$, $g(x) = x^2 - 3$, $x = 0$, $x = 4$



Find the points of intersection.

$$5 - x^2 = x^2 - 3$$

$$8 = 2x^2$$

$$4 = x^2$$

$$\pm 2 = x$$

The curves intersect at $x = 2$ and $x = -2$.

Thus, area is

$$\begin{aligned} & \int_0^2 [(5 - x^2) - (x^2 - 3)] dx \\ & + \int_2^4 [(x^2 - 3) - (5 - x^2)] dx \\ & = \int_0^2 (-2x^2 + 8) dx + \int_2^4 (2x^2 - 8) dx \\ & = \left(\frac{-2x^3}{3} + 8x \right) \Big|_0^2 + \left(\frac{2x^3}{3} - 8x \right) \Big|_2^4 \\ & = \frac{-16}{3} + 16 + \left(\frac{128}{3} - 32 \right) - \left(\frac{16}{3} - 16 \right) \\ & = \frac{32}{3} + \frac{128}{3} - 32 - \frac{16}{3} + 16 \\ & = 32. \end{aligned}$$

67. $\int_1^3 \frac{\ln x}{x} dx$

Trapezoidal Rule:

$n = 4$, $b = 3$, $a = 1$, $f(x) = \frac{\ln x}{x}$

i	x_i	$f(x_i)$
0	1	0
1	1.5	0.27031
2	2	0.34657
3	2.5	0.36652
4	3	0.3662

$$\begin{aligned} \int_1^3 \frac{\ln x}{x} dx & \approx \frac{3-1}{4} \left[\frac{1}{2} (0) + 0.27031 + 0.34657 \right. \\ & \quad \left. + 0.36652 + \frac{1}{2} (0.3662) \right] \\ & = 0.5833 \end{aligned}$$

Exact Value:

$$\begin{aligned} & \int_1^3 \frac{\ln x}{x} dx \\ & = \frac{1}{2} (\ln x)^2 \Big|_1^3 = \frac{1}{2} (\ln 3)^2 - \frac{1}{2} (\ln 1)^2 \\ & \approx 0.6035 \end{aligned}$$

68. $\int_2^{10} \frac{x dx}{x-1}$

Trapezoidal Rule:

$n = 4$, $b = 10$, $a = 2$, $f(x) = \frac{x}{x-1}$

i	x_i	$f(x_i)$
0	2	2
1	4	$\frac{4}{3}$
2	6	$\frac{6}{5}$
3	8	$\frac{8}{7}$
4	10	$\frac{10}{9}$

$$\begin{aligned} & \int_2^{10} \frac{x}{x-1} dx \\ & \approx \frac{10-2}{4} \left[\frac{1}{2} (2) + \frac{4}{3} + \frac{6}{5} + \frac{8}{7} + \frac{1}{2} \left(\frac{10}{9} \right) \right] \\ & \approx 10.46 \end{aligned}$$

Exact Value:

Let $u = x - 1$, so that $du = dx$ and $x = u + 1$.

Then

$$\begin{aligned} & \int_2^{10} \frac{x}{x-1} dx = \int_1^9 \frac{u+1}{u} du \\ & = \int_1^9 \left(1 + \frac{1}{u} \right) du = \int_1^9 du + \int_1^9 \frac{1}{u} du \\ & = u \Big|_1^9 + \ln |u| \Big|_1^9 = (9-1) + (\ln 9 - \ln 1) \\ & = 8 + \ln 9 \approx 10.20. \end{aligned}$$

69. $\int_0^1 e^x \sqrt{e^x + 4} dx$

Trapezoidal Rule:

$$n = 4, b = 1, a = 0, f(x) = e^x \sqrt{e^x + 4}$$

i	x_i	$f(x_i)$
0	0	2.236
1	0.25	2.952
2	0.5	3.919
3	0.75	5.236
4	1	7.046

$$\begin{aligned} \int_0^1 e^x \sqrt{e^x + 4} dx &= \frac{1-0}{4} \left[\frac{1}{2}(2.236) + 2.952 \right. \\ &\quad \left. + 3.919 + 5.236 + \frac{1}{2}(7.046) \right] \\ &\approx 4.187 \end{aligned}$$

Exact value:

$$\begin{aligned} \int_0^1 e^x \sqrt{e^x + 4} dx &= \int_0^1 e^x (e^x + 4)^{1/2} dx \\ &= \frac{2}{3} (e^x + 4)^{3/2} \Big|_0^1 \\ &= \frac{2}{3} (e + 4)^{3/2} - \frac{2}{3} (5)^{3/2} \\ &\approx 4.155 \end{aligned}$$

70. $\int_0^2 x e^{-x^2} dx$

Trapezoidal rule:

$$n = 4, b = 2, a = 0, f(x) = x e^{-x^2}$$

i	x_i	$f(x_i)$
0	0	0
1	0.5	0.3894
2	1	0.3679
3	1.5	0.1581
4	2	0.0366

$$\begin{aligned} \int_0^2 x e^{-x^2} dx &\approx \frac{2-0}{4} \left[\frac{1}{2}(0) + 0.3894 + 0.3679 \right. \\ &\quad \left. + 0.1581 + \frac{1}{2}(0.0366) \right] \\ &= 0.4688 \end{aligned}$$

Exact value:

$$\begin{aligned} \int_0^2 x e^{-x^2} dx &= -\frac{1}{2} \int_0^2 e^{-x^2} (-2x dx) \\ &= -\frac{1}{2} e^{-x^2} \Big|_0^2 \\ &= -\frac{1}{2} e^{-4} + \frac{1}{2} e^0 \\ &= \frac{1}{2} (1 - e^{-4}) \\ &\approx 0.4908 \end{aligned}$$

71. $\int_1^3 \frac{\ln x}{x} dx$

Simpson's rule:

$$n = 4, b = 3, a = 1, f(x) = \frac{\ln x}{x}$$

i	x_i	$f(x_i)$
0	1	0
1	1.5	0.27031
2	2	0.34657
3	2.5	0.36652
4	3	0.3662

$$\begin{aligned} \int_1^3 \frac{\ln x}{x} dx &\approx \frac{3-1}{3(4)} [0 + 4(0.27031) + 2(0.34657) \\ &\quad + 4(0.36652) + 0.3662] \\ &\approx 0.6011 \end{aligned}$$

This answer is close to the value of 0.6035 obtained from the exact integral in Exercise 67.

72. $\int_2^{10} \frac{x}{x-1} dx$

Simpson's Rule:

i	x_i	$f(x_i)$
0	2	2
1	4	$\frac{4}{3}$
2	6	$\frac{6}{5}$
3	8	$\frac{8}{7}$
4	10	$\frac{10}{9}$

$$\begin{aligned} \int_2^{10} \frac{x}{x-1} dx &\approx \frac{10-2}{3(4)} \left[2 + 4\left(\frac{4}{3}\right) + 2\left(\frac{6}{5}\right) + 4\left(\frac{8}{7}\right) + \left(\frac{10}{9}\right) \right] \\ &\approx 10.28 \end{aligned}$$

This answer is close to the answer of 10.20 obtained from the exact integral in Exercise 68.

73. $\int_0^1 e^x \sqrt{e^x + 4} dx$

Simpson's rule:

$$n = 4, b = 1, a = 0, f(x) = e^x \sqrt{e^x + 4}$$

i	x_i	$f(x_i)$
0	0	2.236
1	0.25	2.952
2	0.5	3.919
3	0.75	5.236
4	1	7.046

$$\begin{aligned} \int_0^1 e^x \sqrt{e^x + 4} dx &= \frac{1-0}{3(4)} [2.236 + 4(2.952) + 2(3.919) \\ &\quad + 4(5.236) + 7.046] \\ &\approx 4.156 \end{aligned}$$

This answer is close to the answer of 4.155 obtained from the exact integral in Exercise 69.

74. $\int_0^2 x e^{-x^2} dx$

Simpson's rule:

$$n = 4, b = 2, a = 0, f(x) = x e^{-x^2}$$

i	x_i	$f(x_i)$
0	0	0
1	0.5	0.3894
2	1	0.3679
3	1.5	0.1581
4	2	0.0366

$$\begin{aligned} \int_0^2 x e^{-x^2} dx &\approx \frac{2-0}{3(4)} [0 + 4(0.3894) + 2(0.3679) \\ &\quad + 4(0.1581) + 0.0366] \\ &\approx 0.4937 \end{aligned}$$

The answer is close to the exact value obtained in Exercise 70, which is approximately 0.4908.

75. (a) $\int_1^5 \left[\sqrt{x-1} - \left(\frac{x-1}{2} \right) \right] dx$

$$\begin{aligned} &= \int_1^5 \left(\sqrt{x-1} - \frac{x}{2} + \frac{1}{2} \right) dx \\ &= \left(\frac{2}{3} (x-1)^{3/2} - \frac{x^2}{4} + \frac{x}{2} \right) \Big|_1^5 \\ &= \left(\frac{16}{3} - \frac{25}{4} + \frac{5}{2} \right) - \left(0 - \frac{1}{4} + \frac{1}{2} \right) \\ &= \frac{16}{2} - 6 + 2 = \frac{4}{3} \end{aligned}$$

(b) $n = 4, b = 5, a = 1,$

$$f(x) = \sqrt{x-1} - \frac{x}{2} + \frac{1}{2}$$

i	x_i	$f(x_i)$
0	1	0
1	2	0.5
2	3	0.41421
3	4	0.23205
4	5	0

$$\begin{aligned}
 & \int_1^5 \left(\sqrt{x-1} - \frac{x}{2} + \frac{1}{2} \right) dx \\
 &= \left(\frac{5-1}{4} \right) \left[\frac{1}{2}(0) + 0.5 + 0.41421 \right. \\
 &\quad \left. + 0.23205 + \frac{1}{2}(0) \right] \\
 &= 1.146
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int_1^5 \left(\sqrt{x-1} - \frac{x}{2} + \frac{1}{2} \right) dx \\
 &= \left(\frac{5-1}{3(4)} \right) [0 + 4(0.5) + 2(0.41421) \\
 &\quad + 4(0.23205) + 0] \\
 &= \left(\frac{1}{3} \right) (3.75662) \\
 &= 1.252
 \end{aligned}$$

$$\begin{aligned}
 76. \quad \text{(a)} \quad & \int_0^4 \left(\frac{x+2}{2} - \frac{1}{x+1} \right) dx \\
 &= \left(\frac{(x+2)^2}{4} - \ln|x+1| \right) \Big|_0^4 \\
 &= (9 - \ln 5) - (1 - 0) \\
 &= 8 - \ln 5 \\
 &\approx 6.391
 \end{aligned}$$

(b) Trapezoidal rule:

$$n = 4, b = 4, a = 0, f(x) = \frac{x+2}{2} - \frac{1}{x+1}$$

i	x_i	$f(x_i)$
0	0	0
1	1	1
2	2	1.6667
3	3	2.25
4	4	2.8

$$\begin{aligned}
 & \int_0^4 \left(\frac{x+2}{2} - \frac{1}{x+1} \right) dx \\
 &\approx \frac{4-0}{4} \left[\frac{1}{2}(0) + 1 + 1.6667 + 2.25 + \frac{1}{2}(2.8) \right] \\
 &\approx 6.317
 \end{aligned}$$

(c) Simpson's rule:

$$n = 4, b = 4, a = 0, f(x) = \frac{x+2}{2} - \frac{1}{x+1}$$

i	x_i	$f(x_i)$
0	0	0
1	1	1
2	2	1.6667
3	3	2.25
4	4	2.8

$$\begin{aligned}
 & \int_0^4 \left(\frac{x+2}{2} - \frac{1}{x+1} \right) dx \\
 &\approx \frac{4-0}{4(3)} [0 + 4(1) + 2(1.6667) + 4(2.25) + 2.8] \\
 &\approx 6.378
 \end{aligned}$$

$$77. \quad \int_{-2}^2 [x(x-1)(x+1)(x-2)(x+2)]^2 dx$$

(a) Trapezoidal Rule:

$$n = 4, b = -2, a = 2,$$

$$f(x) = [x(x-1)(x+1)(x-2)(x+2)]^2$$

i	x_i	$f(x_i)$
0	-2	0
1	-1	0
2	0	0
3	1	0
4	2	0

$$\begin{aligned}
 & \int_{-2}^2 [x(x-1)(x+1)(x-2)(x+2)]^2 dx \\
 &\approx \frac{2-(-2)}{4} \left[\frac{1}{2}(0) + 0 + 0 + 0 + \frac{1}{2}(0) \right] \\
 &= 0
 \end{aligned}$$

(b) Simpson's Rule:

$$n = 4, b = 2, a = -2,$$

$$f(x) = [x(x-1)(x+1)(x-2)(x+2)]^2$$

i	x_i	$f(x_i)$
0	-2	0
1	-1	0
2	0	0
3	1	0
4	2	0

$$\begin{aligned}
 & \int_{-2}^2 [x(x-1)(x+1)(x-2)(x+2)]^2 dx \\
 & \approx \frac{2 - (-2)}{3(4)} [0 + 4(0) + 2(0) + 4(0) + 0] \\
 & = 0
 \end{aligned}$$

$$\begin{aligned}
 78. \quad \int_0^2 f(2x) dx &= \frac{1}{2} \int_0^4 f(x) dx \\
 &= \frac{1}{2} \int_0^2 f(x) dx \\
 &\quad + \frac{1}{2} \int_2^4 f(x) dx \\
 &= \frac{1}{2}(3) + \frac{1}{2}(5) \\
 &= 4
 \end{aligned}$$

The answer is c.

$$79. \quad C'(x) = 3\sqrt{2x-1}; \text{ 13 units cost \$270.}$$

$$\begin{aligned}
 C(x) &= \int 3(2x-1)^{1/2} dx \\
 &= \frac{3}{2} \int 2(2x-1)^{1/2} dx
 \end{aligned}$$

Let $u = 2x - 1$, so that

$$\begin{aligned}
 du &= 2dx. \\
 &= \frac{3}{2} \int u^{1/2} du \\
 &= \frac{3}{2} \left(\frac{u^{3/2}}{3/2} \right) + C \\
 &= (2x-1)^{3/2} + C \\
 C(13) &= [2(13)-1]^{3/2} + C
 \end{aligned}$$

Since $C(13) = 270$,

$$\begin{aligned}
 270 &= 25^{3/2} + C \\
 270 &= 125 + C \\
 C &= 145.
 \end{aligned}$$

Thus,

$$C(x) = (2x-1)^{3/2} + 145.$$

$$80. \quad C'(x) = \frac{8}{2x+1}; \text{ fixed cost is \$18.}$$

$$\begin{aligned}
 C(x) &= \int \frac{8}{2x+1} dx \\
 &= 4 \ln |2x+1| + k
 \end{aligned}$$

If $x = 0$, $C(x) = 18$.

Thus

$$\begin{aligned}
 18 &= \ln |1| + k \\
 k &= 18.
 \end{aligned}$$

Thus,

$$C(x) = 4 \ln |2x+1| + 18.$$

81. Read values for the rate of investment income accumulation for every 2 years from year 1 to year 9. These are the heights of rectangles with width $\Delta x = 2$.

Total accumulated income

$$\begin{aligned}
 &= 11,000(2) + 9000(2) + 12,000(2) \\
 &\quad + 10,000(2) + 6000(2) \approx \$96,000
 \end{aligned}$$

82. Total amount

$$\begin{aligned}
 &= \int_0^T 100,000e^{0.03t} dt \\
 &= \left. \frac{100,000e^{0.03t}}{0.03} \right|_0^T \\
 &= \frac{10,000,000}{3} (e^{0.03T} - 1)
 \end{aligned}$$

Set this expression equal to 4,000,000.

$$\begin{aligned}
 \frac{10,000,000}{3} (e^{0.03T} - 1) &= 4,000,000 \\
 e^{0.03T} - 1 &= 1.2 \\
 0.03T &= \ln 2.2 \\
 T &\approx 26.3
 \end{aligned}$$

It will take him 26.3 years to use up the supply.

$$83. \quad S'(x) = 3\sqrt{2x+1} + 3$$

$$\begin{aligned}
 S(x) &= \int_0^4 (3\sqrt{2x+1} + 3) dx \\
 &= [(2x+1)^{3/2} + 3x] \Big|_0^4 \\
 &= (27 + 12) - (1 + 0) = 38
 \end{aligned}$$

Total sales = \$38,000.

$$\begin{aligned}
 84. \quad (a) \quad f(x) &= \int (-0.1624x + 3.4909) dx \\
 &= -\frac{0.1624}{2} x^2 + 3.4909x \\
 &= -0.0812x^2 + 3.4909x
 \end{aligned}$$

Since x is the number of years since 2000, and the productivity in 2000 is 100, we know that $f(0) = 100$, so $f(x) = -0.0812x^2 + 3.4909x + 100$.

$$(b) \quad f(8) = -0.0812(8^2) + 4.4909(8) + 100 \\ \approx 122.7$$

The productivity was approximately 122.7, which is close to the actual value of 122.3.

$$85. \quad S(q) = q^2 + 5q + 100 \\ D(q) = 350 - q^2 \\ S(q) = D(q) \text{ at the equilibrium point.}$$

$$q^2 + 5q + 100 = 350 - q^2 \\ 2q^2 + 5q - 250 = 0 \\ (-2q + 25)(q - 10) = 0 \\ q = -\frac{25}{2} \text{ or } q = 10$$

Since the number of units produced would not be negative, the equilibrium point occurs when $q = 10$.

Equilibrium supply

$$= (10)^2 + 5(10) + 100 = 250$$

Equilibrium demand

$$= 350 - (10)^2 = 250$$

(a) Producers' surplus

$$= \int_0^{10} [250 - (q^2 + 5q + 100)] dq \\ = \int_0^{10} (-q^2 - 5q + 150) dq \\ = \left(-\frac{q^3}{3} - \frac{5q^2}{2} + 150q \right) \Big|_0^{10} \\ = \frac{-1000}{3} - \frac{500}{2} + 1500 \\ = \frac{\$2750}{3} \approx \$916.67$$

(b) Consumers' surplus

$$= \int_0^{10} [(350 - q^2) - 250] dq \\ = \int_0^{10} (100 - q^2) dq \\ = \left(100q - \frac{q^3}{3} \right) \Big|_0^{10} \\ = 1000 - \frac{1000}{3} = \frac{\$2000}{3} \approx \$666.67$$

$$86. \quad S'(x) = 225 - x^2, C'(x) = x^2 + 25x + 150$$

$$S'(x) = C'(x)$$

$$225 - x^2 = x^2 + 25x + 150$$

$$2x^2 + 25x - 75 = 0$$

$$(2x - 5)(x + 15) = 0$$

$$x = \frac{5}{2} = 2.5$$

The company should use the machinery for 2.5 years.

$$\int_0^{2.5} [(225 - x^2) - (x^2 + 25x + 150)] dx \\ = \int_0^{2.5} (-2x^2 - 25x + 75) dx \\ = \left(-\frac{2x^3}{3} - \frac{25x^2}{2} + 75x \right) \Big|_0^{2.5} \\ = \frac{-2(2.5^3)}{3} - \frac{25(2.5^2)}{2} + 75(2.5) \\ \approx 98.95833 \approx 99,000$$

The net savings are about \$99,000.

$$87. \quad (a) \text{ Total amount } \approx \frac{1}{2}(2.131) + 2.118 \\ + 2.097 + 2.073 \\ + 1.983 + 1.890 \\ + 1.862 + 1.848 \\ + 1.812 + \frac{1}{2}(1.938) \\ \approx 17.718$$

The estimate is 17.718 billion barrels.

$$(b) \text{ The left endpoint sum is 17.814 and the right endpoint sum is 17.621. Their average is } \\ \frac{17.814 + 17.621}{2} \approx 17.718.$$

$$(d) \text{ The line of best fit has the equation } \\ y = -0.03545x + 2.13475.$$

$$\int_0^9 (-0.03545x + 2.13475) dx \approx 17.777$$

The integral yields an estimate of 17.777 billion barrels.

88. The inventory must be replenished when

$$\begin{aligned} 19 - S(T) &= 1. \\ 19 - (e^{3T} - 1) &= 1 \\ 19 &= e^{3T} \\ 3T &= \ln 19 \\ T &= \frac{1}{3} \ln 19 \end{aligned}$$

Therefore, the inventory carrying cost from 0 to T week is

$$\begin{aligned} 15 \int_0^T [19 - S(t)] dt &= 15 \int_0^T [19 - (e^{3t} - 1)] dt \\ &= 15 \int_0^T (20 - e^{3t}) dt \\ &= 15 \left(20t - \frac{1}{3} e^{3t} \right) \Big|_0^T \\ &= 15 \left(20T - \frac{1}{3} e^{3T} + \frac{1}{3} \right) \\ &= 15 \left(\frac{20}{3} \ln 19 - \frac{19}{3} + \frac{1}{3} \right) \\ &\approx 15(13.63) \\ &\approx 204. \end{aligned}$$

The correct choice is (c).

89. $f(t) = 100 - t\sqrt{0.4t^2 + 1}$

The total number of additional spiders in the first ten months is

$$\int_0^{10} (100 - t\sqrt{0.4t^2 + 1}) dt,$$

where t is the time in months.

$$= \int_0^{10} 100 dt - \int_0^{10} t\sqrt{0.4t^2 + 1} dt.$$

Let $u = 0.4t^2 + 1$, so that

$$du = 0.8t dt \text{ and } \frac{1}{0.8} du = t dt.$$

When $t = 10$, $u = 41$.

When $t = 0$, $u = 1$.

$$\begin{aligned} &= \int_0^{10} 100 dt - \frac{1}{0.8} \int_1^{41} u^{1/2} du \\ &= 100t \Big|_0^{10} - \frac{5}{4} \cdot \frac{u^{3/2}}{\frac{3}{2}} \Big|_1^{41} = 1000 - \frac{5}{6} u^{3/2} \Big|_1^{41} \\ &\approx 782 \end{aligned}$$

The total number of additional spiders in the first 10 months is about 782.

90. The total number of infected people over the first four months is $\int_0^4 \frac{100t}{t^2 + 1} dt$, where t is time in months.

Let $u = t^2 + 1$, so that

$$du = 2t dt \text{ and } 50 du = 100t dt.$$

If $t = 4$, $u = 17$, so

$$\begin{aligned} &\int_0^4 \frac{100t}{t^2 + 1} dt \\ &= 50 \int_1^{17} \frac{1}{u} du = 50 \ln |u| \Big|_1^{17} \\ &= 50 \ln 17 - 50 \ln |1| \\ &= 50 \ln 17 \\ &\approx 141.66. \end{aligned}$$

Approximately 142 people are infected.

91. (a) The total area is the area of the triangle on $[0, 12]$ with height 0.024 plus the area of the rectangle on $[12, 17.6]$ with height 0.024.

$$\begin{aligned} A &= \frac{1}{2} (12 - 0)(0.024) + (17.6 - 12)(0.024) \\ &= 0.144 + 0.1344 = 0.2784 \end{aligned}$$

- (b) On $[0, 12]$ we define the function $f(x)$ with slope $\frac{0.024-0}{12-0} = 0.002$ and y -intercept 0.

$$f(x) = 0.002x$$

On $[12, 17.6]$, define $g(x)$ as the constant value.

$$g(x) = 0.024.$$

The area is the sum of the integrals of these two functions.

$$\begin{aligned} A &= \int_0^{12} 0.002x dx + \int_{12}^{17.6} 0.024 dx \\ &= 0.001x^2 \Big|_0^{12} + 0.024x \Big|_{12}^{17.6} \\ &= 0.001(12^2 - 0^2) + 0.024(17.6 - 12) \\ &= 0.144 + 0.1344 = 0.2784 \end{aligned}$$

92. Since answers are found by estimating values on the graph, exact answers may vary slightly; however when rounded to the nearest hundred, all answers should be the same. Sample solution:

(a) Left endpoints:

Read the values of the function from the graph, using the open circles for the functional values. The values of $f(x)$ are listed in the table.

x	0	2	5	15	30	45	60
$f(x)$	30	50	60	105	85	70	55

The values give the heights of 6 rectangles. The width of each rectangle is found by subtracting subsequent values of x . We estimate the area under the curve as

$$\begin{aligned}\sum_{i=1}^6 f(x_i) \Delta x_i &= 30(2) + 50(3) + 60(10) \\ &\quad + 105(15) + 85(15) + 70(15) \\ &= 4710.\end{aligned}$$

Right endpoints:

We estimate the area under the curve as

$$\begin{aligned}\sum_{i=1}^6 f(x_i) \Delta x_i &= 50(2) + 60(3) + 105(10) \\ &\quad + 85(15) + 70(15) + 55(15) \\ &= 4480.\end{aligned}$$

Average:

$$\frac{4710 + 4480}{2} = 4595 \approx 4600 \text{ pM.}$$

- (b) Read the values of the function from the graph, using the closed circles for the functional values. The values of x and $g(x)$ are listed in the table.

x	0	2	5	15	30	45	60
$g(x)$	20	42	42	70	52	40	20

The values give the heights of 6 rectangles. The width of each rectangle is found by subtracting subsequent values of x . We estimate the area under the curve as

$$\sum_{i=1}^6 g(x_i) \Delta x_i = 20(2) + 42(3) + 42(10) + 70(15) + 52(15) + 40(15) = 3016.$$

Right endpoints:

We estimate the area under the curve as

$$\sum_{i=1}^6 g(x_i) \Delta x_i = 42(2) + 42(3) + 70(10) + 52(15) + 40(15) + 20(15) = 2590.$$

Average:

$$\frac{3016 + 2590}{2} = 2803 \approx 2800 \text{ pM.}$$

- (c) $\frac{4600 - 2800}{2800} \approx 0.6428$

The area under the curve is about 64% more for the fasting sheep.

93. (a) $\int_0^{321} 1.87t^{1.49}e^{-0.189(\ln t)^2} dt$

Trapezoidal rule:

$$n = 8, b = 321, a = 1,$$

$$f(t) = 1.87t^{1.49}e^{-0.189(\ln t)^2}$$

i	x_i	$f(t_i)$
0	1	1.87
1	41	34.9086
2	81	33.9149
3	121	30.7147
4	161	27.5809
5	161	24.8344
6	201	22.4794
7	281	20.4622
8	321	18.7255

$$\begin{aligned} \text{Total amount} &\approx \frac{321-1}{8} \left[\frac{1}{2}(1.87) + 34.9086 + 33.9149 + 30.7147 + 27.5809 \right. \\ &\quad \left. + 24.8344 + 22.4794 + 20.4622 + \frac{1}{2}(18.7255) \right] \\ &\approx 8208 \end{aligned}$$

The total milk production from $t = 1$ to $t = 321$ is approximately 8208 kg.

(b) Simpson's rule:

$$n = 8, b = 321, a = 1, f(t) = 1.87t^{1.49}e^{-0.189(\ln t)^2}$$

i	t_i	$f(t_i)$
0	1	1.87
1	41	34.9086
2	81	33.9149
3	121	30.7147
4	161	27.5809
5	201	24.8344
6	241	22.4794
7	281	20.4622
8	321	18.7255

$$\begin{aligned} \text{Total amount} &\approx \frac{321-1}{8(3)} \left[1.87 + 4(34.9086) + 2(33.9149) + 4(30.7147) + 2(27.5809) \right. \\ &\quad \left. + 4(24.8344) + 2(22.4794) + 4(20.4622) + 18.7255 \right] \\ &\approx 8430 \end{aligned}$$

The total milk production from $t = 1$ to $t = 321$ is approximately 8430 kg.

(c) Numerical evaluation gives $\int_0^{321} 1.87t^{1.49}e^{-0.189(\ln t)^2} dt \approx 8558$, or 8558 kg

94. (a) Total amount $\approx \frac{1}{2}(309,569) + 317,567$
 $+ 335,869 + 331,055$
 $+ 331,208 + 330,195 + 325,453$
 $+ 313,357 + \frac{1}{2}(278,986)$
 $\approx 2,578,982$

The estimate of the total amount of property damage is \$2,578,982.

(b) The left endpoint sum is 12,594,093 and the right endpoint sum is 2,563,510. Their average is

$$\frac{2,594,273 + 2,563,690}{2} \approx 2,578,982$$

(d) The line of best fit has the equation $y = -2610.9x + 329,694.6$.

$$\int_0^8 (-2610.9x + 329,694.6) dx \approx 2,554,008$$

The integral yields an estimated total property damage of \$2,554,008.

95. $v(t) = t^2 - 2t$

$$s(t) = \int_0^t (t^2 - 2t) dt$$

$$s(t) = \frac{t^3}{3} - t^2 + s_0$$

If $t = 3, s = 8$.

$$8 = 9 - 9 + s_0$$

$$8 = s_0$$

Thus,

$$s(t) = \frac{t^3}{3} - t^2 + 8.$$

Extended Application: Estimating Depletion Dates for Minerals

1. $3,000,000 \div 16,900 \approx 177$

The reserve would last about 177 yr.

2.
$$3,000,000 = \frac{16,900}{0.02} (e^{0.02T_1} - 1)$$

$$\frac{3,000,000(0.02)}{16,900} = e^{0.02T_1} - 1$$

$$3.55 + 1 = e^{0.02T_1}$$

$$4.55 + 1 = e^{0.02T_1}$$

$$\ln 4.55 = 0.02T_1$$

$$T_1 = \frac{\ln 4.55}{0.02}$$

$$T_1 \approx 75.8$$

The reserves would last about 75.8 yr.

3.
$$15,000,000 = \frac{63,000}{0.06} (e^{0.06T_1} - 1)$$

$$\frac{15,000,000(0.06)}{63,000} + 1 = e^{0.06T_1}$$

$$15.286 \approx e^{0.06T_1}$$

$$\ln 15.286 \approx 0.06T_1$$

$$T_1 \approx \frac{\ln 15.286}{0.06}$$

$$= 45.4$$

The depletion time for bauxite is about 45.4 yr.

4.
$$2,000,000 = \frac{2200}{0.04} (e^{0.04T_1} - 1)$$

$$\frac{2,000,000(0.04)}{2200} + 1 = e^{0.04T_1}$$

$$37.36 = e^{0.04T_1}$$

$$\ln 37.36 = 0.04T_1$$

$$T_1 \approx 90.5$$

The depletion time for bituminous coal is about 90.5 yr.

5.
$$k(t) = \frac{0.5}{t + 25}$$

- (a) For $t = 0$,

$$k(t) = \frac{0.5}{25 + 25} = 0.02.$$

This gives a growth rate of 2% for 1970.

For $t = 25$,

$$k(t) = \frac{0.5}{25 + 25} = 0.01.$$

This gives a growth rate of 1% for 1996.

- (b) Use the form of the function $k(t) = \frac{a}{t+b}$, where a and b are both constants. Since $k(0) = 0.03$, $k(t) = \frac{a}{t+b}$, where

$$0.03 = \frac{a}{0+b} = \frac{a}{b}. \text{ Or } a = 0.03b.$$

Also, since $k(25) = 0.02$,

$$0.02 = \frac{a}{25+b}. \text{ Or } a = 0.02(25+b).$$

Solve:

$$0.03b = 0.02(25+b)$$

$$0.03b = 0.5 + 0.02b$$

$$0.01b = 0.5$$

$$b = 50$$

Find a using substitution.

$$a = 0.03b$$

$$a = 0.03(50)$$

$$a = 1.5$$

The function that satisfies these conditions is

$$k(t) = \frac{1.5}{t+50}.$$

6. (a) The function from 5(b) is $k(t) = \frac{1.5}{t+50}$.

The instantaneous growth rate at time t is

$$\text{then } 16,900e^{\frac{1.5t}{t+50}} = 16,900e^{\frac{1.5t}{t+50}}.$$

According to this model, world petroleum consumption from 1970 to time T is represented by the integral

$$\int_0^T 16,900e^{\frac{1.5t}{t+50}} dt.$$

- (b) Trying various values of T and using a numerical integrator, we find that according to this model the current reserve of 3,000,000 million barrels will be exhausted in about 90 years from 1970, that is, in about 2060.

Chapter 8

FURTHER TECHNIQUES AND APPLICATIONS OF INTEGRATION

8.1 Integration by Parts

Your Turn 1

$$\int x e^{-2x} dx$$

Let $u = x$ and $dv = e^{-2x} dx$.

$$\begin{aligned} \text{Then } du &= dx \text{ and } v = \int e^{-2x} dx \\ &= -\frac{1}{2} e^{-2x}. \end{aligned}$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned} \int x e^{-2x} dx &= -\frac{1}{2} x e^{-2x} - \int \left(-\frac{1}{2} e^{-2x} \right) dx \\ &= -\frac{1}{2} x e^{-2x} + \frac{1}{2} \int e^{-2x} dx \\ &= -\frac{1}{2} x e^{-2x} + \frac{1}{2} \left(-\frac{1}{2} \right) e^{-2x} + C \\ &= -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C \\ &= -\frac{1}{4} e^{-2x} (2x + 1) + C \end{aligned}$$

Your Turn 2

$$\int \ln 2x dx$$

Let $u = \ln 2x$ and $dv = dx$.

$$\text{Then } du = \frac{1}{2x}(2) = \frac{1}{x} \text{ and } v = \int dx = x.$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned} \int \ln 2x dx &= (\ln 2x)(x) - \int x \cdot \frac{1}{x} dx \\ &= x \ln 2x - \int dx \\ &= x \ln 2x - x + C \end{aligned}$$

Your Turn 3

$$\int (3x^2 + 4)e^{2x} dx$$

Choose $3x^2 + 4$ as the part to be differentiated and e^{2x} as the part to be integrated.

D	I
$3x^2 + 4$	e^{2x}
$\swarrow +$	$\searrow$
$6x$	$\frac{1}{2} e^{2x}$
$\swarrow -$	$\searrow$
6	$\frac{1}{4} e^{2x}$
$\swarrow +$	$\searrow$
0	$\frac{1}{8} e^{2x}$

$$\begin{aligned} \int (3x^2 + 4)e^{2x} dx &= (3x^2 + 4) \left(\frac{1}{2} e^{2x} \right) - (6x) \left(\frac{1}{4} e^{2x} \right) + \left(\frac{1}{8} e^{2x} \right) + C \\ &= \frac{1}{8} e^{2x} [4(3x^2 + 4) - 2(6x) + 6] + C \\ &= \frac{1}{8} e^{2x} (12x^2 + 16 - 12x + 6) + C \\ &= \frac{1}{4} e^{2x} (6x^2 - 6x + 11) + C \end{aligned}$$

Your Turn 4

$$\int_1^e x^2 \ln x dx$$

Let $u = \ln x$ and $dv = x^2 dx$.

$$\text{Then } du = \frac{1}{x} dx \text{ and } v = \int x^2 dx = \frac{x^3}{3}.$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned}
\int x^2 \ln x \, dx &= \frac{x^3}{3} \ln x - \int \left(\frac{x^3}{3} \cdot \frac{1}{x} \right) dx \\
&= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx \\
&= \frac{x^3}{3} \ln x - \frac{1}{3} \left(\frac{x^3}{3} \right) + C \\
&= \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) + C \\
\int_1^e x^2 \ln x \, dx &= \frac{x^3}{3} \left(\ln x - \frac{1}{3} \right) \Big|_1^e \\
&= \frac{e^3}{3} \left(1 - \frac{1}{3} \right) - \frac{1}{3} \left(0 - \frac{1}{3} \right) \\
&= \frac{2}{9} e^3 + \frac{1}{9} \\
&= \frac{2e^3 + 1}{9}
\end{aligned}$$

Your Turn 5

$$\int \frac{1}{x\sqrt{4+x^2}} dx$$

Use formula 10 from the table of integrals with $a = 2$.

$$\begin{aligned}
\int \frac{1}{x\sqrt{4+x^2}} dx &= \int \frac{1}{x\sqrt{2^2+x^2}} dx \\
&= -\frac{1}{2} \ln \left| \frac{2 + \sqrt{4+x^2}}{x} \right| + C
\end{aligned}$$

8.1 Exercises

1. $\int x e^x dx$

Let $dv = e^x dx$ and $u = x$.

Then $v = \int e^x dx$ and $du = dx$.

$$v = e^x$$

Use the formula

$$\begin{aligned}
\int u \, dv &= uv - \int v \, du. \\
\int x e^x dx &= x e^x - \int e^x dx = x e^x - e^x + C
\end{aligned}$$

2. $\int (x+6)e^x dx$

Let $dv = e^x dx$ and $u = x+6$.

Then $v = \int e^x dx$ and $du = dx$.

$$v = e^x$$

Use the formula

$$\begin{aligned}
\int u \, dv &= uv - \int v \, du. \\
\int (x+6)e^x dx &= (x+6)e^x - \int e^x dx \\
&= x e^x + 6e^x - e^x + C \\
&= (x+5)e^x + C
\end{aligned}$$

3. $\int (4x-12)e^{-8x} dx$

Let $dv = e^{-8x} dx$ and $u = 4x-12$

Then $v = \int e^{-8x} dx$ and $du = 4 dx$.

$$v = \frac{e^{-8x}}{-8}$$

$$\begin{aligned}
\int (4x-12)e^{-8x} dx &= (4x-12) \left(\frac{e^{-8x}}{-8} \right) - \int \left(\frac{e^{-8x}}{-8} \right) \cdot 4 dx \\
&= -\frac{4x}{8} e^{-8x} + \frac{12}{8} e^{-8x} - \left(-\frac{4}{8} \cdot \frac{e^{-8x}}{-8} \right) + C \\
&= -\frac{x}{2} e^{-8x} + \frac{3}{2} e^{-8x} - \frac{1}{16} e^{-8x} + C \\
&= \left(-\frac{x}{2} + \frac{23}{16} \right) e^{-8x} + C
\end{aligned}$$

4. $\int (6x+3)e^{-2x} dx$

Let $dv = e^{-2x} dx$ and $u = 6x+3$

Then $v = \int e^{-2x} dx$ and $du = 6 dx$.

$$v = \frac{e^{-2x}}{-2} + C$$

$$\begin{aligned}
 \int (6x + 3)e^{-2x} dx &= \frac{(6x + 3)e^{-2x}}{-2} - \int \frac{6e^{-2x}}{-2} dx \\
 &= -\frac{1}{2}(6x + 3)e^{-2x} + \frac{3e^{-2x}}{-2} + C \\
 &= -\frac{1}{2}(6x + 3)e^{-2x} - \frac{3}{2}e^{-2x} + C
 \end{aligned}$$

$$5. \int x \ln x \, dx$$

Let $dv = x \, dx$ and $u = \ln x$.

Then $v = \frac{x^2}{2}$ and $du = \frac{1}{x} \, dx$.

$$\begin{aligned}
 \int x \ln x \, dx &= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \\
 &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C
 \end{aligned}$$

$$6. \int x^3 \ln x \, dx$$

Let $dv = x^3 \, dx$ and $u = \ln x$.

Then $v = \frac{x^4}{4}$ and $du = \frac{1}{x} \, dx$.

$$\begin{aligned}
 \int x^3 \ln x \, dx &= \frac{x^4}{4} \cdot \ln x - \int \frac{x^4}{4} \left(\frac{1}{x} \right) dx \\
 &= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 \, dx \\
 &= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C
 \end{aligned}$$

$$\begin{aligned}
 7. \int_0^1 \frac{2x+1}{e^x} dx &= \int_0^1 (2x+1)e^{-x} dx
 \end{aligned}$$

Let $dv = e^{-x} \, dx$ and $u = 2x + 1$.

Then $v = \int e^{-x} \, dx$ and $du = 2 \, dx$.

$$v = -e^{-x}$$

$$\begin{aligned}
 \int \frac{2x+1}{e^x} dx &= -(2x+1)e^{-x} + \int 2e^{-x} dx \\
 &= -(2x+1)e^{-x} - 2e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 \int_0^1 \frac{2x+1}{e^x} dx &= [-(2x+1)e^{-x} - 2e^{-x}] \Big|_0^1 \\
 &= [-(3)e^{-1} - 2e^{-1}] - (-1 - 2) \\
 &= -5e^{-1} + 3 \\
 &\approx 1.161
 \end{aligned}$$

$$8. \int_0^3 \frac{3-x}{3e^x} dx = \frac{1}{3} \int_0^3 (3-x)e^{-x} dx$$

Let $dv = e^{-x} \, dx$ and $u = 3 - x$.

Then $v = -e^{-x}$ and $du = -dx$.

$$\begin{aligned}
 \frac{1}{3} \int (3-x)e^{-x} dx &= \frac{1}{3} \left[-(3-x)e^{-x} - \int e^{-x} dx \right] \\
 &= \frac{1}{3} [-(3-x)e^{-x} + e^{-x} dx] \\
 &= \frac{1}{3} (x-2)e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 \int_0^3 (3-x)e^{-x} dx &= \frac{1}{3} (x-2)e^{-x} \Big|_0^3 \\
 &= \frac{e^{-3} + 2}{3} \approx 0.6833
 \end{aligned}$$

$$9. \int \ln 3x \, dx$$

Let $dv = dx$ and $u = \ln 3x$.

Then $v = x$ and $du = \frac{1}{x} \, dx$.

$$\begin{aligned}
 \int \ln 3x \, dx &= x \ln 3x - \int dx \\
 &= x \ln 3x - x
 \end{aligned}$$

$$\begin{aligned}
 \int \ln 3x \, dx &= (x \ln 3x - x) \Big|_1^9 \\
 &= (9 \ln 27 - 9) - (\ln 3 - 1) \\
 &= 9 \ln 3^3 - 9 - \ln 3 + 1 \\
 &= 27 \ln 3 - \ln 3 - 8 \\
 &= 26 \ln 3 - 8 \approx 20.56
 \end{aligned}$$

10. $\int_1^2 \ln 5x \, dx$

Let $dv = dx$ and $u = \ln 5x$.

Then $v = x$ and $du = \frac{1}{x} dx$.

$$\int \ln 5x \, dx = x \ln 5x - \int x \left(\frac{1}{x} dx \right) = x \ln 5x - x$$

$$\begin{aligned} \int_1^2 \ln 5x \, dx &= (x \ln 5x - x) \Big|_1^2 \\ &= 2 \ln 10 - 2 - \ln 5 + 1 \\ &= \ln \frac{10^2}{5} - 1 = \ln 20 - 1 \approx 1.996 \end{aligned}$$

11. The area is $\int_2^4 (x-2)e^x \, dx$.

Let $dv = e^x dx$ and $u = x-2$.

Then $v = e^x$ and $du = dx$.

$$\int (x-2)e^x \, dx = (x-2)e^x - \int e^x \, dx$$

$$\begin{aligned} \int_1^4 (x-2)e^x \, dx &= [(x-2)e^x - e^x] \Big|_2^4 \\ &= (2e^4 - e^4) - (0 - e^2) \\ &= e^4 + e^2 \approx 61.99 \end{aligned}$$

12. $\int_1^e (x+1) \ln x \, dx$

$$= \int_1^e x \ln x \, dx + \int_1^e \ln x \, dx$$

For the first integral,

let $dv = \frac{1}{x} dx$ and $u = \frac{x^2}{2}$.

Then $v = \ln x$ and $du = x \, dx$.

$$\begin{aligned} \int_1^e (x+1) \ln x \, dx &= \frac{x^2}{2} \ln x \Big|_1^e - \int_1^e \frac{x}{2} \, dx \\ &= \frac{e^2}{2} - \frac{x^2}{4} \Big|_1^e \\ &= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \end{aligned}$$

Using the result of Example 2,

$$\begin{aligned} \int_1^e \ln x \, dx &= (x \ln x - x) \Big|_1^e \\ &= 1 \end{aligned}$$

Thus the original integral is equal to

$$\begin{aligned} \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} + 1 &= \frac{e^2 + 5}{4} \\ &\approx 3.097. \end{aligned}$$

13. $\int x^2 e^{2x} \, dx$

Let $u = x^2$ and $dv = e^{2x} \, dx$.

Use column integration.

D	I
x^2	e^{2x}
$2x$	$\frac{e^{2x}}{2}$
2	$\frac{e^{2x}}{4}$
0	$\frac{e^{2x}}{8}$

$$\begin{aligned} \int x^2 e^{2x} \, dx &= x^2 \left(\frac{e^{2x}}{2} \right) - 2x \left(\frac{e^{2x}}{4} \right) + \frac{2e^{2x}}{8} + C \\ &= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4} + C \end{aligned}$$

14. $\int \frac{x^2 dx}{2x^3 + 1} = \frac{1}{6} \int \frac{6x^2 dx}{2x^3 + 1}$

Let $u = 2x^3 + 1$.

Then $du = 6x^2 \, dx$.

$$\int \frac{x^2 dx}{2x^3 + 1} = \frac{1}{6} \ln |2x^3 + 1| + C$$

15. $\int x^2 \sqrt{x+4} \, dx$

Let $u = x+4$ and $dv = (x+4)^{1/2}$. Use column integration.

D		I
x^2	+	$(x+4)^{1/2}$
$2x$	-	$\frac{2}{3}(x+4)^{3/2}$
2	+	$\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)(x+4)^{5/2}$
0		$\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\left(\frac{2}{7}\right)(x+4)^{7/2}$

$$\begin{aligned}
 & \int x^2 \sqrt{x+4} \, dx \\
 &= x^2(x+4)^{3/2} \left(\frac{2}{3} \right) - 2x(x+4)^{5/2} \left(\frac{2}{3} \right) \left(\frac{2}{5} \right) \\
 & \quad + 2(x+4)^{7/2} \left(\frac{2}{3} \right) \left(\frac{2}{5} \right) \left(\frac{2}{7} \right) + C \\
 &= \frac{2}{3} x^2 (x+4)^{3/2} - \frac{8}{15} x (x+4)^{5/2} \\
 & \quad + \frac{16}{105} (x+4)^{7/2} + C
 \end{aligned}$$

16. $\int (2x-1) \ln(3x) \, dx$

Let $dv = (2x-1) \, dx$ and $u = \ln 3x$.

Then $v = x^2 - x$ and $du = \frac{1}{x} \, dx$.

$$\begin{aligned}
 & \int (2x-1) \ln(3x) \, dx \\
 &= (x^2 - x) \ln 3x - \int (x^2 - x) \left(\frac{1}{x} \, dx \right) \\
 &= (x^2 - x) \ln 3x - \int (x - 1) \, dx \\
 &= (x^2 - x) \ln 3x - \frac{x^2}{2} + x + C
 \end{aligned}$$

17. $\int (8x+10) \ln(5x) \, dx$

Let $dv = (8x+10) \, dx$ and $u = \ln(5x)$.

Then $v = 4x^2 + 10x$ and $du = \frac{1}{x} \, dx$.

$$\begin{aligned}
 & \int (8x+10) \ln(5x) \, dx \\
 &= (4x^2 + 10x) \ln(5x) - \int (4x^2 + 10x) \left(\frac{1}{x} \right) dx \\
 &= (4x^2 + 10x) \ln(5x) - \int (4x + 10) \, dx \\
 &= (4x^2 + 10x) \ln(5x) - 2x^2 - 10x + C
 \end{aligned}$$

18. $\int x^3 e^{x^4} \, dx$

Let $u = x^4$ and $du = 4x^3 \, dx$.

$$\begin{aligned}
 \int x^3 e^{x^4} \, dx &= \frac{1}{4} \int 4x^3 e^{x^4} \, dx \\
 &= \frac{1}{4} \int e^u \, du \\
 &= \frac{1}{4} e^u + C \\
 &= \frac{1}{4} e^{x^4} + C
 \end{aligned}$$

19. $\int_1^2 (1-x^2) e^{2x} \, dx$

Let $u = 1-x^2$ and $dv = e^{2x} \, dx$.

Use column integration.

D		I
$1-x^2$	+	e^{2x}
$-2x$	-	$\frac{e^{2x}}{2}$
-2	+	$\frac{e^{2x}}{4}$
0		$\frac{e^{2x}}{8}$

$$\begin{aligned}
 & \int (1-x^2) e^{2x} \, dx \\
 &= \frac{(1-x^2) e^{2x}}{2} - \frac{(-2x) e^{2x}}{4} + \frac{(-2) e^{2x}}{8} \\
 &= \frac{(1-x^2) e^{2x}}{2} + \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} \\
 &= \frac{e^{2x}}{2} \left(1-x^2 + x - \frac{1}{2} \right) = \frac{e^{2x}}{2} \left(\frac{1}{2} - x^2 + x \right) \\
 & \int_1^2 (1-x^2) e^{2x} \, dx = \frac{e^{2x}}{2} \left(\frac{1}{2} - x^2 + x \right) \Big|_1^2 \\
 &= \frac{e^4}{2} \left(-\frac{3}{2} \right) - \frac{e^2}{2} \left(\frac{1}{2} \right) \\
 &= -\frac{e^2}{4} (3e^2 + 1) \approx -42.80
 \end{aligned}$$

20. $\int_0^1 \frac{x^2 dx}{2x^3 + 1}$

Let $u = 2x^3 + 1$.

Then $du = 6x^2 dx$.

$$\begin{aligned}\int_0^1 \frac{x^2 dx}{2x^3 + 1} &= \frac{1}{6} \int_0^1 \frac{6x^2 dx}{2x^3 + 1} \\ &= \frac{1}{6} \left(\ln |2x^3 + 1| \right) \Big|_0^1 \\ &= \frac{1}{6} (\ln 3) \\ &\approx 0.1831\end{aligned}$$

21. $\int_0^1 \frac{x^3 dx}{\sqrt{3+x^2}} = \int_0^1 x^3 (3+x^2)^{-1/2} dx$

Let $dv = x(3+x^2)^{-1/2} dx$ and $u = x^2$.

Then $v = \frac{2(3+x^2)^{1/2}}{2}$

$v = (3+x^2)^{1/2}$ and $du = 2x dx$.

$$\begin{aligned}\int \frac{x^3 dx}{\sqrt{3+x^2}} &= x^2(3+x^2)^{1/2} - \int 2x(3+x^2)^{1/2} dx \\ &= x^2(3+x^2)^{1/2} - \frac{2}{3}(3+x^2)^{3/2}\end{aligned}$$

$$\begin{aligned}\int_0^1 \frac{x^3 dx}{\sqrt{3+x^2}} &= \left[x^2(3+x^2)^{1/2} - \frac{2}{3}(3+x^2)^{3/2} \right] \Big|_0^1 \\ &= 4^{1/2} - \frac{2}{3}(4^{3/2}) - 0 + \frac{2}{3}(3^{3/2}) \\ &= 2 - \frac{2}{3}(8) + \frac{2}{3}(3^{3/2}) \\ &= -\frac{10}{3} + 2\sqrt{3} \\ &\approx 0.1308\end{aligned}$$

22. $\int_0^5 x\sqrt[3]{x^2+2} dx$

$$\begin{aligned}&= \int_0^5 x(x^2+2)^{1/2} dx \\ &= \frac{1}{2} \int_0^5 2x(x^2+2)^{1/2} dx\end{aligned}$$

Let $u = x^2 + 2$. Then $du = 2x dx$.

If $x = 5$, $u = 27$. If $x = 0$, $u = 2$.

$$\begin{aligned}&= \frac{1}{2} \int_2^{27} u^{1/3} du \\ &= \frac{1}{2} \left(\frac{u^{4/3}}{4/3} \right) \Big|_2^{27} \\ &= \frac{3}{8} (27^{4/3}) - \frac{3}{8} (2^{4/3}) \\ &= \frac{243}{8} - \frac{3(2^{4/3})}{8} \\ &= \frac{243}{8} - \frac{3\sqrt[3]{2}}{4} \approx 29.43\end{aligned}$$

23. $\int \frac{16}{\sqrt{x^2+16}} dx$

Use formula 5 from the table of integrals with $a = 4$.

$$\begin{aligned}\int \frac{16}{\sqrt{x^2+16}} dx &= 16 \int \frac{1}{\sqrt{x^2+4^2}} dx \\ &= 16 \ln \left| x + \sqrt{x^2+16} \right| + C\end{aligned}$$

24. $\int \frac{10}{x^2-25} dx$

Use formula 8 from the table of integrals with $a = 5$.

$$\begin{aligned}\int \frac{10}{x^2-25} dx &= 10 \int \frac{1}{x^2-5^2} dx \\ &= 10 \left[\frac{1}{2(5)} \cdot \ln \left| \frac{x-5}{x+5} \right| \right] + C \\ &= \ln \left| \frac{x-5}{x+5} \right| + C\end{aligned}$$

25. $\int \frac{3}{x\sqrt{121-x^2}} dx = 3 \int \frac{dx}{x\sqrt{11^2-x^2}}$

If $a = 11$, this integral matches formula 9 in the table.

$$\begin{aligned}&= 3 \left(-\frac{1}{11} \ln \left| \frac{11 + \sqrt{121-x^2}}{x} \right| \right) + C \\ &= -\frac{3}{11} \ln \left| \frac{11 + \sqrt{121-x^2}}{x} \right| + C\end{aligned}$$

26. $\int \frac{2}{3x(3x-5)} dx = \frac{2}{3} \int \frac{1}{x(3x-5)} dx$

Use formula 13 from table with $a = 3$, $b = -5$.

$$\begin{aligned} &= \frac{2}{3} \left(-\frac{1}{5} \ln \left| \frac{x}{3x-5} \right| \right) + C \\ &= -\frac{2}{15} \ln \left| \frac{x}{3x-5} \right| + C \end{aligned}$$

27. $\int \frac{-6}{x(4x+6)^2} dx$

Use formula 14 from the table of integrals with $a = 4$ and $b = 6$.

$$\begin{aligned} &\int \frac{-6}{x(4x+6)^2} dx \\ &= -6 \int \frac{1}{x(4x+6)^2} dx \\ &= -6 \left[\frac{1}{6(4x+6)} + \frac{1}{6^2} \ln \left| \frac{x}{4x+6} \right| \right] + C \\ &= \frac{-1}{(4x+6)} - \frac{1}{6} \ln \left| \frac{x}{4x+6} \right| + C \end{aligned}$$

28. $\int \sqrt{x^2 + 15} dx$

Use formula 15 from table of integrals with $a = \sqrt{15}$.

$$\begin{aligned} &\int \sqrt{x^2 + 15} dx \\ &= \int \sqrt{x^2 + (\sqrt{15})^2} dx \\ &= \frac{x}{2} \sqrt{x^2 + (\sqrt{15})^2} \\ &\quad + \frac{(\sqrt{15})^2}{2} \ln \left| x + \sqrt{x^2 + (\sqrt{15})^2} \right| + C \\ &= \frac{x}{2} \sqrt{x^2 + 15} + \frac{15}{2} \ln \left| x + \sqrt{x^2 + 15} \right| + C \end{aligned}$$

31. First find the indefinite integral using integration by parts.

$$\int u dv = uv - \int v du$$

Now substitute the given values.

$$\begin{aligned} \int_0^1 u dv &= uv \Big|_0^1 - \int_0^1 v du \\ &= [u(1)v(1) - u(0)v(0)] - 4 \\ &= (3)(-4) - (2)(1) - 4 = -18 \end{aligned}$$

32. First find the indefinite integral using integration by parts.

$$\int u dv = uv - \int v du$$

Now substitute the given values.

$$\begin{aligned} \int_1^{20} u dv &= uv \Big|_1^{20} - \int_1^{20} v du \\ &= [u(20)v(20) - u(1)v(1)] - (-1) \\ &= (15)(6) - (5)(-2) + 1 = 101 \end{aligned}$$

33. $\int r ds = rs - \int s dr$

$$\begin{aligned} \int_0^2 r ds &= rs \Big|_0^2 - \int_0^2 s dr \\ 10 &= r(2)s(2) - r(0)s(0) - 5 \\ 15 &= r(s)s(2) \end{aligned}$$

34. $\int u dv = uv - \int v du$

$$\begin{aligned} \int_1^3 u dv &= uv \Big|_1^3 - \int_1^3 v du \\ 15 &= u(3)v(3) - u(1)v(1) - 20 \\ -35 &= u(1)v(1) \end{aligned}$$

35. $\int x^n \cdot \ln|x| dx, n \neq -1$

Let $u = \ln|x|$ and $dv = x^n dx$.

Use column integration.

D	I
$\ln x $	$+$
$\frac{1}{x}$	x^n
	$-\frac{1}{n+1}x^{n+1}$

$$\begin{aligned} &\int x^n \cdot \ln|x| dx \\ &= \frac{1}{n+1} x^{n+1} \ln|x| - \int \left[\frac{1}{x} \cdot \frac{1}{n+1} x^{n+1} \right] dx \\ &= \frac{1}{n+1} x^{n+1} \ln|x| - \int \frac{1}{n+1} x^n dx \\ &= \frac{1}{n+1} x^{n+1} \ln|x| - \frac{1}{(n+1)^2} x^{n+1} + C \\ &= x^{n+1} \left[\frac{\ln|x|}{n+1} - \frac{1}{(n+1)^2} \right] + C \end{aligned}$$

36. $\int x^n e^{ax} dx, a \neq 0$

Let $dv = e^{ax} dx$ and $u = x^n$;

then $v = \frac{1}{a} e^{ax}$ and $du = nx^{n-1} dx$.

$$\begin{aligned} \int x^n e^{ax} dx &= \frac{x^n e^{ax}}{a} - \int \left(\frac{1}{a} e^{ax} \cdot nx^{n-1} \right) dx \\ &= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx + C \end{aligned}$$

37. $\int x\sqrt{x+1} dx$

(a) Let $u = x$ and $dv = \sqrt{x+1} dx$.
Use column integration.

D		I
x	+	$\sqrt{x+1}$
1	-	$\left(\frac{2}{3}\right)(x+1)^{3/2}$
0		$\left(\frac{4}{15}\right)(x+1)^{5/2}$

$$\begin{aligned} \int x\sqrt{x+1} dx &= \left(\frac{2}{3}\right)x(x+1)^{3/2} - \left(\frac{4}{15}\right)(x+1)^{5/2} + C \end{aligned}$$

(b) Let $u = x+1$; then $u-1 = x$
and $du = dx$.

$$\begin{aligned} \int x\sqrt{x+1} dx &= \int (u-1)u^{1/2} du = \int (u^{3/2} - u^{1/2}) du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C \end{aligned}$$

(c) Both results factor as $\frac{2}{15}(x+1)^{3/2}(3x-2) + C$, so they are equivalent.

38. The integration constant is missing.

39. $R = \int_0^{12} (x+1) \ln(x+1) dx$

Let $u = \ln(x+1)$ and $dv = (x+1) dx$.

Then $du = \frac{1}{x+1} dx$ and $v = \frac{1}{2}(x+1)^2$.

$$\begin{aligned} \int (x+1) \ln(x+1) dx &= \frac{1}{2}(x+1)^2 \ln(x+1) \\ &\quad - \int \left[\frac{1}{2}(x+1)^2 \cdot \frac{1}{x+1} \right] dx \\ &= \frac{1}{2}(x+1)^2 \ln(x+1) - \int \frac{1}{2}(x+1) dx \\ &= \frac{1}{2}(x+1)^2 \ln(x+1) - \frac{1}{4}(x+1)^2 + C \\ \int_0^{12} (x+1) \ln(x+1) dx &= \left[\frac{1}{2}(x+1)^2 \ln(x+1) - \frac{1}{4}(x+1)^2 \right]_0^{12} \\ &= \frac{169}{2} \ln 13 - 42 \approx \$174.74 \end{aligned}$$

40. $r(t) = \int_1^6 2t^2 e^{-t} dt$

Let $dv = e^{-t} dt$ and $u = 2t^2$.

Then $v = -e^{-t}$ and $du = 4t dt$.

Use column integration.

D		I
$2t^2$	+	e^{-t}
$4t$	-	$-e^{-t}$
4	+	e^{-t}
0		$-e^{-t}$

$$\begin{aligned} \int 2t^2 e^{-t} dt &= 2t^2(-e^{-t}) - 4t(e^{-t}) + 4(-e^{-t}) \\ &= -2t^2 e^{-t} - 4te^{-t} - 4e^{-t} \\ &= -2e^{-t}(t^2 + 2t + 2) \end{aligned}$$

$$\begin{aligned} \int_1^6 2t^2 e^{-t} dt &= -2e^{-t}(t^2 + 2t + 2) \Big|_1^6 \\ &= -100e^{-6} + 10e^{-1} \approx 3.431 \end{aligned}$$

The total reaction to the drug from $t = 1$ to $t = 6$ is about 3.431.

41. The total accumulated growth of the microbe population during the first 2 days is given by

$$\int_0^2 27t e^{3t} dt.$$

Let $dv = e^{3t} dt$ and $u = 27t$.

Then $v = \frac{e^{3t}}{3}$ and $du = 27dt$.

$$\begin{aligned}\int 27te^{3t} dt &= 27t \cdot \frac{e^{3t}}{3} - \int \frac{e^{3t}}{3} \cdot 27dt \\ &= 9te^{3t} - 3e^{3t}\end{aligned}$$

$$\begin{aligned}\int_0^2 27te^{3t} dt &= (9te^{3t} - 3e^{3t}) \Big|_0^2 \\ &= (18e^6 - 3e^6) - (0 - 3) \\ &= 15e^6 + 3 \approx 6054\end{aligned}$$

42. $A = \int_4^9 \sqrt{t} \ln t dt$

Let $u = \ln t$ and $dv = \sqrt{t} dt = t^{1/2} dt$.

Then $du = \frac{1}{t} dt$ and $v = \frac{2}{3} t^{3/2}$.

$$\begin{aligned}\int \sqrt{t} \ln t dt &= \frac{2}{3} t^{3/2} \ln t - \int \left(\frac{2}{3} t^{3/2} \cdot \frac{1}{t} \right) dt \\ &= \frac{2}{3} t^{3/2} \ln t - \int \frac{2}{3} t^{1/2} dt \\ &= \frac{2}{3} t^{3/2} \ln t - \frac{4}{9} t^{3/2} + C\end{aligned}$$

$$\begin{aligned}\int_4^9 \sqrt{t} \ln t dt &= 18 \ln 9 - \frac{16}{3} \ln 4 - \frac{76}{9} \\ &\approx 23.71 \text{ sq cm}\end{aligned}$$

43. $\int_0^6 (-10.28 + 175.9te^{-t/1.3}) dt$

$$= -10.28t + 175.9 \int te^{-t/1.3} dt$$

Evaluate this integral using integration by parts.

Let $u = t$ and $dv = e^{-t/1.3} dt$.

Then $du = dt$ and $v = -1.3e^{-t/1.3}$.

$$\begin{aligned}\int te^{-t/1.3} dt &= (t)(-1.3e^{-t/1.3}) - \int (-1.3e^{-t/1.3}) dt \\ &= -1.3te^{-t/1.3} - 1.69e^{-t/1.3} + C\end{aligned}$$

Substitute this expression in the earlier expression.

$$\begin{aligned}& -10.28t + 175.9(-1.3te^{-t/1.3} - 1.69e^{-t/1.3}) \Big|_0^6 \\ &= -10.28t - 228.67te^{-t/1.3} - 297.271e^{-t/1.3} \Big|_0^6 \\ &= (-61.68 - 1669.291e^{-6/1.3}) - (-297.271) \\ &\approx 219.07\end{aligned}$$

The total thermic energy is about 219 kJ.

44. (a) $\int_0^1 ke^{-kt}(1-t) dt$

Let $u = 1-t$ and $dv = e^{-kt} dt$.

Then $du = -dt$ and $v = -\frac{1}{k} e^{-kt}$

$$\begin{aligned}\int_0^1 ke^{-kt}(1-t) dt &= k \left[(1-t) \left(-\frac{1}{k} e^{-kt} \right) - \int \left(-\frac{1}{k} e^{-kt} \right) (-dt) \right] \Big|_0^1 \\ &= k \left[-\frac{1}{k} e^{-kt}(1-t) - \frac{1}{k} \int e^{-kt} dt \right] \Big|_0^1 \\ &= -e^{-kt}(1-t) + \frac{1}{k} e^{-kt} \Big|_0^1 \\ &= -e^{-kt} \left(1-t - \frac{1}{k} \right) \Big|_0^1 \\ &= \left[-e^{-k} \left(-\frac{1}{k} \right) \right] - \left[-e^0 \left(1 - \frac{1}{k} \right) \right] \\ &= 1 - \frac{1}{k} + \frac{1}{k} e^{-k}\end{aligned}$$

$$k = \frac{1}{12}$$

$$\begin{aligned}\int_0^1 \frac{1}{12} e^{-t/12} (1-t) dt &= 1 - \frac{1}{1/12} + \frac{1}{1/12} e^{-1/12} \\ &= 12e^{-1/12} - 11 \\ &\approx 0.0405\end{aligned}$$

$$k = \frac{1}{24}:$$

$$\begin{aligned} \int_0^1 \frac{1}{24} e^{-t/24} (1-t) dt \\ &= 1 - \frac{1}{1/24} + \frac{1}{1/24} e^{-1/24} \\ &= 24e^{-1/24} - 23 \\ &\approx 0.0205 \end{aligned}$$

$$k = \frac{1}{48}:$$

$$\begin{aligned} \int_0^1 \frac{1}{48} e^{-t/48} (1-t) dt \\ &= 1 - \frac{1}{1/48} + \frac{1}{1/48} e^{-1/48} \\ &= 48e^{-1/48} - 47 \\ &\approx 0.0103 \end{aligned}$$

$$(b) \int_1^6 k e^{-kt} \frac{6-t}{5} dt$$

The integral, easily found by comparing it to the integral in part (a), is

$$\begin{aligned} &-e^{-kt} \left(\frac{6}{5} - \frac{t}{5} - \frac{1}{5k} \right) \Big|_1^6 \\ &= \left[-e^{-6k} \left(\frac{6}{5} - \frac{6}{5} - \frac{1}{5k} \right) \right] \\ &\quad - \left[-e^{-k} \left(\frac{6}{5} - \frac{1}{5} - \frac{1}{5k} \right) \right] \\ &= \frac{1}{5k} e^{-6k} + \left(1 - \frac{1}{5k} \right) e^{-k} \end{aligned}$$

$$k = \frac{1}{12}:$$

$$\begin{aligned} \int_1^6 \frac{1}{12} e^{-t/12} \frac{6-t}{5} dt \\ &= \frac{1}{5(1/12)} e^{-6(1/12)} + \left[1 - \frac{1}{5(1/12)} \right] e^{-1/12} \\ &= \frac{12}{5} e^{-1/2} - \frac{7}{5} e^{-1/12} \approx 0.1676 \end{aligned}$$

$$k = \frac{1}{24}:$$

$$\begin{aligned} \int_1^6 \frac{1}{24} e^{-t/24} \frac{6-t}{5} dt \\ &= \frac{1}{5(1/24)} e^{-6(1/24)} + \left[1 - \frac{1}{5(1/24)} \right] e^{-1/24} \\ &= \frac{24}{5} e^{-1/4} - \frac{19}{5} e^{-1/24} \approx 0.0933 \end{aligned}$$

$$k = \frac{1}{48}:$$

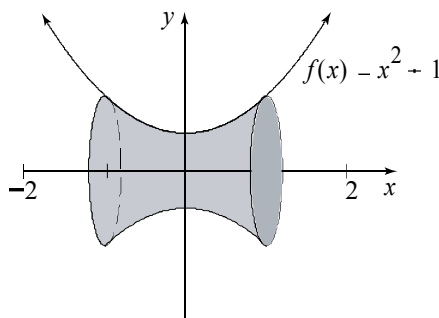
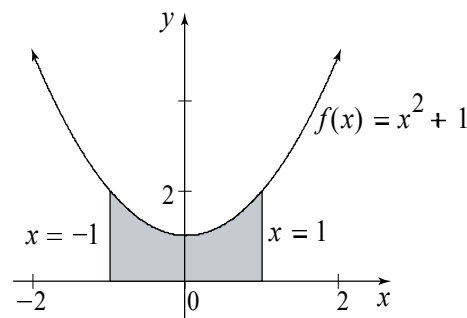
$$\begin{aligned} \int_1^6 \frac{1}{48} e^{-t/48} \frac{6-t}{5} dt \\ &= \frac{1}{5(1/48)} e^{-6(1/48)} + \left[1 - \frac{1}{5(1/48)} \right] e^{-1/48} \\ &= \frac{48}{5} e^{-1/8} - \frac{43}{5} e^{-1/48} \\ &\approx 0.0493 \end{aligned}$$

8.2 Volume and Average Value

Your Turn 1

Find the volume of the solid of revolution formed by rotating about the x -axis the region bounded by $y = x^2 + 1$, $y = 0$, $x = -1$, and $x = 1$.

The region and the solid are shown below.



$$\begin{aligned} V &= \int_{-1}^1 \pi(x^2 + 1)^2 dx \\ &= \pi \int_{-1}^1 (x^4 + 2x^2 + 1) dx \\ &= \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \right) \Big|_{-1}^1 \\ &= \pi \left[\left(\frac{1}{5} + \frac{2}{3} + 1 \right) - \left(-\frac{1}{5} - \frac{2}{3} - 1 \right) \right] \\ &= \pi \left(\frac{56}{15} \right) = \frac{56\pi}{15} \end{aligned}$$

Your Turn 2

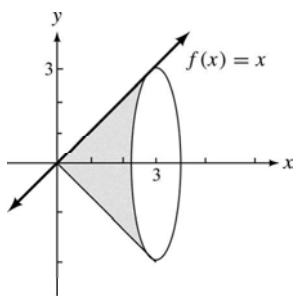
Find the average value of the function $f(x) = x + \sqrt{x}$ on the interval $[1, 4]$.

Use the formula for average value with $a = 1$ and $b = 4$.

$$\begin{aligned}\frac{1}{4-1} \int_1^4 (x + \sqrt{x}) dx &= \frac{1}{2} \left(\frac{x^2}{2} + \frac{2}{3} x^{3/2} \right) \bigg|_1^4 \\ &= \frac{1}{3} \left(8 + \frac{16}{3} - \frac{1}{2} - \frac{2}{3} \right) \\ &= \frac{1}{3} \left(\frac{73}{6} \right) = \frac{73}{18}\end{aligned}$$

8.2 Exercises

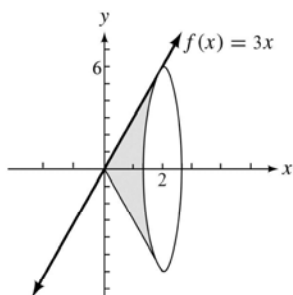
1. $f(x) = x, y = 0, x = 0, x = 3$



$$V = \pi \int_0^3 x^2 dx = \frac{\pi x^3}{3} \bigg|_0^3 = \frac{\pi(27)}{3} - 0 = 9\pi$$

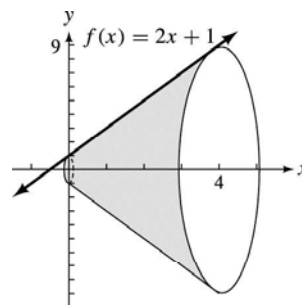
2. $f(x) = 3x, y = 0, x = 0, x = 2$

Graph $f(x) = 3x$. Then show the solid of revolution formed by rotating about the x -axis the region bounded by $f(x)$, $x = 0$, and $x = 2$.



$$\begin{aligned}V &= \pi \int_0^2 (3x)^2 dx = \pi \int_0^2 9x^2 dx \\ &= \frac{9\pi x^3}{3} \bigg|_0^2 = 3\pi(8 - 0) = 24\pi\end{aligned}$$

3. $f(x) = 2x + 1, y = 0, x = 0, x = 4$



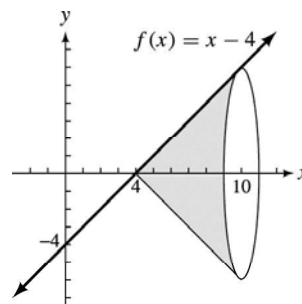
$$V = \pi \int_0^4 (2x + 1)^2 dx$$

Let $u = 2x + 1$. Then $du = 2 dx$.

If $x = 4$, $u = 9$. If $x = 0$, $u = 1$.

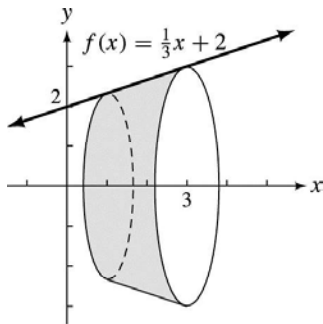
$$\begin{aligned}V &= \frac{1}{2} \pi \int_0^4 2(2x + 1)^2 dx \\ &= \frac{1}{2} \pi \int_1^9 u^2 du \\ &= \frac{\pi}{2} \left(\frac{u^3}{3} \right) \bigg|_1^9 \\ &= \frac{\pi}{2} \left(\frac{729}{3} - \frac{1}{3} \right) \\ &= \frac{728\pi}{6} \\ &= \frac{364\pi}{3}\end{aligned}$$

4. $f(x) = x - 4, y = 0, x = 4, x = 10$



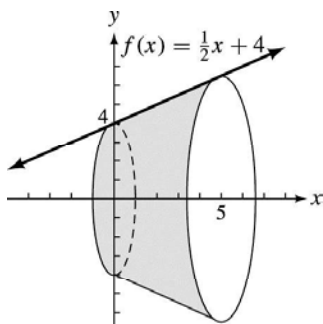
$$\begin{aligned}V &= \pi \int_4^{10} (x - 4)^2 dx \\ &= \frac{\pi (x - 4)^3}{3} \bigg|_4^{10} \\ &= \frac{\pi}{3} [(6)^3 - 0] \\ &= 72\pi\end{aligned}$$

5. $f(x) = \frac{1}{3}x + 2, y = 0, x = 1, x = 3$



$$\begin{aligned} V &= \pi \int_1^3 \left(\frac{1}{3}x + 2 \right)^2 dx \\ &= 3\pi \int_1^3 \frac{1}{3} \left(\frac{1}{3}x + 2 \right)^2 dx \\ &= 3\pi \left. \frac{\left(\frac{1}{3}x + 2 \right)^3}{3} \right|_1^3 \\ &= \pi \left(\frac{1}{3}x + 2 \right)^3 \bigg|_1^3 \\ &= 27\pi - \frac{343\pi}{27} = \frac{386\pi}{27} \end{aligned}$$

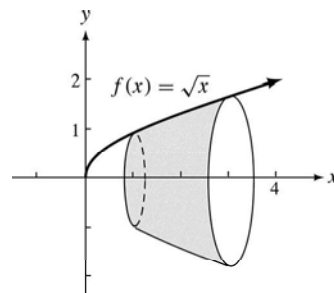
6. $f(x) = \frac{1}{2}x + 4, y = 0, x = 0, x = 5$



$$\begin{aligned} V &= \pi \int_0^5 \left(\frac{1}{2}x + 4 \right)^2 dx \\ &= 2\pi \int_0^5 \frac{1}{2} \left(\frac{1}{2}x + 4 \right)^2 dx \\ &= 2\pi \left. \frac{\left(\frac{1}{2}x + 4 \right)^3}{3} \right|_0^5 \end{aligned}$$

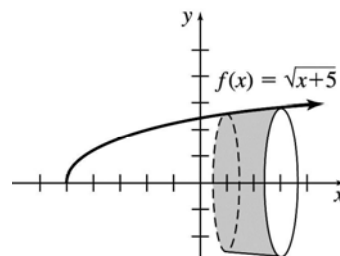
$$\begin{aligned} &= \frac{2\pi}{3} \left[\left(\frac{5}{2} + 4 \right)^3 - (4)^3 \right] \\ &= \frac{2\pi}{3} \left[\left(\frac{13}{2} \right)^3 - 64 \right] \\ &= \frac{2\pi}{3} \left(\frac{2197}{8} - \frac{512}{8} \right) = \frac{1685\pi}{12} \end{aligned}$$

7. $f(x) = \sqrt{x}, y = 0, x = 1, x = 4$



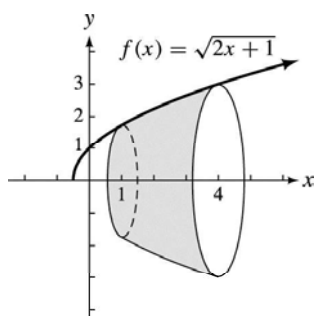
$$\begin{aligned} V &= \pi \int_1^4 (\sqrt{x})^2 dx = \pi \int_1^4 x dx \\ &= \left. \frac{\pi x^2}{2} \right|_1^4 \\ &= 8\pi - \frac{\pi}{2} = \frac{15\pi}{2} \end{aligned}$$

8. $f(x) = \sqrt{x+5}, y = 0, x = 0, x = 3$



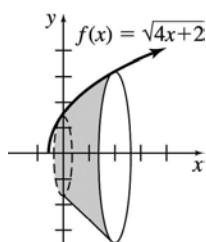
$$\begin{aligned} V &= \pi \int_0^3 (\sqrt{x+5})^2 dx \\ &= \pi \int_0^3 (x+5) dx \\ &= \pi \left(\frac{x^2}{2} + 5x \right) \bigg|_0^3 \\ &= \left[\left(\frac{9}{2} + 15 \right) - \left(\frac{0}{2} + 0 \right) \right] \\ &= \pi \left(\frac{39}{2} - \frac{0}{2} \right) \\ &= 14\pi \end{aligned}$$

9. $f(x) = \sqrt{2x+1}$, $y = 0$, $x = 1$, $x = 4$



$$\begin{aligned} V &= \pi \int_1^4 (\sqrt{2x+1})^2 dx \\ &= \pi \int_1^4 (2x+1) dx \\ &= \pi \left(\frac{2x^2}{2} + x \right) \Big|_1^4 \\ &= \pi[(16+4) - 2] \\ &= 18\pi \end{aligned}$$

10. $f(x) = \sqrt{4x+2}$, $y = 0$, $x = 0$, $x = 2$



$$\begin{aligned} V &= \pi \int_0^2 (\sqrt{4x+2})^2 dx \\ &= \pi \int_0^2 (4x+2) dx \\ &= \pi (2x^2 + 2x) \Big|_0^2 \\ &= \pi(8+4) \\ &= 12\pi \end{aligned}$$

11. $f(x) = e^x$; $y = 0$, $x = 0$, $x = 2$

$$\begin{aligned} V &= \pi \int_0^2 e^{2x} dx = \frac{\pi e^{2x}}{2} \Big|_0^2 \\ &= \frac{\pi e^4}{2} - \frac{\pi}{2} \\ &= \frac{\pi}{2}(e^4 - 1) \\ &\approx 84.19 \end{aligned}$$

12. $f(x) = 2e^x$, $y = 0$, $x = -2$, $x = 1$

$$\begin{aligned} V &= \pi \int_{-2}^1 (2e^x)^2 dx = \pi \int_{-2}^1 4e^{2x} dx \\ &= \frac{4\pi}{2}(e^{2x}) \Big|_{-2}^1 = 2\pi(e^2 - e^{-4}) \\ &\approx 46.31 \end{aligned}$$

13. $f(x) = \frac{2}{\sqrt{x}}$, $y = 0$, $x = 1$, $x = 3$

$$\begin{aligned} V &= \pi \int_1^3 \left(\frac{2}{\sqrt{x}} \right)^2 dx \\ &= \pi \int_1^3 \frac{4}{x} dx \\ &= 4\pi \ln|x| \Big|_1^3 \\ &= 4\pi(\ln 3 - \ln 1) \\ &= 4\pi \ln 3 \approx 13.81 \end{aligned}$$

14. $f(x) = \frac{2}{\sqrt{x+2}}$, $y = 0$, $x = 1$, $x = 2$

$$\begin{aligned} V &= \pi \int_{-1}^2 \left(\frac{2}{\sqrt{x+2}} \right)^2 dx \\ &= \pi \int_{-1}^2 \frac{4}{x+2} dx \\ &= 4\pi \ln|x+2| \Big|_{-1}^2 \\ &= 4\pi(\ln 4 - \ln 1) \\ &= 4\pi \ln 4 \approx 17.42 \end{aligned}$$

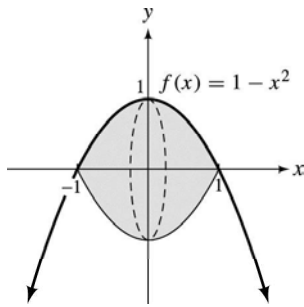
15. $f(x) = x^2$, $y = 0$, $x = 1$, $x = 5$

$$V = \pi \int_1^5 x^4 dx = \frac{\pi x^5}{5} \Big|_1^5 = 625\pi - \frac{\pi}{5} = \frac{3124\pi}{5}$$

16. $f(x) = \frac{x^2}{2}$, $y = 0$, $x = 0$, $x = 4$

$$\begin{aligned} V &= \pi \int_0^4 \left(\frac{x^2}{2} \right)^2 dx \\ &= \pi \int_0^4 \frac{x^4}{4} dx = \frac{\pi}{4} \left(\frac{x^5}{5} \right) \Big|_0^4 \\ &= \frac{\pi}{20}(4^5) = \frac{256\pi}{5} \end{aligned}$$

17. $f(x) = 1 - x^2, y = 0$

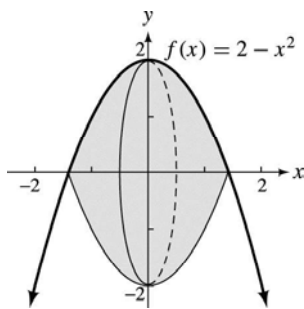


Since $f(x) = 1 - x^2$ intersects $y = 0$ where

$$\begin{aligned} 1 - x^2 &= 0 \\ x &= \pm 1, \\ a &= -1 \text{ and } b = 1. \end{aligned}$$

$$\begin{aligned} V &= \pi \int_{-1}^1 (1 - x^2)^2 dx \\ &= \pi \int_{-1}^1 (1 - 2x^2 + x^4) dx \\ &= \pi \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \Big|_{-1}^1 \\ &= \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) - \pi \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \\ &= 2\pi - \frac{4\pi}{3} + \frac{2\pi}{5} \\ &= \frac{16\pi}{15} \end{aligned}$$

18. $f(x) = 2 - x^2, y = 0$



Since $f(x) = 2 - x^2$ intersects $y = 0$ where

$$\begin{aligned} 2 - x^2 &= 0 \\ x &= \pm\sqrt{2}, \\ a &= -\sqrt{2} \text{ and } b = \sqrt{2}. \end{aligned}$$

$$\begin{aligned} V &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - x^2)^2 dx \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4x^2 + x^4) dx \\ &= \pi \left(4x - \frac{4x^3}{3} + \frac{x^5}{5} \right) \Big|_{-\sqrt{2}}^{\sqrt{2}} \\ &= \pi \left[\left(4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2} \right) - \left(-4\sqrt{2} + \frac{8}{3}\sqrt{2} - \frac{4}{5}\sqrt{2} \right) \right] \\ &= \pi \left(\frac{32}{15}\sqrt{2} + \frac{32}{15}\sqrt{2} \right) \\ &= \frac{64\pi\sqrt{2}}{15} \end{aligned}$$

19. $f(x) = \sqrt{1 - x^2}$
 $r = \sqrt{1} = 1$

$$\begin{aligned} V &= \pi \int_{-1}^1 (\sqrt{1 - x^2})^2 dx \\ &= \pi \int_{-1}^1 (1 - x^2) dx \\ &= \pi \left(x - \frac{x^3}{3} \right) \Big|_{-1}^1 \\ &= \pi \left(1 - \frac{1}{3} \right) - \pi \left(-1 + \frac{1}{3} \right) \\ &= 2\pi - \frac{2}{3}\pi = \frac{4\pi}{3} \end{aligned}$$

20. $f(x) = \sqrt{36 - x^2}$
 $r = \sqrt{36} = 6$

$$\begin{aligned} V &= \pi \int_{-6}^6 (\sqrt{36 - x^2})^2 dx \\ &= \pi \int_{-6}^6 (36 - x^2) dx \\ &= \pi \left(36x - \frac{x^3}{3} \right) \Big|_{-6}^6 \\ &= \pi \left[\left(216 - \frac{216}{3} \right) - \left(-216 + \frac{216}{3} \right) \right] \\ &= 288\pi \end{aligned}$$

21. $f(x) = \sqrt{r^2 - x^2}$

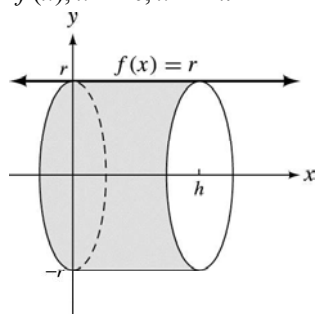
$$\begin{aligned} V &= \pi \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx \\ &= \pi \int_{-r}^r (r^2 - x^2) dx \\ &= \pi \left(r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \pi \left(r^3 - \frac{r^3}{3} \right) - \pi \left(-r^3 + \frac{r^3}{3} \right) \\ &= 2r^3\pi - \left(\frac{2r^3\pi}{3} \right) \\ &= \frac{4\pi r^3}{3} \end{aligned}$$

22. $V = \pi \int_{-a}^a \left[\frac{b}{a} \sqrt{a^2 - x^2} \right]^2 dx$

$$\begin{aligned} &= \pi \int_{-a}^a \frac{b^2}{a^2} (a^2 - x^2) dx \\ &= \frac{\pi b^2}{a^2} \left(a^2 x - \frac{x^3}{3} \right) \Big|_{-a}^a \\ &= \frac{\pi b^2}{a^2} \left[\left(a^3 - \frac{a^3}{3} \right) - \left(-a^3 + \frac{a^3}{3} \right) \right] \\ &= \frac{\pi b^2}{a^2} \left(2a^3 - \frac{2a^3}{3} \right) \\ &= \frac{\pi b^2}{a^2} \left(\frac{4a^3}{3} \right) = \frac{4ab^2\pi}{3} \end{aligned}$$

23. $f(x) = r, x = 0, x = h$

Graph $f(x) = r$; then show the solid of revolution formed by rotating about the x -axis the region bounded by $f(x), x = 0, x = h$.



$$\begin{aligned} \int_0^h \pi r^2 dx &= \pi r^2 x \Big|_0^h \\ &= \pi r^2 h - 0 \\ &= \pi r^2 h \end{aligned}$$

24. $f(x) = 2 - 3x^2; [1, 3]$

Average value

$$\begin{aligned} &= \frac{1}{3-1} \int_1^3 (2 - 3x^2) dx \\ &= \frac{1}{2} (2x - x^3) \Big|_1^3 \\ &= \frac{1}{2} [(6 - 27) - (2 - 1)] \\ &= -11 \end{aligned}$$

25. $f(x) = x^2 - 4; [0, 5]$

Average value

$$\begin{aligned} &= \frac{1}{5-0} \int_0^5 (x^2 - 4) dx \\ &= \frac{1}{5} \left(\frac{x^3}{3} - 4x \right) \Big|_0^5 \\ &= \frac{1}{5} \left[\left(\frac{125}{3} - 20 \right) - 0 \right] \\ &= \frac{13}{3} \approx 4.333 \end{aligned}$$

26. $f(x) = (2x - 1)^{1/3}; [1, 13]$

Average value

$$\begin{aligned} &= \frac{1}{13-1} \int_1^{13} (2x - 1)^{1/2} dx \\ &= \frac{1}{12} \left(\frac{1}{2} \right) \int_1^{13} 2(2x - 1)^{1/2} dx \\ &= \frac{1}{24} \cdot \frac{2}{3} (2x - 1)^{3/2} \Big|_1^{13} \\ &= \frac{1}{36} (25^{3/2} - 1) \\ &= \frac{1}{36} (125) - \frac{1}{36} \\ &= \frac{124}{36} = \frac{31}{9} \approx 3.444 \end{aligned}$$

27. $f(x) = \sqrt{x+1}; [3, 8]$

Average value

$$\begin{aligned} &= \frac{1}{8-3} \int_3^8 \sqrt{x+1} \, dx \\ &= \frac{1}{5} \int_3^8 (x+1)^{1/2} \, dx \\ &= \frac{1}{5} \cdot \frac{2}{3} (x+1)^{3/2} \Big|_3^8 \\ &= \frac{2}{15} (9^{3/2} - 4^{3/2}) \\ &= \frac{2}{15} (27 - 8) = \frac{38}{15} \approx 2.533 \end{aligned}$$

28. $f(x) = e^{0.1x}; [0, 10]$

$$\begin{aligned} \text{Average value} &= \frac{1}{10-0} \int_0^{10} e^{0.1x} \, dx \\ &= \frac{1}{10} \cdot 10(e^{0.1x}) \Big|_0^{10} \\ &= e^1 - 1 = e - 1 \approx 1.718 \end{aligned}$$

29. $f(x) = e^{x/7}; [0, 7]$

Average value

$$\begin{aligned} &= \frac{1}{7-0} \int_0^7 e^{x/7} \, dx \\ &= \frac{1}{7} \cdot 7e^{x/7} \Big|_0^7 \\ &= e^{x/7} \Big|_0^7 = e^1 - e^0 \\ &e - 1 \approx 1.718 \end{aligned}$$

30. $f(x) = x \ln x; [1, e]$

$$\text{Average value} = \frac{1}{e-1} \int_1^e x \ln x \, dx$$

Let $u = \ln x$ and $dv = x \, dx$.

Use column integration.

D	I
$\ln x$	x
$\frac{1}{x}$	$\frac{1}{2}x^2$

$$\int x \ln x \, dx$$

$$\begin{aligned} &= \frac{1}{2}x^2 \ln x - \int \left(\frac{1}{x} \cdot \frac{1}{2}x^2 \right) dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x \, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \end{aligned}$$

$$\begin{aligned} &\frac{1}{e-1} \int_1^e x \ln x \, dx \\ &= \frac{1}{e-1} \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \right) \Big|_1^e \\ &= \frac{1}{e-1} \left(\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \right) \\ &= \frac{1}{e-1} \left(\frac{e^2+1}{4} \right) \\ &= \frac{e^2+1}{4(e-1)} \approx 1.221 \end{aligned}$$

31. $f(x) = x^2 e^{2x}; [0, 2]$

$$\text{Average value} = \frac{1}{2-0} \int_0^2 x^2 e^{2x} \, dx$$

Let $u = x^2$ and $dv = e^{2x} \, dx$.

Use column integration.

D	I
x^2	e^{2x}
$2x$	$\frac{1}{2}e^{2x}$
2	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

$$\begin{aligned} &\frac{1}{2-0} \int_0^2 x^2 e^{2x} \, dx \\ &= \frac{1}{2} \left[(x^2) \left(\frac{1}{2} \right) e^{2x} - (2x) \left(\frac{1}{4} \right) e^{2x} + 2 \left(\frac{1}{8} \right) e^{2x} \right] \Big|_0^2 \\ &= \frac{1}{2} \left(2e^4 - e^4 + \frac{1}{4}e^4 - \frac{1}{4} \right) \\ &= \frac{5e^4 - 1}{8} \approx 34.00 \end{aligned}$$

32. $f(x) = \frac{1}{4+x^2}, y = 0, x = -2, x = 2$

$$V = \pi \int_{-2}^2 \left(\frac{1}{4+x^2} \right)^2 dx = \pi \int_{-2}^2 (4+x^2)^{-2} dx$$

Using an integration feature on a graphing calculator to evaluate the integral, we obtain $0.5047746784 \approx 0.5048$.

33. $f(x) = e^{-x^2}, y = 0, x = -1, x = 1$

$$V = \pi \int_{-1}^1 (e^{-x^2})^2 dx = \pi \int_{-1}^1 e^{-2x^2} dx$$

Using an integration feature on a graphing calculator to evaluate the integral, we obtain $3.758249634 \approx 3.758$.

34. Use the formula for average value with $a = 0$ and $b = 5$. The average price is

$$\begin{aligned} & \frac{1}{5-0} \int_0^5 [t(25-5t) + 18] dt \\ &= \frac{1}{5} \int_0^5 (25t - 5t^2 + 18) dt \\ &= \frac{1}{5} \left(\frac{25t^2}{2} - \frac{5t^3}{3} + 18t \right) \Big|_0^5 \\ &= \frac{1}{5} \left(\frac{625}{2} - \frac{625}{3} + 90 \right) \\ &= \frac{125}{2} - \frac{125}{3} + 18 \approx \$38.83 \end{aligned}$$

35. Use the formula for average value with $a = 0$ and $b = 6$.

$$\begin{aligned} & \frac{1}{6-0} \int_0^6 (37 + 6e^{-0.03t}) dt \\ &= \frac{1}{6} \left(37t + \frac{6}{-0.03} e^{-0.03t} \right) \Big|_0^6 \\ &= \frac{1}{6} (37t - 200e^{-0.03t}) \Big|_0^6 \\ &= \frac{1}{6} [(222 - 200e^{-0.18}) - (0 - 200)] \\ &= \frac{1}{6} (422 - 200e^{-0.18}) \\ &\approx 42.49 \end{aligned}$$

The average price is \$42.49.

36. At the end of any given business day, CFFC has $400 - 80t$ cases of perfume on hand, where $t = 1$ represents Monday, $t = 2$ represents Tuesday, and so on.

The average daily number of cases in inventory is

$$\begin{aligned} \frac{1}{5-0} \int_0^5 (400 - 80t) dt &= \frac{1}{5} \left(400t - \frac{80t^2}{2} \right) \Big|_0^5 \\ &= \frac{1}{5} (400t - 40t^2) \Big|_0^5 \\ &= \frac{1}{5} (2000 - 1000) \\ &= 200 \text{ cases} \end{aligned}$$

37. Use the formula for average value with $a = 0$ and $b = 6$. The average price is

$$\begin{aligned} & \frac{1}{30-0} \int_0^{30} (600 - 20\sqrt{30}t) dt \\ &= \frac{1}{30} \left(600t - 20\sqrt{30} \cdot \frac{2}{3} t^{3/2} \right) \Big|_0^{30} \\ &= \frac{1}{30} \left(600t - \frac{40\sqrt{30}}{3} t^{3/2} \right) \Big|_0^{30} \\ &= \frac{1}{30} (18,000 - 12,000) \\ &= 200 \text{ cases} \end{aligned}$$

38. (a) $\int_0^R 2\pi rk(R^2 - r^2) dr$

$$= 2\pi k \int_0^R r(R^2 - r^2) dr$$

(b) $\int_0^R 2\pi rk(R^2 - r^2) dr$

$$\begin{aligned} &= 2\pi k \int_0^R (rR^2 - r^3) dr \\ &= 2\pi k \left[\frac{r^2 R^2}{2} - \frac{r^4}{4} \right] \Big|_0^R \\ &= 2\pi k \left[\frac{R^4}{2} - \frac{R^4}{4} - 0 \right] \\ &= 2\pi k \left(\frac{R^4}{4} \right) = \frac{\pi k R^4}{2} \end{aligned}$$

39. $R(t) = te^{-0.1t}$

"During the n th hour" corresponds to the interval $(n-1, n)$.

The average intensity during n th hour is

$$\frac{1}{n - (n-1)} \int_{n-1}^n te^{-0.1t} dt = \int_{n-1}^n te^{-0.1t} dt$$

Let $u = t$ and $dv = e^{-0.1t} dt$.

D	I
t	$+$
1	$-$
0	$100e^{-0.1t}$

$$\begin{aligned} \int_{n-1}^n te^{-0.1t} dt \\ = (-10te^{-0.1t} - 100e^{-0.1t}) \Big|_{n-1}^n \end{aligned}$$

(a) Second hour, $n = 2$

Average intensity

$$\begin{aligned} &= -10e^{-0.2}(12) + 10e^{-0.1}(11) \\ &= 110e^{-0.1} - 120e^{-0.2} \\ &\approx 1.284 \end{aligned}$$

(b) Twelfth hour, $n = 12$

Average intensity

$$\begin{aligned} &= -10e^{-1.2}(12 + 10) + 10e^{-1.1}(11 + 10) \\ &= 210e^{-1.1} - 220e^{-1.2} \approx 3.640 \end{aligned}$$

(c) Twenty-fourth hour, $n = 24$

Average intensity

$$\begin{aligned} &= -10e^{-2.4}(24 + 10) + 10e^{-2.3}(23 + 10) \\ &= 330e^{-2.3} - 340e^{-2.4} \approx 2.241 \end{aligned}$$

40. From problem 22, the volume of an ellipsoid with major axis $2a$ and minor axis $2b$ is

$$V = \frac{4}{3}ab^2\pi.$$

(a) i. Canada goose: $a = \frac{8.6}{2} = 4.3$,

$$b = \frac{5.8}{2} = 2.9$$

$$V = \frac{4}{3}(4.3)(2.9)^2\pi \approx 151.51$$

The volume is about 151.5 cubic cm.

ii. Robin: $a = 0.95, b = 0.75$

$$V = \frac{4}{3}(0.95)(0.75)^2\pi \approx 2.238$$

The volume is about 2.238 cubic cm.

iii. Turtledove: $a = 1.55, b = 1.15$

$$V = \frac{4}{3}(1.55)(1.15)^2\pi \approx 8.586$$

The volume is about 8.586 cubic cm.

iv. Hummingbird: $a = 0.5, b = 0.5$

$$V = \frac{4}{3}(0.5)(0.5)^2\pi \approx 0.5236$$

The volume is about 0.5236 cubic cm.

v. Raven: $a = 2.5, b = 1.65$

$$V = \frac{4}{3}(2.5)(1.65)^2\pi \approx 28.51$$

The volume is about 28.51 cubic cm.

(b) Using the formula

$$V = \frac{4}{3}ab^2\pi$$

a is half the length, l , and b half the width, w , of a bird egg.

Substituting these expressions and writing l in terms of w , we have

$$\begin{aligned} V &= \frac{4}{3} \left(\frac{l}{2} \right) \left(\frac{w}{2} \right)^2 \pi \\ &= \frac{4}{3} \left(\frac{1.585w - 0.487}{2} \right) \left(\frac{w^2}{4} \right) \pi \\ &= \frac{\pi(1.585w^3 - 0.487w^2)}{6}. \end{aligned}$$

i. Canada goose: $w = 5.8$

$$\begin{aligned} V &= \frac{\pi[1.585(5.8)^3 - 0.487(5.8)^2]}{6} \\ &\approx 153.3 \end{aligned}$$

The volume is about 153.3 cubic cm.

ii. Robin: $w = 1.5$

$$\begin{aligned} V &= \frac{\pi[1.585(1.5)^3 - 0.487(1.5)^2]}{6} \\ &\approx 2.227 \end{aligned}$$

The volume is about 2.227 cubic cm.

- iii. Turtledove:
- $w = 2.3$

$$V = \frac{\pi[1.585(2.3)^3 - 0.487(2.3)^2]}{6}$$

$$\approx 8.749$$

The volume is about 8.749 cubic cm.

- iv. Hummingbird:
- $w = 1.0$

$$V = \frac{\pi[1.585(1.0)^3 - 0.487(1.0)^2]}{6}$$

$$\approx 0.5749$$

The volume is about 0.5749 cubic cm.

- v. Raven:
- $w = 3.3$

$$V = \frac{\pi[1.585(3.3)^3 - 0.487(3.3)^2]}{6}$$

$$\approx 27.051$$

The volume is about 27.05 cubic cm.

41. For each part below, use

Average value

$$= \frac{1}{b-a} \int_a^b 45 \ln(t+1) dt$$

$$= \frac{45}{b-a} \int_a^b \ln(t+1) dt.$$

Evaluate the integral using integration by parts.

Let $u = \ln(t+1)$ and $dv = dt$.Then $du = \frac{1}{t+1} dt$ and $v = t$.

$$\int \ln(t+1) dt$$

$$= t \ln(t+1) - \int \frac{t}{t+1} dt$$

$$= t \ln(t+1) - \int \left(1 - \frac{1}{t+1}\right) dt$$

$$= t \ln(t+1) - t + \ln(t+1) + C$$

$$= (t+1) \ln(t+1) - t + C$$

Therefore

Average value

$$= \frac{1}{b-a} \int_a^b 45 \ln(t+1) dt$$

$$= \frac{45}{b-a} [(t+1) \ln(t+1) - t] \Big|_a^b.$$

- (a) The average number of items produced daily after 5 days is

$$\frac{45}{5-0} [(t+1) \ln(t+1) - t] \Big|_0^5$$

$$= 9[(6 \ln 6 - 5) - (\ln 1 - 0)]$$

$$= 9(6 \ln 6 - 5) \approx 51.76.$$

- (b) The average number of items produced daily after 9 days is

$$\frac{45}{9-0} [(t+1) \ln(t+1) - t] \Big|_0^9$$

$$= 5(10 \ln 10 - 9) \approx 70.13.$$

- (c) The average number of items produced daily after 30 days is

$$\frac{45}{30-0} [(t+1) \ln(t+1) - t] \Big|_0^{30}$$

$$= \frac{3}{2} (31 \ln 31 - 30) \approx 114.7.$$

42. $W(t) = -3.75t^2 + 30t + 40$

- (a)
- $W(0) = -3.75(0)^2 + 30(0) + 40$
-
- $W(0) = 40$
- words/minute

- (b)
- $W'(t) = -7.50t + 30$
-
- $-7.50t + 30 = 0$
-
- $t = 4$

If $0 \leq t < 4$, $W'(t) > 0$.If $4 < t \leq 5$, $W'(t) < 0$.Therefore, a maximum occurs when $t = 4$.

$$W(4) = -3.75(4)^2 + 30(4) + 40$$

$$W(4) = 100$$

A maximum speed of 100 words per minute occurs when $t = 4$ minutes.

- (c) The average value of
- W
- over
- $[0, 5]$
- is given by

$$\frac{1}{5-0} \int_0^5 (-3.75t^2 + 30t + 40) dt$$

$$= \frac{1}{5} (-1.25t^3 + 15t^2 + 40t) \Big|_0^5$$

$$= \frac{1}{5} [(-156.25 + 375 + 200) - 0]$$

$$= 83.75.$$

The average value is 83.75 words per minute.

43. From Exercise 22, the volume of an ellipsoid with horizontal axis of length $2a$ and vertical axis of length $2b$ is

$$V = \frac{4ab^2\pi}{3}.$$

For the Earth, $a = 6,356,752.3142$ and $b = 6,378,137$.

$$V = \frac{4(6,356,752.3142)(6,378,137)^2\pi}{3} \\ \approx 1.083 \times 10^{21}$$

The volume of the Earth is about 1.083×10^{21} cubic meters (m^3).

8.3 Continuous Money Flow

Your Turn 1

$$f(t) = 810e^{kt}$$

Use $f(1) = 797.94$ to find k .

$$f(1) = 810e^{k(1)}$$

$$797.94 = 810e^k$$

$$e^k = \frac{797.94}{810} \approx 0.9851$$

$$k \approx \ln 0.9851 \approx -0.015$$

$$f(t) = 810e^{-0.015t}$$

$$\text{Total income} = \int_0^2 810e^{-0.015t} dt$$

$$= -\frac{810}{0.015}e^{-0.015t} \Big|_0^2$$

$$= -54,000e^{-0.015t} \Big|_0^2$$

$$= -54,000(e^{-0.03} - 1) \\ \approx 1595.94$$

The total income is \$1595.94

Your Turn 2

Find the present value of an income given by $f(t) = 50,000t$ over the next 5 years if the interest rate is 3.5%.

$$f(t) = 50,000t, \quad 0 \leq t \leq 5$$

$$P = \int_0^5 50,000te^{-0.035t} dt = 50,000 \int_0^5 te^{-0.035t} dt$$

Use integration by parts.

$$\text{Let } u = t \text{ and } dv = e^{-0.035t} dt.$$

$$\text{Then } du = dt \text{ and } v = -\frac{1}{0.035}e^{-0.035t} \\ = -28.57143e^{-0.035t}.$$

$$\int_0^5 te^{-0.035t} dt \\ = -28.57143te^{-0.035t} - \frac{28.57143}{0.035}e^{-0.035t} + C \\ = (-28.57143t - 816.32657)e^{-0.035t} + C$$

$$P = 50,000 \int_0^5 te^{-0.035t} dt \\ = 50,000(-28.57143t - 816.32657)e^{-0.035t} \Big|_0^5 \\ = 50,000[(-28.57143(5) - 816.32657)e^{-0.035(5)} \\ - (0 - 816.32657)] \\ = 50,000(-805.19351 + 816.32657) \\ \approx 556,653$$

The present value of the income is \$556,653.

Your Turn 3

Find the accumulated amount of money flow for an income given by $f(t) = 50,000t$ over the next 5 years if the interest rate is 3.5%.

$$A = e^{0.035(5)} \int_0^5 50,000te^{-0.035t} dt$$

The integral was computed in Your Turn 2. Using this value, the accumulated value is

$$e^{0.175}(556,653) \approx 663,111.$$

The accumulated amount of money flow is \$663,111.

Your Turn 4

Find the present value at the end of 8 years of the continuous flow of money given by

$$f(t) = 200t^2 + 100t + 50$$

at 5% compounded continuously.

$$P = \int_0^8 (200t^2 + 100t + 50)e^{-0.05t} dt$$

Using integration by parts as in Example 5, you can verify that

$$\begin{aligned} & \int (200t^2 + 100t + 50)e^{-0.05t} dt \\ &= -1000e^{-0.05t}(4t^2 + 162t + 3241) + C. \end{aligned}$$

$$\begin{aligned} \text{Thus } P &= -1000e^{-0.05t}(4t^2 + 162t + 3241) \Big|_0^8 \\ &= -1000(e^{-0.4}(4793) - 3241) \\ &\approx 28,156.02 \end{aligned}$$

The present value at the end of 8 years of this flow of money is \$28,156.02.

8.3 Exercises

1. $f(t) = 1000$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 1000e^{-0.08t} dt \\ &= \frac{1000}{-0.08} e^{-0.08t} \Big|_0^{10} \\ &= -12,500(e^{-0.8} - e^0) \\ &= -12,500(e^{-0.8} - 1) \\ &\approx 6883.387949 \end{aligned}$$

(We will use this value for P in part (b). Store it in your calculator without rounding.)

The present value is \$6883.39.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 1000e^{-0.08t} dt \\ &= e^{0.8}P \\ &\approx 15,319.26161 \end{aligned}$$

The accumulated value is \$15,319.26.

2. $f(t) = 300$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} (300)e^{-0.08t} dt \\ &= \frac{300}{-0.08} e^{-0.08t} \Big|_0^{10} \\ &= -3750(e^{-0.8} - e^0) \\ &= -3750(e^{-0.8} - 1) \\ &\approx 2065.016385 \end{aligned}$$

(We will use this value for P in part (b). Store it in your calculator without rounding.)

The present value is \$2065.02.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 300e^{-0.08t} dt \\ &= e^{0.8}P \\ &\approx 4595.778482 \end{aligned}$$

The accumulated value is \$4595.78.

3. $f(t) = 500$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 500e^{-0.08t} dt \\ &= \frac{500}{-0.08} e^{-0.08t} \Big|_0^{10} \\ &= -6250(e^{-0.8} - e^0) \\ &\approx 3441.693974 \end{aligned}$$

The present value is \$3441.69.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 500e^{-0.08t} dt \\ &= e^{0.8}P \approx 7659.630803 \end{aligned}$$

The accumulated value is \$7659.63.

4. $f(t) = 2000$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 2000e^{-0.08t} dt \\ &= \frac{2000}{-0.08} e^{-0.08t} \Big|_0^{10} \\ &= -25,000(e^{-0.8} - e^0) \\ &\approx 13,766.7759 \end{aligned}$$

The present value is \$13,766.78.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 2000e^{-0.08t} dt \\ &= e^{0.8}P \approx 30,638.52321 \end{aligned}$$

The accumulated value is \$30,638.52.

5. $f(t) = 400e^{0.03t}$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 400e^{0.03t} e^{-0.08t} dt \\ &= 400 \int_0^{10} e^{-0.05t} dt = \frac{400}{-0.05} e^{-0.05t} \Big|_0^{10} \\ &= -8000(e^{-0.5} - e^0) \\ &\approx 3147.754722 \end{aligned}$$

The present value is \$3147.75.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 400e^{0.03t} e^{-0.08t} dt \\ &= e^{0.8} P \approx 7005.456967 \\ \text{The accumulated value is } \$7005.46. \end{aligned}$$

$$6. \quad f(t) = 800e^{0.05t}$$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 800e^{0.05t} e^{-0.08t} dt \\ &= 800 \int_0^{10} e^{-0.03t} dt \\ &= \frac{800}{-0.03} e^{-0.03t} \Big|_0^{10} \\ &= -\frac{800}{0.03} (e^{-0.03} - e^0) \\ &\approx 6911.514115 \end{aligned}$$

The present value is \$6911.51.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 800e^{0.05t} e^{-0.08t} dt \\ &= e^{0.8} P \approx 15,381.85754 \\ \text{The accumulated value is } \$15,381.86. \end{aligned}$$

$$7. \quad f(t) = 5000e^{-0.01t}$$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 5000e^{-0.01t} e^{-0.08t} dt \\ &= 5000 \int_0^{10} e^{-0.09t} dt \\ &= \frac{5000}{-0.09} e^{-0.09t} \Big|_0^{10} \\ &= -\frac{5000}{0.09} (e^{-0.9} - e^0) \\ &\approx 32,968.35224 \end{aligned}$$

The present value is \$32,968.35.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 5000e^{-0.01t} e^{-0.08t} dt \\ &= e^{0.8} P \\ &\approx 73,372.41725 \end{aligned}$$

The accumulated value is \$73,372.42.

$$8. \quad f(t) = 1000e^{-0.02t}$$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 1000e^{-0.02t} e^{-0.08t} dt \\ &= 1000 \int_0^{10} e^{-0.1t} dt \\ &= \frac{1000}{-0.1} e^{-0.1t} \Big|_0^{10} \\ &= -10,000(e^{-1} - e^0) \\ &\approx 6321.205588 \end{aligned}$$

The present value is \$6321.21.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 1000e^{-0.02t} e^{-0.08t} dt \\ &= e^{0.8} P \\ &\approx 14,068.10175 \end{aligned}$$

The accumulated value is \$14,068.10.

$$9. \quad f(t) = 25t$$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} 25te^{-0.08t} dt \\ &= 25 \int_0^{10} te^{-0.08t} dt \end{aligned}$$

Find the antiderivative using integration by parts.

$$\begin{aligned} \text{Let } u &= t \quad \text{and} \quad dv = e^{-0.08t} dt. \\ \text{Then } du &= dt \quad \text{and} \quad v = \frac{1}{-0.08} e^{-0.08t} \\ &= -12.5e^{-0.08t}. \end{aligned}$$

$$\begin{aligned} &\int te^{-0.08t} dt \\ &= t(-12.5e^{-0.08t}) - \int (-12.5e^{-0.08t}) dt \\ &= -12.5te^{-0.08t} + 12.5 \int e^{-0.08t} dt \\ &= -12.5te^{-0.08t} + \frac{12.5}{-0.08} e^{-0.08t} + C \\ &= -(12.5t + 156.25)e^{-0.08t} + C \end{aligned}$$

Therefore

$$\begin{aligned} P &= 25[-(12.5t + 156.25)e^{-0.08t}] \Big|_0^{10} \\ &= [-25(12.5t + 156.25)e^{-0.08t}] \Big|_0^{10} \\ &= (-7031.25e^{-0.8}) - (-3906.25e^0) \\ &\approx 746.9057211. \end{aligned}$$

The present value is \$746.91.

$$\begin{aligned}
 \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 25te^{-0.08t} dt \\
 &= e^{0.8} P \\
 &\approx 1662.269252
 \end{aligned}$$

The accumulated value is \$1662.27.

$$10. \quad f(t) = 50t$$

$$\begin{aligned}
 \text{(a)} \quad P &= \int_0^{10} 50te^{-0.08t} dt \\
 &= 50 \int_0^{10} te^{-0.08t} dt
 \end{aligned}$$

Find the antiderivative using integration by parts.

$$\text{Let } u = t \text{ and } dv = e^{-0.08t} dt.$$

$$\begin{aligned}
 \text{Then } du &= dt \text{ and } v = \frac{1}{-0.08} e^{-0.08t} \\
 &= -12.5e^{-0.08t}.
 \end{aligned}$$

$$\begin{aligned}
 \int te^{-0.08t} dt &= t(-12.5e^{-0.08t}) - \int (-12.5e^{-0.08t}) dt \\
 &= -12.5te^{-0.08t} + 12.5 \int e^{-0.08t} dt \\
 &= -12.5te^{-0.08t} + \frac{12.5}{-0.08} e^{-0.08t} + C \\
 &= -(12.5t + 156.25)e^{-0.08t} + C
 \end{aligned}$$

Therefore:

$$\begin{aligned}
 P &= 50[-(12.5t + 156.25)e^{-0.08t}]_0^{10} \\
 &= [-50(12.5t + 156.25)e^{-0.08t}]_0^{10} \\
 &= (-14,062.5e^{-0.8}) - (-7812.5e^0) \\
 &\approx 1493.811442.
 \end{aligned}$$

The present value is \$1493.81.

$$\begin{aligned}
 \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} 50te^{-0.08t} dt \\
 &= e^{0.8} P \\
 &\approx 3324.538504
 \end{aligned}$$

The accumulated value is \$3324.54.

$$11. \quad f(t) = 0.01t + 100$$

$$\begin{aligned}
 \text{(a)} \quad P &= \int_0^{10} (0.01t + 100)e^{-0.08t} dt \\
 &= \int_0^{10} 0.01te^{-0.08t} dt \\
 &\quad + \int_0^{10} 100e^{-0.08t} dt \\
 &= 0.01 \int_0^{10} te^{-0.08t} dt \\
 &\quad + 100 \int_0^{10} e^{-0.08t} dt
 \end{aligned}$$

From Exercise 9, we know that

$$\int te^{-0.08t} dt = -(12.5t + 156.25)e^{-0.08t} + C$$

From Exercise 1, we know that

$$\int e^{-0.08t} dt = -12.5e^{-0.08t} + C$$

Substitute the given expressions and simplify.

$$\begin{aligned}
 P &= \{0.01[-(12.5t + 156.25)e^{-0.08t}] \\
 &\quad + 100(-12.5e^{-0.08t})\}_0^{10} \\
 &= [-(0.125t + 1251.5625)e^{-0.08t}]_0^{10} \\
 &= (-1252.8125e^{-0.8}) - (-1251.5625e^0) \\
 &\approx 688.6375571
 \end{aligned}$$

The Present value is \$688.64.

$$\begin{aligned}
 \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} (0.01t + 100)e^{-0.08t} dt \\
 &= e^{0.8} P \\
 &\approx 1532.591068
 \end{aligned}$$

The accumulated value is \$1532.59.

$$12. \quad f(t) = 0.05t + 500$$

$$\begin{aligned}
 \text{(a)} \quad P &= \int_0^{10} (0.05t + 500)e^{-0.08t} dt \\
 &= \int_0^{10} 0.05te^{-0.08t} dt + \int_0^{10} 500e^{-0.08t} dt \\
 &= 0.05 \int_0^{10} te^{-0.08t} dt + 500 \int_0^{10} e^{-0.08t} dt
 \end{aligned}$$

From Exercise 10, we know that

$$\int te^{-0.08t} dt = -(12.5t + 156.25)e^{-0.08t} + C$$

From Exercise 2, we know that

$$\int e^{-0.08t} dt = -12.5e^{-0.08t} + C$$

Substitute the given expression and simplify.

$$\begin{aligned} P &= \{0.05[-(12.5t + 156.25)e^{-0.08t}] \\ &\quad + 500(-12.5e^{-0.08t})\} \Big|_0^{10} \\ &= [-(0.625t + 6257.8125)e^{-0.08t}] \Big|_0^{10} \\ &= (-6264.0625e^{-0.8}) - (-6257.8125e^0) \\ &\approx 3443.187786 \end{aligned}$$

The present value is \$3443.19.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} (0.05t + 500)e^{-0.08t} dt \\ &= e^{0.8} P \\ &\approx 7662.955342 \end{aligned}$$

The accumulated value is \$7662.96.

$$13. \quad f(t) = 1000t - 100t^2$$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} (1000t - 100t^2)e^{-0.08t} dt \\ &= 1000 \int_0^{10} te^{-0.08t} dt \\ &\quad - 100 \int_0^{10} t^2 e^{-0.08t} dt \end{aligned}$$

From Exercise 9, we know that

$$\int te^{-0.08t} dt = -(12.5t + 156.25)e^{-0.08t} + C$$

Evaluate the antiderivative $\int t^2 e^{-0.08t} dt$ using column integration. (Note that $\frac{1}{-0.08} = -12.5$.)

D		I
t^2	+	$e^{-0.08t}$
$2t$	-	$12.5e^{-0.08t}$
2	+	$156.25e^{-0.08t}$
0		$-1953.125e^{-0.08t}$

Thus,

$$\begin{aligned} \int_0^{10} t^2 e^{-0.08t} dt &= (t^2)(-12.5e^{-0.08t}) \\ &\quad - (2t)(156.25e^{-0.08t}) \\ &\quad + (2)(-1953.125e^{-0.08t}) + C \\ &= -(12.5t^2 + 312.5t \\ &\quad + 3906.25)e^{-0.08t} + C. \end{aligned}$$

Therefore:

$$\begin{aligned} P &= \{1000[-(12.5t + 156.25)e^{-0.08t}] \\ &\quad - 100[-(12.5t^2 + 312.5t \\ &\quad + 3906.25)e^{-0.08t}]\} \Big|_0^{10} \end{aligned}$$

Collect like terms and simplify.

$$\begin{aligned} P &= [(1250t^2 + 18,750t + 234,375)e^{-0.08t}] \Big|_0^{10} \\ &= (546,875e^{-0.8}) - (234,375e^0) \\ &\approx 11,351.77725 \end{aligned}$$

The principal value is \$11,351.78.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} (1000t - 100t^2)e^{-0.08t} dt \\ &= e^{0.8} P \\ &\approx 25,263.84488 \end{aligned}$$

The accumulated value is \$25,263.84.

$$14. \quad f(t) = 2000t - 150t^2$$

$$\begin{aligned} \text{(a)} \quad P &= \int_0^{10} (2000t - 150t^2)e^{-0.08t} dt \\ &= 2000 \int_0^{10} te^{-0.08t} dt \\ &\quad - 150 \int_0^{10} t^2 e^{-0.08t} dt \end{aligned}$$

From Exercise 10, we know that

$$\int t e^{-0.08t} dt = -(12.5t + 156.25)e^{-0.08t} + C$$

Evaluate the antiderivative $\int t^2 e^{-0.08t} dt$ using column integration. (Note that

$$\frac{1}{-0.08} = -12.5.)$$

D		I
t^2	+	$e^{-0.08t}$
$2t$	-	$-12.5e^{-0.08t}$
2	+	$156.25e^{-0.08t}$
0		$-1953.125e^{-0.08t}$

Thus,

$$\begin{aligned} \int_0^{10} t^2 e^{-0.08t} dt &= (t^2)(-12.5e^{-0.08t}) - (2t)(156.25e^{-0.08t}) \\ &\quad + (2)(-1953.125e^{-0.08t}) + C \\ &= -(12.5t^2 + 312.5t + 3906.25)e^{-0.08t} + C. \end{aligned}$$

Therefore:

$$\begin{aligned} P &= \{2000[-(12.5t + 156.25)e^{-0.08t}] \\ &\quad - 150[-(12.5t^2 + 312.5t + 3906.25)e^{-0.08t}]\}_{0}^{10} \end{aligned}$$

Collect like terms and simplify.

$$\begin{aligned} P &= [(1875t^2 + 21,875t + 273,437.5)e^{-0.08t}]_{0}^{10} \\ &= (679,687.5e^{-0.8}) - (273,437.5e^0) \\ &\approx 31,965.7803 \end{aligned}$$

The principal value is \$31,965.78.

$$\begin{aligned} \text{(b)} \quad A &= e^{0.08(10)} \int_0^{10} (2000t - 150t^2)e^{-0.08t} dt \\ &= e^{0.8} P \\ &\approx 71,141.1524 \end{aligned}$$

The accumulated value is \$71,141.15.

$$\begin{aligned} 15. \quad A &= e^{0.04(3)} \int_0^3 20,000e^{-0.04t} dt \\ &= e^{0.12} \left(\frac{20,000}{-0.04} e^{-0.04t} \right)_{0}^3 \\ &= e^{0.12} \left(\frac{20,000}{-0.04} e^{-0.12} + \frac{20,000}{0.04} \right) \\ &\approx \$63,748.43 \end{aligned}$$

16. (a) Present value

$$\begin{aligned} &= \int_0^6 8000e^{-0.02t} dt \\ &= \frac{8000}{-0.02} e^{-0.02t} \Big|_0^6 \\ &= -400,000(e^{-0.12} - 1) \\ &\approx \$45,231.83 \end{aligned}$$

(b) Present value

$$\begin{aligned} &= \int_0^6 8000e^{-0.05t} dt \\ &= \frac{8000}{-0.05} e^{-0.05t} \Big|_0^6 \\ &= -160,000(e^{-0.3} - 1) \\ &\approx \$41,469.08 \end{aligned}$$

(c) Present value

$$\begin{aligned} &= \int_0^6 8000e^{-0.08t} dt \\ &= \frac{8000}{-0.08} e^{-0.08t} \Big|_0^6 \\ &= -100,000(e^{-0.48} - 1) \\ &\approx \$38,121.66 \end{aligned}$$

17. (a) Present value

$$\begin{aligned} &= \int_0^8 5000e^{-0.01t} e^{-0.08t} dt \\ &= \int_0^8 5000e^{-0.09t} dt \\ &= \left(\frac{5000}{-0.09} e^{-0.09t} \right)_{0}^8 \\ &= \frac{5000e^{-0.72}}{-0.09} + \frac{5000}{0.09} \\ &= \$28,513.76 \end{aligned}$$

(b) Final amount

$$\begin{aligned} &= e^{0.08(8)} \int_0^8 5000e^{-0.01t} e^{-0.08t} dt \\ &\approx e^{0.64} (28,513.76) \\ &\approx \$54,075.81 \end{aligned}$$

18. (a) Present Value

$$\begin{aligned}
&= \int_0^4 1000e^{0.05t}e^{-0.035t} dt \\
&= 1000 \int_0^4 e^{0.015t} dt \\
&= \frac{1000}{0.015} e^{0.015t} \Big|_0^4 \\
&= \frac{1000}{0.015} (e^{0.06} - 1) \\
&\approx 4122.436 \\
&\approx \$4122.44
\end{aligned}$$

(b) Final amount

$$\begin{aligned}
&= e^{0.035(4)} \int_0^4 1000e^{0.05t}e^{-0.035t} dt \\
&= e^{0.14} (4122.436) \\
&\approx \$4741.93
\end{aligned}$$

$$\begin{aligned}
19. \quad P &= \int_0^5 (1500 - 60t^2)e^{-0.05t} dt \\
&= \int_0^5 1500e^{-0.05t} dt - \int_0^5 60t^2e^{-0.05t} dt \\
&= 1500 \int_0^5 e^{-0.05t} dt - 60 \int_0^5 t^2e^{-0.05t} dt
\end{aligned}$$

Find the second integral by column integration.

D	I
t^2	$e^{-0.05t}$
$+$	$\swarrow$
$2t$	$\frac{e^{-0.05t}}{-0.05}$
$-$	$\swarrow$
2	$\frac{e^{-0.05t}}{0.0025}$
$+$	$\swarrow$
0	$\frac{e^{-0.05t}}{-0.000125}$

$$\begin{aligned}
\int t^2 e^{-0.05t} dt &= \frac{t^2 e^{-0.05t}}{-0.05} - \frac{2t e^{-0.05t}}{0.0025} + \frac{2e^{-0.05t}}{-0.000125} + C \\
&= -e^{-0.05t} \left(\frac{t^2}{0.05} + \frac{2t}{0.0025} + \frac{2}{0.000125} \right) + C
\end{aligned}$$

Now add the first integral to this result.

$$\begin{aligned}
&1500 \int_0^5 e^{-0.05t} dt - 60 \int_0^5 t^2 e^{-0.05t} dt \\
&= \frac{1500}{-0.05} e^{-0.05t} \Big|_0^5 \\
&\quad + 60 e^{-0.05t} \left(\frac{t^2}{0.05} + \frac{2t}{0.0025} + \frac{2}{0.000125} \right) \Big|_0^5 \\
&= \frac{1500}{-0.05} (e^{-0.25} - 1) \\
&\quad + 60 \left[e^{-0.25} \left(\frac{25}{0.05} + \frac{10}{0.0025} + \frac{2}{0.000125} \right) - \frac{2}{0.000125} \right] \\
&\approx 6636.977 - 2075.037 \\
&= \$4560.94
\end{aligned}$$

$$\begin{aligned}
20. \quad A &= e^{0.05(3)} \int_0^3 (1000 - t^2)e^{-0.05t} dt \\
&= e^{0.15} \int_0^3 1000e^{-0.05t} dt - e^{0.15} \int_0^3 t^2 e^{-0.05t} dt
\end{aligned}$$

Using the integral we found in Exercise 19, we can write this as follows:

$$\begin{aligned}
&\left(1000e^{0.15} \right) \frac{e^{-0.05t}}{-0.05} \Big|_0^3 \\
&\quad - (e^{0.15}) \left[-e^{-0.05t} \left(\frac{t^2}{0.05} + \frac{2t}{0.0025} + \frac{2}{0.000125} \right) \right] \Big|_0^3 \\
&\approx 3236.685 - 9.348 \\
&\approx \$3227.34
\end{aligned}$$

8.4 Improper Integrals

Your Turn 1

$$\begin{aligned}
(a) \quad \int_8^\infty \frac{1}{x^{1/3}} dx &= \lim_{b \rightarrow \infty} \int_8^b \frac{1}{x^{1/3}} dx \\
&= \lim_{b \rightarrow \infty} \left(\frac{3}{2} x^{2/3} \right) \Big|_8^b \\
&= \lim_{b \rightarrow \infty} \left(\frac{3}{2} b^{2/3} - \frac{3}{2} (4) \right)
\end{aligned}$$

As $b \rightarrow \infty$, $b^{2/3} \rightarrow \infty$, so the limit above does not exist and the integral diverges.

$$\begin{aligned}
 \text{(b)} \quad \int_8^\infty \frac{1}{x^{4/3}} dx &= \lim_{b \rightarrow \infty} \int_8^b \frac{1}{x^{4/3}} dx \\
 &= \lim_{b \rightarrow \infty} \left(-3x^{-1/3} \right) \Big|_8^b \\
 &= \lim_{b \rightarrow \infty} \left[-3b^{-1/3} - \left(-\frac{3}{8^{1/3}} \right) \right] \\
 \text{As } b \rightarrow \infty, b^{-1/3} &= \frac{1}{b^{1/3}} \rightarrow 0, \text{ so the} \\
 \text{integral is convergent and its value is} \\
 \frac{3}{8^{1/3}} &= \frac{3}{2}.
 \end{aligned}$$

Your Turn 2

$$\begin{aligned}
 \int_0^\infty 5e^{-2x} dx &= \lim_{b \rightarrow \infty} \int_0^b 5e^{-2x} dx \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{5}{2} e^{-2x} \right) \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{5}{2} e^{-2b} - \left(-\frac{5}{2} e^{-2(0)} \right) \right]
 \end{aligned}$$

As $b \rightarrow \infty$, $e^{-2b} = \frac{1}{e^b} \rightarrow 0$, so the integral is convergent and its value is $\frac{5}{2}e^0 = \frac{5}{2}$.

8.4 Exercises

$$\begin{aligned}
 1. \quad \int_3^\infty \frac{1}{x^2} dx &= \lim_{b \rightarrow \infty} \int_3^b x^{-2} dx \\
 &= \lim_{b \rightarrow \infty} -x^{-1} \Big|_3^b \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + \frac{1}{3} \right) \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{b} \right) + \lim_{b \rightarrow \infty} \frac{1}{3}
 \end{aligned}$$

As $b \rightarrow \infty$, $-\frac{1}{b} \rightarrow 0$. The integral is convergent.

$$\int_3^\infty \frac{1}{x^2} dx = 0 + \frac{1}{3} = \frac{1}{3}$$

$$\begin{aligned}
 2. \quad \int_3^\infty \frac{1}{(x+1)^3} dx &= \lim_{b \rightarrow \infty} \int_3^b \frac{1}{(x+1)^3} dx \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{2} \frac{1}{(x+1)^2} \right) \Big|_3^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2(b+1)^2} - \left(-\frac{1}{2(3+1)^2} \right) \right]
 \end{aligned}$$

As $b \rightarrow \infty$, $\frac{1}{(b+1)^2} \rightarrow 0$. The integral is convergent and its value is $\frac{1}{2(3+1)^2} = \frac{1}{32}$.

$$\begin{aligned}
 3. \quad \int_4^\infty \frac{2}{\sqrt{x}} dx &= \lim_{b \rightarrow \infty} \int_4^b 2x^{-1/2} dx \\
 &= \lim_{b \rightarrow \infty} 4x^{1/2} \Big|_4^b \\
 &= \lim_{b \rightarrow \infty} (4\sqrt{b} - 4\sqrt{4}) \\
 &= \lim_{b \rightarrow \infty} 4\sqrt{b} - 8
 \end{aligned}$$

As $b \rightarrow \infty$, $4\sqrt{b} \rightarrow \infty$. The integral diverges.

$$\begin{aligned}
 4. \quad \int_{100}^\infty \frac{-3}{\sqrt{x}} dx &= \lim_{b \rightarrow \infty} \int_{100}^b -3x^{-1/2} dx \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{3x^{1/2}}{\frac{1}{2}} \right) \Big|_{100}^b \\
 &= \lim_{b \rightarrow \infty} (-6\sqrt{b} - 6\sqrt{100}) \\
 &= \lim_{b \rightarrow \infty} (-6\sqrt{b} - 60)
 \end{aligned}$$

As $b \rightarrow \infty$, $(-6\sqrt{b} - 60) \rightarrow -\infty$, so the integral is divergent.

$$\begin{aligned}
 5. \quad \int_{-\infty}^{-1} \frac{2}{x^3} dx &= \int_{-\infty}^{-1} 2x^{-3} dx \\
 &= \lim_{a \rightarrow -\infty} \int_a^{-1} 2x^{-3} dx \\
 &= \lim_{a \rightarrow -\infty} \left(\frac{2x^{-2}}{-2} \right) \Big|_a^{-1} \\
 &= \lim_{a \rightarrow -\infty} \left(-1 + \frac{1}{a^2} \right)
 \end{aligned}$$

As $a \rightarrow -\infty$, $\frac{1}{a^2} \rightarrow 0$. The integral is convergent.

$$\int_{-\infty}^{-1} \frac{2}{x^3} dx = -1 + 0 = -1$$

$$\begin{aligned} 6. \quad \int_{-\infty}^{-4} \frac{3}{x^4} dx &= \lim_{a \rightarrow -\infty} \int_a^{-4} 3x^{-4} dx \\ &= \lim_{a \rightarrow -\infty} \left(\frac{3x^{-3}}{-3} \right) \Big|_a^{-4} \\ &= \lim_{a \rightarrow -\infty} \left(-\frac{1}{x^3} \right) \Big|_a^{-4} \\ &= \lim_{a \rightarrow -\infty} \left(\frac{1}{64} + \frac{1}{a^3} \right) \end{aligned}$$

As $a \rightarrow \infty$, $\frac{1}{a^3} \rightarrow 0$. The integral is convergent.

$$\int_{-\infty}^{-4} \frac{3}{x^4} dx = \frac{1}{64} + 0 = \frac{1}{64}$$

$$\begin{aligned} 7. \quad \int_1^{\infty} \frac{1}{x^{1.0001}} dx &= \int_1^{\infty} x^{-1.0001} dx \\ &= \lim_{b \rightarrow \infty} \int_1^b x^{-1.0001} dx \\ &= \lim_{b \rightarrow \infty} \left(\frac{x^{-0.0001}}{-0.0001} \right) \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{(0.0001)b^{0.0001}} + \frac{1}{0.0001} \right) \end{aligned}$$

As $b \rightarrow \infty$, $-\frac{1}{0.0001b^{0.0001}} \rightarrow 0$.

The integral is convergent.

$$\int_1^{\infty} \frac{1}{x^{1.0001}} dx = 0 + \frac{1}{0.0001} = 10,000$$

$$\begin{aligned} 8. \quad \int_1^{\infty} \frac{1}{x^{0.999}} dx &= \lim_{b \rightarrow \infty} \int_1^b x^{-0.999} dx \\ &= \lim_{b \rightarrow \infty} \left(\frac{x^{0.001}}{0.001} \right) \Big|_1^b \\ &= \lim_{b \rightarrow \infty} (1000x^{0.001}) \Big|_1^b \\ &= \lim_{b \rightarrow \infty} (1000b^{0.001} - 1000) \end{aligned}$$

As $b \rightarrow \infty$, $(1000b^{0.001} - 1000) \rightarrow \infty$.

The integral is divergent.

$$\begin{aligned} 9. \quad \int_{-\infty}^{-10} x^{-2} dx &= \lim_{a \rightarrow -\infty} \int_a^{-10} x^{-2} dx \\ &= \lim_{a \rightarrow -\infty} (-x^{-1}) \Big|_a^{-10} \\ &= \lim_{a \rightarrow -\infty} \left(\frac{1}{10} + \frac{1}{a} \right) \\ &= \frac{1}{10} + 0 = \frac{1}{10} \end{aligned}$$

The integral is convergent and its value is $\frac{1}{10}$.

$$\begin{aligned} 10. \quad \int_{-\infty}^{-1} (x-2)^{-3} dx &= \lim_{a \rightarrow -\infty} \int_a^{-1} (x-2)^{-3} dx \\ &= \lim_{a \rightarrow -\infty} \left(-\frac{1}{2} \frac{1}{(x-2)^2} \right) \Big|_a^{-1} \\ &= \lim_{a \rightarrow -\infty} \left[-\frac{1}{18} - \left(-\frac{1}{2(a-2)^2} \right) \right] \end{aligned}$$

As $a \rightarrow -\infty$, $\frac{1}{2(a-2)^2} \rightarrow 0$. The integral is

convergent and its value is $-\frac{1}{18}$.

$$\begin{aligned} 11. \quad \int_{-\infty}^{-1} x^{-8/3} dx &= \lim_{a \rightarrow -\infty} \int_a^{-1} x^{-8/3} dx \\ &= \lim_{a \rightarrow -\infty} \left(-\frac{3}{5} x^{-5/3} \right) \Big|_a^{-1} \\ &= \lim_{a \rightarrow -\infty} \left(\frac{3}{5} + \frac{3}{5a^{5/3}} \right) \\ &= \frac{3}{5} + 0 = \frac{3}{5} \end{aligned}$$

The integral is convergent, and its value is $\frac{3}{5}$.

$$\begin{aligned}
 12. \quad \int_{-\infty}^{-27} x^{-5/3} dx &= \lim_{a \rightarrow -\infty} \int_a^{-27} x^{-5/3} dx \\
 &= \lim_{a \rightarrow -\infty} \left(\frac{x^{-2/3}}{-\frac{2}{3}} \right) \Big|_a^{-27} \\
 &= \lim_{a \rightarrow -\infty} \left(-\frac{3}{2} x^{-2/3} \right) \Big|_a^{-27} \\
 &= \lim_{a \rightarrow -\infty} \left[-\frac{3}{2} (-27)^{-2/3} + \frac{3}{2} (a)^{-2/3} \right] \\
 &= \lim_{a \rightarrow -\infty} \left(-\frac{1}{6} + \frac{3}{2a^{2/3}} \right)
 \end{aligned}$$

As $a \rightarrow -\infty$, $\frac{3}{2a^{2/3}} \rightarrow 0$. The integral is convergent.

$$\int_{-\infty}^{-27} x^{-5/3} dx = -\frac{1}{6} + 0 = -\frac{1}{6}$$

$$\begin{aligned}
 13. \quad \int_0^{\infty} 8e^{-8x} dx &= \lim_{b \rightarrow \infty} \int_0^b 8e^{-8x} dx \\
 &= \lim_{b \rightarrow \infty} \left(\frac{8e^{-8x}}{-8} \right) \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} (-e^{-8b} + 1) \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{e^{8b}} + 1 \right) \\
 &= 0 + 1 = 1
 \end{aligned}$$

The integral is convergent, and its value is 1.

$$\begin{aligned}
 14. \quad \int_0^{\infty} 50e^{-50x} dx &= \lim_{b \rightarrow \infty} \int_0^b 50e^{-50x} dx \\
 &= \lim_{b \rightarrow \infty} (-e^{-50x}) \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} (-e^{-50b} + e^0) \\
 &= \lim_{b \rightarrow \infty} (-e^{-50b} + 1)
 \end{aligned}$$

As $b \rightarrow \infty$, $\frac{-1}{e^{50b}} \rightarrow 0$. The integral is convergent.

$$\int_0^{\infty} 50e^{-50x} dx = 0 + 1 = 1$$

$$\begin{aligned}
 15. \quad \int_{-\infty}^0 1000e^x dx &= \lim_{a \rightarrow -\infty} \int_a^0 1000e^x dx \\
 &= \lim_{a \rightarrow -\infty} (1000e^x) \Big|_a^0 \\
 &= \lim_{a \rightarrow -\infty} (1000 - 1000e^a)
 \end{aligned}$$

As $a \rightarrow \infty$, $-1000e^a \rightarrow 0$. The integral is convergent.

$$\int_{-\infty}^0 1000e^x dx = 1000 - 0 = 1000$$

$$\begin{aligned}
 16. \quad \int_{-\infty}^0 5e^{60x} dx &= \lim_{a \rightarrow -\infty} \int_a^0 5e^{60x} dx \\
 &= \lim_{a \rightarrow -\infty} \left(\frac{5}{60} e^{60x} \right) \Big|_a^0 \\
 &= \lim_{a \rightarrow -\infty} \left(\frac{1}{12} - \frac{1}{12} e^{60a} \right)
 \end{aligned}$$

As $a \rightarrow -\infty$, $\frac{1}{12} e^{60a} \rightarrow 0$. The integral converges.

$$\int_{-\infty}^0 5e^{60x} dx = \frac{1}{12} - 0 = \frac{1}{12}$$

$$\begin{aligned}
 17. \quad \int_{-\infty}^{-1} \ln |x| dx &= \lim_{a \rightarrow -\infty} \int_a^{-1} \ln |x| dx \\
 \text{Let } u &= \ln |x| \text{ and } dv = dx. \\
 \text{Then } du &= \frac{1}{x} dx \text{ and } v = x.
 \end{aligned}$$

$$\begin{aligned}
 \int \ln |x| dx &= x \ln |x| - \int \frac{x}{x} dx \\
 &= x \ln |x| - x + C
 \end{aligned}$$

$$\begin{aligned}
 \int_{-\infty}^{-1} \ln |x| dx &= \lim_{a \rightarrow -\infty} (x \ln |x| - x) \Big|_a^{-1} \\
 &= \lim_{a \rightarrow -\infty} (-\ln 1 + 1 - a \ln |a| + a) \\
 &= \lim_{a \rightarrow -\infty} (1 + a - a \ln |a|)
 \end{aligned}$$

The integral is divergent, since as $a \rightarrow -\infty$,

$$(a - a \ln |a|) = -a(-1 + \ln |a|) \rightarrow \infty.$$

$$\begin{aligned}
 18. \quad \int_1^{\infty} \ln |x| dx &= \lim_{b \rightarrow \infty} \int_1^b \ln |x| dx \\
 \text{Let } u &= \ln |x| \text{ and } dv = dx. \\
 \text{Then } du &= \frac{1}{x} dx \text{ and } v = x.
 \end{aligned}$$

$$\int \ln |x| dx = x \ln |x| - \int \frac{x}{x} dx = x \ln |x| - x + C$$

$$\begin{aligned} \int_1^\infty \ln |x| dx &= \lim_{b \rightarrow \infty} (x \ln |x| - x) \Big|_1^b \\ &= \lim_{b \rightarrow \infty} [(b \ln b - b) - (-1)] \\ &= \lim_{b \rightarrow \infty} [b \ln b - b + 1] \end{aligned}$$

As $b \rightarrow \infty$, $(b \ln b - b + 1) \rightarrow \infty$.

The integral is divergent.

$$\begin{aligned} 19. \quad \int_0^\infty \frac{dx}{(x+1)^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(x+1)^2} \quad \text{Use substitution} \\ &= \lim_{b \rightarrow \infty} -(x+1)^{-1} \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left(\frac{-1}{b+1} + 1 \right) \end{aligned}$$

As $b \rightarrow \infty$, $-\frac{1}{b+1} \rightarrow 0$. The integral is convergent.

$$\int_0^\infty \frac{dx}{(x+1)^2} = 0 + 1 = 1$$

$$\begin{aligned} 20. \quad \int_0^\infty \frac{dx}{(4x+1)^3} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(4x+1)^3} \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{4} \int_0^b 4(4x+1)^{-3} dx \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{4} \cdot \frac{(4x+1)^{-2}}{-2} \right] \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{8} (4x+1)^{-2} \right] \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{8} (4b+1)^{-2} + \frac{1}{8} (1)^{-2} \right] \end{aligned}$$

As $b \rightarrow \infty$, $-\frac{1}{8} (4b+1)^{-2} \rightarrow 0$. The integral converges.

$$\int_0^\infty \frac{dx}{(4x+1)^3} = 0 + \frac{1}{8} = \frac{1}{8}$$

$$\begin{aligned} 21. \quad \int_{-\infty}^{-1} \frac{2x-1}{x^2-x} dx &= \lim_{a \rightarrow -\infty} \int_0^{-1} \frac{2x-1}{x^2-x} dx \quad \text{Use substitution} \\ &= \lim_{a \rightarrow -\infty} \ln |x^2-x| \Big|_a^{-1} \\ &= \lim_{a \rightarrow -\infty} (\ln 2 - \ln |a^2-a|) \end{aligned}$$

As $a \rightarrow -\infty$, $\ln |a^2-a| \rightarrow \infty$. The integral is divergent.

$$22. \quad \int_1^\infty \frac{4x+6}{x^2+3x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{4x+6}{x^2+3x} dx$$

Note that if $u = x^2 + 3x$ then

$du = (2x+3)dx$. Also, $4x+6 = 2(2x+3)$.

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_1^b \frac{4x+6}{x^2+3x} dx &= \lim_{b \rightarrow \infty} 2 \int_1^b \frac{2x+3}{x^2+3x} dx \\ &= \lim_{b \rightarrow \infty} (2 \ln |x^2+3x|) \Big|_1^b \\ &= \lim_{b \rightarrow \infty} (2 \ln |b^2+3b| - 2 \ln 4) \end{aligned}$$

As $b \rightarrow \infty$, $\ln |b^2+3b| \rightarrow \infty$. The integral diverges.

$$\begin{aligned} 23. \quad \int_2^\infty \frac{1}{x \ln x} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x \ln x} dx \quad \text{Use substitution} \\ &= \lim_{b \rightarrow \infty} \left[\ln(\ln x) \right]_2^b \\ &= \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln 2)] \end{aligned}$$

As $b \rightarrow \infty$, $\ln(\ln b) \rightarrow \infty$. The integral is divergent.

$$\begin{aligned}
 24. \quad & \int_2^\infty \frac{1}{x(\ln x)^2} dx \\
 &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx \quad \text{Use substitution} \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{\ln x} \Big|_2^b \right) \\
 &= \lim_{b \rightarrow \infty} \left(\frac{-1}{\ln b} + \frac{1}{\ln 2} \right)
 \end{aligned}$$

As $b \rightarrow \infty$, $-\frac{1}{\ln b} \rightarrow 0$. The integral is convergent.

$$\begin{aligned}
 \int_2^\infty \frac{1}{x(\ln x)^2} dx &= 0 + \frac{1}{\ln 2} \\
 &= \frac{1}{\ln 2}
 \end{aligned}$$

$$25. \quad \int_0^\infty xe^{4x} dx = \lim_{b \rightarrow \infty} \int_0^b xe^{4x} dx$$

Let $dv = e^{4x} dx$ and $u = x$.

Then $v = \frac{1}{4}e^{4x}$ and $du = dx$.

$$\begin{aligned}
 \int xe^{4x} dx &= \frac{x}{4}e^{4x} - \int \frac{1}{4}e^{4x} dx \\
 &= \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C \\
 &= \frac{1}{16}(4x - 1)e^{4x} + C
 \end{aligned}$$

$$\begin{aligned}
 \int_0^\infty xe^{4x} dx &= \lim_{b \rightarrow \infty} \left[\frac{1}{16}(4x - 1)e^{4x} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{16}(4b - 1)e^{4b} - \frac{1}{16}(-1)(1) \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{16}(4b - 1)e^{4b} + \frac{1}{16} \right]
 \end{aligned}$$

As $b \rightarrow \infty$, $\frac{1}{16}(4b - 1)e^{4b} \rightarrow \infty$. The integral is divergent.

$$26. \quad \int_{-\infty}^0 xe^{0.2x} dx = \lim_{a \rightarrow -\infty} \int_a^0 xe^{0.2x} dx$$

Let $u = x$ and $dv = e^{0.2x} dx$.

Then $du = dx$ and $v = \frac{1}{0.2}e^{0.2x} = 5e^{0.2x}$.

$$\begin{aligned}
 \int xe^{0.2x} dx &= 5xe^{0.2x} - 5 \int e^{0.2x} dx \\
 &= 5xe^{0.2x} - 25e^{0.2x} + C \\
 &= 5(x - 5)e^{0.2x} + C
 \end{aligned}$$

$$\begin{aligned}
 \lim_{a \rightarrow -\infty} \int_a^0 xe^{0.2x} dx &= \lim_{a \rightarrow -\infty} [5(x - 5)e^{0.2x}]_a^0 \\
 &= -25e^0 - \lim_{a \rightarrow -\infty} [5(a - 5)e^{0.2a}]
 \end{aligned}$$

As $a \rightarrow -\infty$, $e^{0.2a}$ is in the denominator of a fraction so that $5(a - 5)e^{0.2a} \rightarrow 0$. The integral converges.

$$\int_{-\infty}^0 xe^{0.2x} dx = -25 - 0 = -25.$$

$$\begin{aligned}
 27. \quad & \int_{-\infty}^\infty x^3 e^{-x^4} dx \\
 &= \int_{-\infty}^0 x^3 e^{-x^4} dx + \int_0^\infty x^3 e^{-x^4} dx
 \end{aligned}$$

We evaluate each of two improper integrals on the right.

$$\begin{aligned}
 \int_{-\infty}^0 x^3 e^{-x^4} dx &= \lim_{b \rightarrow -\infty} \int_b^0 x^3 e^{-x^4} dx \quad \text{Use substitution} \\
 &= \lim_{b \rightarrow -\infty} \left[-\frac{1}{4}e^{-x^4} \Big|_b^0 \right] \\
 &= \lim_{b \rightarrow -\infty} \left[-\frac{1}{4} + \frac{1}{4e^{b^4}} \right]
 \end{aligned}$$

As $b \rightarrow -\infty$, $\frac{1}{4e^{b^4}} \rightarrow 0$. The integral is convergent.

$$\int_{-\infty}^0 x^3 e^{-x^4} dx = -\frac{1}{4} + 0 = -\frac{1}{4}$$

$$\begin{aligned}\int_0^{\infty} x^3 e^{-x^4} dx &= \lim_{b \rightarrow \infty} \int_0^b x^3 e^{-x^4} dx \text{ Use substitution} \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{4} e^{-x^4} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{4e^{b^4}} + \frac{1}{4} \right]\end{aligned}$$

As $b \rightarrow -\infty$, $-\frac{1}{4e^{b^4}} \rightarrow 0$. The integral is convergent.

$$\int_0^{\infty} x^3 e^{-x^4} dx = 0 + \frac{1}{4} = \frac{1}{4}$$

Since each of the improper integrals converges, the original improper integral converges.

$$\int_{-\infty}^{\infty} x^3 e^{-x^4} dx = -\frac{1}{4} + \frac{1}{4} = 0$$

$$28. \int_{-\infty}^{\infty} e^{-|x|} dx = \int_{-\infty}^0 e^{-|x|} dx + \int_0^{\infty} e^{-|x|} dx$$

We evaluate each of the two improper integrals on the right.

$$\begin{aligned}\int_{-\infty}^0 e^{-|x|} dx &= \int_{-\infty}^0 e^x dx = \lim_{b \rightarrow -\infty} \int_b^0 e^x dx \\ &= \lim_{b \rightarrow -\infty} \left[e^x \right]_b^0 = \lim_{b \rightarrow -\infty} (1 - e^b)\end{aligned}$$

As $b \rightarrow -\infty$, $e^b \rightarrow 0$. The integral is convergent.

$$\int_{-\infty}^0 e^{-|x|} dx = 1 - 0 = 1$$

$$\begin{aligned}\int_0^{\infty} e^{-|x|} dx &= \int_0^{\infty} e^{-x} dx = \lim_{x \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = \lim_{b \rightarrow \infty} [-e^{-b} + 1]\end{aligned}$$

As $b \rightarrow \infty$, $e^{-b} \rightarrow 0$. The integral is convergent.

$$\int_0^{\infty} e^{-|x|} dx = -0 + 1 = 1$$

Since each of the improper integrals converges, the original improper integral converges.

$$\int_{-\infty}^{\infty} e^{-|x|} dx = 1 + 1 = 2$$

$$\begin{aligned}29. \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx \\ = \int_{-\infty}^0 \frac{x}{x^2 + 1} dx + \int_0^{\infty} \frac{x}{x^2 + 1} dx\end{aligned}$$

We evaluate the first improper integrals on the right.

$$\begin{aligned}\int_{-\infty}^0 \frac{x}{x^2 + 1} dx \\ = \lim_{b \rightarrow -\infty} \int_b^0 \frac{x}{x^2 + 1} dx \text{ Use substitution} \\ = \lim_{b \rightarrow -\infty} \left[\frac{1}{2} \ln(x^2 + 1) \right]_b^0 \\ = \lim_{b \rightarrow -\infty} \left[0 - \frac{1}{2} \ln(b^2 + 1) \right]\end{aligned}$$

As $b \rightarrow -\infty$, $\ln(b^2 + 1) \rightarrow \infty$. The integral is divergent. Since one of the two improper integrals on the right diverges, the original improper integral diverges.

$$30. \int_{-\infty}^{\infty} \frac{2x + 4}{x^2 + 4x + 5} dx$$

This is an improper integral at each limit of integration. Therefore, rewrite the doubly improper integral as two separate improper integrals. Then rewrite each of the improper integrals as limits of proper integrals.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{2x + 4}{x^2 + 4x + 5} dx \\ = \int_{-\infty}^0 \frac{2x + 4}{x^2 + 4x + 5} dx \\ + \int_0^{\infty} \frac{2x + 4}{x^2 + 4x + 5} dx \\ = \lim_{a \rightarrow -\infty} \int_a^0 \frac{2x + 4}{x^2 + 4x + 5} dx \\ + \lim_{b \rightarrow \infty} \int_0^b \frac{2x + 4}{x^2 + 4x + 5} dx\end{aligned}$$

Evaluate the first of the two proper integrals in the last equation first. If we let $u = x^2 + 4x + 5$, then $du = (2x + 4)dx$ so that the integral is of the form $\int \frac{du}{u} = \ln|u| + C$.

$$\begin{aligned}
& \lim_{x \rightarrow -\infty} \int_a^0 \frac{2x+4}{x^2+4x+5} dx \\
&= \lim_{a \rightarrow -\infty} \left(\ln |x^2+4x+5| \Big|_a^0 \right) \\
&= \ln 5 - \lim_{a \rightarrow -\infty} \ln |a^2+4a+5|
\end{aligned}$$

As $a \rightarrow -\infty$, $\ln(a^2+4a+5) \rightarrow \infty$. The integral diverges. Since one of the two integrals diverges, the original improper integral diverges.

31. $f(x) = \frac{1}{x-1}$ for $(-\infty, 0]$

$$\begin{aligned}
\int_{-\infty}^0 \frac{1}{x-1} dx &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{x-1} \ln |x-1| \\
&= \lim_{a \rightarrow -\infty} \left(\ln |x-1| \Big|_a^0 \right) \\
&= \lim_{a \rightarrow -\infty} (\ln |-1| - \ln |a-1|)
\end{aligned}$$

But $\lim_{a \rightarrow -\infty} (\ln |a-1|) = \infty$.

The integral is divergent, so the area cannot be found.

32. $f(x) = e^{-x}$ for $(-\infty, e]$

$$\begin{aligned}
\int_{-\infty}^e e^{-x} dx &= \lim_{a \rightarrow -\infty} \int_a^e e^{-x} dx \\
&= \lim_{a \rightarrow -\infty} (-e^{-x}) \Big|_a^e \\
&= \lim_{a \rightarrow -\infty} (-e^{-e} + e^{-a})
\end{aligned}$$

As $a \rightarrow -\infty$, $e^{-a} \rightarrow \infty$ and

$(e^{-e} + e^{-a}) \rightarrow \infty$. The integral is divergent, so the area cannot be found.

33. $f(x) = \frac{1}{(x-1)^2}$ for $(-\infty, 0]$

$$\begin{aligned}
& \int_{-\infty}^0 \frac{1}{(x-1)^2} dx \\
&= \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{(x-1)^2} dx \quad \text{Use substitution} \\
&= \lim_{a \rightarrow -\infty} -(x-1)^{-1} \Big|_a^0 \\
&= \lim_{a \rightarrow -\infty} \left(-\frac{1}{-1} + \frac{1}{a-1} \right)
\end{aligned}$$

As $a \rightarrow -\infty$, $\frac{1}{a-1} \rightarrow 0$. The integral is convergent.

$$= 1 + 0 = 1$$

Therefore, the area is 1.

34. $f(x) = \frac{3}{(x-1)^3}$ for $(-\infty, 0]$

$$\begin{aligned}
& \int_{-\infty}^0 \frac{3}{(x-1)^3} dx \\
&= \lim_{a \rightarrow -\infty} \int_a^0 \frac{3}{(x-1)^3} dx \\
&= \lim_{a \rightarrow -\infty} \left(\frac{3(x-1)^{-2}}{-2} \right) \Big|_a^0 \\
&= \lim_{a \rightarrow -\infty} \left(-\frac{3}{2(x-1)^2} \right) \Big|_a^0 \\
&= \lim_{a \rightarrow -\infty} \left(-\frac{3}{2} + \frac{3}{2(a-1)^2} \right) \\
&= -\frac{3}{2}
\end{aligned}$$

Since area is positive, the area is $\left| -\frac{3}{2} \right| = \frac{3}{2}$.

35. $\int_{-\infty}^{\infty} xe^{-x^2} dx$

Let $u = -x^2$, so that $du = -2x dx$.

$$\begin{aligned}
&= \lim_{a \rightarrow -\infty} \left(-\frac{1}{2} \int_a^0 -2xe^{-x^2} dx \right) \\
&+ \lim_{b \rightarrow \infty} \left(-\frac{1}{2} \int_0^b -2xe^{-x^2} dx \right) \\
&= \lim_{a \rightarrow -\infty} \left(-\frac{1}{2} e^{-x^2} \right) \Big|_a^0 \\
&+ \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-x^2} \right) \Big|_0^b \\
&= \lim_{a \rightarrow -\infty} \left(-\frac{1}{2} + \frac{1}{2e^{-a^2}} \right) \\
&+ \lim_{b \rightarrow \infty} \left(-\frac{1}{2e^{b^2}} + \frac{1}{2} \right) \\
&= -\frac{1}{2} + \frac{1}{2} = 0
\end{aligned}$$

$$\begin{aligned}
36. \quad & \int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2} dx \\
&= \int_{-\infty}^0 \frac{x}{(1+x^2)^2} dx + \int_0^{\infty} \frac{x}{(1+x^2)^2} dx \\
&= \lim_{a \rightarrow -\infty} \int_a^{\infty} \frac{xdx}{(1+x^2)^2} + \lim_{b \rightarrow \infty} \int_0^b \frac{xdx}{(1+x^2)^2} \\
&= \lim_{a \rightarrow -\infty} \left(\frac{-1}{2(1+x^2)} \right) \bigg|_a^0 + \lim_{b \rightarrow \infty} \left(\frac{-1}{2(1+x^2)} \right) \bigg|_0^b \\
&= \lim_{a \rightarrow -\infty} \left[\frac{-1}{2(1+0^2)} + \frac{1}{2(1+a^2)} \right] \\
&\quad + \lim_{b \rightarrow \infty} \left[\frac{-1}{2(1+b^2)} + \frac{1}{2(1+0^2)} \right] \\
&= -\frac{1}{2} + 0 + 0 + \frac{1}{2} = 0
\end{aligned}$$

$$37. \quad \int_1^{\infty} \frac{1}{x^p} dx$$

Case 1a $p < 1$:

$$\begin{aligned}
& \int_1^{\infty} \frac{1}{x^p} dx \\
&= \int_1^{\infty} x^{-p} dx \\
&= \lim_{a \rightarrow \infty} \int_1^a x^{-p} dx \\
&= \lim_{a \rightarrow \infty} \left[\frac{x^{-p+1}}{(-p+1)} \bigg|_1^a \right] \\
&= \lim_{a \rightarrow \infty} \left[\frac{1}{(-p+1)} (a^{-p+1} - 1) \right] \\
&= \lim_{a \rightarrow \infty} \left[\frac{1}{(-p+1)} a^{1-p} - \frac{1}{(-p+1)} \right]
\end{aligned}$$

Since $p < 1$, $1-p$ is positive and, as $a \rightarrow \infty$, $a^{1-p} \rightarrow \infty$. The integral diverges.

Case 1b $p = 1$:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} dx &= \int_1^{\infty} \frac{1}{x} dx \\
&= \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x} dx \\
&= \lim_{a \rightarrow \infty} \left(\ln |x| \bigg|_1^a \right) \\
&= \lim_{a \rightarrow \infty} (\ln |a| - \ln 1) \\
&= \lim_{a \rightarrow \infty} \ln |a|
\end{aligned}$$

As $a \rightarrow \infty$, $\ln |a| \rightarrow \infty$. The integral diverges.

Therefore, $\int_1^{\infty} \frac{1}{x^p}$ diverges when $p \leq 1$.

Case 2 $p > 1$:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} dx &= \lim_{x \rightarrow \infty} \int_1^a x^{-p} dx \\
&= \lim_{a \rightarrow \infty} \left(\frac{x^{-p+1}}{-p+1} \bigg|_1^a \right) \\
&= \lim_{a \rightarrow \infty} \left[\frac{a^{-p+1}}{(-p+1)} - \frac{1}{(-p+1)} \right]
\end{aligned}$$

Since $p > 1$, $-p+1 < 0$; thus as $a \rightarrow \infty$, $\frac{a^{-p+1}}{(-p+1)} \rightarrow 0$.

Hence,

$$\begin{aligned}
\lim_{a \rightarrow \infty} \left[\frac{a^{-p+1}}{(-p+1)} - \frac{1}{(-p+1)} \right] &= 0 - \frac{1}{(-p+1)} \\
&= \frac{-1}{-p+1} \\
&= \frac{1}{p-1}.
\end{aligned}$$

The integral converges.

39. (a) Use the *fnInt* feature on a graphing utility to obtain

$$\int_1^{20} \frac{1}{\sqrt{1+x^2}} dx \approx 2.808;$$

$$\int_1^{50} \frac{1}{\sqrt{1+x^2}} dx \approx 3.724;$$

$$\int_1^{100} \frac{1}{\sqrt{1+x^2}} dx \approx 4.417;$$

$$\int_1^{1000} \frac{1}{\sqrt{1+x^2}} dx \approx 6.720;$$

$$\int_1^{10,000} \frac{1}{\sqrt{1+x^2}} dx \approx 9.022.$$

- (b) Since the values of the integrals in part a do not appear to be approaching some fixed finite number but get bigger, the integral

$$\int_1^{\infty} \frac{1}{\sqrt{1+x^2}} dx \text{ appears to be divergent.}$$

- (c) Use the *fnInt* feature on a graphing utility to obtain

$$\int_1^{20} \frac{1}{\sqrt{1+x^4}} dx \approx 0.8770;$$

$$\int_1^{50} \frac{1}{\sqrt{1+x^4}} dx \approx 0.9070;$$

$$\int_1^{100} \frac{1}{\sqrt{1+x^4}} dx \approx 0.9170;$$

$$\int_1^{1000} \frac{1}{\sqrt{1+x^4}} dx \approx 0.9260;$$

$$\int_1^{10,000} \frac{1}{\sqrt{1+x^4}} dx \approx 0.9269.$$

- (d) Since the values of the integrals in part c appear to be approaching some fixed finite number, the integral

$$\int_1^{\infty} \frac{1}{\sqrt{1+x^4}} dx \text{ appears to be convergent.}$$

- (e) For large x , we may consider $1+x^2 \approx x^2$ and $1+x^4 \approx x^4$.

Thus,

$$\frac{1}{\sqrt{1+x^2}} \approx \frac{1}{\sqrt{x^2}} = \frac{1}{x} \text{ and}$$

$$\frac{1}{\sqrt{1+x^4}} \approx \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}.$$

In Example 1(a) we showed that $\int_1^{\infty} \frac{1}{x} dx$ diverges. Thus, we might guess that $\int_1^{\infty} \frac{1}{\sqrt{1+x^2}} dx$ diverges as well. In Exercise 1, we saw that $\int_2^{\infty} \frac{1}{x^2} dx$ converges. Thus, we might guess that $\int_1^{\infty} \frac{1}{\sqrt{1+x^4}} dx$ converges as well.

40. (a) Use the *fnInt* feature on a graphing utility to obtain

$$\int_0^1 e^{-x^2} dx \approx 0.7468; \int_0^5 e^{-x^2} dx \approx 0.8862;$$

$$\int_0^{10} e^{-x^2} dx \approx 0.8862; \int_0^{20} e^{-x^2} dx \approx 0.8862.$$

- (b) Since the approximate values of the last three integrals in part a are all 0.8862, it appears that the integral $\int_0^{\infty} e^{-x^2} dx$ converges with an approximate value of 0.8862.

- (c) For $x > 1$, $e^{-x^2} < e^{-x}$. Thus, the area between the graph of $y = e^{-x^2}$ and the x -axis on the interval $[1, \infty]$ is less than the area between $y = e^{-x}$ and the x -axis. Since

$$\begin{aligned} \int_1^{\infty} e^{-x} dx &= \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} [-e^{-x}]_1^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{e^b} + \frac{1}{e} \right] \\ &= 0 + \frac{1}{e} = \frac{1}{e}, \end{aligned}$$

the area between $y = e^{-x}$ and the x -axis on the interval $[1, \infty)$ is finite. It follows that the area between $y = e^{-x^2}$ and the x -axis, being smaller, must also be finite, so the integral $\int_1^{\infty} e^{-x^2} dx$ converges. Since the area between $y = e^{-x^2}$ and the x -axis on the interval $[0, 1]$ is finite as well, the integral $\int_0^{\infty} e^{-x^2} dx$ converges.

41. (a) Use the *fnInt* feature on a graphing utility to obtain

$$\int_0^{10} e^{-.00001x} dx \approx 9.9995;$$

$$\int_0^{50} e^{-.00001x} dx \approx 49.9875;$$

$$\int_0^{100} e^{-.00001x} dx \approx 99.9500;$$

$$\int_0^{1000} e^{-.00001x} dx \approx 995.0166.$$

- (b) Since the values of the integrals in part a do not appear to be approaching some fixed finite number, the integral $\int_0^{\infty} e^{-0.00001x} dx$ appears to be divergent.

$$\begin{aligned} \text{(c)} \quad & \int_0^{\infty} e^{-0.00001x} dx \\ &= \lim_{b \rightarrow \infty} \int_0^b e^{-0.00001x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{e^{-0.00001x}}{-0.00001} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{0.00001e^{0.00001b}} + \frac{1}{0.00001} \right] \\ &= 0 + 100,000 = 100,000 \end{aligned}$$

$$\begin{aligned} 42. \quad & \int_0^{\infty} 225,000e^{-0.06t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b 225,000e^{-0.06t} dt \\ &= \lim_{b \rightarrow \infty} \left(\frac{225,000}{-0.06} e^{-0.06t} \right) \Big|_0^b \\ &= -3,750,000 \left[\lim_{b \rightarrow \infty} (e^{-0.06b}) - e^0 \right] \end{aligned}$$

As $b \rightarrow \infty$, $\frac{1}{e^{0.06b}} \rightarrow 0$. The integral converges.

$$\begin{aligned} & \int_0^{\infty} 225,000e^{-0.06t} dt \\ &= -3,750,000(0 - 1) \\ &= 3,750,000 \end{aligned}$$

The capital value is \$3,750,000.

$$\begin{aligned} 43. \quad & \int_0^{\infty} 1,000,000e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b 1,000,000e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \left(\frac{1,000,000}{-0.05} e^{-0.05t} \right) \Big|_0^b \\ &= -20,000,000 \left[\lim_{b \rightarrow \infty} (e^{-0.05b}) - e^0 \right] \end{aligned}$$

As $b \rightarrow \infty$, $e^{-0.05b} = \frac{1}{e^{0.05b}} \rightarrow 0$. The integral converges.

$$\begin{aligned} & \int_0^{\infty} 1,000,000e^{-0.05t} dt \\ &= -20,000,000(0 - 1) \\ &= 20,000,000 \end{aligned}$$

The capital value is \$20,000,000.

$$\begin{aligned} 44. \quad \text{(a)} \quad & \int_0^{\infty} 7200e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b 7200e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \left(\frac{7200}{-0.05} e^{-0.05t} \right) \Big|_0^b \\ &= -144,000 \left[\lim_{b \rightarrow \infty} (e^{-0.05b}) - e^0 \right] \end{aligned}$$

As $b \rightarrow \infty$, $e^{-0.05b} = \frac{1}{e^{0.05b}} \rightarrow 0$.

The integral converges.

$$\begin{aligned} \int_0^{\infty} 7200e^{-0.05t} dt &= -144,000(0 - 1) \\ &= 144,000 \end{aligned}$$

The capital value is \$144,000.

$$\begin{aligned} \text{(b)} \quad & \int_0^{\infty} 7200e^{-0.10t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b 7200e^{-0.10t} dt \\ &= \lim_{b \rightarrow \infty} \left(\frac{7200}{-0.10} e^{-0.10t} \right) \Big|_0^b \\ &= -72,000 \left[\lim_{b \rightarrow \infty} (e^{-0.10b}) - e^0 \right] \end{aligned}$$

As $b \rightarrow \infty$, $e^{-0.10b} = \frac{1}{e^{0.10b}} \rightarrow 0$.

The integral converges.

$$\begin{aligned}\int_0^{\infty} 7200e^{-0.10t} dt &= -72,000(0 - 1) \\ &= 72,000\end{aligned}$$

The capital value is \$72,000.

$$\begin{aligned}45. \quad \int_0^{\infty} 1200e^{0.03t}e^{-0.07t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b 1200e^{-0.04t} dt \\ &= \lim_{b \rightarrow \infty} \left(\frac{1200}{-0.04} e^{-0.04t} \right) \Big|_0^b \\ &= -30,000 \left[\lim_{b \rightarrow \infty} (e^{-0.04b}) - e^0 \right]\end{aligned}$$

As $b \rightarrow \infty$, $e^{-0.04b} = \frac{1}{e^{0.04b}} \rightarrow 0$. The integral converges.

$$\begin{aligned}\int_0^{\infty} 1200e^{0.03t}e^{-0.07t} dt &= -30,000(0 - 1) \\ &= 30,000\end{aligned}$$

The capital value is \$30,000.

$$46. \quad R(t) = 6000 + 200t, r = 0.05$$

The capital value is given by

$$\begin{aligned}\int_0^{\infty} (6000 + 200t)e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b (6000 + 200t)e^{-0.05t} dt.\end{aligned}$$

Let $u = 6000 + 200t$ and $dv = e^{-0.05t} dt$.

Then $du = 200dt$ and

$$v = \frac{1}{-0.05} e^{-0.05t} = -20e^{-0.05t}.$$

$$\begin{aligned}\int (6000 + 200t)e^{-0.05t} dt \\ &= -20e^{-0.05t}(6000 + 200t) + \int 4000e^{-0.05t} dt \\ &= -20e^{-0.05t}(6000 + 200t) - 80,000e^{-0.05t} + C\end{aligned}$$

$$\begin{aligned}\int_0^{\infty} (6000 + 200t)e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b (6000 + 200t)e^{-0.05t} dt \\ &= \lim_{b \rightarrow \infty} [-20e^{-0.05t}(6000 + 200t) - 80,000e^{-0.05t}] \Big|_0^b \\ &= \lim_{b \rightarrow \infty} [e^{-0.05t}((-20)(6000 + 200t) - 80,000)] \Big|_0^b \\ &= \lim_{b \rightarrow \infty} [e^{-0.05b}(-120,000 - 4000b - 80,000) \\ &\quad + 200,000] \\ &= 0 + 200,000 = \$200,000\end{aligned}$$

$$\begin{aligned}47. \quad \int_0^{\infty} 3000e^{-0.1t} dt \\ &= \lim_{b \rightarrow \infty} \int_0^b 3000e^{-0.1t} dt \\ &= \lim_{b \rightarrow \infty} \frac{3000e^{-0.1b}}{-0.1} \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left(\frac{3000e^{-0.1b}}{-0.1} + \frac{3000}{0.1} \right) \\ &= 0 + 30,000 \\ &= \$30,000\end{aligned}$$

$$\begin{aligned}48. \quad r'(x) &= 2x^2e^{-x} \\ r(x) &= \int_0^{\infty} 2x^2e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b 2x^2e^{-x} dx\end{aligned}$$

Let $u = 2x^2$ and $dv = e^{-x} dx$.

Use column integration to obtain

$$\begin{aligned}\lim_{b \rightarrow \infty} \int_0^b 2x^2e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[(-2x^2e^{-x} - 4xe^{-x} - 4e^{-x}) \Big|_0^b \right] \\ &= \lim_{b \rightarrow \infty} [(-2b^2e^{-b} - 4be^{-b} - 4e^{-b}) - (0 - 4 \cdot 1)] \\ &= -(-4) \quad \text{Use hint to evaluate limits} \\ &= 4.\end{aligned}$$

$$\begin{aligned}
49. \quad S &= N \int_0^\infty \frac{a(1 - e^{-kt})}{k} e^{-bt} dt \\
&= \frac{Na}{k} \lim_{c \rightarrow \infty} \int_0^c (1 - e^{-kt})(e^{-bt}) dt \\
&= \frac{Na}{k} \lim_{c \rightarrow \infty} \int_0^c (e^{-bt} - e^{-(b+k)t}) dt \\
&= \frac{Na}{k} \lim_{c \rightarrow \infty} \left[-\frac{1}{b} e^{-bt} + \frac{1}{b+k} e^{-(b+k)t} \right] \Big|_0^c \\
&= \frac{Na}{k} \lim_{c \rightarrow \infty} \left[\left(-\frac{1}{b} e^{-bc} + \frac{1}{b+k} e^{-(b+k)c} \right) - \left(-\frac{1}{b} e^0 + \frac{1}{b+k} e^0 \right) \right] \\
&= \frac{Na}{k} \left(0 + 0 + \frac{1}{b} - \frac{1}{b+k} \right) \\
&= \frac{Na}{k} \cdot \frac{(b+k) - b}{b(b+k)} \\
&= \frac{Nak}{kb(b+k)} \\
&= \frac{Na}{b(b+k)}
\end{aligned}$$

$$\begin{aligned}
50. \quad P &= \int_0^\infty e^{-rt}(at + b)K dt \\
&= K \lim_{c \rightarrow \infty} \int_0^c (at + b)e^{-rt} dt
\end{aligned}$$

Evaluate $\int_0^c (at + b)e^{-rt} dt$ using integration by parts.

Let $u = at + b$ and $dv = e^{-rt} dt$.

Then $du = a dt$ and $v = -\frac{1}{r} e^{-rt}$.

$$\begin{aligned}
&\int_0^c (at + b)e^{-rt} dt \\
&= \left[(at + b) \left(-\frac{1}{r} e^{-rt} \right) - \int \left(-\frac{1}{r} e^{-rt} \right) a dt \right] \Big|_0^c \\
&= \left[-\frac{at + b}{r} e^{-rt} + \frac{a}{r} \int e^{-rt} dt \right] \Big|_0^c \\
&= \left[-\frac{at + b}{r} e^{-rt} - \frac{a}{r^2} e^{-rt} \right] \Big|_0^c \\
&= \left(-\frac{ac + b}{r} e^{-rc} - \frac{a}{r^2} e^{-rc} \right) - \left(-\frac{b}{r} e^0 - \frac{a}{r^2} e^0 \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&K \lim_{c \rightarrow \infty} \int_0^c (at + b)e^{-rt} dt \\
&= K \lim_{c \rightarrow \infty} \left(-\frac{ac + b}{r} e^{-rc} - \frac{a}{r^2} e^{-rc} + \frac{b}{r} + \frac{a}{r^2} \right) \\
&= K \left(0 - 0 + \frac{b}{r} + \frac{a}{r^2} \right) \\
&= \frac{K(a + br)}{r^2}
\end{aligned}$$

$$\begin{aligned}
51. \quad \int_0^\infty 50e^{-0.06t} dt &= 50 \lim_{b \rightarrow \infty} \int_0^b e^{-0.06t} dt \\
&= 50 \lim_{b \rightarrow \infty} \frac{e^{-0.06t}}{-0.06} \Big|_0^b \\
&= \frac{50}{-0.06} \lim_{b \rightarrow \infty} (e^{-0.06b} - e^0) \\
&= -\frac{50}{0.06} (0 - 1) = \frac{50}{0.06} \\
&\approx 833.3
\end{aligned}$$

$$\begin{aligned}
52. \quad \int_0^\infty 50e^{-0.04t} dt &= 50 \lim_{b \rightarrow \infty} \int_0^b e^{-0.04t} dt \\
&= 50 \lim_{b \rightarrow \infty} \frac{e^{-0.04t}}{-0.04} \Big|_0^b \\
&= \frac{50}{-0.04} \lim_{b \rightarrow \infty} \left(\frac{1}{e^{0.04b}} - 1 \right) \\
&\text{As } b \rightarrow \infty, \frac{1}{e^{0.04b}} \rightarrow 0. \\
\int_0^\infty 50e^{-0.04t} dt &= -\frac{50}{-0.04} \\
&= 1250
\end{aligned}$$

Chapter 8 Review Exercises

- False: This integral is best evaluated by substitution.
- True
- False: Using the substitution $u = x^2$, $dv = xe^{-x^2}$ this integral requires only one integration by parts.

4. True

5. False: The integrand should be just $2x^2 + 3$.6. False: The integrand should be $\pi(x^2 + 1)$.

7. True

8. True

9. True

10. False: We must write the integral as $\lim_{a \rightarrow -\infty} \int_a^c x e^{-2x} dx + \lim_{b \rightarrow \infty} \int_c^b x e^{-2x} dx$. The first of these integrals diverges so $\int_{-\infty}^{\infty} x e^{-2x} dx$ diverges.

$$14. \int x(8-x)^{3/2} dx$$

Let $u = x$ and $dv = (8-x)^{3/2}$.

Then $du = dx$ and $v = -\frac{2}{5}(8-x)^{5/2}$.

$$\begin{aligned} \int x(8-x)^{3/2} dx &= -\frac{2}{5}x(8-x)^{5/2} + \int \frac{2}{5}(8-x)^{5/2} dx \\ &= -\frac{2}{5}x(8-x)^{5/2} - \frac{2}{5} \left(\frac{2}{7} \right) (8-x)^{7/2} + C \\ &= -\frac{2x}{5}(8-x)^{5/2} - \frac{4}{35}(8-x)^{7/2} + C \end{aligned}$$

$$15. \int \frac{3x}{\sqrt{x-2}} dx = \int 3x(x-2)^{-1/2} dx$$

Let $u = 3x$ and $dv = (x-2)^{-1/2} dx$.

Then $du = 3 dx$ and $v = 2(x-2)^{1/2}$.

$$\begin{aligned} \int \frac{3x}{\sqrt{x-2}} dx &= 6x(x-2)^{1/2} - 6 \int (x-2)^{1/2} dx \\ &= 6x(x-2)^{1/2} - \frac{6(x-2)^{3/2}}{\frac{3}{2}} + C \\ &= 6x(x-2)^{1/2} - 4(x-2)^{3/2} + C \end{aligned}$$

$$16. \int x e^x dx$$

Let $u = x$ and $dv = e^x dx$.

Then $du = dx$ and $v = e^x$.

$$\begin{aligned} \int x e^x dx &= x e^x - \int e^x dx \\ &= x e^x - e^x + C \end{aligned}$$

$$17. \int (3x+6)e^{-3x} dx$$

Let $u = 3x+6$ and $dv = e^{-3x} dx$.

Then $du = 3 dx$ and $v = \frac{1}{-3} e^{-3x}$.

$$\begin{aligned} \int (3x+6)e^{-3x} dx &= (3x+6) \left(-\frac{1}{3} e^{-3x} \right) - \int \left(-\frac{1}{3} e^{-3x} \right) 3 dx \\ &= -(x+2)e^{-3x} + \int e^{-3x} dx \\ &= -(x+2)e^{-3x} - \frac{1}{3} e^{-3x} + C \end{aligned}$$

$$18. \int \ln |4x+5| dx$$

First, use substitution.

Let $w = 4x+5$. Then $dw = 4 dx$.

$$\int \ln |4x+5| dx = \frac{1}{4} \int \ln |w| dw$$

Now use integration parts.

Let $u = \ln |w|$ and $dv = dw$.

Then $du = \frac{1}{w} dw$ and $v = w$.

$$\begin{aligned} \int \ln |w| dw &= w \ln |w| - \int w \frac{1}{w} dw \\ &= w \ln |w| - \int dw \\ &= w \ln |w| - w + C \end{aligned}$$

Finally, resubstitute $4x+5$ for w .

$$\begin{aligned} \frac{1}{4} \int \ln |4x+5| dx &= \frac{1}{4} [(4x+5) \ln |4x+5| - (4x+5)] + C \\ &= \frac{1}{4} (4x+5)(\ln |4x+5| - 1) + C \end{aligned}$$

19. $\int (x-1) \ln |x| dx$

Let $u = \ln |x|$ and $dv = (x-1) dx$.

Then $du = \frac{1}{x} dx$ and $v = \frac{x^2}{2} - x$.

$$\begin{aligned} \int (x-1) \ln |x| dx &= \left(\frac{x^2}{2} - x \right) \ln |x| - \int \left(\frac{x}{2} - 1 \right) dx \\ &= \left(\frac{x^2}{2} - x \right) \ln |x| - \frac{x^2}{4} + x + C \end{aligned}$$

20. $\int \frac{x}{25-9x^2} dx$

Use substitution.

Let $u = 25 - 9x^2$. Then $du = -18x dx$.

$$\begin{aligned} \int \frac{x}{25-9x^2} dx &= -\frac{1}{18} \int \frac{-18x}{25-9x^2} dx \\ &= -\frac{1}{18} \int \frac{du}{u} \\ &= -\frac{1}{18} \ln |u| + C \\ &= -\frac{1}{18} \ln |25 - 9x^2| + C \end{aligned}$$

21. $\int \frac{x}{\sqrt{16+8x^2}} dx$

Use substitution.

Let $u = 16 + 8x^2$. Then $du = 16x dx$.

$$\begin{aligned} \int \frac{x}{\sqrt{16+8x^2}} dx &= \frac{1}{16} \int \frac{16x}{\sqrt{16+8x^2}} dx \\ &= \frac{1}{16} \int \frac{1}{\sqrt{u}} du \\ &= \frac{1}{16} \int u^{-1/2} du \\ &= \frac{1}{16} (2)u^{1/2} + C \\ &= \frac{1}{8} (16 + 8x^2)^{1/2} + C \\ &= \frac{1}{8} \sqrt{16 + 8x^2} + C \end{aligned}$$

22. $\int_1^e x^3 \ln x dx$

Let $u = \ln x$ and $dv = x^3 dx$.

Use column integration.

D	I
$\ln x$	x^3
$\frac{1}{x}$	$\frac{1}{4}x^4$

$$\begin{aligned} \int x^3 \ln x dx &= \frac{1}{4}x^4 \ln x - \int \left(\frac{1}{4}x^4 \cdot \frac{1}{x} \right) dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C \end{aligned}$$

$$\begin{aligned} \int_1^e x^3 \ln x dx &= \left(\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \right) \Big|_1^e \\ &= \left(\frac{e^4}{4} \right) (1) - \frac{e^4}{16} - 0 + \frac{1}{16} \\ &= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} \\ &= \frac{3e^4 + 1}{16} \\ &\approx 10.30 \end{aligned}$$

23. $\int_0^1 x^2 e^{x/2} dx$

Let $u = x^2$ and $dv = e^{x/2} dx$.

Use column integration.

D	I
x^2	$e^{x/2}$
$2x$	$2e^{x/2}$
2	$4e^{x/2}$
0	$8e^{x/2}$

$$\begin{aligned} \int_0^1 x^2 e^{x/2} dx &= (2x^2 e^{x/2} - 8x e^{x/2} + 16e^{x/2}) \Big|_0^1 \\ &= 2e^{1/2} - 8e^{1/2} + 16e^{1/2} - 16 \\ &= 10e^{1/2} - 16 \approx 0.4872 \end{aligned}$$

$$\begin{aligned}
 24. \quad A &= \int_0^1 (3 + x^2)e^{2x} dx \\
 &= \int_0^1 3e^{2x} dx + \int_0^1 x^2 e^{2x} dx \\
 \int 3e^{2x} dx &= \frac{3}{2}e^{2x} + C
 \end{aligned}$$

For the second integral, $\int x^2 e^{2x} dx$, let $u = x^2$

and $dv = e^{2x} dx$.

Use column integration.

D		I
x^2	+	e^{2x}
$2x$	-	$\frac{e^{2x}}{2}$
2	+	$\frac{e^{2x}}{4}$
0	-	$\frac{e^{2x}}{8}$

$$\begin{aligned}
 \int x^2 e^{2x} dx &= x^2 \frac{e^{2x}}{2} - 2x \frac{e^{2x}}{4} + 2 \frac{e^{2x}}{8} \\
 &= \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4}
 \end{aligned}$$

$$\begin{aligned}
 A &= \left(\frac{3}{2}e^{2x} + \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4} \right) \Big|_0^1 \\
 &= \frac{3}{2}e^2 + \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \left(\frac{3}{2} + \frac{1}{4} \right) \\
 &= \left(\frac{6 + 2 - 2 + 1}{4} \right) e^2 - \left(\frac{7}{4} \right) \\
 &= \frac{7}{4}(e^2 - 1) \approx 11.18
 \end{aligned}$$

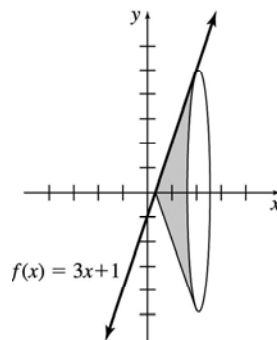
$$25. \quad A = \int_1^3 x^3 (x^2 - 1)^{1/3} dx$$

Let $u = x^2$ and $dv = x(x^2 - 1)^{1/3} dx$.

Then $du = 2x dx$ and $v = \frac{3}{8}(x^2 - 1)^{4/3}$.

$$\begin{aligned}
 &\int x^3 (x^2 - 1)^{1/3} dx \\
 &= \frac{3x^2}{8} (x^2 - 1)^{4/3} - \frac{3}{4} \int x (x^2 - 1)^{4/3} dx \\
 &= \frac{3x^2}{8} (x^2 - 1)^{4/3} - \frac{3}{4} \left[\frac{1}{2} \cdot \frac{3}{7} (x^2 - 1)^{7/3} \right] \\
 &= \frac{3x^2}{8} (x^2 - 1)^{4/3} - \frac{9}{56} (x^2 - 1)^{7/3} + C \\
 A &= \left[\frac{3x^2}{8} (x^2 - 1)^{4/3} - \frac{9}{56} (x^2 - 1)^{7/3} \right] \Big|_0^3 \\
 &= \frac{3}{8}(144) - \frac{9}{56}(128) \\
 &= 54 - \frac{144}{7} = \frac{234}{7} \approx 33.43
 \end{aligned}$$

$$26. \quad f(x) = 3x - 1, y = 0, x = 2$$

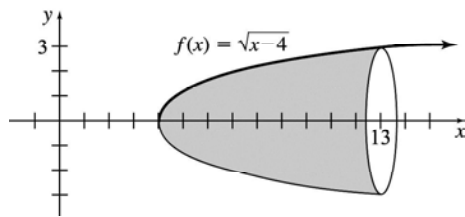


Since $f(x) = 3x - 1$ intersects $y = 0$

at $x = \frac{1}{3}$, the integral has a lower bound $a = \frac{1}{3}$.

$$\begin{aligned}
 V &= \pi \int_{1/3}^2 (3x - 1)^2 dx \\
 &= \pi \int_{1/3}^2 (9x^2 - 6x + 1) dx \\
 &= \pi (3x^3 - 3x^2 + x) \Big|_{1/3}^2 \\
 &= \pi \left[(24 - 12 + 2) - \left(\frac{1}{9} - \frac{1}{3} + \frac{1}{3} \right) \right] \\
 &= \pi \left(14 - \frac{1}{9} \right) = \frac{125\pi}{9} \approx 43.63
 \end{aligned}$$

27. $f(x) = \sqrt{x-4}$; $y = 0$; $x = 13$



Since $f(x) = \sqrt{x-4}$ intersects $y = 0$ at $x = 4$, the integral has lower bound $a = 4$.

$$\begin{aligned} V &= \pi \int_4^{13} (\sqrt{x-4})^2 dx \\ &= \pi \int_4^{13} (x-4) dx \\ &= \pi \left[\frac{x^2}{2} - 4x \right]_4^{13} \\ &= \pi \left[\left(\frac{169}{2} - 52 \right) - (8 - 16) \right] \\ &= \pi \left(\frac{65}{2} + 8 \right) = \frac{81}{2} \pi \approx 127.2 \end{aligned}$$

28. $f(x) = e^{-x}$, $y = 0$, $x = -2$, $x = 1$

$$\begin{aligned} V &= \pi \int_{-2}^1 e^{-2x} dx = \frac{\pi e^{-2x}}{-2} \Big|_{-2}^1 \\ &= \frac{\pi e^{-2}}{-2} + \frac{\pi e^4}{2} = \frac{\pi(e^4 - e^{-2})}{2} \\ &\approx 85.55 \end{aligned}$$

29. $f(x) = \frac{1}{\sqrt{x-1}}$, $y = 0$, $x = 2$, $x = 4$

$$\begin{aligned} V &= \pi \int_2^4 \left(\frac{1}{\sqrt{x-1}} \right)^2 dx \\ &= \pi \int_2^4 \frac{dx}{x-1} \\ &= \pi (\ln|x-1|) \Big|_2^4 \\ &= \pi \ln 3 \approx 3.451 \end{aligned}$$

30. $f(x) = 4 - x^2$, $y = 0$, $x = -1$, $x = 1$

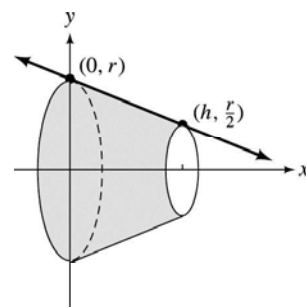
$$\begin{aligned} V &= \pi \int_{-1}^1 (4 - x^2)^2 dx \\ &= \pi \int_{-1}^1 (16 - 8x^2 + x^4) dx \\ &= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-1}^1 \\ &= \pi \left(16 - \frac{8}{3} + \frac{1}{5} + 16 - \frac{8}{3} + \frac{1}{5} \right) \\ &= \pi \left(32 - \frac{16}{3} + \frac{2}{5} \right) \\ &= \frac{406\pi}{15} \approx 85.03 \end{aligned}$$

31. $f(x) = \frac{x^2}{4}$, $y = 0$, $x = 4$

Since $f(x) = \frac{x^2}{4}$ intersects $y = 0$ at $x = 0$, the integral has a lower bound, $a = 0$.

$$\begin{aligned} V &= \pi \int_0^4 \left(\frac{x^2}{4} \right)^2 dx = \pi \int_0^4 \frac{x^4}{16} dx \\ &= \frac{\pi}{16} \left[\frac{x^5}{5} \right]_0^4 = \frac{\pi}{16} \left(\frac{1024}{5} \right) \\ &= \frac{64\pi}{5} \approx 40.21 \end{aligned}$$

32. The frustum may be shown as follows.



Use the two points given to find

$$f(x) = -\frac{r}{2h}x + r.$$

$$\begin{aligned}
 V &= \pi \int_0^h \left(-\frac{r}{2h}x + r \right)^2 dx \\
 &= -\frac{2\pi h}{3r} \left(-\frac{r}{2h}x + r \right)^3 \Big|_0^h \\
 &= -\frac{2\pi h}{3r} \left[\left(-\frac{r}{2} + r \right)^3 - (0 + r)^3 \right] \\
 &= -\frac{2\pi h}{3r} \left[\left(\frac{r}{2} \right)^3 - r^3 \right] \\
 &= -\frac{2\pi h}{3r} \left(\frac{r^3}{8} - r^3 \right) \\
 &= -\frac{2\pi h}{3r} \left(-\frac{7r^3}{8} \right) \\
 &= \frac{7\pi r^2 h}{12}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad f(x) &= \sqrt{x+1} \\
 \frac{1}{b-a} \int_a^b f(x) dx &= \frac{1}{8-0} \int_0^8 \sqrt{x+1} dx \\
 &= \frac{1}{8} \int_0^8 (x+1)^{1/2} dx \\
 &= \frac{1}{8} \left(\frac{2}{3} \right) (x+1)^{3/2} \Big|_0^8 \\
 &= \frac{1}{12} (9)^{3/2} - \frac{1}{12} (1) \\
 &= \frac{27}{12} - \frac{1}{12} = \frac{26}{12} \\
 &= \frac{13}{6}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \text{Average value} &= \frac{1}{2-0} \int_0^2 7x^2(x^3+1)^6 dx \\
 &= \frac{7}{2} \int_0^2 x^2(x^3+1)^6 dx
 \end{aligned}$$

Let $u = x^3 + 1$. Then $du = 3x^2 dx$.

$$\begin{aligned}
 \int x^2(x^3+1)^6 dx &= \frac{1}{3} \int 3x^2(x^3+1)^6 dx \\
 &= \frac{1}{3} \int u^6 du \\
 &= \frac{1}{3} \cdot \frac{1}{7} u^7 + C \\
 &= \frac{1}{21} (x^3+1)^7 + C
 \end{aligned}$$

$$\begin{aligned}
 \frac{7}{2} \int_0^2 x^2(x^3+1)^6 dx &= \frac{7}{2} \cdot \frac{1}{21} (x^3+1)^7 \Big|_0^2 \\
 &= \frac{1}{6} (9^7 - 1^7) = \frac{1}{6} (4,782,969 - 1) \\
 &= \frac{2,391,484}{3}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \int_{10}^{\infty} x^{-1} dx &= \lim_{b \rightarrow \infty} \int_{10}^b x^{-1} dx \\
 &= \lim_{b \rightarrow \infty} \ln |x| \Big|_{10}^b \\
 &= \lim_{b \rightarrow \infty} \ln b - \ln 10
 \end{aligned}$$

As $b \rightarrow \infty$, $\ln |b| \rightarrow \infty$. The integral diverges.

$$\begin{aligned}
 37. \quad \int_{-\infty}^{-5} x^{-2} dx &= \lim_{a \rightarrow -\infty} \int_a^{-5} x^{-2} dx \\
 &= \lim_{a \rightarrow -\infty} \left(\frac{x^{-1}}{-1} \right) \Big|_a^{-5} \\
 &= \lim_{a \rightarrow -\infty} \left(-\frac{1}{x} \right) \Big|_a^{-5} \\
 &= \frac{1}{5} + \lim_{a \rightarrow -\infty} \left(\frac{1}{a} \right)
 \end{aligned}$$

As $a \rightarrow -\infty$, $\frac{1}{a} \rightarrow 0$. The integral converges.

$$\int_{-\infty}^{-5} x^{-2} dx = \frac{1}{5} + 0 = \frac{1}{5}$$

$$38. \quad \int_0^{\infty} \frac{dx}{(3x+1)^2} = \lim_{b \rightarrow \infty} \int_0^b (3x+1)^{-2} dx$$

$$\begin{aligned}
 &= \lim_{b \rightarrow \infty} \left[\frac{1}{3} \cdot \frac{(3x+1)^{-1}}{-1} \right] \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} \left[\frac{-1}{3(3x+1)} \right] \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} \left(\frac{-1}{3(3b+1)} \right) + \frac{1}{3}
 \end{aligned}$$

As $b \rightarrow \infty$, $\frac{-1}{3(3b+1)} \rightarrow 0$. The integral converges.

$$\int_0^{\infty} \frac{dx}{(3x+1)^2} = 0 + \frac{1}{3} = \frac{1}{3}$$

$$\begin{aligned}
 39. \quad \int_1^{\infty} 6e^{-x} dx &= \lim_{b \rightarrow \infty} \int_1^b 6e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} -6e^{-x} \Big|_1^b \\
 &= \lim_{b \rightarrow \infty} (-6e^{-b} + 6e^{-1}) \\
 &= \lim_{b \rightarrow \infty} \left(\frac{-6}{e^b} + \frac{6}{e} \right)
 \end{aligned}$$

As $b \rightarrow \infty$, $e^b \rightarrow \infty$, so $\frac{-6}{e^b} \rightarrow 0$. The integral converges.

$$\int_1^{\infty} 6e^{-x} dx = 0 + \frac{6}{e} = \frac{6}{e} \approx 2.207$$

$$\begin{aligned}
 40. \quad \int_{-\infty}^0 \frac{x}{x^2 + 3} dx &= \lim_{a \rightarrow -\infty} \frac{1}{2} \int_a^0 \frac{2x dx}{x^2 + 3} \\
 &= \lim_{a \rightarrow -\infty} \frac{1}{2} (\ln |x^2 + 3|) \Big|_a^0 \\
 &= \lim_{a \rightarrow -\infty} \frac{1}{2} (\ln 3 - \ln |a^2 + 3|)
 \end{aligned}$$

As $a \rightarrow -\infty$, $\frac{1}{2}(\ln 3 - \ln |a^2 + 3|) \rightarrow \infty$.

The integral is divergent.

$$\begin{aligned}
 41. \quad \int_4^{\infty} \ln(5x) dx &= \lim_{b \rightarrow \infty} \int_4^b \ln(5x) dx \\
 \text{Let } u &= \ln(5x) \text{ and } dv = dx. \\
 \text{Then } du &= \frac{1}{x} dx \text{ and } v = x.
 \end{aligned}$$

$$\begin{aligned}
 \int \ln(5x) dx &= x \ln(5x) - \int x \cdot \frac{1}{x} dx \\
 &= x \ln(5x) - \int dx \\
 &= x \ln(5x) - x + C
 \end{aligned}$$

$$\begin{aligned}
 \lim_{b \rightarrow \infty} \int_4^b \ln(5x) dx &= \lim_{b \rightarrow \infty} [x \ln(5x) - x] \Big|_4^b \\
 &= \lim_{b \rightarrow \infty} [b \ln(5b) - b] - (4 \ln 20 - 4)
 \end{aligned}$$

As $b \rightarrow \infty$, $b \ln(5b) - b \rightarrow \infty$. The integral diverges.

$$\begin{aligned}
 42. \quad A &= \int_{-\infty}^1 \frac{5}{(x-2)^2} dx \\
 &= \lim_{a \rightarrow -\infty} \int_a^1 5(x-2)^{-2} dx \\
 &= \lim_{a \rightarrow -\infty} \left[\frac{5(x-2)^{-1}}{-1} \right] \Big|_a^1 = \lim_{a \rightarrow -\infty} \left[\frac{-5}{x-2} \right] \Big|_a^1 \\
 &= 5 + \lim_{a \rightarrow -\infty} \left(\frac{5}{a-2} \right)
 \end{aligned}$$

As $a \rightarrow -\infty$, $\frac{5}{a-2} \rightarrow 0$. The integral converges.

$$A = 5 + 0 = 5$$

$$43. \quad f(x) = 3e^{-x} \text{ for } [0, \infty)$$

$$\begin{aligned}
 A &= \int_0^{\infty} 3e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b 3e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} (-3e^{-x}) \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} \left(\frac{-3}{e^b} + 3 \right)
 \end{aligned}$$

As $b \rightarrow \infty$, $\frac{-3}{e^b} \rightarrow 0$.

$$A = 0 + 3 = 3$$

$$\begin{aligned}
 45. \quad R' &= x(x-50)^{1/2} \\
 R &= \int_{50}^{75} x(x-50)^{1/2} dx
 \end{aligned}$$

Let $u = x$ and $dv = (x-50)^{1/2}$.

Then $du = dx$ and $v = \frac{2}{3}(x-50)^{3/2}$.

$$\begin{aligned}
 \int x(x-50)^{1/2} dx &= \frac{2}{3}x(x-50)^{3/2} - \frac{2}{3} \int (x-50)^{3/2} dx \\
 &= \frac{2}{3}x(x-50)^{3/2} - \frac{2}{3} \cdot \frac{2}{5}(x-50)^{5/2} \\
 R &= \left[\frac{2}{3}x(x-50)^{3/2} - \frac{4}{15}(x-50)^{5/2} \right] \Big|_{50}^{75} \\
 &= \frac{2}{3}(75)(25^{3/2}) - \frac{4}{15}(25^{5/2}) \\
 &= 6250 - \frac{2500}{3} \\
 &= \frac{16,250}{3} \approx \$5416.67
 \end{aligned}$$

- 46.
- $f(x) = 5000; 8 \text{ yr}; 9\%$

$$P = \int_0^8 5000e^{-0.09x} dx = \frac{5000}{-0.09} e^{-0.09x} \Big|_0^8$$

$$= \frac{-5000}{0.09} e^{-0.72} + \frac{5000}{0.09} \approx \$28,513.76$$

- 47.
- $f(t) = 25,000; 12 \text{ yr}; 10\%$

$$P = \int_0^{12} 25,000e^{-0.10t} dt$$

$$= 25,000 \left(\frac{e^{-0.10t}}{-0.10} \right) \Big|_0^{12}$$

$$\approx 250,000(-0.3012 + 1)$$

$$\approx \$174,701.45$$

- 48.
- $f(t) = 150e^{0.04t}; 5 \text{ yr}; 6\%$

$$P = \int_0^5 150e^{0.04t} e^{-0.06t} dt$$

$$= \int_0^5 150e^{-0.02t} dt$$

$$= \frac{150}{-0.02} e^{-0.02t} \Big|_0^5$$

$$= -7500(e^{-0.1} - e^0)$$

$$= -7500(e^{-0.1} - 1)$$

$$\approx 713.7193647$$

The present value is \$713.72.

- 49.
- $f(t) = 15t; 18 \text{ mo}; 8\%$

$$P = \int_0^{1.5} 15te^{-0.08t} dt$$

$$= 15 \int_0^{1.5} te^{-0.08t} dt$$

Find the antiderivative using integration by parts.

Let $u = t$ and $dv = e^{-0.08t} dt$.

Then $du = dt$ and $v = \frac{1}{-0.08} e^{-0.08t}$

$= -12.5e^{-0.08t}$

$$\int te^{-0.08t} dt = -12.5te^{-0.08t} - \int (-12.5e^{-0.08t}) dt$$

$$= -12.5te^{-0.08t} - 156.25e^{-0.08t} + C$$

$$P = 15 \int_0^{1.5} te^{-0.08t} dt$$

$$= 15(-12.5te^{-0.08t} - 156.25e^{-0.08t}) \Big|_0^{1.5}$$

$$= 15[(-18.75e^{-0.12} - 156.25e^{-0.12}) - (0 - 156.25)]$$

$$= 15(-175e^{-0.12} + 156.25)$$

$$\approx 15.58385362$$

The present value is \$15.58.

- 50.
- $f(t) = 1000; 5 \text{ yr}; 6\% \text{ per yr}$

$$A = e^{0.06(5)} \int_0^5 1000e^{-0.06t} dt$$

$$= e^{0.3} \left(\frac{1000}{-0.06} e^{-0.06t} \right) \Big|_0^5$$

$$= e^{0.3} \left[\frac{1000}{-0.06} (e^{-0.3} - 1) \right]$$

$$\approx 5830.980126$$

The accumulated value is \$5830.98.

- 51.
- $f(t) = 500e^{-0.04t}; 8 \text{ yr}; 10\% \text{ per yr}$

$$A = e^{0.1(8)} \int_0^8 500e^{-0.04t} \cdot e^{-0.1t} dt$$

$$= e^{0.8} \int_0^8 500e^{-0.14t} dt$$

$$= e^{0.8} \left(\frac{500}{-0.14} e^{-0.14t} \right) \Big|_0^8$$

$$= e^{0.8} \left[\frac{500}{-0.14} (e^{-1.12} - 1) \right]$$

$$\approx 5354.971041$$

The accumulated value is \$5354.97.

- 52.
- $f(t) = 20t; 6 \text{ yr}; 4\% \text{ per yr}$

$$A = e^{0.04(6)} \int_0^6 20te^{-0.04t} dt$$

$$= 20e^{0.24} \int_0^6 te^{-0.04t} dt$$

Find the antiderivative using integration by parts.

Let $u = t$ and $dv = e^{-0.04t} dt$.

Then $du = dt$ and $v = \frac{1}{-0.04} e^{-0.04t}$

$= -25e^{-0.04t}$

$$\begin{aligned}\int te^{-0.04t} dt &= -25te^{-0.04t} - \int (-25e^{-0.04t}) dt \\ &= -25te^{-0.04t} - 625e^{-0.04t} + C\end{aligned}$$

$$\begin{aligned}20e^{0.24} \int_0^6 te^{-0.04t} dt &= 20e^{0.24} (-25te^{-0.04t} - 625e^{-0.04t}) \Big|_0^6 \\ &= 20e^{0.24} [(-150te^{-0.24} - 625e^{-0.24}) - (0 - 625)] \\ &= 20(625e^{0.24} - 775) \\ &\approx 390.614379\end{aligned}$$

The accumulated value is \$390.61.

53. $f(t) = 1000 + 200t$; 10 yr; 9% per yr

$$\begin{aligned}e^{(0.09)(10)} \int_0^{10} (1000 + 200t)e^{-0.09t} dt &= e^{0.9} \left[\frac{1000}{-0.09} e^{0.09t} + \frac{200}{(0.09)^2} (-0.09t - 1)e^{-0.09t} \right] \Big|_0^{10} \\ &= e^{0.9} \left[\frac{1000}{-0.09} (e^{-0.9} - 1) + \frac{200}{(0.09)^2} (-1.9e^{-0.9} + 1) \right] \\ &\approx \$30,035.17\end{aligned}$$

54. $f(t) = Ce^{kt}$ where $C = 1000$, $k = 0.05$

$$f(t) = 1000e^{0.05t}$$

Use $P = \int_0^T f(t)e^{-rt} dt$ with

$$r = 0.11 \quad T = 7.$$

$$\begin{aligned}P &= \int_0^7 1000e^{0.05t} \cdot e^{-0.11t} dt \\ &= 1000 \int_0^7 e^{-0.06t} dt \\ &= 1000 \left(\frac{1}{-0.06} e^{-0.06t} \right) \Big|_0^7 \\ &= \frac{1000}{-0.06} (e^{-0.42} - 1) \approx \$5715.89\end{aligned}$$

55. $e^{0.105(10)} \int_0^{10} 10,000e^{-0.105t} dt$

$$\begin{aligned}&= e^{1.05} \left(\frac{10,000e^{-0.105t}}{-0.105} \right) \Big|_0^{10} \\ &= \frac{10,000e^{1.05}}{-0.105} (e^{-1.05} - 1) \\ &\approx -272,157.25(-0.65006) \\ &\approx \$176,919.15\end{aligned}$$

56. $\int_0^b Re^{-kt} dt$

$$\begin{aligned}&= \int_0^\infty 50,000e^{-0.09t} dt \\ &= \lim_{b \rightarrow \infty} \left(\frac{50,000e^{-0.09t}}{-0.09} \right) \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{50,000}{-0.09} e^{-0.09(b)} + \frac{50,000}{0.09} (1) \right] \\ &= \lim_{b \rightarrow \infty} \left[\frac{-50,000}{0.09e^{0.09b}} + \frac{50,000}{0.09} \right] \\ &\approx \$555,555.56\end{aligned}$$

As $b \rightarrow \infty$, $e^{0.09b} \rightarrow \infty$, so $\frac{-50,000}{0.09e^{0.09b}} \rightarrow 0$.

57. $\int_0^5 0.5te^{-t} dt = 0.5 \int_0^5 te^{-t} dt$

Let $u = t$ and $dv = e^{-t} dt$.

Then $du = dt$ and $v = \frac{e^{-t}}{-1}$.

$$\begin{aligned}\int te^{-t} dt &= \frac{te^{-t}}{-1} + \int e^{-t} dt \\ &= -te^{-t} + \frac{e^{-t}}{-1}\end{aligned}$$

$$\begin{aligned}0.5 \int_0^5 te^{-t} dt &= 0.5(-te^{-t} - e^{-t}) \Big|_0^5 \\ &= 0.5(-5e^{-5} - e^{-5} + e^0) \\ &\approx 0.4798\end{aligned}$$

The total reaction over the first 5 hr is 0.4798.

58. $f(t) = 125e^{-0.025t}$

The total amount of oil is found by evaluating the improper integral.

$$\begin{aligned} \int_0^{\infty} 125e^{-0.025t} dt &= \lim_{b \rightarrow \infty} \int_0^b 125e^{-0.025t} dt \\ &= \lim_{b \rightarrow \infty} \left(\frac{125}{-0.025} e^{-0.025t} \right) \bigg|_0^b \\ &= \lim_{b \rightarrow \infty} (-5000)(e^{-0.025b} - e^0) \\ &= -5000 \left(\lim_{b \rightarrow \infty} \frac{1}{e^{0.025b}} - 1 \right) \end{aligned}$$

As $b \rightarrow \infty$, $\frac{1}{e^{0.025b}} \rightarrow 0$. The integral converges.

$$\int_0^{\infty} 125e^{-0.025t} dt = -5000(0 - 1) = 5000$$

The total amount of oil that will enter the bay is 5000 gallons.

59. (a) $\bar{T} = \frac{1}{10 - 0} \int_0^{10} (160 - 0.05x^2) dx$

$$\begin{aligned} &= \frac{1}{10} \left(160x - \frac{0.05x^3}{3} \right) \bigg|_0^{10} \\ &= \frac{1}{10} \left[160(10) - \frac{0.05}{3}(10)^3 \right] \\ &\approx \frac{1}{10}(1583.3) \approx 158.3^\circ \end{aligned}$$

(b) $\bar{T} = \frac{1}{40 - 10} \int_{10}^{40} (160 - 0.05x^2) dx$

$$\begin{aligned} &= \frac{1}{30} \left(160x - \frac{0.05x^3}{3} \right) \bigg|_{10}^{40} \\ &= \frac{1}{30} \left[\left(160(40) - \frac{0.05(40)^3}{3} \right) - \left(160(10) - \frac{0.05(10)^3}{3} \right) \right] \\ &\approx \frac{1}{30}(5333.33 - 1583.33) \\ &= 125^\circ \end{aligned}$$

$$\begin{aligned} \bar{T} &= \frac{1}{40 - 0} \int_0^{40} (160 - 0.05x^2) dx \\ &= \frac{1}{40} \left(160x - \frac{0.05x^3}{3} \right) \bigg|_0^{40} \\ &= \frac{1}{40} \left[(160)(40) - \frac{(0.05)(40)^3}{3} \right] \\ &\approx \frac{1}{40}(5,333.33) \\ \text{(c)} \quad &\approx 133.3^\circ \end{aligned}$$

Extended Application: Estimating Learning Curves in Manufacturing with Integrals

1. $C(280) \approx C(1) \cdot 280^{-0.152}$

$$\begin{aligned} &\approx 271,000 \cdot 280^{-0.152} \\ &\approx \$115,000 \end{aligned}$$

2. No; using the Change of Base formula for logarithms, we have

$$b = \frac{\ln r}{\ln 2} = \log_2 r = \frac{\log r}{\log 2}.$$

3. If $C(x) = ax^b$ is a solution to the function equation $C(2n) = r \cdot C(n)$, then

$$\begin{aligned} a(2n)^b &= r \cdot an^b \\ a \cdot 2^b \cdot n^b &= a \cdot r \cdot n^b \\ 2^b &= r, a \neq 0, n \neq 0 \\ b \ln 2 &= \ln r \\ b &= \frac{\ln r}{\ln 2} \end{aligned}$$

Thus, choose $b = \frac{\ln r}{\ln 2}$. The only condition on a is that it be nonzero. Thus, choose $a = 1$.

4. (a) The sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is the area of the region comprising the four rectangles in the picture. The integral $\int_1^5 \frac{1}{x} dx$ is the area of the part of the region lying below the graph of $y = \frac{1}{x}$. The graph at the right in Figure 16 shows the parts of the region above the graph of $y = \frac{1}{x}$ stacked vertically inside a rectangle of height 1. The lowest point in the shaded region is at a

height $\frac{1}{5}$; thus, the height of the shaded region is $1 - \frac{1}{5}$. Since the width of the shaded region is 1 at its widest points, the shaded region lies in a rectangle of area $1 - \frac{1}{5}$. By observation, we see that slightly more than half of that rectangle is shaded, so $\frac{1-\frac{1}{5}}{2}$ is a slight underestimate of the area of the shaded region.

Thus,

$$\int_1^5 \frac{1}{x} dx + \frac{1 - \frac{1}{5}}{2}$$

is a slight underestimate of the area of the rectangles, which is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.

$$\begin{aligned} \text{(b)} \quad \int_1^5 \frac{1}{x} dx + \frac{1 - \frac{1}{5}}{2} &= \ln x \Big|_1^5 + \frac{2}{5} \\ &= \ln 5 + \frac{2}{5} \approx 2.009 \end{aligned}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \approx 2.083$$

The percent error is approximately

$$\frac{2.009 - 2.083}{2.083} \approx -0.036 \approx -3.6\%.$$

MULTIVARIABLE CALCULUS

9.1 Functions of Several Variables

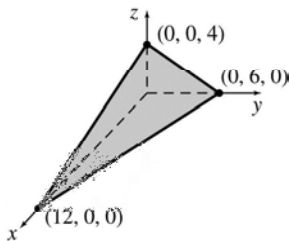
Your Turn 1

$$\begin{aligned}
 f(x, y) &= 4x^2 + 2xy + \frac{3}{y} \\
 f(2, 3) &= 4(2)^2 + 2(2)(3) + \frac{3}{3} \\
 &= 16 + 12 + 1 \\
 &= 29
 \end{aligned}$$

Your Turn 2

$$\begin{aligned}
 f(x, y, z) &= 4xz - 3x^2y + 2z^2 \\
 f(1, 2, 3) &= 4(1)(3) - 3(1)^2(2) + 2(3)^2 \\
 &= 12 - 6 + 18 \\
 &= 24
 \end{aligned}$$

Your Turn 3



Your Turn 4

Use the Cobb-Douglas production function

$$z = x^{1/4}y^{3/4}.$$

$$\begin{aligned}
 27 &= x^{1/4}y^{3/4} \\
 \frac{27}{x^{1/4}} &= y^{3/4} \\
 \left(\frac{27}{x^{1/4}}\right)^4 &= \left(y^{3/4}\right)^4 \\
 y^3 &= \frac{(27)^4}{x} \\
 (y^3)^{1/3} &= \left(\frac{(27)^4}{x}\right)^{1/3} \\
 y &= \frac{(27)^{4/3}}{x^{1/3}} = \frac{81}{x^{1/3}}
 \end{aligned}$$

9.1 Exercises

1. $f(x, y) = 2x - 3y + 5$
 - (a) $f(2, -1) = 2(2) - 3(-1) + 5 = 12$
 - (b) $f(-4, 1) = 2(-4) - 3(1) + 5 = -6$
 - (c) $f(-2, -3) = 2(-2) - 3(-3) + 5 = 10$
 - (d) $f(0, 8) = 2(0) - 3(8) + 5 = -19$

2. $g(x, y) = x^2 - 2xy + y^3$
 - (a) $g(-2, 4) = (-2)^2 - 2(-2)(4) + (4)^3 = 84$
 - (b) $g(-1, -2) = (-1)^2 - 2(-1)(-2) + (-2)^3 = -11$
 - (c) $g(-2, 3) = (-2)^2 - 2(-2)(3) + (3)^3 = 43$
 - (d) $g(5, 1) = (5)^2 - 2(5)(1) + (1)^3 = 16$

3. $h(x, y) = \sqrt{x^2 + 2y^2}$
 - (a) $h(5, 3) = \sqrt{25 + 2(9)} = \sqrt{43}$
 - (b) $h(2, 4) = \sqrt{4 + 32} = 6$
 - (c) $h(-1, -3) = \sqrt{1 + 18} = \sqrt{19}$
 - (d) $h(-3, -1) = \sqrt{9 + 2} = \sqrt{11}$

4. $f(x, y) = \frac{\sqrt{9x + 5y}}{\log x}$
 - (a) $f(10, 2) = \frac{\sqrt{9(10) + 5(2)}}{\log 10} = \frac{\sqrt{100}}{1} = 10$

$$\begin{aligned} \text{(b)} \quad f(100, 1) &= \frac{\sqrt{9(100) + 5(1)}}{\log 100} \\ &= \frac{\sqrt{905}}{2} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad f(1000, 0) &= \frac{\sqrt{9(1000) + 5(0)}}{\log 1000} \\ &= \frac{\sqrt{9000}}{3} \\ &= 10\sqrt{10} \end{aligned}$$

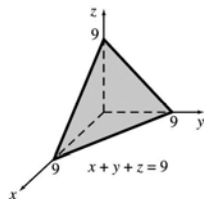
$$\begin{aligned} \text{(d)} \quad f\left(\frac{1}{10}, 5\right) &= \frac{\sqrt{9\left(\frac{1}{10}\right) + 5(5)}}{\log \frac{1}{10}} \\ &= \frac{\sqrt{25.9}}{-1} \\ &= -\sqrt{25.9} \end{aligned}$$

5. $x + y + z = 9$

If $x = 0$ and $y = 0$, $z = 9$.

If $x = 0$ and $z = 0$, $y = 9$.

If $y = 0$ and $z = 0$, $x = 9$.



6. $x + y + z = 15$

To find x -intercept, let $y = 0$, $z = 0$.

$$\begin{aligned} x + 0 + 0 &= 15 \\ x &= 15 \end{aligned}$$

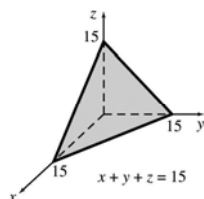
To find y -intercept, let $x = 0$, $z = 0$.

$$y = 15$$

To find z -intercept, let $x = 0$, $y = 0$.

$$z = 15$$

Sketch the portion of the plane in the first octant that contains these intercepts.

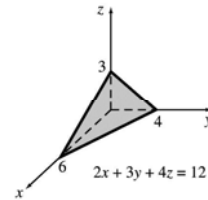


7. $2x + 3y + 4z = 12$

If $x = 0$ and $y = 0$, $z = 3$.

If $x = 0$ and $z = 0$, $y = 4$.

If $y = 0$ and $z = 0$, $x = 6$.



8. $4x + 2y + 3z = 24$

x -intercept: $y = 0$, $z = 0$

$$4x = 24$$

$$x = 6$$

y -intercept: $x = 0$, $z = 0$

$$2y = 24$$

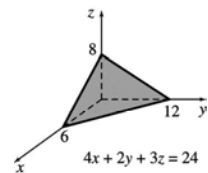
$$y = 12$$

z -intercept: $x = 0$, $y = 0$

$$3z = 24$$

$$z = 8$$

Sketch the portion of the plane in the first octant that contains these intercepts.

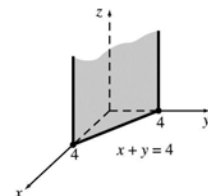


9. $x + y = 4$

If $x = 0$, $y = 4$.

If $y = 0$, $x = 4$.

There is no z -intercept.



10. $y + z = 5$

x -intercept: $y = 0$, $z = 0$

$$0 = 5$$

Impossible, so no x -intercept

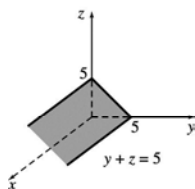
y-intercept: $x = 0, z = 0$

$$y = 5$$

z-intercept: $x = 0, y = 0$

$$z = 5$$

Sketch the portion of the plane in the first octant that contains these intercepts and is parallel to the x -axis.

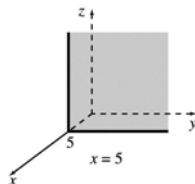


11. $x = 5$

The point $(5, 0, 0)$ is on the graph.

There are no y - or z -intercepts.

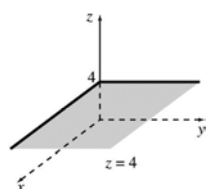
The plane is parallel to the yz -plane.



12. $z = 4$

No x -intercept, no y -intercept

Sketch the portion of the plane in the first octant that passes through $(0, 0, 4)$ parallel to the xy -plane.

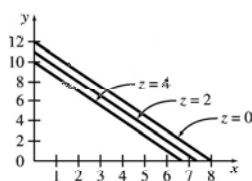


13. $3x + 2y + z = 24$

For $z = 0$, $3x + 2y = 24$. Graph the line $3x + 2y = 24$ in the xy -plane.

For $z = 2$, $3x + 2y = 22$. Graph the line $3x + 2y = 22$ in the plane $z = 2$.

For $z = 4$, $3x + 2y = 20$. Graph the line $3x + 2y = 20$ in the plane $z = 4$.

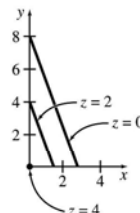


14. $3x + y + 2z = 8$

For $z = 0$, $3x + y = 8$. Graph the line $3x + y = 8$ in the xy -plane.

For $z = 2$, $3x + y = 4$. Graph the line $3x + y = 4$ in the plane $z = 2$.

For $z = 4$, $3x + y = 0$. Graph the line $3x + y = 0$ in the plane $z = 4$.

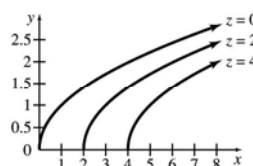


15. $y^2 - x = -z$

For $z = 0$, $x = y^2$. Graph $x = y^2$ in the xy -plane.

For $z = 2$, $x = y^2 + 2$. Graph $x = y^2 + 2$ in the plane $z = 2$.

For $z = 4$, $x = y^2 + 4$. Graph $x = y^2 + 4$ in the plane $z = 4$.



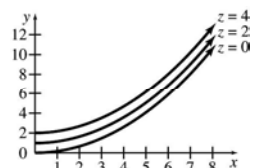
16. $2y - \frac{x^2}{3} = z$

If $z = 0$, we have $y = \frac{1}{6}x^2$. The level curve is a parabola in the xy -plane with vertex at the origin.

If $z = 2$, we have $y = \frac{1}{6}x^2 + 1$. The level curve is a parabola in the plane $z = 2$ with vertex at the point $(0, 1, 2)$.

If $z = 4$, we have $y = \frac{1}{6}x^2 + 2$. The level curve is a parabola in the plane $z = 4$ with vertex at the point $(0, 2, 4)$.

Sketch portions of these curves in the first octant.



18. x -intercept: $y = 0, z = 0$ ($a \neq 0$)

$$ax = d$$

$$x = \frac{d}{a}$$

- y -intercept: $x = 0, z = 0$ ($b \neq 0$)

$$by = d$$

$$y = \frac{d}{b}$$

- z -intercept: $x = 0, y = 0$ ($c \neq 0$)

$$cz = d$$

$$z = \frac{d}{c}$$

If $d \neq 0$, then for the plane to have a portion in the first octant, one of $\frac{d}{a}$, $\frac{d}{b}$, or $\frac{d}{c}$ must be positive, so at least one of a , b , or c must have the same sign as d .

If $d = 0$, then the trace in the xy -plane is the line $x = -\frac{b}{a}y$, the trace in the yz -plane is

$y = -\frac{c}{b}z$, and the trace in the xz -plane is

$x = -\frac{c}{a}z$. $-\frac{c}{a}$, $-\frac{c}{b}$, or $-\frac{b}{a}$ must be positive, so a , b , and c cannot all have the same sign.

21. $z = x^2 + y^2$

The xz -trace is

$$z = x^2 + 0 = x^2.$$

The yz -trace is

$$z = 0 + y^2 = y^2.$$

Both are parabolas with vertices at the origin that open upward.

The xy -trace is

$$0 = x^2 + y^2.$$

This is a point, the origin.

The equation is represented by a paraboloid, as shown in (c).

22. $z^2 - y^2 - x^2 = 1$

If $z = 0$,

$$-(y^2 + x^2) = 1.$$

This is impossible so there is no xy -trace.

If $x = 0$,

$$z^2 - y^2 = 1.$$

yz -trace: hyperbola

If $y = 0$,

$$z^2 - x^2 = 1.$$

xz -trace: hyperbola

The equation is represented by a hyperboloid of two sheets, as shown in (f).

23. $x^2 - y^2 = z$

The xz -trace is

$$x^2 = z,$$

which is a parabola with vertex at the origin that opens upward.

The yz -trace is

$$-y^2 = z,$$

which is a parabola with vertex at the origin that opens downward.

The xy -trace is

$$x^2 - y^2 = 0$$

$$x^2 = y^2$$

$$x = y \quad \text{or} \quad x = -y,$$

which are two lines that intersect at the origin.

The equation is represented by a hyperbolic paraboloid, as shown in (e).

24. $z = y^2 - x^2$

If $z = 0$,

$$x^2 = y^2$$

$$x = \pm y.$$

xy -trace: two intersecting lines.

If $x = 0$,

$$z = y^2.$$

yz -trace: parabola, opening upward

If $y = 0$,

$$z = -x^2.$$

xz -trace: parabola, opening downward

Hyperbolic paraboloid

Both (a) and (e) are hyperbolic paraboloids, but only (a) has traces described by this function.

$$25. \quad \frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{4} = 1$$

xz-trace:

$$\frac{x^2}{16} + \frac{z^2}{4} = 1, \text{ an ellipse}$$

yz-trace:

$$\frac{y^2}{25} + \frac{z^2}{4} = 1, \text{ an ellipse}$$

xy-trace:

$$\frac{x^2}{16} + \frac{y^2}{25} = 1, \text{ an ellipse}$$

The graph is an ellipsoid, as shown in (b).

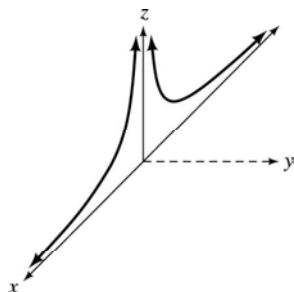
$$26. \quad z = 5(x^2 + y^2)^{-1/2} = \frac{5}{\sqrt{x^2 + y^2}}$$

Note that $z > 0$ for all values of x and y

xz-trace: $y = 0$

$$z = \frac{5}{\sqrt{x^2}} = \frac{5}{|x|}$$

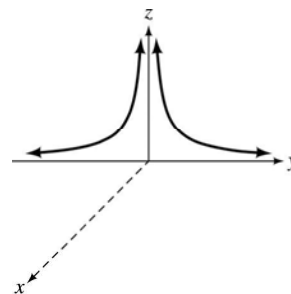
This gives one branch of the hyperbola $xz = 5$, where $z > 0$ and $x > 0$, and one branch of the hyperbola $xz = -5$, where $z > 0$ and $x < 0$.



yz-trace: $x = 0$

$$z = \frac{5}{\sqrt{y^2}} = \frac{5}{|y|}$$

This gives one branch of the hyperbola $yz = 5$, where $z > 0$ and $y > 0$, and one branch of the hyperbola $yz = -5$, where $z > 0$ and $y < 0$.



Level curves on planes $z = k$, where $k > 0$, are

$$k = \frac{5}{\sqrt{x^2 + y^2}}$$

$$x^2 + y^2 = \frac{25}{k^2}.$$

These are circles with centers $(0, 0, k)$ and radii $\frac{5}{k}$.

The graph of $z = 5(x^2 + y^2)^{-1/2}$ is (d).

$$27. \quad f(x, y) = 4x^2 - 2y^2$$

$$\begin{aligned} \text{(a)} \quad & \frac{f(x+h, y) - f(x, y)}{h} \\ &= \frac{[4(x+h)^2 - 2y^2] - [4x^2 - 2y^2]}{h} \\ &= \frac{4x^2 + 8xh + 4h^2 - 2y^2 - 4x^2 + 2y^2}{h} \\ &= \frac{h(8x + 4h)}{h} = 8x + 4h \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \frac{f(x, y+h) - f(x, y)}{h} \\ &= \frac{[4x^2 - 2(y+h)^2] - [4x^2 - 2y^2]}{h} \\ &= \frac{4x^2 - 2y^2 - 4yh - 2h^2 - 4x^2 + 2y^2}{h} \\ &= \frac{h(-4y - 2h)}{h} \\ &= -4y - 2h \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \\ &= \lim_{h \rightarrow 0} (8x + 4h) \\ &= 8x + 4(0) = 8x \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad & \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} \\ &= \lim_{h \rightarrow 0} (-4y - 2h) \\ &= -4y - 2(0) = -4y \end{aligned}$$

28. $f(x, y) = 5x^3 + 3y^2$

(a)

$$\begin{aligned} & \frac{f(x+h, y) - f(x, y)}{h} \\ &= \frac{[5(x+h)^3 + 3y^2] - (5x^3 + 3y^2)}{h} \\ &= \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 + 3y^2 - 5x^3 - 3y^2}{h} \\ &= \frac{h(15x^2 + 15xh + 5h^2)}{h} \\ &= 15x^2 + 15xh + 5h^2 \end{aligned}$$

(b) $\frac{f(x, y+h) - f(x, y)}{h}$

$$\begin{aligned} &= \frac{[5x^3 + 3(y+h)^2] - [5x^3 + 3y^2]}{h} \\ &= \frac{5x^3 + 3y^2 + 6yh + 3h^2 - 5x^3 - 3y^2}{h} \\ &= \frac{h(6y + 3h)}{h} \\ &= 6y + 3h \end{aligned}$$

(c) $\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} (15x^2 + 15xh + 5h^2) \\ &= 15x^2 + 15x(0) + 5(0)^2 \\ &= 15x^2 \end{aligned}$$

(d) $\lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} (6y + 3h) \\ &= 6y + 3(0) = 6y \end{aligned}$$

29. $f(x, y) = xye^{x^2+y^2}$

(a) $\lim_{h \rightarrow 0} \frac{f(1+h, 1) - f(1, 1)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{(1+h)(1)e^{1+2h+h^2+1} - (1)(1)e^{1+1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)e^{2+2h+h^2} - e^2}{h} \\ &= e^2 \lim_{h \rightarrow 0} \frac{(1+h)e^{2h+h^2} - 1}{h} \end{aligned}$$

h	$f(1+h, 1) - f(1, 1)$
0.001	3.005
1E-4	3.0005
1E-5	3.0001
1E-6	3.0000
-1E-5	2.9995
-1E-4	2.9995
-.001	2.995

$(1+h)e^{2h+h^2} - 1$

The graphing calculator indicates that

$$\lim_{h \rightarrow 0} \frac{(1+h)e^{2h+h^2} - 1}{h} = 3, \text{ thus}$$

$$\lim_{h \rightarrow 0} \frac{f(1+h, 1) - f(1, 1)}{h} = 3e^2.$$

The slope of the tangent line in the direction of x at $(1, 1)$ is $3e^2$.

(b) $\lim_{h \rightarrow 0} \frac{f(1, 1+h) - f(1, 1)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{(1)(1+h)e^{1+1+2h+h^2} - (1)(1)e^{1+1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)e^{2+2h+h^2} - e^2}{h} \\ &= e^2 \lim_{h \rightarrow 0} \frac{(1+h)e^{2h+h^2} - 1}{h} \end{aligned}$$

So, this limit reduces to the exact same limit as in part a. Therefore, since

$$\lim_{h \rightarrow 0} \frac{(1+h)e^{2h+h^2} - 1}{h} = 3,$$

then

$$\lim_{h \rightarrow 0} \frac{f(1, 1+h) - f(1, 1)}{h} = 3e^2.$$

The slope of the tangent line in the direction of y at $(1, 1)$ is $3e^2$.

30. A linear regression on the $y = 0$ column gives

$$f_0(x) = 4.019 + 1.989x.$$

The $y = 1$ column gives

$$f_1(x) = 7.045 + 1.98x.$$

The $y = 2$ column gives

$$f_2(x) = 9.978 + 2.013x.$$

The $y = 3$ column gives

$$f_3(x) = 12.989 + 2.019x.$$

Use the nearest integer values.

$$\begin{aligned} f_0(x) &= f(x, 0) = 4 + 2x \\ &= a + bx + c(0) \\ &= a + bx \end{aligned}$$

$$\begin{aligned}f_1(x) &= f(x, 1) = 7 + 2x \\&= a + bx + c(1) \\&= a + bx + c\end{aligned}$$

$$\begin{aligned}f_2(x) &= f(x, 2) = 10 + 2x \\&= a + bx + c(2) \\&= a + bx + 2c\end{aligned}$$

$$\begin{aligned}f_3(x) &= f(x, 3) = 13 + 2x \\&= a + bx + c(3) \\&= a + bx + 3c\end{aligned}$$

Thus, $b = 2$ and we have

$$\begin{aligned}4 &= a \\7 &= a + c \\10 &= a + 2c \\13 &= a + 3c\end{aligned}$$

so $a = 4$ and $c = 3$.

$$f(x, y) = 4 + 2x + 3y$$

$$31. \quad P(x, y) = 100 \left[\frac{3}{5}x^{-2/5} + \frac{2}{5}y^{-2/5} \right]^{-5}$$

(a) $P(32, 1)$

$$\begin{aligned}&= 100 \left[\frac{3}{5}(32)^{-2/5} + \frac{2}{5}(1)^{-2/5} \right]^{-5} \\&= 100 \left[\frac{3}{5} \left(\frac{1}{4} \right) + \frac{2}{5}(1) \right]^{-5} \\&= 100 \left(\frac{11}{20} \right)^{-5} \\&= 100 \left(\frac{20}{11} \right)^5 \\&\approx 1986.95\end{aligned}$$

The production is approximately 1987 cameras.

(b) $P(1, 32)$

$$\begin{aligned}&= 100 \left[\frac{3}{5}(1)^{-2/5} + \frac{2}{5}(32)^{-2/5} \right]^{-5} \\&= 100 \left[\frac{3}{5}(1) + \frac{2}{5} \left(\frac{1}{4} \right) \right]^{-5} \\&= 100 \left(\frac{7}{10} \right)^{-5} \\&= 100 \left(\frac{10}{7} \right)^5 \approx 595\end{aligned}$$

The production is approximately 595 cameras.

(c) 32 work hours means that $x = 32$. 243 units of capital means that $y = 243$.

$P(32, 243)$

$$\begin{aligned}&= 100 \left[\frac{3}{5}(32)^{-2/5} + \frac{2}{5}(243)^{-2/5} \right]^{-5} \\&= 100 \left[\frac{3}{5} \left(\frac{1}{4} \right) + \frac{2}{5} \left(\frac{1}{9} \right) \right]^{-5} \\&= 100 \left(\frac{7}{36} \right)^{-5} \\&= 100 \left(\frac{36}{7} \right)^5 \\&\approx 359,767.81\end{aligned}$$

The production is approximately 359,768 cameras.

$$32. \quad M = f(25, 0.05, 0.33)$$

$$\begin{aligned}&= \frac{(1 + 0.05)^{25}(1 - 0.33) + 0.33}{[1 + (1 - 0.33)(0.05)]^{25}} \\&= \frac{(1.05)^{25}(0.67) + 0.33}{[1 + (0.67)(0.05)]^{25}} \\&\approx 1.1403\end{aligned}$$

The multiplier is 1.14. Since $M > 1$, the IRA account grows faster.

$$33. \quad M = f(40, 0.06, 0.28)$$

$$\begin{aligned}&= \frac{(1 + 0.06)^{40}(1 - 0.28) + 0.28}{[1 + (1 - 0.28)(0.06)]^{40}} \\&= \frac{(1.06)^{40}(0.72) + 0.28}{[1 + (0.72)(0.06)]^{40}} \\&\approx 1.416\end{aligned}$$

The multiplier is 1.416. Since $M > 1$, the IRA account grows faster.

$$34. \quad z = 1.01x^{3/4}y^{1/4}$$

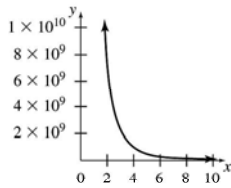
$$500 = 1.01x^{3/4}y^{1/4}$$

$$y^{1/4} = \frac{500}{1.01x^{3/4}}$$

$$y = \frac{(500)^4}{(1.01x^{3/4})^4}$$

$$= \left(\frac{500}{1.01} \right)^4 \frac{1}{x^3}$$

$$\approx \frac{6 \cdot 10^{10}}{x^3}$$



35. $z = x^{0.6}y^{0.4}$ where $z = 500$

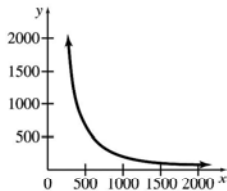
$$500 = x^{3/5}y^{2/5}$$

$$\frac{500}{x^{3/5}} = y^{2/5}$$

$$\left(\frac{500}{x^{3/5}}\right)^{5/2} = (y^{2/5})^{5/2}$$

$$y = \frac{(500)^{5/2}}{x^{3/2}}$$

$$y \approx \frac{5,590,170}{x^{3/2}}$$



36. $z = x^{0.7}y^{0.3}$

If x is doubled, z is multiplied by $2^{0.75}$.

If y is doubled, z is multiplied by $2^{0.25}$.

If both are doubled, z is multiplied by $2^{0.75} \cdot 2^{0.25} = 2^{1.0} = 2$. Thus, z is doubled.

37. The cost function, C , is the sum of the products of the unit costs times the quantities x , y , and z .
Therefore,

$$C(x, y, z) = 250x + 150y + 75z.$$

38. $H(m, t, A) = \frac{15.2m^{0.67}(T - A)}{10.23 \ln m - 10.74}$

(a) $H = \frac{15.2(21)^{0.67}(36 - 4)}{10.23 \ln 21 - 10.74}$
 $\approx 183 \text{ W}$

(b) $H = \frac{15.2(29)^{0.67}(38 - 16)}{10.23 \ln 29 - 10.74}$
 $\approx 135 \text{ W}$

39. $A = 0.024265h^{0.3964}m^{0.5378}$

Chapter 9 MULTIVARIABLE CALCULUS

(a) $A = 0.024265(178)^{0.3964}(72)^{0.5378}$
 $\approx 1.89 \text{ m}^2$

(b) $A = 0.024265(140)^{0.3964}(65)^{0.5378}$
 $\approx 1.62 \text{ m}^2$

(c) $A = 0.024265(160)^{0.3964}(70)^{0.5378}$
 $\approx 1.78 \text{ m}^2$

(d) Answers will vary.

40. $F = \frac{v^2}{gl}$

(a) $2.56 = \frac{v^2}{9.81 \cdot 0.09}$
 $v = 1.5 \text{ m/sec}$

$$2.56 = \frac{v^2}{9.81 \cdot 1.2}$$

$$v = 5.5 \text{ m/sec}$$

(b) $2.56 = \frac{v^2}{9.81 \cdot 4}$
 $v = 1 \text{ m/sec}$

41.

$$P(W, R, A) = 48 - 2.43W - 1.81R - 1.22A$$

(a) $P(5, 15, 0)$
 $= 48 - 2.43(5) - 1.81(15) - 1.22(0)$
 $= 8.7$

8.7% of fish will be intolerant to pollution.

- (b) The maximum percentage will occur when the variable factors are a minimum, or when $W = 0$, $R = 0$, and $A = 0$.

$$P(0, 0, 0) = 48 - 2.43(0) - 1.81(0) - 1.22(0)$$

$$= 48$$

48% of fish will be intolerant to pollution.

- (c) Any combination of values of W , R , and A that result in $P = 0$ is a scenario that will drive the percentage of fish intolerant to pollution to zero.

If $R = 0$ and $A = 0$:

$$P(W, 0, 0) = 48 - 2.43W - 1.81(0) - 1.22(0)$$

$$= 48 - 2.43W.$$

$$48 - 2.43W = 0$$

$$W = \frac{48}{2.43} \\ \approx 19.75$$

So $W = 19.75$, $R = 0$, $A = 0$ is one scenario.

If $W = 10$ and $R = 10$:

$$P(10, 10, A) = 48 - 2.43(10) - 1.81(10) - 1.22A \\ = 5.6 - 1.22A$$

$$5.6 - 1.22A = 0$$

$$A = \frac{5.6}{1.22} \\ \approx 4.59$$

So $W = 10$, $R = 10$, $A = 4.59$ is another scenario.

- (d) Since the coefficient of W is greater than the coefficients of R and A , a change in W will affect the value of P more than an equal change in R or A . Thus, the percentage of wetland (W) has the greatest influence on P .

$$42. \quad I(p, a, m, n, e) = (25.54 + 0.04p - 7.92a \\ + 2.62m + 4.46n + 0.15e)^2$$

- (a) Replace the variables with the given values.

$$I(80, 23, 34, 16, 50) \\ = [25.54 + 0.04(80) - 7.92(23) \\ + 2.62(34) + 4.46(16) + 0.15(50)]^2 \\ = (14.52)^2 \\ = 210.8304$$

The incidence of a dengue fever outbreak is about 211.

- (b) Since the coefficient of a , the average temperature, is negative, it has a negative influence on the incidence of dengue fever. If all other variables are held constant, an increase in a results in a decrease in I , and vice-versa.

$$43. \quad A(L, T, U, C) = 53.02 + 0.383L + 0.0015T \\ + 0.0028U - 0.0003C$$

$$(a) \quad A(266, 107, 484, 31, 697, 24, 870) \\ = 53.02 + 0.383(266) + 0.0015(107, 484) \\ + 0.0028(31, 697) - 0.0003(24, 870) \\ \approx 397$$

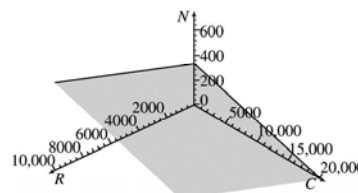
The estimated number of accidents is 397.

$$44. \quad N(R, C) = 329.32 + 0.0377R - 0.0171C$$

$$(a) \quad N(141, 319, 37, 960) \\ = 329.32 + 0.0377(141, 319) \\ - 0.0171(37, 960) \\ \approx 5008$$

Approximately 5008 deer were harvested in Tuscarawas County.

(b)



$$45. \quad \ln(T) = 5.49 - 3.00 \ln(F) + 0.18 \ln(C)$$

$$(a) \quad e^{\ln(T)} = e^{5.49 - 3.00 \ln(F) + 0.18 \ln(C)} \\ T = e^{5.49} e^{-3.00 \ln(F)} e^{0.18 \ln(C)} \\ = \frac{e^{5.49} e^{\ln(C^{0.18})}}{e^{\ln(F^3)}} \\ T \approx \frac{242.257 C^{0.18}}{F^3}$$

- (b) Replace F with 2 and C with 40 in the preceding formula.

$$T \approx \frac{242.257(40)^{0.18}}{(2)^3} \approx 58.82$$

T is about 58.8%. In other words, a tethered sow spends nearly 59% of the time doing repetitive behavior when she is fed 2 kg of food per day and neighboring sows spend 40% of the time doing repetitive behavior.

46. The girth is $2H + 2W$. Thus,

$$f(L, W, H) = L + 2H + 2W.$$

47. Let the area be given by $g(L, W, H)$.

Then,

$$g(L, W, H) = 2LW + 2WH + 2LH \text{ ft}^2.$$

48. $f(H, D) = \sqrt{H^2 + D^2}$ with $D = 3.75$ in

(a) $\frac{H}{D} = \frac{3}{4}$

$$H = \frac{3}{4}D$$

$$H = \frac{3}{4}(3.75)$$

$$H = 2.8125$$

$$f(2.8125, 3.75)$$

$$= \sqrt{(2.8125)^2 + (3.75)^2}$$

$$\approx 4.69$$

The length of the ellipse is approximately 4.69 inches, and its width is 3.75 inches.

(b) $\frac{H}{D} = \frac{2}{5}$

$$H = \frac{2}{5}D$$

$$H = \frac{2}{5}(3.75)$$

$$H = 1.5$$

$$f(1.5, 3.75)$$

$$= \sqrt{(1.5)^2 + (3.75)^2}$$

$$\approx 4.04$$

The length of the ellipse is approximately 4.04 inches, and its width is 3.75 inches.

9.2 Partial Derivatives

Your Turn 1

$$f(x, y) = 2x^2y^3 + 6x^5y^4$$

$$f_x(x, y) = 4xy^3 + 30x^4y^4$$

$$f_y(x, y) = 6x^2y^2 + 24x^5y^3$$

Your Turn 2

$$f(x, y) = e^{3x^2y}$$

$$f_x(x, y) = e^{3x^2y} \cdot \frac{\partial}{\partial x}(3x^2y)$$

$$= 6xye^{3x^2y}$$

$$f_y(x, y) = e^{3x^2y} \cdot \frac{\partial}{\partial y}(3x^2y)$$

$$= 3x^2e^{3x^2y}$$

Your Turn 3

$$f(x, y) = xy e^{x^2+y^3}$$

Use the product rule.

$$f_x(x, y) = xy \cdot \frac{\partial}{\partial x}(e^{x^2+y^3}) + (e^{x^2+y^3}) \frac{\partial}{\partial x}(xy)$$

$$= xy(2xe^{x^2+y^3}) + (e^{x^2+y^3})y$$

$$= (2x^2y + y)e^{x^2+y^3}$$

$$f_x(2, 1) = [2(2)^2(1) + 1]e^{2^2+1^3}$$

$$= 9e^5$$

$$f_y(x, y) = xy \cdot \frac{\partial}{\partial y}(e^{x^2+y^3}) + (e^{x^2+y^3}) \frac{\partial}{\partial y}(xy)$$

$$= xy(3y^2e^{x^2+y^3}) + (e^{x^2+y^3})x$$

$$= (3xy^3 + x)e^{x^2+y^3}$$

$$f_y(2, 1) = [3(2)(1)^3 + 2]e^{2^2+1^3}$$

$$= 8e^5$$

Your Turn 4

$$f(x, y) = x^2e^{7y} + x^4y^5$$

$$f_x(x, y) = 2xe^{7y} + 4x^3y^5$$

$$f_y(x, y) = 7x^2e^{7y} + 5x^4y^4$$

$$\begin{aligned}
 f_{xx}(x, y) &= \frac{\partial}{\partial x} f_x(x, y) \\
 &= \frac{\partial}{\partial x} (2xe^{7y} + 4x^3y^5) \\
 &= 2e^{7y} + 12x^2y^5 \\
 f_{yy}(x, y) &= \frac{\partial}{\partial y} f_y(x, y) \\
 &= \frac{\partial}{\partial y} (7x^2e^{7y} + 5x^4y^4) \\
 &= 49x^2e^{7y} + 20x^4y^3 \\
 f_{xy}(x, y) &= \frac{\partial}{\partial y} f_x(x, y) \\
 &= \frac{\partial}{\partial y} (2xe^{7y} + 4x^3y^5) \\
 &= 14xe^{7y} + 20x^3y^4 \\
 f_{yx}(x, y) &= \frac{\partial}{\partial x} f_y(x, y) \\
 &= \frac{\partial}{\partial x} (7x^2e^{7y} + 5x^4y^4) \\
 &= 14xe^{7y} + 20x^3y^4
 \end{aligned}$$

Note that, as in Example 6, $f_{xy}(x, y) = f_{yx}(x, y)$.

9.2 Exercises

1. $z = f(x, y) = 6x^2 - 4xy + 9y^2$

(a) $\frac{\partial z}{\partial x} = 12x - 4y$

(b) $\frac{\partial z}{\partial y} = -4x + 18y$

(c) $\frac{\partial f}{\partial x}(2, 3) = 12(2) - 4(3) = 12$

(d) $f_y(1, -2) = -4(1) + 18(-2) = -40$

2. $z = g(x, y) = 8x + 6x^2y + 2y^2$

(a) $\frac{\partial g}{\partial x} = 8 + 12xy$

(b) $\frac{\partial g}{\partial y} = 6x^2 + 4y$

(c) $\frac{\partial z}{\partial y}(-3, 0) = \frac{\partial g}{\partial y}(-3, 0) = 6(-3)^2 + 4(0) = 54$

(d) $g_x(2, 1) = \frac{\partial g}{\partial x}(2, 1) = 8 + 12(2)(1) = 32$

3. $f(x, y) = -4xy + 6y^3 + 5$

$$f_x(x, y) = -4y$$

$$f_y(x, y) = -4x + 18y^2$$

$$f_x(2, -1) = -4(-1) = 4$$

$$\begin{aligned}
 f_y(-4, 3) &= -4(-4) + 18(3)^2 \\
 &= 16 + 18(9) \\
 &= 178
 \end{aligned}$$

4. $f(x, y) = 9x^2y^2 - 4y^2$

$$f_x(x, y) = 18xy^2$$

$$f_y(x, y) = 18x^2y - 8y$$

$$f_x(2, -1) = 18(2)(-1)^2 = 36$$

$$f_y(-4, 3) = 18(-4)^2(3) - 8(3) = 840$$

5. $f(x, y) = 5x^2y^3$

$$f_x(x, y) = 10xy^3$$

$$f_y(x, y) = 15x^2y^2$$

$$f_x(2, -1) = 10(2)(-1)^3 = -20$$

$$f_y(-4, 3) = 15(-4)^2(3)^2 = 2160$$

6. $f(x, y) = -3x^4y^3 + 10$

$$f_x(x, y) = -12x^3y^3$$

$$f_y(x, y) = -9x^4y^2$$

$$f_x(2, -1) = -12(2)^3(-1)^3 = 96$$

$$f_y(-4, 3) = -9(-4)^4(3)^2 = -20,736$$

7. $f(x, y) = e^{x+y}$

$$f_x(x, y) = e^{x+y}$$

$$f_y(x, y) = e^{x+y}$$

$$f_x(2, -1) = e^{2-1}$$

$$= e^1 = e$$

$$f_y(-4, 3) = e^{-4+3}$$

$$= e^{-1}$$

$$= \frac{1}{e}$$

$$\begin{aligned}
 8. \quad f(x, y) &= 4e^{3x+2y} \\
 f_x(x, y) &= 12e^{3x+2y} \\
 f_y(x, y) &= 8e^{3x+2y} \\
 f_x(2, -1) &= 12e^{3(2)+2(-1)} = 12e^4 \\
 f_y(-4, 3) &= 8e^{3(-4)+2(3)} = 8e^{-6}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad f(x, y) &= -6e^{4x-3y} \\
 f_x(x, y) &= -24e^{4x-3y} \\
 f_y(x, y) &= 18e^{4x-3y} \\
 f_x(2, -1) &= -24e^{4(2)-3(-1)} = -24e^{11} \\
 f_y(-4, 3) &= 18e^{4(-4)-3(3)} = 18e^{-25}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad f(x, y) &= 8e^{7x-y} \\
 f_x(x, y) &= 56e^{7x-y} \\
 f_y(x, y) &= -8e^{7x-y} \\
 f_x(2, -1) &= 56e^{7(2)-(-1)} = 56e^{15} \\
 f_y(-4, 3) &= -8e^{7(-4)-3} = -8e^{-31}
 \end{aligned}$$

$$\begin{aligned}
 11. \quad f(x, y) &= \frac{x^2 + y^3}{x^3 - y^2} \\
 f_x(x, y) &= \frac{2x(x^3 - y^2) - 3x^2(x^2 + y^3)}{(x^3 - y^2)^2} \\
 &= \frac{2x^4 - 2xy^2 - 3x^4 - 3x^2y^3}{(x^3 - y^2)^2} \\
 &= \frac{-x^4 - 2xy^2 - 3x^2y^3}{(x^3 - y^2)^2} \\
 f_y(x, y) &= \frac{3y^2(x^3 - y^2) - (-2y)(x^2 + y^3)}{(x^3 - y^2)^2} \\
 &= \frac{3x^3y^2 - 3y^4 + 2x^2y + 2y^4}{(x^3 - y^2)^2} \\
 &= \frac{3x^3y^2 - y^4 + 2x^2y}{(x^3 - y^2)^2} \\
 f_x(2, -1) &= \frac{-2^4 - 2(2)(-1)^2 - 3(2^2)(-1)^3}{[2^3 - (-1)^2]^2} \\
 &= -\frac{8}{49} \\
 f_y(-4, 3) &= \frac{3(-4)^3(3)^2 - 3^4 + 2(-4)^2(3)}{[(-4)^3 - 3^2]^2} \\
 &= -\frac{1713}{5329}
 \end{aligned}$$

$$\begin{aligned}
 12. \quad f(x, y) &= \frac{3x^2y^3}{x^2 + y^2} \\
 f_x(x, y) &= \frac{(x^2 + y^2) \cdot 6xy^3 - 3x^2y^3 \cdot 2x}{(x^2 + y^2)^2} \\
 &= \frac{6xy^5}{(x^2 + y^2)^2} \\
 f_y(x, y) &= \frac{(x^2 + y^2) \cdot 9x^2y^2 - 3x^2y^3 \cdot 2y}{(x^2 + y^2)^2} \\
 &= \frac{9x^4y^2 + 3x^2y^4}{(x^2 + y^2)^2} \\
 f_x(2, -1) &= \frac{6(2)(-1)^5}{[(2)^2 + (-1)^2]^2} \\
 &= -\frac{12}{25} \\
 f_y(-4, 3) &= \frac{9(-4)^4(3)^2 + 3(-4)^2(3)^4}{[(-4)^2 + (3)^2]^2} \\
 &= \frac{24,624}{625}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad f(x, y) &= \ln|1 + 5x^3y^2| \\
 f_x(x, y) &= \frac{1}{1 + 5x^3y^2} \cdot 15x^2y^2 = \frac{15x^2y^2}{1 + 5x^3y^2} \\
 f_y(x, y) &= \frac{1}{1 + 5x^3y^2} \cdot 10x^3y = \frac{10x^3y}{1 + 5x^3y^2} \\
 f_x(2, -1) &= \frac{15(2)^2(-1)^2}{1 + 5(2)^3(-1)^2} = \frac{60}{41} \\
 f_y(-4, 3) &= \frac{10(-4)^3(3)}{1 + 5(-4)^3(3)^2} = \frac{1920}{2879}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad f(x, y) &= \ln|4x^4 - 2x^2y^2| \\
 f_x(x, y) &= \frac{1}{4x^4 - 2x^2y^2} \cdot (16x^3 - 4xy^2) \\
 &= \frac{16x^3 - 4xy^2}{4x^4 - 2x^2y^2} \\
 &= \frac{8x^2 - 2y^2}{2x^3 - xy^2} \\
 f_y(x, y) &= \frac{1}{4x^4 - 2x^2y^2} \cdot (-4x^2y) \\
 &= -\frac{4x^2y}{4x^4 - 2x^2y^2} \\
 &= -\frac{2y}{2x^2 - y^2}
 \end{aligned}$$

$$\begin{aligned} f_x(2, -1) &= \frac{8(2)^2 - 2(-1)^2}{2(2)^3 - (-2)(-1)^2} \\ &= \frac{15}{7} \end{aligned}$$

$$\begin{aligned} f_y(-4, 3) &= -\frac{2(3)}{2(-4)^2 - (3)^2} \\ &= -\frac{6}{23} \end{aligned}$$

$$\begin{aligned} 15. \quad f(x, y) &= xe^{x^2y} \\ f_x(x, y) &= e^{x^2y} \cdot 1 + x(2xy)(e^{x^2y}) \\ &= e^{x^2y}(1 + 2x^2y) \\ f_y(x, y) &= x^3e^{x^2y} \\ f_x(2, -1) &= e^{-4}(1 - 8) = -7e^{-4} \\ f_y(-4, 3) &= -64e^{48} \end{aligned}$$

$$\begin{aligned} 16. \quad f(x, y) &= y^2e^{x+3y} \\ f_x(x, y) &= y^2e^{x+3y} \\ f_y(x, y) &= y^2 \cdot 3e^{x+3y} + e^{x+3y} \cdot 2y \\ &= ye^{x+3y}(3y + 2) \\ f_x(2, -1) &= (-1)^2e^{2+3(-1)} = e^{-1} \\ f_y(-4, 3) &= 3e^{-4+3(3)}[3(3) + 2] \\ &= 33e^5 \end{aligned}$$

$$\begin{aligned} 17. \quad f(x, y) &= \sqrt{x^4 + 3xy + y^4 + 10} \\ f_x(x, y) &= \frac{4x^3 + 3y}{2\sqrt{x^4 + 3xy + y^4 + 10}} \\ f_y(x, y) &= \frac{3x + 4y^3}{2\sqrt{x^4 + 3xy + y^4 + 10}} \\ f_x(2, -1) &= \frac{4(2)^3 + 3(-1)}{2\sqrt{2^4 + 3(2)(-1) + (-1)^4 + 10}} \\ &= \frac{29}{2\sqrt{21}} \\ f_y(-4, 3) &= \frac{3(-4) + 4(3)^3}{2\sqrt{(-4)^4 + 3(-4)(3) + 3^4 + 10}} \\ &= \frac{48}{\sqrt{311}} \end{aligned}$$

$$\begin{aligned} 18. \quad f(x, y) &= (7x^2 + 18xy^2 + y^3)^{1/3} \\ f_x(x, y) &= \frac{14x + 18y^2}{3(7x^2 + 18xy^2 + y^3)^{2/3}} \\ f_y(x, y) &= \frac{36xy + 3y^2}{3(7x^2 + 18xy^2 + y^3)^{2/3}} \\ f_x(2, -1) &= \frac{14(2) + 18(-1)^2}{3(7(2)^2 + 18(2)(-1)^2 + (-1)^3)^{2/3}} \\ &= \frac{46}{3(63)^{2/3}} \\ f_y(-4, 3) &= \frac{36(-4)(3) + 3(3)^2}{3(7(-4)^2 + 18(-4)(3)^2 + (3)^3)^{2/3}} \\ &= -\frac{135}{(509)^{2/3}} \end{aligned}$$

$$\begin{aligned} 19. \quad f(x, y) &= \frac{3x^2y}{e^{xy} + 2} \\ f_x(x, y) &= \frac{6xy(e^{xy} + 2) - ye^{xy}(3x^2y)}{(e^{xy} + 2)^2} \\ &= \frac{6xy(e^{xy} + 2) - 3x^2y^2e^{xy}}{(e^{xy} + 2)^2} \\ f_y(x, y) &= \frac{3x^2(e^{xy} + 2) - xe^{xy}(3x^2y)}{(e^{xy} + 2)^2} \\ &= \frac{3x^2(e^{xy} + 2) - 3x^3ye^{xy}}{(e^{xy} + 2)^2} \\ f_x(2, -1) &= \frac{6(2)(-1)(e^{2(-1)} + 2) - 3(2)^2(-1)^2e^{2(-1)}}{(e^{2(-1)} + 2)^2} \\ &= \frac{-12e^{-2} - 24 - 12e^{-2}}{(e^{-2} + 2)^2} \\ &= \frac{-24(e^{-2} + 1)}{(e^{-2} + 2)^2} \\ f_y(-4, 3) &= \frac{3(-4)^2(e^{(-4)(3)} + 2) - 3(-4)^3(3)e^{(-4)(3)}}{(e^{(-4)(3)} + 2)^2} \\ &= \frac{48e^{-12} + 96 + 576e^{-12}}{(e^{-12} + 2)^2} \\ &= \frac{624e^{-12} + 96}{(e^{-12} + 2)^2} \end{aligned}$$

$$\begin{aligned}
 20. \quad f(x, y) &= (7e^{x+2y} + 4)(e^{x^2} + y^2 + 2) \\
 f_x(x, y) &= 7e^{x+2y}(e^{x^2} + y^2 + 2) \\
 &\quad + 2xe^{x^2}(7e^{x+2y} + 4) \\
 f_y(x, y) &= 14e^{x+2y}(e^{x^2} + y^2 + 2) \\
 &\quad + 2y(7e^{x+2y} + 4) \\
 f_x(2, -1) &= 7e^{2+2(-1)}(e^{2^2} + (-1)^2 + 2) \\
 &\quad + 2(2)e^{2^2}(7e^{2+2(-1)} + 4) \\
 &= 7e^0(e^4 + 3) + 4e^4(7e^0 + 4) \\
 &= 51e^4 + 21 \\
 f_y(-4, 3) &= 14e^{-4+2(3)}(e^{(-4)^2} + 3^2 + 2) \\
 &\quad + 2(3)(7e^{-4+2(3)} + 4) \\
 &= 14e^2(e^{16} + 11) + 6(7e^2 + 4)
 \end{aligned}$$

$$\begin{aligned}
 21. \quad f(x, y) &= 4x^2y^2 - 16x^2 + 4y \\
 f_x(x, y) &= 8xy^2 - 32x \\
 f_y(x, y) &= 8x^2y + 4 \\
 f_{xx}(x, y) &= 8y^2 - 32 \\
 f_{yy}(x, y) &= 8x^2 \\
 f_{xy}(x, y) &= f_{yx}(x, y) = 16xy
 \end{aligned}$$

$$\begin{aligned}
 22. \quad g(x, y) &= 5x^4y^2 + 12y^3 - 9x \\
 g_x(x, y) &= 20x^3y^2 - 9 \\
 g_y(x, y) &= 10x^4y + 36y^2 \\
 g_{xx}(x, y) &= 60x^2y^2 \\
 g_{yy}(x, y) &= 10x^4 + 72y \\
 g_{xy}(x, y) &= g_{yx}(x, y) = 40x^3y
 \end{aligned}$$

$$\begin{aligned}
 23. \quad R(x, y) &= 4x^2 - 5xy^3 + 12y^2x^2 \\
 R_x(x, y) &= 8x - 5y^3 + 24y^2x \\
 R_y(x, y) &= -15xy^2 + 24yx^2 \\
 R_{xx}(x, y) &= 8 + 24y^2 \\
 R_{yy}(x, y) &= -30xy + 24x^2 \\
 R_{xy}(x, y) &= -15y^2 + 48xy \\
 &= R_{yx}(x, y)
 \end{aligned}$$

$$\begin{aligned}
 24. \quad h(x, y) &= 30y + 5x^2y + 12xy^2 \\
 h_x(x, y) &= 10xy + 12y^2 \\
 h_{xx}(x, y) &= 10y \\
 h_{xy}(x, y) &= 10x + 24y \\
 h_y(x, y) &= 30 + 5x^2 + 24xy \\
 h_{yy}(x, y) &= 24x \\
 h_{yx}(x, y) &= 10x + 24y
 \end{aligned}$$

$$\begin{aligned}
 25. \quad r(x, y) &= \frac{6y}{x+y} \\
 r_x(x, y) &= \frac{(x+y)(0) - 6y(1)}{(x+y)^2} \\
 &= -6y(x+y)^{-2} \\
 r_y(x, y) &= \frac{(x+y)(6) - 6y(1)}{(x+y)^2} \\
 &= 6x(x+y)^{-2} \\
 r_{xx}(x, y) &= -6y(-2)(x+y)^{-3}(-1) \\
 &= \frac{12y}{(x+y)^3} \\
 r_{yy}(x, y) &= 6x(-2)(x+y)^{-3}(1) \\
 &= -\frac{12x}{(x+y)^3} \\
 r_{xy}(x, y) &= r_{yx}(x, y) \\
 &= -6y(-2)(x+y)^{-3}(1) + (x+y)^{-2}(-6) \\
 &= \frac{12y - 6(x+y)}{(x+y)^3} \\
 &= \frac{6y - 6x}{(x+y)^3}
 \end{aligned}$$

$$\begin{aligned}
 26. \quad k(x, y) &= \frac{-7x}{2x+3y} \\
 k_x(x, y) &= \frac{(2x+3y)(-7) - (-7x)(2)}{(2x+3y)^2} \\
 &= -21y(2x+3y)^{-2} \\
 k_y(x, y) &= \frac{(2x+3y)(0) - (-7x)(3)}{(2x+3y)^2} \\
 &= 21x(2x+3y)^{-2} \\
 k_{xx}(x, y) &= -21y(-2)(2x+3y)^{-3}(2) \\
 &= \frac{84y}{(2x+3y)^3}
 \end{aligned}$$

$$\begin{aligned}
 k_{yy}(x, y) &= 21x(-2)(2x + 3y)^{-3}(3) \\
 &= -\frac{126x}{(2x + 3y)^3} \\
 k_{xy}(x, y) &= k_{yx}(x, y) \\
 &= -21y(-2)(2x + 3y)^{-3}(3) \\
 &\quad + (2x + 3y)^{-2}(-21) \\
 &= \frac{126y + (-21)(2x + 3y)}{(2x + 3y)^3} \\
 &= \frac{63y - 42x}{(2x + 3y)^3}
 \end{aligned}$$

$$\begin{aligned}
 27. \quad z &= 9ye^x \\
 z_x &= 9ye^x \\
 z_y &= 9e^x \\
 z_{xx} &= 9ye^x \\
 z_{yy} &= 0 \\
 z_{xy} &= z_{yx} = 9e^x
 \end{aligned}$$

$$\begin{aligned}
 28. \quad z &= -6xe^y \\
 z_x &= -6e^y \\
 z_y &= -6xe^y \\
 z_{xx} &= 0 \\
 z_{yy} &= -6xe^y \\
 z_{xy} &= z_{yx} = -6e^y
 \end{aligned}$$

$$\begin{aligned}
 29. \quad r &= \ln |x + y| \\
 r_x &= \frac{1}{x + y} \\
 r_y &= \frac{1}{x + y} \\
 r_{xx} &= \frac{-1}{(x + y)^2} \\
 r_{yy} &= \frac{-1}{(x + y)^2} \\
 r_{xy} &= r_{yx} = \frac{-1}{(x + y)^2}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad k &= \ln |5x - 7y| \\
 k_x(x, y) &= \frac{5}{5x - 7y} = 5(5x - 7y)^{-1} \\
 k_y(x, y) &= \frac{-7}{5x - 7y} = -7(5x - 7y)^{-1} \\
 k_{xx}(x, y) &= -5(5x - 7y)^{-2} \cdot 5 \\
 &= -25(5x - 7y)^{-2} \text{ or } \frac{-25}{(5x - 7y)^2} \\
 k_{yy}(x, y) &= 7(5x - 7y)^{-2} \cdot (-7) \\
 &= -49(5x - 7y)^{-2} \text{ or } \frac{-49}{(5x - 7y)^2} \\
 k_{xy}(x, y) &= -5(5x - 7y)^{-2} \cdot (-7) \\
 &= 35(5x - 7y)^{-2} \text{ or } \frac{35}{(5x - 7y)^2} \\
 k_{yx}(x, y) &= 7(5x - 7y)^{-2} \cdot (5) \\
 &= 35(5x - 7y)^{-2} \text{ or } \frac{35}{(5x - 7y)^2}
 \end{aligned}$$

$$\begin{aligned}
 31. \quad z &= x \ln |xy| \\
 z_x &= \ln |xy| + 1 \\
 z_y &= \frac{x}{y} \\
 z_{xx} &= \frac{1}{x} \\
 z_{yy} &= -xy^{-2} = \frac{-x}{y^2} \\
 z_{xy} &= z_{yx} = \frac{1}{y}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad z &= (y + 1) \ln |x^3 y| \\
 &= (y + 1)(3 \ln |x| + \ln |y|) \\
 z_x(x, y) &= (y + 1) \cdot \left(3 \cdot \frac{1}{x} \right) \\
 &= \frac{3(y + 1)}{x} \\
 &= 3x^{-1}(y + 1) \\
 z_y(x, y) &= (y + 1) \cdot \left(\frac{1}{y} \right) + (3 \ln |x| + \ln |y|) \cdot 1 \\
 &= \frac{y + 1}{y} + 3 \ln |x| + \ln |y|
 \end{aligned}$$

$$z_{xx}(x, y) = -3x^{-2}(y+1) = \frac{-3(y+1)}{x^2}$$

$$z_{yy}(x, y) = \frac{y \cdot 1 - (y+1) \cdot 1}{y^2} + \frac{1}{y}$$

$$= -\frac{1}{y^2} + \frac{1}{y}$$

$$z_{xy}(x, y) = 3x^{-1} = \frac{3}{x}$$

$$z_{yx}(x, y) = 3 \cdot \frac{1}{x} = \frac{3}{x}$$

33. $f(x, y) = 6x^2 + 6y^2 + 6xy + 36x - 5$

First, $f_x = 12x + 6y + 36$ and

$f_y = 12y + 6x$.

We must solve the system

$$12x + 6y + 36 = 0$$

$$12y + 6x = 0.$$

Multiply both sides of the first equation by -2 and add.

$$\begin{array}{r} -24x - 12y - 72 = 0 \\ 6x + 12y = 0 \\ \hline -18x - 72 = 0 \\ x = -4 \end{array}$$

Substitute into either equation to get $y = 2$.

The solution is $x = -4, y = 2$.

34. $f(x, y) = 50 + 4x - 5y + x^2 + y^2 + xy$

$f_x(x, y) = 4 + 2x + y,$

$f_y(x, y) = -5 + 2y + x$

Solve the system

$$4 + 2x + y = 0$$

$$-5 + x + 2y = 0.$$

Multiply the second equation by -2 and add.

$$\begin{array}{r} 4 + 2x + y = 0 \\ 10 - 2x - 4y = 0 \\ \hline 14 - 3y = 0 \\ y = \frac{14}{3} \end{array}$$

Substitute into the second equation to get

$x = -\frac{13}{3}$. The solution is $x = -\frac{13}{3}, y = \frac{14}{3}$.

35. $f(x, y) = 9xy - x^3 - y^3 - 6$

First, $f_x = 9y - 3x^2$ and $f_y = 9x - 3y^2$.

We must solve the system

$$9y - 3x^2 = 0$$

$$9x - 3y^2 = 0.$$

From the first equation, $y = \frac{1}{3}x^2$.

Substitute into the second equation to get

$$9x - 3\left(\frac{1}{3}x^2\right)^2 = 0$$

$$9x - 3\left(\frac{1}{9}x^4\right) = 0$$

$$9x - \frac{1}{3}x^4 = 0.$$

Multiply by 3 to get

$$27x - x^4 = 0.$$

Now factor.

$$x(27 - x^3) = 0$$

Set each factor equal to 0.

$$x = 0 \quad \text{or} \quad 27 - x^3 = 0$$

$$x = 3$$

Substitute into $y = \frac{x^2}{3}$.

$$y = 0 \quad \text{or} \quad y = 3$$

The solutions are $x = 0, y = 0$ and

$x = 3, y = 3$.

36. $f(x, y) = 2200 + 27x^3 + 72xy + 8y^2$

$f_x(x, y) = 81x^2 + 72y, f_y(x, y) = 72x + 16y$

Solve the system

$$81x^2 + 72y = 0 \quad (1)$$

$$72x + 16y = 0. \quad (2)$$

Divide equation (1) by 9 and equation (2) by 8.

$$9x^2 + 8y = 0 \quad (3)$$

$$9x + 2y = 0 \quad (4)$$

From equation (4), $y = -\frac{9}{2}x$.

Substitute into equation (3).

$$\begin{aligned} 9x^2 - 36x &= 0 \\ 9x(x - 4) &= 0 \\ x = 0 \quad \text{or} \quad x &= 4 \end{aligned}$$

Since $y = -\frac{9}{2}x$,

if $x = 0, y = 0$;

if $x = 4, y = -\frac{9}{2}(4) = -18$.

The solutions are $x = 0, y = 0$, and $x = 4, y = -18$.

37. $f(x, y, z) = x^4 + 2yz^2 + z^4$

$$f_x(x, y, z) = 4x^3$$

$$f_y(x, y, z) = 2z^2$$

$$f_z(x, y, z) = 4yz + 4z^3$$

$$f_{yz}(x, y, z) = 4z$$

38. $f(x, y, z) = 6x^3 - x^2y^2 + y^5$

$$f_x(x, y, z) = 18x^2 - 2xy^2$$

$$f_y(x, y, z) = -2x^2y + 5y^4$$

$$f_z(x, y, z) = 0$$

$$f_{yz}(x, y, z) = 0$$

39. $f(x, y, z) = \frac{6x - 5y}{4z + 5}$

$$f_x(x, y, z) = \frac{6}{4z + 5}$$

$$f_y(x, y, z) = \frac{-5}{4z + 5}$$

$$f_z(x, y, z) = \frac{-4(6x - 5y)}{(4z + 5)^2}$$

$$f_{yz}(x, y, z) = \frac{20}{(4z + 5)^2}$$

40. $f(x, y, z) = \frac{2x^2 + xy}{yz - 2}$

$$= \frac{1}{yz - 2}(2x^2 + xy)$$

$$= (2x^2 + xy)(yz - 2)^{-1}$$

$$f_x(x, y, z) = \frac{1}{yz - 2}(4x + y)$$

$$= \frac{4x + y}{yz - 2}$$

$$f_y(x, y, z) = \frac{(yz - 2) \cdot x - (2x^2 + xy) \cdot z}{(yz - 2)^2}$$

$$= \frac{-2x - 2x^2z}{(yz - 2)^2}$$

$$f_z(x, y, z)$$

$$= -(2x^2 + xy)(yz - 2)^{-2} \cdot y$$

$$= -\frac{(2x^2y + xy^2)}{(yz - 2)^2}$$

$$f_{yz}(x, y, z)$$

$$= \frac{(yz - 2)^2(-2x^2) - (-2x - 2x^2z) \cdot 2(yz - 2) \cdot y}{(yz - 2)^4}$$

$$= \frac{4x^2 + 4xy + 2x^2yz}{(yz - 2)^3}$$

41. $f(x, y, z) = \ln|x^2 - 5xz^2 + y^4|$

$$f_x(x, y, z) = \frac{2x - 5z^2}{x^2 - 5xz^2 + y^4}$$

$$f_y(x, y, z) = \frac{4y^3}{x^2 - 5xz^2 + y^4}$$

$$f_z(x, y, z) = \frac{-10xz}{x^2 - 5xz^2 + y^4}$$

$$f_{yz}(x, y, z) = \frac{4y^3(10xz)}{(x^2 - 5xz^2 + y^4)^2}$$

$$= \frac{40xy^3z}{(x^2 - 5xz^2 + y^4)^2}$$

42. $f(x, y, z) = \ln|8xy + 5yz - x^3|$

$$f_x(x, y, z) = \frac{1}{8xy + 5yz - x^3} \cdot (8y - 3x^2)$$

$$= \frac{8y - 3x^2}{8xy + 5yz - x^3}$$

$$f_y(x, y, z) = \frac{1}{8xy + 5yz - x^3} \cdot (8x + 5z)$$

$$= \frac{8x + 5z}{8xy + 5yz - x^3}$$

$$\begin{aligned}
 f_z(x, y, z) &= \frac{1}{8xy + 5yz - x^3} \cdot 5y \\
 &= \frac{5y}{8xy + 5yz - x^3} \\
 f_{yz}(x, y, z) &= \frac{(8xy + 5yz - x^3) \cdot 5 - (8x + 5z) \cdot 5y}{(8xy + 5yz - x^3)^2} \\
 &= \frac{-5x^3}{(8xy + 5yz - x^3)^2}
 \end{aligned}$$

43. $f(x, y) = \left(x + \frac{y}{2}\right)^{x+y/2}$

(a) $f_x(1, 2) = \lim_{h \rightarrow 0} \frac{f(1+h, 2) - f(1, 2)}{h}$

We will use a small value for h . Let $h = 0.00001$.

$$\begin{aligned}
 f_x(1, 2) &\approx \frac{f(1.00001, 2) - f(1, 2)}{0.00001} \\
 &\approx \frac{\left(1.00001 + \frac{2}{2}\right)^{1.00001+2/2} - \left(1 + \frac{2}{2}\right)^{1+2/2}}{0.00001} \\
 &\approx \frac{2.00001^{2.00001} - 2^2}{0.00001} \\
 &\approx 6.773
 \end{aligned}$$

(b) $f_y(1, 2) = \lim_{h \rightarrow 0} \frac{f(1, 2+h) - f(1, 2)}{h}$

Again, let $h = 0.00001$.

$$\begin{aligned}
 f_y(1, 2) &\approx \frac{f(1, 2.00001) - f(1, 2)}{0.00001} \\
 &\approx \frac{\left(1 + \frac{2.00001}{2}\right)^{1+2.00001/2} - \left(1 + \frac{2}{2}\right)^{1+2/2}}{0.00001} \\
 &\approx \frac{2.000005^{2.000005} - 2^2}{0.00001} \\
 &\approx 3.386
 \end{aligned}$$

44. $f(x, y) = (x + y^2)^{2x+y}$

(a) $f_x(2, 1) = \lim_{h \rightarrow 0} \frac{f(2+h, 1) - f(2, 1)}{h}$

We will use a small value for h .
Let $h = 0.00001$.

$$\begin{aligned}
 f_x(2, 1) &\approx \frac{f(2.00001, 1) - f(2, 1)}{0.00001} \\
 &\approx \frac{(2.00001 + 1^2)^{2(2.00001)+1} - (2 + 1^2)^{2(2)+1}}{0.00001} \\
 &\approx \frac{(3.00001)^{5.00002} - 3^5}{0.00001} \\
 &\approx 938.9
 \end{aligned}$$

(b) $f_y(2, 1) = \lim_{h \rightarrow 0} \frac{f(2, 1+h) - f(2, 1)}{h}$

Again, let $h = 0.00001$.

$$\begin{aligned}
 f_y(2, 1) &\approx \frac{f(2, 1.00001) - f(2, 1)}{0.00001} \\
 &\approx \frac{[2 + (1.00001)^2]^{2(2)+1.00001} - (2 + 1^2)^{2(2)+1}}{0.00001} \\
 &\approx \frac{[2 + (1.00001)^2]^{5.00001} - 3^5}{0.00001} \\
 &\approx 1077
 \end{aligned}$$

45. $M(x, y) = 45x^2 + 40y^2 - 20xy + 50$

(a) $M_y(x, y) = 80y - 20x$

$$M_y(4, 2) = 80(2) - 20(4) = 80$$

(b) $M_x(x, y) = 90x - 20y$

$$M_x(3, 6) = 90(3) - 20(6) = 150$$

(c) $\frac{\partial M}{\partial x}(2, 5) = 90(2) - 20(5) = 80$

(d) $\frac{\partial M}{\partial y}(6, 7) = 80(7) - 20(6) = 440$

46. $R(x, y) = 5x^2 + 9y^2 - 4xy$

(a) $R(9, 5) = 5(9)^2 + 9(5)^2 - 4(9)(5)$
 $= 450$

$$R_x(x, y) = 10x - 4y$$

$$R_x(9, 5) = 10(9) - 4(5) = 70$$

R would increase by \$70.

(b) $R_y(x, y) = 18y - 4x$

$$R_y(9, 5) = 18(5) - 4(9) = 54$$

R would increase by \$54.

47. $f(p, i) = 99p - 0.5pi - 0.0025p^2$

(a) $f(19, 400, 8)$
 $= 99(19, 400) - 0.5(19, 400)(8)$
 $- 0.0025(19, 400)^2$
 $= \$902,100$

The weekly sales are \$902,100.

- (b) $f_p(p, i) = 99 - 0.5i - 0.005p$, which represents the rate of change in weekly sales revenue per unit change in price when the interest rate remains constant.

$f_i(p, i) = -0.5p$, which represents the rate of change in weekly sales revenue per unit change in interest rate when the list price remains constant.

- (c) $p = 19,400$ remains constant and i changes by 1 unit from 8 to 9.

$$\begin{aligned} f_i(p, i) &= f_i(19, 400, 8) \\ &= -0.5(19, 400) \\ &= -9700 \end{aligned}$$

Therefore, sales revenue declines by \$9700.

48. $P(x, y) = 250\sqrt{x^2 + y^2}$
 $= 250(x^2 + y^2)^{1/2}$

(a) $\frac{\partial P}{\partial x} = 250\left(\frac{1}{2}\right)(x^2 + y^2)^{-1/2}(2x)$
 $= \frac{250x}{\sqrt{x^2 + y^2}}$

$$\begin{aligned} \frac{\partial P}{\partial x}(6, 8) &= \frac{250(6)}{\sqrt{(6)^2 + (8)^2}} \\ &= \frac{1500}{10} = 150 \end{aligned}$$

(b) $\frac{\partial P}{\partial x} = 250\left(\frac{1}{2}\right)(x^2 + y^2)^{-1/2}(2y)$
 $= \frac{250y}{\sqrt{x^2 + y^2}}$

$$\begin{aligned} \frac{\partial P}{\partial y}(6, 8) &= \frac{250(8)}{\sqrt{(6)^2 + (8)^2}} \\ \frac{2000}{10} &= 200 \end{aligned}$$

49. $f(x, y) = \left(\frac{1}{4}x^{-1/4} + \frac{3}{4}y^{-1/4}\right)^{-4}$

(a) $f(16, 81) = \left[\frac{1}{4}(16)^{-1/4} + \frac{3}{4}(81)^{-1/4}\right]^{-4}$
 $= \left(\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{3}\right)^{-4}$
 $= \left(\frac{3}{8}\right)^{-4} \approx 50.56790123$

50.57 hundred units are produced.

(b)

$$\begin{aligned} f_x(x, y) &= -4\left(\frac{1}{4}x^{-1/4} + \frac{3}{4}y^{-1/4}\right)^{-5}\left[\frac{1}{4}\left(-\frac{1}{4}\right)x^{-5/4}\right] \\ &= \frac{1}{4}x^{-5/4}\left(\frac{1}{4}x^{-1/4} + \frac{3}{4}y^{-1/4}\right)^{-5} \end{aligned}$$

$$\begin{aligned} f_x(16, 81) &= \frac{1}{4}(16)^{-5/4}\left[\frac{1}{4}(16)^{-1/4} + \frac{3}{4}(81)^{-1/4}\right]^{-5} \\ &= \frac{1}{4}\left(\frac{1}{32}\right)\left(\frac{3}{8}\right)^{-5} = \frac{256}{243} \\ &\approx 1.053497942 \end{aligned}$$

$f_x(16, 81) = 1.053$ hundred units and is the rate at which production is changing when labor changes by one unit (from 16 to 17) and capital remains constant.

$$\begin{aligned} f_y(x, y) &= -4\left(\frac{1}{4}x^{-1/4} + \frac{3}{4}y^{-1/4}\right)^{-5}\left[\frac{3}{4}\left(-\frac{1}{4}\right)y^{-5/4}\right] \\ &= \frac{3}{4}y^{-5/4}\left(\frac{1}{4}x^{-1/4} + \frac{3}{4}y^{-1/4}\right)^{-5} \end{aligned}$$

$$\begin{aligned} f_y(16, 81) &= \frac{3}{4}(81)^{-5/4}\left[\frac{1}{4}(16)^{-1/4} + \frac{3}{4}(81)^{-1/4}\right]^{-5} \\ &= \frac{3}{4}\left(\frac{1}{243}\right)\left(\frac{3}{8}\right)^{-5} = \frac{8192}{19,683} \\ &\approx 0.4161967180 \end{aligned}$$

$f_y(16, 81) = 0.4162$ hundred units and is the rate at which production is changing when capital changes by one unit (from 81 to 82) and labor remains constant.

- (c) Using the value of $f_x(16, 81)$ found in (b), production would increase by approximately 105 units.

50. $z = x^{0.7}y^{0.3}$, where x is labor, y is capital.

Marginal productivity of labor is

$$\frac{\partial z}{\partial x} = 0.7x^{-0.3}y^{0.3}.$$

Marginal productivity of capital is

$$\frac{\partial z}{\partial y} = 0.3x^{0.7}y^{-0.7}.$$

51. $z = x^{0.4}y^{0.6}$

The marginal productivity of labor is

$$\begin{aligned}\frac{\partial z}{\partial x} &= 0.4x^{-0.6}y^{0.6} + x^{0.4} \cdot 0 \\ &= 0.4x^{-0.6}y^{0.6}.\end{aligned}$$

The marginal productivity of capital is

$$\begin{aligned}\frac{\partial z}{\partial y} &= x^{0.4}(0.6y^{-0.4}) + y^{0.6} \cdot 0 \\ &= 0.6x^{0.4}y^{-0.4}.\end{aligned}$$

52. $f(x, y) = 3x^{1/3}y^{2/3}$, where x is labor, y is capital.

$$\begin{aligned}\text{(a)} \quad f_x(x, y) &= x^{-2/3}y^{2/3} \\ &= \frac{y^{2/3}}{x^{2/3}} = \left(\frac{y}{x}\right)^{2/3}\end{aligned}$$

$$\begin{aligned}f_x(64, 125) &= \left(\frac{125}{64}\right)^{2/3} \\ &= \frac{25}{16} = 1.5625,\end{aligned}$$

which is the approximate change in production (in thousands of units) for a 1 unit change in labor.

$$f_y(x, y) = 2x^{1/3}y^{-1/3}$$

$$= 2\frac{x^{1/3}}{y^{1/3}} = 2\left(\frac{x}{y}\right)^{1/3}$$

$$f_y(64, 125) = 2\left(\frac{64}{125}\right)^{1/3} = \frac{8}{5} = 1.6,$$

which is the approximate change in production (in thousands of units) for a 1 unit change in capital.

- (b) Increasing to 65 units of labor would result in an increase of approximately

$$\frac{25}{16}(1000) \approx 1563 \text{ batteries.}$$

- (c) An increase of approximately

$$\frac{16}{5}(1000) = 3200 \text{ batteries}$$

would be the effect of increasing capital to 126 while holding labor at 64. Increasing capital is the better option.

$$53. \quad f(w, v) = 25.92w^{0.68} + \frac{3.62w^{0.75}}{v}$$

$$\begin{aligned}\text{(a)} \quad f(300, 10) &= 25.92(300)^{0.68} + \frac{3.62(300)^{0.75}}{10} \\ &\approx 1279.46\end{aligned}$$

The value is about 1279 kcal/hr.

$$\begin{aligned}\text{(b)} \quad f_w(w, v) &= 25.92(0.68)w^{-0.32} \\ &\quad + \frac{3.62(0.75)w^{-0.25}}{v} \\ &= \frac{17.6256}{w^{0.32}} + \frac{2.715}{w^{0.25}v} \\ f_w(300, 10) &= \frac{17.6256}{(300)^{0.32}} + \frac{2.715}{(300)^{0.25}(10)} \\ &\approx 2.906\end{aligned}$$

The value is about 2.906 kcal/hr/g. This means the instantaneous rate of change of energy usage for a 300 kg animal traveling at 10 kilometers per hour to walk or run 1 kilometer is about 2.906 kcal/hr/g.

$$54. \quad H(m, T, A) = \frac{15.2m^{0.67}(T - A)}{10.23 \ln m - 10.74}$$

$$\text{(a)} \quad H_T(m, T, A) = \frac{15.2m^{0.67}}{10.23 \ln m - 10.74}$$

$$\begin{aligned}H_T(24, 37, 8) &= \frac{15.2(24)^{0.67}}{10.23 \ln 24 - 10.74} \\ &\approx 5.87\end{aligned}$$

The approximate change in the rate of heat loss is 5.87 W.

$$\text{(b)} \quad H_A(m, T, A) = \frac{-15.2m^{0.67}}{10.23 \ln m - 10.74}$$

$$\begin{aligned}H_A(26, 37, 10) &= \frac{-15.2(26)^{0.67}}{10.23 \ln 26 - 10.74} \\ &\approx -5.97\end{aligned}$$

The approximate change in the rate of heat loss is -5.97 W.

55. $A = 0.024265h^{0.3964}m^{0.5378}$

(a) $A_m = (0.024265)(0.5378)h^{0.3964}m^{(0.5378-1)}$
 $= 0.013050h^{0.3964}m^{-0.4622}$

When the mass m increases from 72 to 73 while the height h remains at 180 cm, the approximate change in body surface area is

$$0.013050(180)^{0.3964}(72)^{-0.4622} \approx 0.0142$$

or about 0.0142 m^2 .

(b) $A_h = (0.024265)(0.3964)h^{(0.3964-1)}m^{0.5378}$
 $= 0.0096186h^{-0.6036}m^{0.5378}$

When the height h increases from 160 to 161 while the mass m remains at 70, the approximate change in body surface area is

$$0.0096186(160)^{-0.6036}(70)^{0.5378} \approx 0.00442$$

or about 0.00442 m^2 .

56. $C(a, b, v) = \frac{b}{a - v} = b(a - v)^{-1}$

(a) $C(160, 200, 125) = \frac{200}{160 - 125}$
 $= \frac{200}{35}$
 ≈ 5.714

(b) $C_a(a, b, v) = -b(a - v)^{-2} \cdot 1$
 $= -\frac{b}{(a - v)^2}$

$$C_a(160, 200, 125) = -\frac{200}{(160 - 125)^2}$$

$$= -\frac{200}{35^2}$$

$$\approx -0.1633$$

(c) $C_b(a, b, v) = (a - v)^{-1}$

$$C_b(160, 200, 125) = (160 - 125)^{-1}$$

$$= \frac{1}{35}$$

$$\approx 0.0286$$

(d) $C_v(a, b, v) = -b(a - v)^{-2} \cdot (-1)$
 $= \frac{b}{(a - v)^2}$

$$C_v(160, 200, 125) = \frac{200}{(160 - 125)^2}$$

$$= \frac{200}{35^2}$$

$$\approx 0.1633$$

(e) Changing a by 1 unit produces the greatest decrease in the liters of blood pumped, while changing v by 1 unit produces the same amount of increase in the liters of blood pumped.

57. $f(n, c) = \frac{1}{8}n^2 - \frac{1}{5}c + \frac{1937}{8}$

(a) $f(4, 1200) = \frac{1}{8}(4) - \frac{1}{5}(1200) + \frac{1937}{8}$
 $= 2 - 240 + \frac{1937}{8} = 4.125$

The client could expect to lose 4.125 lb.

(b) $\frac{\partial f}{\partial n} = \frac{1}{8}(2n) - \frac{1}{5}(0) + 0 = \frac{1}{4}n$,

which represents the rate of change of weight loss per unit change in number of workouts.

(c) $f_n(3, 1100) = \frac{1}{4}(3) = \frac{3}{4} \text{ lb}$

represents an additional weight loss by adding the fourth workout.

58. $B(w, h) = \frac{703w}{h^2}$

(a) $B(317, 79) = \frac{703(317)}{(79)^2}$
 ≈ 35.7

(b) $\frac{\partial B}{\partial w} = \frac{703}{h^2}$
 $\frac{\partial B}{\partial h} = \frac{-2(703)w}{h^3} = -\frac{1406w}{h^3}$

Since $\frac{\partial B}{\partial w}$ is positive, as w increases, so does B . Since $\frac{\partial B}{\partial h}$ is negative, B decreases as h increases.

(c) If w_m and h_m represent a person's weight and height, in kilograms and meters, respectively, then $w_m = 0.4536w$, where w is weight in pounds, and $h_m = 0.0254h$,

where h is height in inches. To transform the formula $B = \frac{703w}{h^2}$ to handle metric inputs,

make the substitutions $w = \frac{w_m}{0.4536}$ and

$$h = \frac{h_m}{0.0254}.$$

$$\begin{aligned} B_m &= \frac{703 \frac{w_m}{0.4536}}{\left(\frac{h_m}{0.0254}\right)^2} \\ &= \frac{703(0.0254)^2}{0.4536} \cdot \frac{w_m}{h_m^2}. \end{aligned}$$

Since $\frac{703(0.0254)^2}{0.4536} \approx 1$, use $B_m = \frac{w_m}{h_m^2}$.

$$59. \quad R(x, t) = x^2(a - x)t^2e^{-t} = (ax^2 - x^3)t^2e^{-t}$$

$$(a) \quad \frac{\partial R}{\partial x} = (2ax - 3x^2)t^2e^{-t}$$

$$(b) \quad \begin{aligned} \frac{\partial R}{\partial t} &= x^2(a - x) \cdot [t^2 \cdot (-e^{-t}) + e^{-t} \cdot 2t] \\ &= x^2(a - x)(-t^2 + 2t)e^{-t} \end{aligned}$$

$$(c) \quad \frac{\partial^2 R}{\partial x^2} = (2a - 6x)t^2e^{-t}$$

$$(d) \quad \frac{\partial^2 R}{\partial x \partial t} = (2ax - 3x^2)(-t^2 + 2t)e^{-t}$$

(e) $\frac{\partial R}{\partial x}$ gives the rate of change of the reaction per unit of change in the amount of drug administered.

$\frac{\partial R}{\partial t}$ gives the rate of change of the reaction for a 1-hour change in the time after the drug is administered.

$$60. \quad p(l, t) = 1 + \frac{l}{33}(1 - 2^{-t/5})$$

$$\begin{aligned} (a) \quad p(33, 10) &= 1 + \frac{33}{33}(1 - 2^{-10/5}) \\ &= 1 + (1)(1 - 2^{-2}) \\ &= 1 + \frac{3}{4} \\ &= 1.75 \end{aligned}$$

The pressure at 33 feet for a 10-minute dive is 1.75 atmospheres.

$$(b) \quad p_l(l, t) = \frac{1}{33}(1 - 2^{-t/5})$$

$$\begin{aligned} p_t(l, t) &= \frac{l}{33}(-(\ln 2))\left(-\frac{1}{5}2^{-t/5}\right) \\ &= \frac{l}{165}(\ln 2)(2^{-t/5}) \end{aligned}$$

$$\begin{aligned} p_l(33, 10) &= \frac{1}{33}(1 - 2^{-10/5}) = \frac{1}{33}\left(\frac{3}{4}\right) \\ &\approx 0.0227 \text{ atm/ft} \end{aligned}$$

Increasing the depth of the dive by 1 foot (to 34 feet), while keeping the dive length at 10 minutes, increases the pressure by approximately 0.0227 atmospheres.

$$\begin{aligned} p_t(33, 10) &= \frac{33}{165}(\ln 2)(2^{-10/5}) \\ &= \frac{1}{20} \ln 2 \\ &\approx 0.0347 \text{ atm/min} \end{aligned}$$

Increasing the dive time by 1 minute (to 11 minutes), while keeping the depth of the dive to 33 feet increases the pressure by approximately 0.0347 atmospheres.

(c) Solve

$$p(66, t) = 2.15$$

$$1 + \frac{66}{33}(1 - 2^{-t/5}) = 2.15$$

$$2(1 - 2^{-t/5}) = 1.15$$

$$1 - 2^{-t/5} = 0.575$$

$$2^{-t/5} = 0.425$$

$$-\frac{t}{5} \ln 2 = \ln 0.425$$

$$t = \frac{5 \ln 0.425}{\ln 2}$$

$$t \approx 6.172$$

The maximum dive length is 6.172 minutes.

$$61. \quad W(V, T)$$

$$= 91.4 - \frac{(10.45 + 6.69\sqrt{V} - 0.447V)(91.4 - T)}{22}$$

(a)

$$W(20, 10)$$

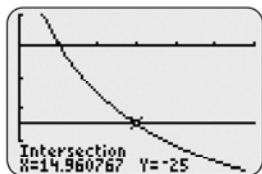
$$\begin{aligned} &= 91.4 - \frac{(10.45 + 6.69\sqrt{20} - 0.447(20))(91.4 - 10)}{22} \\ &\approx -24.9 \end{aligned}$$

The wind chill is -24.9°F when the wind speed is 20 mph and the temperature is 10°F .

(b) Solve

$$-25 = 91.4 - \frac{(10.45 + 6.69\sqrt{V} - 0.447V)(91.4 - 5)}{22}$$

for V .



The wind speed is approximately 15 mph.

$$\begin{aligned} \text{(c)} \quad W_V &= -\frac{1}{22} \left(\frac{6.69}{2\sqrt{V}} - 0.447 \right) (91.4 - T) \\ W_T &= -\frac{1}{22} (10.45 + 6.69\sqrt{V} - 0.447V)(-1) \\ &= \frac{1}{22} (10.45 + 6.69\sqrt{V} - 0.447V) \\ W_V(20, 10) &= \frac{1}{22} \left(\frac{6.69}{2\sqrt{20}} - 0.447 \right) (91.4 - 10) \\ &\approx -1.114 \end{aligned}$$

When the temperature is held fixed at 10°F , the wind chill decreases approximately 1.1 degrees when the wind velocity increases by 1 mph.

$$\begin{aligned} W_T(20, 10) &= \\ \frac{1}{22} [10.45 + 6.69\sqrt{20} - 0.447(20)] &\approx 1.429 \end{aligned}$$

When the wind velocity is held fixed at 20 mph, the wind chill increases approximately 1.429°F when the temperature increases from 10°F to 11°F .

(d) A sample table is

$T \backslash V$	5	10	15	20
30	27	16	9	4
20	16	3	-5	-11
10	6	-9	-18	-25
0	-5	-21	-32	-39

62. (a) $f(90, 30) \approx 90$

(b) $f(90, 75) \approx 109$

(c) $f(80, 75) \approx 86$

(d)
$$\begin{aligned} f_T(90, 30) &\approx \frac{f(95, 30) - f(90, 30)}{5} \\ &\approx \frac{95 - 90}{5} = 1 \end{aligned}$$

(e)
$$\begin{aligned} f_H(90, 30) &\approx \frac{f(90, 35) - f(90, 30)}{5} \\ &\approx \frac{92 - 90}{5} = 0.4 \end{aligned}$$

(f)
$$\begin{aligned} f_T(90, 75) &\approx \frac{f(95, 75) - f(90, 75)}{5} \\ &\approx \frac{130 - 109}{5} = 4.2 \end{aligned}$$

(g)
$$\begin{aligned} f_H(90, 75) &\approx \frac{f(90, 80) - f(90, 75)}{5} \\ &\approx \frac{114 - 109}{5} = 1 \end{aligned}$$

63. The rate of change in lung capacity with respect to age can be found by comparing the change in two lung capacity measurements to the difference in the respective ages when the height is held constant. So for a woman 58 inches tall, at age 20 the measured lung capacity is 1900 ml, and at age 25 the measured lung capacity is 1850 ml. So the rate of change in lung capacity with respect to age is

$$\begin{aligned} \frac{1900 - 1850}{20 - 25} &= \frac{50}{-5} \\ &= -10 \text{ ml per year.} \end{aligned}$$

The rate of change in lung capacity with respect to height can be found by comparing the change in two lung capacity measurements to the difference in the respective heights when the age is held constant. So for a 20-year old woman the measured lung capacity for a woman 58 inches tall is 1900 ml and the measured lung capacity for a woman 60 inches tall is 2100 ml. So the rate of change in lung capacity with respect to height is

$$\begin{aligned} \frac{1900 - 2100}{58 - 60} &= \frac{-200}{-2} \\ &= 100 \text{ ml per in.} \end{aligned}$$

The two rates of change remain constant throughout the table.

$$64. \quad p = f(s, n, a) = 0.05a + 6(sn)^{1/2}$$

$$\begin{aligned} \text{(a)} \quad p &= f(6, 4, 460) \\ &= 0.05(460) + 6[(6)(4)]^{1/2} \\ &= 23 + 6(24)^{1/2} \\ &\approx 52.39387691 \end{aligned}$$

The student's probability of passing is 52.39%.

$$\begin{aligned} \text{(b)} \quad p &= f(4, 5, 300) \\ &= 0.05(300) + 6[(4)(5)]^{1/2} \\ &= 15 + 6(20)^{1/2} \\ &\approx 41.83281573 \end{aligned}$$

The student's probability of passing is 41.83%.

$$\begin{aligned} \text{(c)} \quad f_n(s, n, a) &= 6\left(\frac{1}{2}\right)(sn)^{-1/2}(s) \\ &= 3s(sn)^{-1/2} \\ f_n(4, 5, 480) &= 3(4)[(4)(5)]^{-1/2} \\ &= 12(20)^{-1/2} \\ &\approx 2.683281573 \end{aligned}$$

$f_n(4, 5, 480) \approx 2.683\%$ is the rate at which the probability of success is changing when the number of semesters of mathematics passed in high school change by one unit (from 5 to 6) and the other quantities remain constant.

$$f_a(s, n, a) = 0.05$$

$f_a(4, 5, 480) = 0.05\%$ is the rate at which the probability of success is changing when the student's SAT score changes by one unit (from 480 to 481) and the other quantities remain constant.

$$65. \quad F = \frac{mgR^2}{r^2} = mgR^2r^{-2}$$

$$\text{(a)} \quad F_m = \frac{gR^2}{r^2} \text{ is the approximate rate of change in gravitational force per unit change in mass while distance is held constant.}$$

$F_r = \frac{-2mgR^2}{r^3}$ is the approximate rate of change in gravitational force per unit change in distance while mass is held constant.

$$\text{(b)} \quad F_m = \frac{gR^2}{r^2}, \text{ where all quantities are positive. Therefore, } F_m > 0.$$

$F_r = \frac{-2mgR^2}{r^3}$, where m , g , R^2 , and r^3 are positive.

Therefore, $F_r < 0$.

These results are reasonable since gravitational force increases when mass increases (m is in the numerator) and gravitational force decreases when distance increases (r is in the denominator).

$$66. \quad w(x, y) = \frac{x + y}{1 + \frac{xy}{c^2}}$$

$$\begin{aligned} \text{(a)} \quad w(50,000, 150,000) &= \frac{50,000 + 150,000}{1 + \frac{(50,000)(150,000)}{(186,282)^2}} \\ &\approx 164,456 \end{aligned}$$

The rocket is traveling 164,500 m/sec relative to the stationary observer.

$$\begin{aligned} \text{(b)} \quad w_x &= \frac{\left(1 + \frac{xy}{c^2}\right) - \frac{y}{c^2}(x + y)}{\left(1 + \frac{xy}{c^2}\right)^2} \\ &= \frac{1 + \frac{xy}{c^2} - \frac{xy}{c^2} - \frac{y^2}{c^2}}{\left(1 + \frac{xy}{c^2}\right)^2} \\ &= \frac{1 - \left(\frac{y}{c}\right)^2}{\left(1 + \frac{xy}{c^2}\right)^2} \end{aligned}$$

$$W_x(50,000, 150,000)$$

$$\begin{aligned} &= \frac{1 - \left(\frac{150,000}{186,282}\right)^2}{\left(1 + \frac{(50,000)(150,000)}{(186,282)^2}\right)^2} \\ &\approx 0.2377 \end{aligned}$$

The instantaneous rate of change is 0.238 m/sec per m/sec.

$$\text{(c)} \quad w(c, c) = \frac{c + c}{1 + \frac{(c)(c)}{c^2}} = \frac{2c}{2} = c$$

The speed is the speed of light, c .

$$67. \quad T = (s, w) = 105 + 265 \log_2 \left(\frac{2s}{w} \right)$$

$$\begin{aligned} \text{(a)} \quad T(3, 0.5) &= 105 + 265 \log_2 \left[\frac{2(3)}{0.5} \right] \\ &= 105 + 265 \log_2 12 \\ &\approx 1055 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad T(s, w) &= 105 + 265 \frac{\ln \left(\frac{2s}{w} \right)}{\ln 2} \\ &= 105 + \frac{265}{\ln 2} [\ln(2s) - \ln(w)] \end{aligned}$$

$$T_s(s, w) = \frac{265}{\ln 2} \left(\frac{1}{s} \right)$$

$$T_w(s, w) = -\frac{265}{\ln 2} \left(\frac{1}{w} \right)$$

$$T_s(3, 0.5) = \frac{265}{3 \ln 2} \approx 127.4 \text{ msec/ft}$$

If the distance the object is being moved increases from 3 feet to 4 feet, while keeping w fixed at 0.5 foot, the time to move the object increases by approximately 127.4 msec.

$$\begin{aligned} T_w(3, 0.5) &= -\frac{265}{0.5 \ln 2} \\ &\approx -764.5 \text{ msec/ft} \end{aligned}$$

If the width of the target area is increased by 1 foot, while keeping the distance fixed at 3 feet, the movement time decreases by approximately 764.5 msec.

9.3 Maxima and Minima

Your Turn 1

$$f(x, y) = 4x^3 + 3xy + 4y^3$$

$$f_x(x, y) = 12x^2 + 3y, \quad f_y(x, y) = 3x + 12y^2$$

$$12x^2 + 3y = 0$$

$$3x + 12y^2 = 0$$

Solve this system by substitution. From the first equation,

$$y = -4x^2.$$

Substituting for y in the second equation gives

$$3x + 12(-4x^2)^2 = 0.$$

$$x + 64x^4 = 0$$

$$x(1 + 64x^3) = 0$$

$$x = 0 \quad \text{or} \quad 1 + 64x^3 = 0$$

$$x = 0 \quad \text{or} \quad x = -\frac{1}{4}$$

$$\text{If } x = 0, y = -4(0)^2 = 0.$$

$$\text{If } x = -\frac{1}{4}, y = -4\left(-\frac{1}{4}\right)^2 = -\frac{1}{4}.$$

So the critical points are

$$(0, 0) \quad \text{and} \quad \left(-\frac{1}{4}, -\frac{1}{4}\right).$$

Your Turn 2

Use the information from Your Turn 1:

$$f(x, y) = 4x^3 + 3xy + 4y^3$$

$$f_x(x, y) = 12x^2 + 3y$$

$$f_y(x, y) = 3x + 12y^2$$

The critical points are $(0, 0)$ and $(-1/4, -1/4)$.

Now compute the second partial derivatives.

$$f_{xx}(x, y) = 24x, \quad f_{yy}(x, y) = 24y, \quad f_{xy}(x, y) = 3$$

$$\begin{aligned} D(a, b) &= f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 \\ &= (24a)(24b) - 9 \\ &= 576ab - 9 \end{aligned}$$

At the critical point $(0, 0)$,

$D(0, 0) = 576(0)(0) - 9 = -9 < 0$, so this critical point is a saddle point.

At the critical point $(-1/4, -1/4)$,

$$\begin{aligned} D\left(-\frac{1}{4}, -\frac{1}{4}\right) &= 576\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) - 9 \\ &= 36 - 9 \\ &= 24 \end{aligned}$$

and

$$f_{xx}\left(-\frac{1}{4}, -\frac{1}{4}\right) = 24\left(-\frac{1}{4}\right) = -\frac{1}{6}.$$

Since $D > 0$ and $f_{xx} < 0$, there is a relative maximum at $(-1/4, -1/4)$.

9.3 Exercises

$$1. \quad f(x, y) = xy + y - 2x$$

$$f_x(x, y) = y - 2, f_y(x, y) = x + 1$$

$$\text{If } f_x(x, y) = 0, y = 2.$$

$$\text{If } f_y(x, y) = 0, x = -1.$$

Therefore, $(-1, 2)$ is the critical point.

$$f_{xx}(x, y) = 0$$

$$f_{yy}(x, y) = 0$$

$$f_{xy}(x, y) = 1$$

For $(-1, 2)$,

$$D = 0 \cdot 0 - 1^2 = -1 < 0.$$

A saddle point is at $(-1, 2)$.

$$2. \quad f(x, y) = 3xy + 6y - 5x$$

$$f_x(x, y) = 3y - 5$$

$$f_y(x, y) = 3x + 6$$

$$\text{If } f_x(x, y) = 0, y = \frac{5}{3}.$$

$$\text{If } f_y(x, y) = 0, x = -2.$$

Therefore, $(-2, \frac{5}{3})$ is a critical point.

$$f_{xx}(x, y) = 0$$

$$f_{yy}(x, y) = 0$$

$$f_{xy}(x, y) = 3$$

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

$$= 0 \cdot 0 - 3^2 = -9 < 0$$

There is a saddle point at $(-2, \frac{5}{3})$.

$$3. \quad f(x, y) = 3x^2 - 4xy + 2y^2 + 6x - 10$$

$$f_x(x, y) = 6x - 4y + 6$$

$$f_y(x, y) = -4x + 4y$$

Solve the system $f_x(x, y) = 0, f_y(x, y) = 0$.

$$6x - 4y + 6 = 0$$

$$\frac{-4x + 4y}{2x} = 0$$

$$2x + 6 = 0$$

$$x = -3$$

$$-4(-3) + 4y = 0$$

$$y = -3$$

Therefore, $(-3, -3)$ is a critical point.

$$f_{xx}(x, y) = 6$$

$$f_{yy}(x, y) = 4$$

$$f_{xy}(x, y) = -4$$

$$D = 6 \cdot 4 - (-4)^2 = 8 > 0$$

Since $f_{xx}(x, y) = 6 > 0$, there is a relative minimum at $(-3, -3)$.

$$4. \quad f(x, y) = x^2 + xy + y^2 - 6x - 3$$

$$f_x(x, y) = 2x + y - 6, f_y(x, y) = x + 2y$$

$$2x + y - 6 = 0$$

$$\frac{x + 2y}{2x + y - 6} = 0$$

$$2x + y - 6 = 0$$

$$\frac{-2x - 4y}{2x + y - 6} = 0$$

$$-3y - 6 = 0$$

$$y = -2$$

$$x + 2(-2) = 0$$

$$x = 4$$

$$f_{xx}(x, y) = 2, f_{yy}(x, y) = 2, f_{xy}(x, y) = 1$$

$$D = 2 \cdot 2 - 1^2 = 3 > 0 \text{ and } f_{xx}(x, y) > 0$$

Relative minimum at $(4, -2)$

$$5. \quad f(x, y) = x^2 - xy + y^2 + 2x + 2y + 6$$

$$f_x(x, y) = 2x - y + 2,$$

$$f_y(x, y) = -x + 2y + 2$$

Solve the system $f_x(x, y) = 0, f_y(x, y) = 0$.

$$2x - y + 2 = 0$$

$$\frac{-x + 2y + 2}{2x - y + 2} = 0$$

$$2x - y + 2 = 0$$

$$\frac{-2x + 4y + 4}{2x - y + 2} = 0$$

$$3y + 6 = 0$$

$$y = -2$$

$$-x + 2(-2) + 2 = 0$$

$$x = -2$$

$(-2, -2)$ is the critical point.

$$f_{xx}(x, y) = 2$$

$$f_{yy}(x, y) = 2$$

$$f_{xy}(x, y) = -1$$

For $(-2, -2)$,

$$D = (2)(2) - (-1)^2 = 3 > 0.$$

Since $f_{xx}(x, y) > 0$, a relative minimum is at $(-2, -2)$.

$$\begin{aligned} 6. \quad f(x, y) &= 2x^2 + 3xy + 2y^2 - 5x + 5y \\ f_x(x, y) &= 4x + 3y - 5 \\ f_y(x, y) &= 3x + 4y + 5 \end{aligned}$$

Solve the system $f_x(x, y) = 0$, $f_y(x, y) = 0$.

$$\begin{array}{r} 4x + 3y - 5 = 0 \\ 3x + 4y + 5 = 0 \\ -12x - 9y + 15 = 0 \\ 12x + 16y + 20 = 0 \\ \hline 7y + 35 = 0 \\ y = -5 \\ 4x + 3(-5) - 5 = 0 \\ 4x = 20 \\ x = 5 \end{array}$$

Therefore, $(5, -5)$ is a critical point.

$$\begin{aligned} f_{xx}(x, y) &= 4 \\ f_{yy}(x, y) &= 4 \\ f_{xy}(x, y) &= 3 \end{aligned}$$

$$D = 4 \cdot 4 - 3^2 = 7 > 0$$

Since $f_{xx}(x, y) = 4 > 0$, there is a relative minimum at $(5, -5)$.

$$\begin{aligned} 7. \quad f(x, y) &= x^2 + 3xy + 3y^2 - 6x + 3y \\ f_x(x, y) &= 2x + 3y - 6, \\ f_y(x, y) &= 3x + 6y + 3 \end{aligned}$$

Solve the system $f_x(x, y) = 0$, $f_y(x, y) = 0$.

$$\begin{array}{r} 2x + 3y - 6 = 0 \\ 3x + 6y + 3 = 0 \\ -4x - 6y + 12 = 0 \\ 3x + 6y + 3 = 0 \\ \hline -x + 15 = 0 \\ x = 15 \end{array}$$

$$\begin{aligned} 3(15) + 6y + 3 &= 0 \\ 6y &= -48 \\ y &= -8 \end{aligned}$$

$(15, -8)$ is the critical point.

$$\begin{aligned} f_{xx}(x, y) &= 2 \\ f_{yy}(x, y) &= 6 \\ f_{xy}(x, y) &= 3 \end{aligned}$$

For $(15, -8)$,

$$D = 2 \cdot 6 - 9 = 3 > 0.$$

Since $f_{xx}(x, y) > 0$, a relative minimum is at $(15, -8)$.

$$\begin{aligned} 8. \quad f(x, y) &= 5xy - 7x^2 - y^2 + 3x - 6y - 4 \\ f_x(x, y) &= 5y - 14x + 3 \\ f_y(x, y) &= 5x - 2y - 6 \end{aligned}$$

$$\begin{array}{r} 5y - 14x + 3 = 0 \\ -2y + 5x - 6 = 0 \\ 10y - 28x + 6 = 0 \\ -10y + 25x - 30 = 0 \\ \hline -3x - 24 = 0 \\ x = -8 \\ -2y + 5(-8) - 6 = 0 \\ -2y = 46 \\ y = -23 \end{array}$$

$$\begin{aligned} f_{xx}(x, y) &= -14, f_{yy}(x, y) = -2, \\ f_{xy}(x, y) &= 5 \end{aligned}$$

$$\begin{aligned} D &= (-14)(-2) - 5^2 \\ &= 3 > 0 \text{ and } f_{xx}(x, y) < 0 \end{aligned}$$

Relative maximum at $(-8, -23)$

$$\begin{aligned} 9. \quad f(x, y) &= 4xy - 10x^2 \\ &\quad - 4y^2 + 8x + 8y + 9 \\ f_x(x, y) &= 4y - 20x + 8 \\ f_y(x, y) &= 4x - 8y + 8 \end{aligned}$$

$$\begin{array}{r} 4y - 20x + 8 = 0 \\ 4x - 8y + 8 = 0 \\ 4y - 20x + 8 = 0 \\ -4y + 2x + 4 = 0 \\ \hline -18x + 12 = 0 \\ x = \frac{2}{3} \end{array}$$

$$4y - 20\left(\frac{2}{3}\right) + 8 = 0$$

The critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$.

$$f_{xx}(x, y) = -20$$

$$f_{yy}(x, y) = -8$$

$$f_{xy}(x, y) = 4$$

For $\left(\frac{2}{3}, \frac{4}{3}\right)$,

$$D = (-20)(-8) - 16 = 144 > 0.$$

Since $f_{xx}(x, y) < 0$, a relative maximum is at $\left(\frac{2}{3}, \frac{4}{3}\right)$.

$$10. \quad f(x, y) = 4y^2 + 2xy + 6x + 4y - 8$$

$$f_x(x, y) = 2y + 6$$

$$f_y(x, y) = 8y + 2x + 4$$

If $f_x(x, y) = 0$, $y = -3$. Substitute -3 for y in $f_y = (x, y) = 0$ and solve for x .

$$8(-3) + 2x + 4 = 0$$

$$2x = 20$$

$$x = 10$$

Therefore, $(10, -3)$ is a critical point.

$$f_{xx}(x, y) = 0$$

$$f_{yy}(x, y) = 8$$

$$f_{xy}(x, y) = 2$$

$$D = 0 \cdot 8 - 2^2 = -4 < 0$$

There is a saddle point at $(10, -3)$.

$$11. \quad f(x, y) = x^2 + xy - 2x - 2y + 2$$

$$f_x(x, y) = 2x + y - 2$$

$$f_y(x, y) = x - 2$$

$$2x + y - 2 = 0$$

$$x - 2 = 0$$

$$x = 2$$

$$2(2) + y - 2 = 0$$

$$y = -2$$

The critical point is $(2, -2)$.

$$f_{xx}(x, y) = 2$$

$$f_{yy}(x, y) = 0$$

$$f_{xy}(x, y) = 1$$

For $(2, -2)$,

$$D = 2 \cdot 0 - 1^2 = -1 < 0.$$

A saddle point is at $(2, -2)$.

$$12. \quad f(x, y) = x^2 + xy + y^2 - 3x - 5$$

$$f_x(x, y) = 2x + y - 3$$

$$f_y(x, y) = x + 2y$$

$$2x + y - 3 = 0$$

$$x + 2y = 0$$

$$2x + y - 3 = 0$$

$$\frac{-2x - 4y = 0}{-3y - 3 = 0}$$

$$y = -1$$

$$x + 2(-1) = 0$$

$$x = 2$$

$$f_{xx}(x, y) = 2, f_{yy}(x, y) = 2, f_{xy}(x, y) = 1$$

$$D = 2 \cdot 2 - 1^2 = 3 > 0 \text{ and } f_{xx}(x, y) > 0$$

Relative minimum at $(2, -1)$

$$13. \quad f(x, y) = 3x^2 + 2y^3 - 18xy + 42$$

$$f_x(x, y) = 6x - 18y$$

$$f_y(x, y) = 6y^2 - 18x$$

If $f_x(x, y) = 0$, $6x - 18y = 0$, or $x = 3y$.

Substitute $3y$ for x in $f_y(x, y) = 0$ and solve for y .

$$6y^2 - 18(3y) = 0$$

$$6y(y - 9) = 0$$

$$y = 0 \text{ or } y = 9$$

$$\text{Then } x = 0 \text{ or } x = 27.$$

Therefore, $(0, 0)$ and $(27, 9)$ are critical points.

$$f_{xx}(x, y) = 6$$

$$f_{yy}(x, y) = 12y$$

$$f_{xy}(x, y) = -18$$

For $(0, 0)$,

$$D = 6 \cdot 12(0) - (-18)^2 = -324 < 0.$$

There is a saddle point at $(0, 0)$.

For $(27, 9)$,

$$D = 6 \cdot 12(9) - (-18)^2 = 324 > 0.$$

Since $f_{xx}(x, y) = 6 > 0$, there is a relative minimum at $(27, 9)$.

14. $f(x, y) = 7x^3 + 3y^2 - 126xy - 63$

$$f_x(x, y) = 21x^2 - 126y$$

$$f_y(x, y) = 6y - 126x$$

If $f_y(x, y) = 0$, $6y - 126x = 0$, or $y = 21x$.

Substitute $21x$ for y in $f_x(x, y) = 0$ and solve for x .

$$21x^2 - 126(21x) = 0$$

$$21x(x - 126) = 0$$

$$x = 0 \quad \text{or} \quad x = 126$$

Then $y = 0$ or $y = 2646$.

Therefore, $(0, 0)$ and $(126, 2646)$ are critical points.

$$f_{xx}(x, y) = 42x$$

$$f_{yy}(x, y) = 6$$

$$f_{xy}(x, y) = -126$$

For $(0, 0)$,

$$D = 42(0) \cdot 6 - (-126)^2 = -15,876 < 0.$$

There is a saddle point at $(0, 0)$.

For $(126, 2646)$,

$$\begin{aligned} D &= 42(126) \cdot 2646 - (-126)^2 \\ &= 13,986,756 > 0. \end{aligned}$$

Since $f_{xx}(126, 2646) = 42(126) = 5292 > 0$, there is a relative minimum at $(126, 2646)$.

15. $f(x, y) = x^2 + 4y^3 - 6xy - 1$

$$f_x(x, y) = 2x - 6y, \quad f_y(x, y) = 12y^2 - 6x$$

Solve $f_x(x, y) = 0$ for x .

$$2x + 6 = 0$$

$$x = 3y$$

Substitute for x in $12y^2 - 6x = 0$.

$$12y^2 - 6(3y) = 0$$

$$6y(2y - 3) = 0$$

$$y = 0 \quad \text{or} \quad y = \frac{3}{2}$$

Then $x = 0$ or $x = \frac{9}{2}$.

The critical points are $(0, 0)$ and $(\frac{9}{2}, \frac{3}{2})$.

$$f_{xx}(x, y) = 2$$

$$f_{yy}(x, y) = 24y$$

$$f_{xy}(x, y) = -6$$

For $(0, 0)$,

$$\begin{aligned} D &= 2 \cdot 24(0) - (-6)^2 \\ &= -36 < 0. \end{aligned}$$

A saddle point is at $(0, 0)$.

For $(\frac{9}{2}, \frac{3}{2})$,

$$\begin{aligned} D &= 2 \cdot 24\left(\frac{3}{2}\right) - (-6)^2 \\ &= 36 > 0. \end{aligned}$$

Since $f_{xx}(x, y) > 0$, a relative minimum is at $(\frac{9}{2}, \frac{3}{2})$.

16. $f(x, y) = 3x^2 + 7y^3 - 42xy + 5$

$$f_x(x, y) = 6x - 42y$$

$$f_y(x, y) = 21y^2 - 42x$$

$$6x - 42x = 0 \quad (1)$$

$$21y^2 - 42x = 0 \quad (2)$$

Divide equation (1) by 6 and equation (2) by 21.

$$x - 7y = 0 \quad (3)$$

$$y^2 - 2x = 0 \quad (4)$$

Solve equation (3) for x .

$$x = 7y$$

Substituting in equation (4), we have

$$y^2 - 2(7y) = 0$$

$$y^2 - 14y = 0$$

$$y = 0 \quad \text{or} \quad y = 14$$

$$x = 0 \quad \text{or} \quad x = 98.$$

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 42y, \quad f_{xy}(x, y) = -42$$

At $(0, 0)$,

$$D = 6 \cdot 0 - (-42)^2 = -1764 < 0.$$

Saddle point at $(0, 0)$

At (98, 14),

$$D = 6 \cdot 588 - (-42)^2 \\ = 1764 > 0 \text{ and } f_{xx}(x, y) > 0.$$

Relative minimum at (98, 14)

$$17. \quad f(x, y) = e^{x(y+1)} \\ f_x(x, y) = (y+1)e^{x(y+1)} \\ f_y(x, y) = xe^{x(y+1)}$$

$$\text{If } f_x(x, y) = 0 \\ (y+1)e^{x(y+1)} = 0 \\ y+1 = 0 \\ y = -1.$$

$$\text{If } f_y(x, y) = 0 \\ xe^{x(y+1)} = 0 \\ x = 0.$$

Therefore, (0, -1) is a critical point.

$$f_{xx}(x, y) = (y+1)^2 e^{x(y+1)} \\ f_{yy}(x, y) = x^2 e^{x(y+1)} \\ f_{xy}(x, y) = (y+1)e^{x(y+1)} \cdot x + e^{x(y+1)} \cdot 1 \\ = (xy + x + 1)e^{x(y+1)}$$

For (0, -1),

$$f_{xx}(0, -1) = (0)^2 e^0 = 0 \\ f_{yy}(0, -1) = (0)^2 e^0 = 0 \\ f_{xy}(0, -1) = (0 + 0 + 1)e^0 = 1 \\ D = 0 \cdot 0 - 1^2 = -1 < 0$$

There is a saddle point at (0, -1).

$$18. \quad f(x, y) = y^2 + 2e^x \\ f_x(x, y) = 2e^x \\ f_y(x, y) = 2y$$

If $f_y(x, y) = 0$, $y = 0$. But $f_x(x, y) = 2e^x = 0$ has no solution. There are no extrema and no saddle points.

$$21. \quad z = -3xy + x^3 - y^3 + \frac{1}{8} \\ f_x(x, y) = -3y + 3x^2, f_y(x, y) = -3x - 3y^2$$

Solve the system $f_x = 0$, $f_y = 0$.

$$-3y + 3x^2 = 0 \\ -3x - 3y^2 = 0 \\ -y + x^2 = 0 \\ -x - y^2 = 0$$

Solve the first equation for y , substitute into the second, and solve for x .

$$y = x^2 \\ -x - x^4 = 0 \\ x(1 + x^3) = 0 \\ x = 0 \text{ or } x = -1 \\ \text{Then } y = 0 \text{ or } y = 1.$$

The critical points are (0, 0) and (-1, 1).

$$f_{xx}(x, y) = 6x \\ f_{yy}(x, y) = -6y \\ f_{xy}(x, y) = -3$$

For (0, 0),

$$D = 0 \cdot 0 - (-3)^2 = -9 < 0.$$

A saddle point is at (0, 0).

For (-1, 1),

$$D = -6(-6) - (-3)^2 = 27 > 0. \\ f_{xx}(x, y) = 6(-1) = -6 < 0. \\ f(-1, 1) = -3(-1)(1) + (-1)^3 - 1^3 + \frac{1}{8} \\ = \frac{9}{8}$$

A relative maximum of $\frac{9}{8}$ is at (-1, 1).

The equation matches graph (a).

$$22. \quad z = \frac{3}{2}y - \frac{1}{2}y^3 - x^2y + \frac{1}{16} \\ \frac{\partial z}{\partial x} = -2xy, \quad \frac{\partial z}{\partial y} = \frac{3}{2} - \frac{3}{2}y^2 - x^2 \\ -2xy = 0 \\ x = 0 \text{ or } y = 0 \\ \frac{3}{2} - \frac{3}{2}y^2 - x^2 = 0$$

If $x = 0$,

$$\frac{3}{2} - \frac{3}{2}y^2 = 0 \\ y = \pm 1.$$

If $y = 0$,

$$\frac{3}{2} - x^2 = 0$$

$$x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}.$$

$$\frac{\partial^2 z}{\partial x^2} = -2y, \frac{\partial^2 z}{\partial y^2} = -3y, \frac{\partial^2 z}{\partial y \partial x} = -2x$$

$$D = (-2y)(-3y) - (-2x)^2 = 6y^2 - 4x^2$$

At $(0, 1)$,

$$D = 6 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = -2 < 0.$$

$$z = \frac{3}{2} - \frac{1}{2} + \frac{1}{16} = 1\frac{1}{16}$$

Relative maximum of $1\frac{1}{16}$ at $(0, 1)$

At $(0, -1)$,

$$D = 6 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = 2 > 0.$$

$$z = -\frac{3}{2} + \frac{1}{2} + \frac{1}{16} = -\frac{15}{16}$$

Relative minimum of $-\frac{15}{16}$ at $(0, -1)$

At $\left(\pm\frac{\sqrt{6}}{2}, 0\right)$, $D = -6 < 0$.

Saddle points at $\left(\frac{\sqrt{6}}{2}, 0\right)$ and $\left(-\frac{\sqrt{6}}{2}, 0\right)$

The correct graph is (d).

23. $z = y^4 - 2y^2 + x^2 - \frac{17}{16}$

$$f_x(x, y) = 2x, f_y(x, y) = 4y^3 - 4y$$

Solve the system $f_x = 0, f_y = 0$.

$$2x = 0 \quad (1)$$

$$4y^3 - 4y = 0 \quad (2)$$

$$4y(y^2 - 1) = 0$$

$$4y(y + 1)(y - 1) = 0$$

Equation (1) gives $x = 0$ and equation (2) gives $y = 0, y = -1$, or $y = 1$.

The critical points are $(0, 0)$, $(0, -1)$ and $(0, 1)$.

$$f_{xx}(x, y) = 2,$$

$$f_{yy}(x, y) = 12y^2 - 4,$$

$$f_{xy}(x, y) = 0$$

For $(0, 0)$,

$$D = 2(12 \cdot 0^2 - 4) - 0 = -8 < 0.$$

A saddle point is at $(0, 0)$.

For $(0, -1)$,

$$D = 2[12(-1)^2 - 4] - 0 = 16 > 0.$$

$$f_{xx}(x, y) = 2 > 0$$

$$\begin{aligned} f(0, -1) &= (-1)^4 - 2(-1)^2 + 0^2 - \frac{17}{16} \\ &= -2\frac{1}{16} \end{aligned}$$

A relative minimum of $-2\frac{1}{16}$ is at $(0, -1)$.

For $(0, 1)$,

$$D = 2(12 \cdot 1^2 - 4) - 0 = 16 > 0$$

$$f_{xx}(x, y) = 2 > 0$$

$$\begin{aligned} f(0, 1) &= 1^4 - 2 \cdot 1^2 + 0^2 - \frac{17}{16} \\ &= -\frac{33}{16} \end{aligned}$$

A relative minimum of $-\frac{33}{16}$ is at $(0, 1)$.

The equation matches graph (b).

24. $z = -2x^3 - 3y^4 + 6xy^2 + \frac{1}{16}$

$$\frac{\partial z}{\partial x} = -6x^2 + 6y^2, \frac{\partial z}{\partial y} = -12y^3 + 12xy$$

$$-6x^2 + 6y^2 = 0 \quad (1)$$

$$-12y^3 + 12xy = 0 \quad (2)$$

Divide equation (1) by 6 and equation (2) by 12.

$$-x^2 + y^2 = 0 \quad (3)$$

$$-y^3 + xy = 0 \quad (4)$$

Solve equation (3) for y . Substitute $y = \pm x$ in equation (4).

If $y = x$,

$$-x^3 + x^2 = 0$$

$$-x^2(-x + 1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1.$$

If $x = 0, y = 0$.

If $x = 1, y = 1$.

If $y = -x$,

$$x^3 - x^2 = 0$$

$$x^2(x - 1) = 0$$

$$x = 0 \text{ or } x = 1.$$

If $x = 0, y = 0$

If $x = 1, y = -1$.

$$\frac{\partial^2 z}{\partial x^2} = -12x, \quad \frac{\partial^2 z}{\partial y^2} = -36y^2 + 12x,$$

$$\frac{\partial^2 z}{\partial y \partial x} = 12y$$

$$D = (-12x)(-36y^2 + 12x) - (12y^2)$$

At $(0, 0)$, $D = 0$, which gives no information.

At $(1, 1)$,

$$D = -12(-36 + 12) - 144$$

$$= 144 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = -12 < 0.$$

$$z = -2 - 3 + 6\frac{1}{16} = 1\frac{1}{16}$$

so there is a relative maximum of $1\frac{1}{16}$ at $(1, 1)$.

At $(1, -1)$,

$$D = -12(-36 + 12) - 144$$

$$= 144 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = -12 < 0,$$

so there is a relative maximum of $1\frac{1}{16}$ at $(1, -1)$.

The correct graph is (c).

25. $z = -x^4 + y^4 + 2x^2 - 2y^2 + \frac{1}{16}$

$$f_x(x, y) = -4x^3 + 4x, \quad f_y(x, y) = 4y^3 - 4y$$

Solve $f_x(x, y) = 0, f_y(x, y) = 0$.

$$-4x^3 + 4x = 0 \quad (1)$$

$$4y^3 - 4y = 0 \quad (2)$$

$$-4x(x^2 - 1) = 0 \quad (1)$$

$$-4x(x + 1)(x - 1) = 0$$

$$4y(y^2 - 1) = 0 \quad (2)$$

$$4y(y + 1)(y - 1) = 0$$

Equation (1) gives $x = 0, -1$, or 1 .

Equation (2) gives $y = 0, -1$, or 1 .

Critical points are $(0, 0), (0, -1), (0, 1), (-1, 0), (-1, -1), (-1, 1), (1, 0), (1, -1), (1, 1)$.

$$f_{xx}(x, y) = -12x^2 + 4,$$

$$f_{yy}(x, y) = 12y^2 - 4$$

$$f_{xy}(x, y) = 0$$

For $(0, 0)$,

$$D = 4(-4) - 0 = -16 < 0.$$

For $(0, -1)$,

$$D = 4(8) - 0 = 32 > 0,$$

and $f_{xx}(x, y) = 4 > 0$.

$$f(0, -1) = -\frac{15}{16}$$

For $(0, 1)$,

$$D = 4(8) - 0 = 32 > 0,$$

and $f_{xx}(x, y) = 4 > 0$.

$$f(0, 1) = -\frac{15}{16}$$

For $(-1, 0)$,

$$D = -8(-4) - 0 = 32 > 0,$$

and $f_{xx}(x, y) = -8 < 0$.

$$f(-1, 0) = \frac{17}{16}$$

For $(-1, -1)$,

$$D = -8(8) - 0 = -64 < 0.$$

For $(-1, 1)$,

$$D = -8(8) - 0 = -64 < 0.$$

For $(1, 0)$,

$$D = -8(-4) = 32 > 0,$$

and $f_{xx}(x, y) = -8 < 0$.

$$f(1, 0) = 1\frac{1}{16}$$

For $(1, -1)$,

$$D = -8(8) - 0 = -64 < 0.$$

For $(1, 1)$,

$$D = -8(8) - 0 = -64 < 0.$$

Saddle points are at $(0, 0)$, $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$.

Relative maximum of $\frac{17}{16}$ is at $(-1, 0)$ and $(1, 0)$.

Relative minimum of $-\frac{15}{16}$ is at $(0, -1)$ and $(0, 1)$.

The equation matches graph (e).

26. $z = -y^4 + 4xy - 2x^2 + \frac{1}{16}$

$$\frac{\partial z}{\partial x} = 4y - 4x, \quad \frac{\partial z}{\partial y} = -4y^3 + 4x$$

$$4y - 4x = 0 \quad (1)$$

$$-4y^3 + 4x = 0 \quad (2)$$

Divide equation (1) by 4 and equation (2) by -4 .

$$y - x = 0 \quad (3)$$

$$-y^3 + x = 0 \quad (4)$$

Solve equation (3) for y .

$$y = x$$

Substituting, we have

$$-x^3 + x = 0$$

$$x(-x^2 + 1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -1.$$

If $x = 0$, $y = 0$.

If $x = 1$, $y = 1$.

If $x = -1$, $y = -1$.

$$\frac{\partial^2 z}{\partial x^2} = -4, \quad \frac{\partial^2 z}{\partial y^2} = -12y^2, \quad \frac{\partial^2 z}{\partial y \partial x} = 4$$

$$D = -4(-12y^2) - 4^2 = 48y^2 - 16$$

At $(0, 0)$, $D = -16 < 0$.

Saddle point at $(0, 0)$

At $(1, 1)$ and $(-1, -1)$,

$$D = 48 - 16 = 32 > 0 \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} = -4 < 0.$$

Relative maximum of $1\frac{1}{16}$ at $(1, 1)$ and at $(-1, -1)$.

The correct graph is (f).

27. $f(x, y) = 1 - x^4 - y^4$

$$f_x(x, y) = -4x^3, \quad f_y(x, y) = -4y^3$$

The system

$$f_x(x, y) = -4x^3 = 0, \quad f_y(x, y) = -4y^3 = 0$$

gives the critical point $(0, 0)$.

$$f_{xx}(x, y) = -12x^2$$

$$f_{yy}(x, y) = -12y^3$$

$$f_{xy}(x, y) = 0$$

For $(0, 0)$,

$$D = 0 \cdot 0 - 0^2 = 0.$$

Therefore, the test gives no information. Examine a graph of the function drawn by using level curves.

If $f(x, y) = 1$, then $x^4 + y^4 = 0$. The level curve is the point $(0, 0, 1)$.

If $f(x, y) = 0$, then $x^4 + y^4 = 1$. The level curve is the circle with center $(0, 0, 0)$ and radius 1.

If $f(x, y) = -15$, then $x^4 + y^4 = 16$. The level curve is the curve with center $(0, 0, -15)$ and radius 2.

The xz -trace is

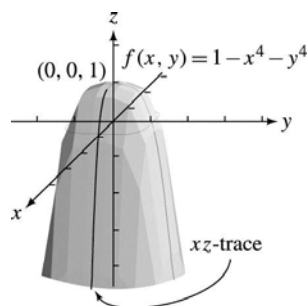
$$z = 1 - x^4.$$

This curve has a maximum at $(0, 0, 1)$ and opens downward.

The yz -trace is

$$z = 1 - y^4.$$

This curve also has a maximum at $(0, 0, 1)$ and opens downward.



If $f(x, y) > 1$, then $x^4 + y^4 < 0$, which is impossible, so the function does not exist. Thus, the function has a relative maximum of 1 at $(0, 0)$.

28. $f(x, y) = x^3 + (x - y)^2$

$$f_x(x, y) = 3x^2 + 2(x - y)$$

$$f_y(x, y) = -2(x - y)$$

$$3x^2 + 2x - 2y = 0$$

$$-2x + 2y = 0$$

$$3x^2 = 0$$

$$x = 0$$

$$0 + 2y = 0$$

$$y = 0$$

$$f_{xx}(x, y) = 6x + 2,$$

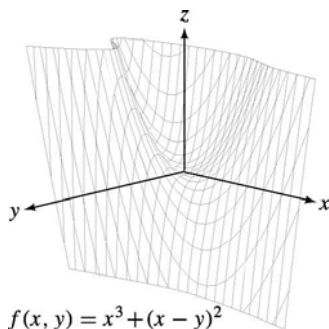
$$f_{yy}(x, y) = 2, f_{xy}(x, y) = -2$$

$$D = (6x + 2)(2) - (-2)^2 = 12x$$

At $(0, 0)$, $D = 0$, which gives no information.

Examine the graph of $z = x^3 + (x - y)^2$: in the yz -plane, the trace is $z = y^2$, which has a minimum at $(0, 0, 0)$; in the xz -plane, the trace is $z = x^3 + x^2$, which has a minimum at $(0, 0, 0)$.

But in the plane $y = x$, the trace is $z = x^3$, which has neither a maximum nor a minimum at $(0, 0, 0)$. So the function has no relative extrema. Notice the orientation of axes in the following figure: This is a back view.



30. $f(x, y) = y^2 - 2x^2y + 4x^3 + 20x^2$

$$f_x(x, y) = -4xy + 12x^2 + 40x$$

$$f_y(x, y) = 2y - 2x^2$$

$$f_{xx}(x, y) = -4y + 24x + 40$$

$$f_{yy}(x, y) = 2$$

$$f_{xy}(x, y) = -4x$$

For $(-2, 4)$,

$$f_{xx}(-2, 4) = -4(4) + 24(-2) + 40 = -24$$

$$f_{yy}(-2, 4) = 2$$

$$f_{xy}(-2, 4) = -4(-2) = 8$$

$$D = (-24) \cdot (2) - 8^2 = -112 < 0$$

There is a saddle point at $(-2, 4)$.

For $(0, 0)$,

$$f_{xx}(0, 0) = -4(0) + 24(0) + 40 = 40$$

$$f_{yy}(0, 0) = 2$$

$$f_{xy}(0, 0) = -4(0) = 0$$

$$D = (40) \cdot (2) - 0^2 = 80 > 0$$

Since $f_{xx}(0, 0) = 40 > 0$, there is a relative minimum at $(0, 0)$.

For $(5, 25)$,

$$f_{xx}(5, 25) = -4(25) + 24(5) + 40 = 60$$

$$f_{yy}(5, 25) = 2$$

$$f_{xy}(5, 25) = -4(5) = -20$$

$$D = (60) \cdot (2) - (-20)^2 = -280 < 0$$

There is a saddle point at $(5, 25)$.

The correct answer choice is (e).

31. $f(x, y) = x^2(y + 1)^2 + k(x + 1)^2y^2$

(a) $f_x(x, y) = 2x + 2ky^2(x + 1)$

$$f_y(x, y) = 2x^2(y + 1) + 2k(x + 1)^2y$$

$$\begin{aligned} f_x(0, 0) &= 2(0) + 2k(0)^2(0 + 1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_y(0, 0) &= 2(0)^2(0 + 1) + 2k(0 + 1)^2(0) \\ &= 0 \end{aligned}$$

Thus, $(0, 0)$ is a critical point for all values of k .

(b) $f_{xx}(x, y) = 2 + 2ky^2$

$$f_{yy}(x, y) = 2x^2 + 2k(x + 1)^2$$

$$f_{xy}(x, y) = 4ky(x + 1)$$

$$f_{xx}(0, 0) = 2 + 2k(0)^2 = 2$$

$$f_{yy}(0, 0) = 2(0)^2 = 2k(0 + 1)^2 = 2k$$

$$f_{xy}(0, 0) = 4k(0)(0 + 1) = 0$$

$$D = 2 \cdot 2k - 0^2 = 4k$$

$(0, 0)$ is a relative minimum when $4k > 0$, hence when $k > 0$. When $k = 0$, $D = 0$ so the test for relative extrema gives no information. But if $k = 0$, $f(xy) = x^2(y + 1)^2$, which is always greater than or equal to $f(0, 0) = 0$. So $(0, 0)$ is a relative minimum for $k \geq 0$.

$$\begin{aligned}
 32. \quad S(m, b) &= \sum (mx + b - y)^2 \\
 S_m(m, b) &= \sum 2(mx + b - y)^1(x) \\
 &= 2 \sum (mx^2 + bx - xy) \\
 S_b(m, b) &= \sum 2(mx + b - y)(1) \\
 &= 2 \sum (mx + b - y)
 \end{aligned}$$

If $S_m(m, b) = 0$, then

$$\begin{aligned}
 2 \sum (mx^2 + bx - xy) &= 0 \\
 \sum (mx^2) + \sum (bx) &= \sum (xy) \\
 \left(\sum x^2 \right) m + \sum (x)b &= \sum (xy). \quad (1)
 \end{aligned}$$

If $S_b(m, b) = 0$, then

$$\begin{aligned}
 2 \sum (mx + b - y) &= 0 \\
 \sum (mx) + \sum (b) - \sum y &= 0 \\
 \left(\sum x \right) m + nb &= \sum y. \quad (2)
 \end{aligned}$$

Equations (1) and (2) give us the following system of equations.

$$\begin{aligned}
 \left(\sum x \right) b + \left(\sum x^2 \right) m &= \sum xy \\
 nb + \left(\sum x \right) m &= \sum y
 \end{aligned}$$

$$\begin{aligned}
 34. \quad P(x, y) &= 1500 + 36x - 1.5x^2 + 120y - 2y^2 \\
 P_x(x, y) &= 36 - 3x \\
 P_y(x, y) &= 120 - 4y
 \end{aligned}$$

If $P_x = 0$, $x = 12$. If $P_y = 0$, $y = 30$.

Therefore, (12, 30) is a critical point.

$$P_{xx}(x, y) = -3$$

$$P_{yy}(x, y) = -4$$

$$P_{xy}(x, y) = 0$$

$$D = (-3) \cdot (-4) - 0^2 = 12 > 0$$

Since $P_{xx} = -3 < 0$, there is a relative maximum at (12, 30).

$$\begin{aligned}
 P(12, 30) &= 1500 + 36(12) - 1.5(12)^2 \\
 &\quad + 120(30) - 2(30)^2 \\
 &= 3516 \text{ (hundred dollars)}
 \end{aligned}$$

The maximum profit is \$351,600 when the cost of a unit of labor is 12 and the cost of a unit of goods is 30.

$$35. \quad L(x, y) = \frac{3}{2}x^2 + y^2 - 2x - 2y - 2xy + 68,$$

where x is the number of skilled hours and y is the number of semiskilled hours.

$$L_x(x, y) = 3x - 2 - 2y,$$

$$L_y(x, y) = 2y - 2 - 2x$$

$$3x - 2 - 2y = 0$$

$$\frac{-2x - 2 + 2y = 0}{x - 4} = 0$$

$$x = 4$$

$$-2(4) - 2 + 2y = 0$$

$$2y = 10$$

$$y = 5$$

Let $L_{xx}(x, y) = 3$, $L_{yy}(x, y) = 2$,

$$L_{xy}(x, y) = -2.$$

$$D = 3(2) - (-2)^2 = 2 > 0 \text{ and } L_{xx}(x, y) > 0.$$

Relative minimum at (4, 5) is

$$\begin{aligned}
 L(4, 5) &= \frac{3}{2}(4)^2 + (5)^2 - 2(4) - 2(5) \\
 &\quad - 2(4)(5) + 68 \\
 &= 59.
 \end{aligned}$$

So \$59 is a minimum cost, when $x = 4$ and $y = 5$.

$$\begin{aligned}
 36. \quad C(x, y) &= 2x^2 + 2y^2 - 3xy \\
 &\quad + 4x - 94y + 4200
 \end{aligned}$$

$$C_x(x, y) = 4x - 3y + 4$$

$$C_y(x, y) = 4y - 3x - 94$$

Solve the system $C_x(x, y) = 0, C_y(x, y) = 0$.

$$4x - 3y + 4 = 0$$

$$-3x + 4y - 94 = 0$$

$$12x - 9y + 12 = 0$$

$$\frac{-12x + 16y - 376 = 0}{7y - 364 = 0}$$

$$y = 52$$

$$4x - 3(52) + 4 = 0$$

$$4x = 152$$

$$x = 38$$

Therefore, (38, 52) is a critical point.

$$C_{xx} = 4$$

$$C_{yy} = 4$$

$$C_{xy} = -3$$

$$D = (4)(4) - (-3)^2 = 7 > 0$$

Since $C_{xx} = 4 > 0$, there is a relative minimum at (38, 52).

$$\begin{aligned} C(38, 52) &= 2(38)^2 + 2(52)^2 - 3(38)(52) \\ &\quad + 4(38) - 94(52) + 4200 \\ &= 1832 \end{aligned}$$

38 units of electrical tape and 52 units of packing tape should be produced to yield a minimum cost of \$1832.

$$\begin{aligned} 37. \quad R(x, y) &= 15 + 169x + 182y \\ &\quad - 5x^2 - 7y^2 - 7xy \end{aligned}$$

$$R_x(x, y) = 169 - 10x - 7y$$

$$R_y(x, y) = 182 - 14y - 7x$$

Solve the system $R_x = 0, R_y = 0$.

$$-10x - 7y + 169 = 0$$

$$-7x - 14y + 182 = 0$$

$$20x + 14y - 338 = 0$$

$$\frac{-7x - 14y + 182 = 0}{13x \quad -156 = 0}$$

$$x = 12$$

$$-10(12) - 7y + 169 = 0$$

$$-7y = -49$$

$$y = 7$$

(12, 7) is a critical point.

$$R_{xx} = -10$$

$$R_{yy} = -14$$

$$R_{xy} = -7$$

$$D = (-10)(-14) - (-7)^2 = 91 > 0$$

Since $R_{xx} = -10 < 0$, there is a relative maximum at (12, 7).

$$\begin{aligned} R(12, 7) &= 15 + 169(12) + 182(7) - 5(12)^2 \\ &\quad - 7(7)^2 - 7(12)(7) \\ &= 1666 \text{ (hundred dollars)} \end{aligned}$$

12 spas and 7 solar heaters should be sold to produce a maximum revenue of \$166,600.

$$38. \quad P(x, y) = 36xy - x^3 - 8y^3$$

$$P_x(x, y) = 36y - 3x^2$$

$$P_y(x, y) = 36x - 24y^2$$

$$P_x(x, y) = 0$$

$$36y - 3x^2 = 0$$

$$36y = 3x^2$$

$$y = \frac{1}{12}x^2$$

$$P_y(x, y) = 0$$

$$36x - 24y^2 = 0$$

$$36x = 24y^2$$

$$x = \frac{2}{3}y^2$$

Use substitution to solve the system of equations

$$y = \frac{1}{12}x^2$$

$$x = \frac{2}{3}y^2$$

$$y = \frac{1}{12} \left(\frac{2}{3}y^2 \right)^2$$

$$y = \frac{1}{12} \left(\frac{4}{9} \right) y^4$$

$$y = \frac{1}{27}y^4$$

$$\frac{1}{27}y^4 - y = 0$$

$$\left(\frac{1}{27}y^3 - 1 \right) y = 0$$

$$\frac{1}{27}y^3 - 1 = 0 \quad \text{or} \quad y = 0$$

$$\frac{1}{27}y^3 = 1 \quad \text{or} \quad y = 0$$

$$y^3 = 27 \quad \text{or} \quad y = 0$$

$$y = 3 \quad \text{or} \quad y = 0$$

$$\text{If } y = 3, x = \frac{2}{3}(3)^2 = 6.$$

$$\text{If } y = 0, x = \frac{2}{3}(0)^2 = 0.$$

The critical points are (6, 3) and (0, 0).

$$P_{xx}(x, y) = -6x$$

$$P_{yy}(x, y) = -48y$$

$$P_{xy}(x, y) = 36$$

$$P_{xx}(6, 3) = -36$$

$$P_{yy}(6, 3) = -144$$

$$P_{xy}(6, 3) = 36$$

$$D = (-36)(-144) - (36)^2 = 3888$$

Here $D > 0$ and $P_{xx} < 0$, so there is a relative maximum at $(6, 3)$.

$$P_{xx}(0, 0) = 0$$

$$P_{yy}(0, 0) = 0$$

$$P_{xy}(0, 0) = 36$$

$$D = 0 \cdot 0 - 36^2 = -1296$$

Since $D < 0$, there is a saddle point at $(0, 0)$.

$$\begin{aligned} P(6, 3) &= 36(6)(3) - (6)^3 - 8(3)^3 \\ &= 648 - 216 - 216 \\ &= 216 \end{aligned}$$

So 6 tons of steel and 3 tons of aluminum produce a maximum profit of \$216,000.

39. $T(x, y) = x^4 + 16y^4 - 32xy + 40$

$$T_x(x, y) = 4x^3 - 32y$$

$$T_y(x, y) = 64y^3 - 32x$$

$$T_x(x, y) = 0$$

$$4x^3 - 32y = 0$$

$$4x^3 = 32y$$

$$\frac{1}{8}x^3 = y$$

$$T_y(x, y) = 0$$

$$64y^3 - 32x = 0$$

$$64y^3 = 32x$$

$$2y^3 = x$$

Use substitution to solve the system of equations

$$\frac{1}{8}x^3 = y$$

$$2y^3 = x$$

$$y = \frac{1}{8}(2y^3)^3$$

$$y = \frac{1}{8}(8)y^9$$

$$y = y^9$$

$$y^9 - y = 0$$

$$y(y^8 - 1) = 0$$

$$y = 0 \quad \text{or} \quad y^8 - 1 = 0$$

$$y = 0 \quad \text{or} \quad y^8 = 1$$

$$y = 0 \quad \text{or} \quad y = 1$$

$$\text{If } y = 0, x = 2(0)^3 = 0.$$

$$\text{If } y = 1, x = 2(1)^3 = 2.$$

The critical points are $(0, 0)$ and $(2, 1)$.

$$T_{xx}(x, y) = 12x^2$$

$$T_{yy}(x, y) = 192y^2$$

$$T_{xy}(x, y) = -32$$

$$T_{xx}(0, 0) = 0$$

$$T_{yy}(0, 0) = 0$$

$$T_{xy}(0, 0) = -32$$

$$D = 0 \cdot 0 - (-32)^2 = -1024$$

Since $D < 0$, there is a saddle point at $(0, 0)$.

$$T_{xx}(2, 1) = 48$$

$$T_{yy}(2, 1) = 192$$

$$T_{xy}(2, 1) = -32$$

$$D = 48 \cdot 192 - (-32)^2 = 8192$$

Since $D > 0$ and $T_{xx} > 0$, there is a relative minimum at $(2, 1)$.

$$\begin{aligned} T(2, 1) &= 2^4 + 16(1)^4 - 32(2)(1) + 40 \\ &= 16 + 16 - 64 + 40 \\ &= 8 \end{aligned}$$

Spend \$2000 on quality control and \$1000 on consulting, for a minimum time of 8 hours.

40.
$$P(\alpha, r, s) = \alpha(3r^2(1-r) + r^3) + (1-\alpha)(3s^2(1-s) + s^3)$$

(a)
$$\begin{aligned} P(0.9, 0.5, 0.6) &= 0.9[3(0.5)^2(1-0.5) + (0.5)^3] \\ &\quad + (1-0.9)[3(0.6)^2(1-0.6) + (0.6)^3] \\ &= 0.5148 \end{aligned}$$

$$\begin{aligned}
 P(0.1, 0.8, 0.4) &= 0.1[3(0.8)^2(1 - 0.8) + (0.8)^3] \\
 &\quad + (1 - 0.1)[3(0.4)^2(1 - 0.4) + (0.4)^3] \\
 &= 0.4064
 \end{aligned}$$

The jury is less likely to make the correct decision in the second situation.

- (b) If $r = s = 1$ then $P(\alpha, 1, 1) = 1$, so the jury always makes a correct decision. These values do not depend on α , but in a real-life situation α is likely to influence r and s .
- (c) When P reaches a maximum, P_α, P_r , and P_s equal 0.

$$\begin{aligned}
 P_\alpha(\alpha, r, s) &= 3r^2(1 - r) + r^3 \\
 &\quad - (3s^2(1 - s) + s^3) \\
 &= 3r^2 - 2r^3 - (3s^2 - 2s^3) \\
 P_r(\alpha, r, s) &= \alpha(6r(1 - r) - 3r^2 + 3r^2) \\
 &= 6\alpha r(1 - r) \\
 P_s(\alpha, r, s) &= (1 - \alpha)(6s(1 - s) - 3s^2 + 3s^2) \\
 &= 6s(1 - \alpha)(1 - s) \\
 P_\alpha(\alpha, r, s) &= 0 \quad \text{when} \quad r = s.
 \end{aligned}$$

Since $P_r(\alpha, r, s) = 6\alpha r(1 - r)$, and $P_s(\alpha, r, s) = 6(1 - \alpha)s(1 - s)$, then P_α, P_r , and P_s are simultaneously 0 at the points $(\alpha, 1, 1)$ and $(\alpha, 0, 0)$. So $(\alpha, 1, 1)$ and $(\alpha, 0, 0)$ are critical points.

$$P(\alpha, 0, 0) = 0 \quad \text{while} \quad P(\alpha, 1, 1) = 1$$

Since $P(\alpha, r, s)$ represents a probability, $0 \leq P(\alpha, r, s) \leq 1$. Thus, $P(\alpha, 1, 1) = 1$ is a maximum value of the function.

41. Using the procedure suggested, take the natural logarithm of the number-of-transistors data.

Time in years since 1985,	Natural log of number of transistors in millionS		
t	w		
0	$\ln(0.275)$	$\approx$	-1.291
4	$\ln(1.2)$	$\approx$	0.182
8	$\ln(3.1)$	$\approx$	1.131
12	$\ln(7.5)$	$\approx$	2.015
14	$\ln(9.5)$	$\approx$	2.251
15	$\ln(42)$	$\approx$	3.738
20	$\ln(291)$	$\approx$	5.673
22	$\ln(820)$	$\approx$	6.709
24	$\ln(1900)$	$\approx$	7.550

- (a) A linear fit of the data in the table above gives

$$\begin{aligned}
 w &= r + st \\
 w &= -1.722 + 0.3653t
 \end{aligned}$$

where w is the natural logarithm of the number of transistors in millions and t is the number of years since 1985. To convert this back to an exponential fit for the original data, compute

$$\begin{aligned}
 a &= e^r = e^{-1.722} = 0.1787 \\
 \text{and} \\
 b &= e^s = e^{0.3652} = 1.441.
 \end{aligned}$$

Thus an exponential fit to the original data has the form $y = a \cdot b^t$ with $a = 0.1787$ and $b = 1.441$:

$$y = 0.1787(1.441)^t$$

- (b) Same as (a).
(c) Same as (a).

$$\begin{aligned}
 42. \quad E(t, T) &= 436.16 - 10.57t - 5.46T - 0.02t^2 \\
 &\quad + 0.02T^2 + 0.08tT
 \end{aligned}$$

- (a) $E(0, 0) = 436.16$.

The value of E before cooking is 436.16 kJ/mol.

- (b) $E(10, 180)$
 $= 436.16 - 10.57(10)$
 $\quad - 5.46(180) - 0.02(10)^2$
 $\quad + 0.02(180)^2 + 0.08(10)(180)$
 $= 137.66$

After cooking for 10 minutes at 180°C , the total change in color is 137.66 kJ/mol.

- (c) $E_t = -10.57 - 0.04t + 0.08T$
 $E_T = -5.46 + 0.04T + 0.08t$

Solve the system $E_t = 0, E_T = 0$.

$$\begin{aligned}
 -0.04t + 0.08T - 10.57 &= 0 \\
 0.08t + 0.04T - 5.46 &= 0 \\
 -0.04t + 0.08T - 10.57 &= 0 \\
 -0.16t - 0.08T + 10.92 &= 0 \\
 \hline
 -0.20t &\quad + 0.35 = 0 \\
 t &= 1.75
 \end{aligned}$$

$$\begin{aligned}-0.04(1.75) + 0.08T - 10.57 &= 0 \\ 0.08T - 10.64 &= 0 \\ T &= 133\end{aligned}$$

(1.75, 133) is a critical point.

$$\begin{aligned}E_{tt} &= -0.04 \\ E_{TT} &= 0.04 \\ E_{tT} &= 0.08 \\ D &= (-0.04)(0.04) - (0.08)^2 \\ &= -0.008 < 0\end{aligned}$$

(1.75, 133) is a saddle point.

$$\begin{aligned}x + 3y + 1 &= \lambda \\ 2x + 6y + 1 &= \lambda \\ x + 3y &= 2x + 6y \\ x + 3y &= 0 \\ x &= -3y\end{aligned}$$

Now substitute for x in the last equation.

$$\begin{aligned}-2(-3y) - 3y + 12 &= 0 \\ 6y - 3y &= -12 \\ 3y &= -12 \\ y &= -4\end{aligned}$$

Since $x = -3y$ we have $x = 12$ and $y = -4$. So a candidate for a minimum value of the function f will be $f(12, -4) = 12$.

To see if this is really a minimum we can evaluate f at a few nearby points that satisfy the constraint $2x + 3y = 12$. For example, let $x = 11$. Then $2(11) + 3y = 12$, so $y = -10/3$. $f(11, -10/3) = 13$, which is larger than our candidate. We conclude that the minimum value of f subject to the given constraint is 12.

9.4 Lagrange Multipliers

Your Turn 1

Find the minimum value of $f(x, y) = x^2 + 2x + 9y^2 + 3y + 6xy$ subject to the constraint $2x + 3y = 12$.

Step 1 Rewrite the constraint as $g(x, y) = 2x + 3y - 12$.

Step 2 Form the Lagrange function.

$$\begin{aligned}F(x, y, \lambda) &= f(x, y) - \lambda \cdot g(x, y) \\ &= x^2 + 2x + 9y^2 + 3y \\ &\quad + 6xy - \lambda(2x + 3y - 12) \\ &= x^2 + 9y^2 + 6xy - 2x\lambda \\ &\quad - 3y\lambda + 2x + 3y + 12\lambda\end{aligned}$$

Step 3 Find the first partial derivatives of F .

$$\begin{aligned}F_x(x, y, \lambda) &= 2x + 6y - 2\lambda + 2 \\ F_y(x, y, \lambda) &= 6x + 18y - 3\lambda + 3 \\ F_\lambda(x, y, \lambda) &= -2x - 3y + 12\end{aligned}$$

Step 4 Form the system of equations

$$\begin{aligned}F_x(x, y, \lambda) &= 0, F_y(x, y, \lambda) = 0, F_\lambda(x, y, \lambda) = 0. \\ 2x + 6y - 2\lambda + 2 &= 0 \\ 6x + 18y - 3\lambda + 3 &= 0 \\ -2x - 3y + 12 &= 0\end{aligned}$$

Step 5 Solve the system of equations from Step 4.

First solve the first two equations for λ and then set these results equal to each other (remove the common factors of 2 from the first equation and 3 from the second equation).

Your Turn 2

As in Example 3, let x , y , and z be the dimensions of the box. If the front and top are missing, the surface area is $2xy + xz + yz$ so the constraint is $2xy + xz + yz = 6$. As in Example 3, the volume is xyz . The problem is thus to maximize $f(x, y, z) = xyz$ subject to $2xy + xz + yz = 6$.

Step 1 Rewrite the constraint as

$$g(x, y, z) = 2xy + xz + yz - 6.$$

Step 2 Form the Lagrange function.

$$\begin{aligned}F(x, y, z, \lambda) &= f(x, y, z) - \lambda \cdot g(x, y, z) \\ &= xyz - \lambda(2xy + xz + yz - 6) \\ &= xyz - 2xy\lambda - xz\lambda - yz\lambda + 6\lambda\end{aligned}$$

Step 3 Find the first partial derivatives of F .

$$\begin{aligned}F_x(x, y, z, \lambda) &= yz - 2y\lambda - z\lambda \\ F_y(x, y, z, \lambda) &= xz - 2x\lambda - z\lambda \\ F_z(x, y, z, \lambda) &= xy - x\lambda - y\lambda \\ F_\lambda(x, y, z, \lambda) &= -2xy - xz - yz + 6\end{aligned}$$

Step 4 Form the system of equations

$$\begin{aligned}F_x(x, y, z, \lambda) &= 0, \\F_y(x, y, z, \lambda) &= 0, \\F_z(x, y, z, \lambda) &= 0, \\F_\lambda(x, y, z, \lambda) &= 0. \\yz - 2y\lambda - z\lambda &= 0 \\xz - 2x\lambda - z\lambda &= 0 \\xy - x\lambda - y\lambda &= 0 \\-2xy - xz - yz + 6 &= 0\end{aligned}$$

Step 5 Solve the system of equations from Step 4. First solve each of the first three equations for λ . This gives three expressions for λ .

$$\lambda = \frac{yz}{2y + z}, \quad \lambda = \frac{xz}{2x + z}, \quad \lambda = \frac{xy}{x + y}$$

Set the second and third expressions equal.

$$\begin{aligned}\frac{xz}{2x + z} &= \frac{xy}{x + y} \\x^2z + xyz &= 2x^2y + xyz \\x^2z &= x^2(2y)\end{aligned}$$

Since x is not zero, this shows that $z = 2y$.

Exactly the same calculation using the first and third expressions will show that $z = 2x$. Thus $x = y$. Now use the fourth equation:

$$\begin{aligned}-2xy - xz - yz + 6 &= 0 \\-2x^2 - x(2x) - x(2x) + 6 &= 0 \\6x^2 &= 6 \\x &= 1 \text{ (since } x \text{ is positive)}\end{aligned}$$

Now we know that $x = 1$, $y = 1$, and $z = 2$. The box should be 2 ft. wide and 1 ft. high and long.

9.4 Exercises

1. Maximize $f(x, y) = 4xy$,
subject to $x + y = 16$.

$$\begin{aligned}1. \quad g(x, y) &= x + y - 16 \\2. \quad F(x, y, \lambda) &= 4xy - \lambda(x + y - 16). \\3. \quad F_x(x, y, \lambda) &= 4y - \lambda \\F_y(x, y, \lambda) &= 4x - \lambda \\F_\lambda(x, y, \lambda) &= -(x + y - 16) \\4. \quad 4y - \lambda &= 0 \quad (1) \\4x - \lambda &= 0 \quad (2) \\x + y - 16 &= 0 \quad (3)\end{aligned}$$

5. Equations (1) and (2) give $\lambda = 4y$ and $\lambda = 4x$. Thus,

$$\begin{aligned}4y &= 4x \\y &= x.\end{aligned}$$

Substituting into equation (3),

$$\begin{aligned}x + (x) - 16 &= 0 \\x &= 8.\end{aligned}$$

$$\text{So } y = 8.$$

Maximum is $f(8, 8) = 4(8)(8) = 256$.

2. Maximize $f(x, y) = 2xy + 4$,
subject to $x + y = 20$.

$$\begin{aligned}1. \quad g(x, y) &= x + y - 20 \\2. \quad F(x, y, \lambda) &= 2xy + 4 - \lambda(x + y - 20) \\3. \quad F_x(x, y, \lambda) &= 2x - \lambda \\F_y(x, y, \lambda) &= 2y - \lambda \\F_\lambda(x, y, \lambda) &= -(x + y - 20) \\4. \quad 2y - \lambda &= 0 \quad (1) \\2x - \lambda &= 0 \quad (2) \\x + y - 20 &= 0 \quad (3)\end{aligned}$$

5. Equations (1) and (2) give $\lambda = 2y$ and $\lambda = 2x$. Thus,

$$\begin{aligned}2y &= 2x \\y &= x\end{aligned}$$

Substitution into equation (3),

$$\begin{aligned}x + (x) - 20 &= 0 \\x &= 10.\end{aligned}$$

$$\text{So } y = 10.$$

Maximum is $f(10, 10) = 2(10)(10) + 4 = 204$.

3. Maximize $f(x, y) = xy^2$,
subject to $x + 2y = 15$.

$$\begin{aligned}1. \quad g(x, y) &= x + 2y - 15 \\2. \quad F(x, y, \lambda) &= xy^2 - \lambda(x + 2y - 15) \\3. \quad F_x(x, y, \lambda) &= y^2 - \lambda \\F_y(x, y, \lambda) &= 2xy - 2\lambda \\F_\lambda(x, y, \lambda) &= -(x + 2y - 15)\end{aligned}$$

$$4. \quad y^2 - \lambda = 0 \quad (1)$$

$$2xy - 2\lambda = 0 \quad (2)$$

$$x + 2y - 15 = 0 \quad (3)$$

5. Equations (1) and (2) give $\lambda = y^2$ and $\lambda = xy$. Thus,

$$y^2 = xy$$

$$y(y - x) = 0$$

$$y = 0 \quad \text{or} \quad y = x$$

Substituting $y = 0$ into equation (3),

$$x + 2(0) - 15 = 0$$

$$x = 15.$$

Substituting $y = x$ into equation (3)

$$x + 2(x) - 15 = 0$$

$$x = 5.$$

So $y = x = 5$.

Thus,

$$f(15, 0) = 15(0)^2 = 0, \quad \text{and}$$

$$f(5, 5) = 5(5)^2 = 125.$$

Since $f(5, 5) > f(15, 0)$, $f(5, 5) = 125$ is a maximum.

4. Maximize $f(x, y) = 8x^2y$,

subject to $3x - y = 9$.

$$1. \quad g(x, y) = 3x - y - 9$$

$$2. \quad F(x, y, \lambda) = 8x^2y - \lambda(3x - y - 9)$$

$$3. \quad F_x(x, y, \lambda) = 16xy - 3\lambda$$

$$F_y(x, y, \lambda) = 8x^2 + \lambda$$

$$F_\lambda(x, y, \lambda) = -(3x - y - 9)$$

$$4. \quad 16xy - 3\lambda = 0 \quad (1)$$

$$8x^2 + \lambda = 0 \quad (2)$$

$$3x - y - 9 = 0 \quad (3)$$

5. Equations (1) and (2) give $\lambda = \frac{16xy}{3}$ and

$$\lambda = -8x^2. \quad \text{Thus,}$$

$$\frac{16xy}{3} = -8x^2$$

$$8x^2 + \frac{16xy}{3} = 0$$

$$8x \left(x + \frac{2y}{3} \right) = 0$$

$$x = 0 \quad \text{or} \quad x = -\frac{2y}{3}.$$

Substituting $x = 0$ into equation (3),

$$3(0) - y - 9 = 0$$

$$y = -9.$$

Substituting $x = -\frac{2y}{3}$ into equation (3),

$$3 \left(-\frac{2y}{3} \right) - y - 9 = 0$$

$$y = -3.$$

$$\text{So} \quad x = -\frac{2(-3)}{3} = 2.$$

Thus,

$$f(0, -9) = 8(0)^2(-9) = 0, \quad \text{and}$$

$$f(2, -3) = 8(2)^2(-3) = -96.$$

Since $f(0, -9) > f(2, -3)$, $f(0, -9) = 0$ is a maximum.

5. Minimize $f(x, y) = x^2 + 2y^2 - xy$,
subject to $x + y = 8$.

$$1. \quad g(x, y) = x + y - 8$$

$$2. \quad F(x, y, \lambda) = x^2 + 2y^2 - xy - \lambda(x + y - 8)$$

$$3. \quad F_x(x, y, \lambda) = 2x - y - \lambda$$

$$F_y(x, y, \lambda) = 4y - x - \lambda$$

$$F_\lambda(x, y, \lambda) = -(x + y - 8)$$

$$4. \quad 2x - y - \lambda = 0$$

$$4y - x - \lambda = 0$$

$$x + y - 8 = 0$$

5. Subtracting the second equation from the first equation to eliminate λ gives the new system of equations

$$x + y = 8$$

$$3x - 5y = 0.$$

Solve this system.

$$5x + 5y = 40$$

$$3x - 5y = 0$$

$$8x = 40$$

$$x = 5$$

But $x + y = 8$, so $y = 3$.

Thus, $f(5, 3) = 25 + 18 - 15 = 28$ is a minimum.

6. Minimize $f(x, y) = 3x^2 + 4y^2 - xy - 2$,
subject to $2x + y = 21$.

1. $g(x, y) = 2x + y - 21$

2. $F(x, y, \lambda)$
 $= 3x^2 + 4y^2 - xy$
 $- 2 - \lambda(2x + y - 21)$

3. $F_x(x, y, \lambda) = 6x - y - 2\lambda$
 $F_y(x, y, \lambda) = 8y - x - \lambda$
 $F_\lambda(x, y, \lambda) = -(2x + y - 21)$

4. $6x - y - 2\lambda = 0$
 $8y - x - \lambda = 0$
 $2x + y - 21 = 0$

5. $\lambda = \frac{6x - y}{2}$ and $\lambda = 8y - x$

$$\frac{6x - y}{2} = 8y - x$$

$$y = \frac{8}{17}x$$

Substituting into $2x + y - 21 = 0$,
we have

$$2x + \frac{8}{17}x - 21 = 0,$$

so $x = \frac{17}{2}$ and $y = 4$.

Minimum is

$$f\left(\frac{17}{2}, 4\right) = 3\left(\frac{17}{2}\right)^2 + 4(4)^2 - \frac{17}{2}(4) - 2$$

$$= \frac{979}{4} = 244.75.$$

7. Maximize $f(x, y) = x^2 - 10y^2$,
subject to $x - y = 18$.

1. $g(x, y) = x - y - 18$

2. $F(x, y, \lambda)$
 $= x^2 - 10y^2 - \lambda(x - y - 18)$

3. $F_x(x, y, \lambda) = 2x - \lambda$
 $F_y(x, y, \lambda) = -20y - \lambda$
 $F_\lambda(x, y, \lambda) = -(x - y - 18)$

4. $2x - \lambda = 0$
 $-20y + \lambda = 0$
 $x - y - 18 = 0$

5. Adding the first two equations to eliminate λ gives

$$2x - 20y = 0$$

$$x = 10y.$$

Substituting $x = 10y$ in the third equation gives

$$10y - y = 18$$

$$y = 2$$

$$x = 20.$$

Thus,

$$f(20, 2) = 20^2 - 10(2)^2$$

$$= 400 - 40 = 360.$$

$f(20, 2) = 360$ is a maximum.

8. Maximize $f(x, y) = 12xy - x^2 - 3y^2$,
subject to $x + y = 16$.

1. $g(x, y) = x + y - 16$

2. $F(x, y, \lambda)$
 $= 12xy - x^2 - 3y^2 - \lambda(x + y - 16)$

3. $F_x(x, y, \lambda) = 12y - 2x - \lambda$
 $F_y(x, y, \lambda) = 12x - 6y - \lambda$
 $F_\lambda(x, y, \lambda) = -(x + y - 16)$

4. $12y - 2x - \lambda = 0$
 $12x - 6y - \lambda = 0$
 $x + y - 16 = 0$

5. $\lambda = 12y - 2x$ and $\lambda = 12x - 6y$

$$12y - 2x = 12x - 6y$$

$$y = \frac{7}{9}x$$

Substituting into $x + y - 16 = 0$, we have

$$x + \frac{7}{9}x - 16 = 0,$$

so $x = 9$ and $y = 7$.

Maximum is

$$f(9, 7) = 12(9)(7) - (9)^2 - 3(7)^2$$

$$= 528.$$

9. Maximize $f(x, y, z) = xyz^2$,
subject to $x + y + z = 6$.

1. $g(x, y, z) = x + y + z - 6$

2. $F(x, y, \lambda)$
 $= xyz^2 - \lambda(x + y + z - 6)$

$$\begin{aligned}
3. \quad F_x(x, y, z, \lambda) &= yz^2 - \lambda \\
F_y(x, y, z, \lambda) &= xz^2 - \lambda \\
F_z(x, y, z, \lambda) &= 2xyz - \lambda \\
F_\lambda(x, y, z, \lambda) &= -(x + y + z - 6)
\end{aligned}$$

4. Setting F_x, F_y, F_z and F_λ equal to zero yields

$$yz^2 - \lambda = 0 \quad (1)$$

$$xz^2 - \lambda = 0 \quad (2)$$

$$2xyz - \lambda = 0 \quad (3)$$

$$x + y + z - 6 = 0. \quad (4)$$

$$5. \quad \lambda = yz^2, \lambda = xz^2, \text{ and } \lambda = 2xyz$$

$$yz^2 = xz^2$$

$$z^2(y - x) = 0$$

$$x = y \quad \text{or} \quad z = 0$$

$$yz^2 = 2xyz$$

$$2xyz - yz^2 = 0$$

$$yz(2x - z) = 0$$

$$y = 0 \quad \text{or} \quad z = 0 \quad \text{or} \quad z = 2x$$

In a similar way, the third equation

$$xz^2 = 2xyz$$

implies that $x = 0$ or $z = 0$ or $z = 2y$.

By the nature of the function to be maximized, $f(x, y, z) = xyz^2$, a nonzero maximum can come only from those points with nonzero coordinates.

Therefore, assume $y = x$ and $z = 2y = 2x$.

If $y = x$ and $z = 2x$ are substituted into equation (4), then

$$x + x + 2x - 6 = 0$$

$$x = \frac{3}{2}.$$

Thus, $y = \frac{3}{2}$ and $z = 3$, and

$$\begin{aligned}
f\left(\frac{3}{2}, \frac{3}{2}, 3\right) &= \frac{3}{2} \cdot \frac{3}{2} \cdot 9 \\
&= \frac{81}{4} > 0.
\end{aligned}$$

So, $f\left(\frac{3}{2}, \frac{3}{2}, 3\right) = \frac{81}{4} = 20.25$ is a maximum.

10. Maximize $f(x, y, z) = xy + 2xz + 2yz$, subject to $xyz = 32$.

$$1. \quad g(x, y, z) = xyz - 32$$

$$2. \quad F(x, y, z, \lambda) = xy + 2xz + 2yz - \lambda(xyz - 32)$$

$$\begin{aligned}
3. \quad F_x(x, y, z, \lambda) &= y + 2z - \lambda yz \\
F_y(x, y, z, \lambda) &= x + 2z - \lambda xz \\
F_z(x, y, z, \lambda) &= 2x + 2y - \lambda xy \\
F_\lambda(x, y, z, \lambda) &= -(xyz - 32)
\end{aligned}$$

$$\begin{aligned}
4. \quad y + 2z - \lambda yz &= 0 \\
x + 2z - \lambda xz &= 0 \\
2x + 2y - \lambda xy &= 0 \\
xyz - 32 &= 0
\end{aligned}$$

$$\begin{aligned}
5. \quad \lambda &= \frac{y + 2z}{yz}, \lambda = \frac{x + 2z}{xz}, \text{ and} \\
\lambda &= \frac{2x + 2y}{xy}
\end{aligned}$$

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz}$$

$$xyz + 2xz^2 = xyz + 2yz^2$$

$$2yz^2 - 2xz^2 = 0$$

$$2z^2(y - x) = 0$$

$$z = 0 \quad \text{or} \quad y = x$$

and

$$\frac{y + 2z}{yz} = \frac{2x + 2y}{xy}$$

$$xy^2 + 2xyz = 2xyz + 2y^2z$$

$$2y^2z - xy^2 = 0$$

$$y^2(2z - x) = 0$$

$$y = 0 \quad \text{or} \quad z = \frac{x}{2}$$

Since $xyz = 32$, $z = 0$ and $y = 0$ are impossible. Substituting $y = x$ and $z = \frac{x}{2}$ into $xyz - 32 = 0$, we have

$$x(x)\left(\frac{x}{2}\right) = 32, \text{ so } x = 4, y = 4, \text{ and } z = 2.$$

Maximum is

$$\begin{aligned}
f(4, 4, 2) &= 4(4) + 2(4)(2) + 2(4)(2) \\
&= 48.
\end{aligned}$$

11. The problem can be restated as

$$\text{Maximize } f(x, y) = 3xy^2,$$

$$\text{subject to } x + y = 24, x > 0, y > 0.$$

$$1. \quad g(x, y) = x + y - 24$$

$$2. \quad F(x, y, \lambda) = 3xy^2 - \lambda(x + y - 24)$$

$$\begin{aligned} 3. \quad F_x(x, y, \lambda) &= 3y^2 - \lambda \\ F_y(x, y, \lambda) &= 6xy - \lambda \\ F_\lambda(x, y, \lambda) &= -(x + y - 24) \end{aligned}$$

$$4. \quad 3y^2 - \lambda = 0 \quad (1)$$

$$6xy - \lambda = 0 \quad (2)$$

$$x + y - 24 = 0 \quad (3)$$

5. Equations (1) and (2) give $\lambda = 3y^2$ and $\lambda = 6xy$. Thus,

$$3y^2 = 6xy$$

$$3y^2 - 6xy = 0$$

$$3y(y - 2x) = 0$$

$$y = 0 \quad \text{or} \quad y = 2x.$$

Substituting $y = 0$ into equation (3),

$$x + (0) - 24 = 0$$

$$x = 24.$$

Substituting $y = 2x$ into equation (3),

$$x + (2x) - 24 = 0$$

$$3x - 24 = 0$$

$$x = 8.$$

$$\text{So} \quad y = 2x = 16.$$

Thus,

$$f(24, 0) = 3(24)(0)^2 = 0, \text{ and}$$

$$f(8, 16) = 3(8)(16)^2 = 6144.$$

Since $f(8, 16) > f(24, 0)$, $x = 8$ and $y = 16$ will maximize $f(x, y) = 3xy^2$.

12. The problem can be restated as

$$\text{Maximize } f(x, y) = 5x^2y + 10,$$

$$\text{subject to } x + y = 48, x > 0, y > 0.$$

$$1. \quad g(x, y) = x + y - 48$$

$$2. \quad F(x, y, \lambda) = 5x^2y + 10 - \lambda(x + y - 48)$$

$$3. \quad F_x(x, y, \lambda) = 10xy - \lambda$$

$$F_y(x, y, \lambda) = 5x^2 - \lambda$$

$$F_\lambda(x, y, \lambda) = -(x + y - 48)$$

$$4. \quad 10xy - \lambda = 0 \quad (1)$$

$$5x^2 - \lambda = 0 \quad (2)$$

$$x + y - 48 = 0 \quad (3)$$

5. Equations (1) and (2) give $\lambda = 10xy$ and $\lambda = 5x^2$. Thus,

$$5x^2 = 10xy$$

$$5x^2 - 10xy = 0$$

$$5x(x - 2y) = 0$$

$$x = 0 \text{ or } x = 2y.$$

Substituting $x = 0$ into equation (3),

$$(0) + y - 48 = 0$$

$$y = 48.$$

Substituting $x = 2y$ into equation (3),

$$(2y) + y - 48 = 0$$

$$3y - 48 = 0$$

$$y = 16.$$

$$\text{So} \quad x = 2y = 32.$$

Thus,

$$f(0, 48) = 5(0)^2(48) + 10 = 10, \text{ and}$$

$$f(32, 16) = 5(32)^2(16) + 10 = 81,930.$$

Since $f(32, 16) > f(0, 48)$, $x = 32$ and

$y = 16$ will maximize $f(x, y) = 5x^2y + 10$.

13. Let x, y , and z be three number such that

$$x + y + z = 90$$

$$\text{and} \quad f(x, y, z) = xyz.$$

$$1. \quad g(x, y, z) = x + y + z - 90$$

$$\begin{aligned} 2. \quad F(x, y, z) &= xyz - \lambda(x + y + z - 90) \\ &= xyz - \lambda(x + y + z - 90) \end{aligned}$$

$$3. \quad F_x(x, y, z, \lambda) = yz - \lambda$$

$$F_y(x, y, z, \lambda) = xz - \lambda$$

$$F_z(x, y, z, \lambda) = xy - \lambda$$

$$F_\lambda(x, y, z, \lambda) = -(x + y + z - 90)$$

$$4. \quad yz - \lambda = 0 \quad (1)$$

$$xz - \lambda = 0 \quad (2)$$

$$xy - \lambda = 0 \quad (3)$$

$$x + y + z - 90 = 0 \quad (4)$$

$$5. \quad \lambda = yz, \lambda = xz, \text{ and } \lambda = xy$$

$$yz = xz$$

$$yz - xz = 0$$

$$(y - x)z = 0$$

$$y - x = 0 \quad \text{or} \quad z = 0$$

$$xz - xy = 0$$

$$x(z - y) = 0$$

$$x = 0 \quad \text{or} \quad z - y = 0$$

Since $x = 0$ or $z = 0$ would not maximize

$f(x, y, z) = xyz$, then $y - x = 0$ and

$z - y = 0$ imply that $y = x = z$.

Substituting into equation (4) gives

$$x + x + x - 90 = 0$$

$$x = 30.$$

$x = y = z = 30$ will maximize $f(x, y, z)$

$= xyz$. The numbers are 30, 30, and 30.

- 14.** Let x, y , and z be three positive numbers such that $x + y + z = 240$. Maximize $f(x, y, z) = xyz$, subject to $x + y + z = 240$.

$$1. \quad g(x, y) = x + y + z - 240$$

$$2. \quad F(x, y, z, \lambda) = xyz - \lambda(x + y + z - 240)$$

$$3. \quad F_x(x, y, z, \lambda) = yz - \lambda$$

$$F_y(x, y, z, \lambda) = xz - \lambda$$

$$F_z(x, y, z, \lambda) = xy - \lambda$$

$$F_\lambda(x, y, z, \lambda) = -(x + y + z - 240)$$

$$4. \quad yz - \lambda = 0$$

$$xz - \lambda = 0$$

$$xy - \lambda = 0$$

$$x + y + z - 240 = 0$$

$$5. \quad \lambda = yz, \lambda = xz, \lambda = xy$$

$$yz = xz$$

$$z = 0 \text{ (impossible) or } x = y$$

$$xz = xy$$

$$x = 0 \text{ (impossible) or } z = y$$

Thus, $x = y = z$.

$$x + x + x - 240 = 0$$

$$x = 80$$

Thus,

$$x = y = z = 80.$$

The three numbers are 80, 80, and 80.

- 15.** Find the maximum and minimum of $f(x, y) = x^3 + 2xy + 4y^2$ subject to $x + 2y = 12$.

$$1. \quad g(x, y) = x + 2y - 12$$

$$2. \quad F(x, y) = x^3 + 2xy + 4y^2 - \lambda(x + 2y - 12)$$

$$3. \quad F_x(x, y, \lambda) = 3x^2 + 2y - \lambda$$

$$F_y(x, y, \lambda) = 2x + 8y - 2\lambda$$

$$F_\lambda(x, y, \lambda) = -x - 2y + 12$$

$$4. \quad 3x^2 + 2y - \lambda = 0$$

$$2x + 8y - 2\lambda = 0$$

$$-x - 2y + 12 = 0$$

5. Solve the second equation for λ and substitute into the first equation.

$$2x + 8y - 2\lambda = 0$$

$$\lambda = x + 4y$$

The first equation is now

$$3x^2 + 2y - (x + 4y) = 0$$

or

$$3x^2 - x - 2y = 0$$

Solve the last equation for $-2y$ and substitute into this new equation.

$$-x - 2y + 12 = 0$$

$$-2y = x - 12$$

$$3x^2 - x + (x - 12) = 0$$

$$3x^2 - 12 = 0$$

Now solve for x .

$$3x^2 - 12 = 0$$

$$x^2 = 4$$

$$x = 2 \text{ or } x = -2$$

Find the corresponding y .

When $x = 2$, $-2 - 2y + 12 = 0$ so $y = 5$.

When $x = -2$, $-(-2) - 2y + 12 = 0$ so $y = 7$.

Thus our candidates for the locations of maxima or minima of f subject to the given constraint are $(2, 5)$ and $(-2, 7)$.

$f(2, 5) = 128$ and $f(-2, 7) = 160$, so probably the maximum is 160 at $(-2, 7)$ and the minimum is 128 at $(2, 5)$.

Try some nearby points that satisfy the constraints to check.

When $x = -2.2$, $y = (12 + 2.2)/2 = 7.1$;
 $f(-2.2, 7.1) = 159.752$.

When $x = 2.2$, $y = (12 - 2.2)/2 = 4.9$;
 $f(2.2, 4.9) = 128.248$.

This confirms our answer: There is a maximum value of 160 at $(-2, 7)$ and a minimum value of 128 at $(2, 5)$.

- 18.** Consider the constraint and solve for x in terms of y .

$$\begin{aligned}x + 2y &= 15 \\x &= 15 - 2y\end{aligned}$$

Then

$$\begin{aligned}f(x, y) &= xy^2 \\&= (15 - 2y)y^2 \\&= -2y^3 + 15y^2\end{aligned}$$

So, $f(x, y) = -2y^3 + 15y^2 = f(y)$. Notice that f is unbounded; more specifically,

$$\begin{aligned}\lim_{y \rightarrow \infty} f(y) &= -\infty \\ \text{and } \lim_{y \rightarrow -\infty} f(y) &= \infty.\end{aligned}$$

Therefore f , subject to the given constraint, has neither an absolute maximum nor an absolute minimum.

- 19.** Consider the constraint and solve for y in terms of x .

$$\begin{aligned}3x - y &= 9 \\y &= 3x - 9\end{aligned}$$

Then

$$\begin{aligned}f(x, y) &= 8x^2y \\&= 8x^2(3x - 9) \\&= 24x^3 - 72x^2\end{aligned}$$

So, $f(x, y) = 24x^3 - 72x^2 = f(x)$. Notice that f is unbounded; more specifically,

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \infty \\ \text{and } \lim_{x \rightarrow -\infty} f(x) &= -\infty.\end{aligned}$$

Therefore f , subject to the given constraint, has neither an absolute maximum nor an absolute minimum.

- 21.** Minimize

$f(x, y) = x^2 + 2x + 9y^2 + 4y + 8xy$ subject to $x + y = 1$.

(a) 1. $g(x, y) = x + y = 1$

2. $F(x, y) = x^2 + 2x + 9y^2 + 4y + 8xy - \lambda(x + y - 1)$

3. $\begin{aligned}F_x(x, y, \lambda) &= 2x + 8y - \lambda + 2 \\F_y(x, y, \lambda) &= 18y + 8x - \lambda + 4 \\F_\lambda(x, y, \lambda) &= -x - y + 1\end{aligned}$

4. $\begin{aligned}2x + 8y - \lambda + 2 &= 0 \\18y + 8x - \lambda + 4 &= 0 \\-x - y + 1 &= 0\end{aligned}$

5. Solve the system in Step 4. Solve the first two equations for λ and eliminate λ .

$$\begin{aligned}2x + 8y + 2 &= 18y + 8x + 4 \\6x + 10y + 2 &= 0 \\3x + 5y + 1 &= 0\end{aligned}$$

Now combine this equation in x and y with the last equation to form a system.

$$\begin{aligned}3x + 5y + 1 &= 0 \\-x - y + 1 &= 0\end{aligned}$$

The solution of this system is $(3, -2)$.

$f(3, -2) = -5$. Test nearby points that satisfy the constraint to see if -5 is indeed a minimum.

When $x = 3.2$, $y = 1 - 3.2 = -2.2$.

When $x = 2.8$, $y = 1 - 2.8 = -1.8$.

$f(3.2, -2.2) = f(2.8, -1.8) = -4.92$, so subject to the given constraint f has a minimum of -5 at $(3, -2)$.

- (d) $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 18$,
 $f_{xy}(x, y) = 8$.
 $D(3, -2) = 18 \cdot 2 - 64 = -28$, so $(3, 2)$ is
 a saddle point of the function f .

23. Maximize $f(x, y) = xy^2$ subject to
 $x + 2y = 60$.

1. $g(x, y) = x + 2y - 60$
2. $F(x, y) = xy^2 - \lambda(x + 2y - 60)$
3. $F_x(x, y, \lambda) = y^2 - \lambda$
 $F_y(x, y, \lambda) = 2xy - 2\lambda$
 $F_\lambda(x, y, \lambda) = -x - 2y + 60$
4. $y^2 - \lambda = 0$
 $2xy - 2\lambda = 0$
 $-x - 2y + 60 = 0$
5. Solve the first two equations for λ and eliminate λ .

$$\begin{aligned} y^2 &= \lambda \\ xy &= \lambda \\ y^2 &= xy \end{aligned}$$

Either $y = 0$ or $x = y$. If $y = 0$, then $x = 60$; if $x = y$, then $-3x = 60$ and $x = y = 20$. So our candidates for a maximum of f are $(60, 0)$ and $(20, 20)$. But $f(60, 0) = 0$ so the maximum must be $f(20, 20) = 8000$.

We can confirm this by solving the constraint for x .

$$\begin{aligned} x &= 60 - 2y \\ f(x, y) &= xy^2 = (60 - 2y)y^2 = 60y^2 - 2y^3. \end{aligned}$$

This is now a function of one variable. The second derivative with respect to y is $120 - 12y$ which is negative when $y = 20$, so the value of 8000 found above is the maximum utility, obtained by purchasing 20 units of x and 20 units of y .

24. Maximize $f(x, y) = x^2y^3$ subject to $2x + y = 80$.

1. $g(x, y) = 2x + y - 80$
2. $F(x, y) = x^2y^3 - \lambda(2x + y - 80)$

3. $F_x(x, y, \lambda) = 2xy^3 - 2\lambda$
 $F_y(x, y, \lambda) = 3x^2y^2 - \lambda$
 $F_\lambda(x, y, \lambda) = -2x - y + 80$

4. $2xy^3 - 2\lambda = 0$
 $3x^2y^2 - \lambda = 0$
 $-2x - y + 80 = 0$

5. Solve the first two equations for λ and eliminate λ .

$$\begin{aligned} xy^3 &= \lambda \\ 3x^2y^2 &= \lambda \\ xy^3 &= 3x^2y^2 \end{aligned}$$

If either x or y equals 0, the utility will have a minimum value of 0. So we can assume $xy \neq 0$ and divide by xy^2 to find that $y = 3x$. Substitute for y in the last equation.

$$\begin{aligned} -2x - 3x + 80 &= 0 \\ x = 16, y = 3x &= 48. \\ f(16, 48) &= 28,311,552. \end{aligned}$$

The maximum utility is 28,311,552, obtained by purchasing 16 units of x and 48 units of y .

25. Maximize $f(x, y) = x^4y^2$ subject to
 $2x + 4y = 60$.

1. $g(x, y) = 2x + 4y - 60$
2. $F(x, y) = x^4y^2 - \lambda(2x + 4y - 60)$
3. $F_x(x, y, \lambda) = 4x^3y^2 - 2\lambda$
 $F_y(x, y, \lambda) = 2x^4y - 4\lambda$
 $F_\lambda(x, y, \lambda) = -2x - 4y + 60$

4. $4x^3y^2 - 2\lambda = 0$
 $2x^4y - 4\lambda = 0$
 $-2x - 4y + 60 = 0$

5. Solve the first two equations for 2λ and eliminate 2λ .

$$\begin{aligned} 4x^3y^2 &= 2\lambda \\ x^4y &= 2\lambda \\ 4x^3y^2 &= x^4y \end{aligned}$$

If either x or y equals 0, the utility will have a minimum value of 0. So we can assume $xy \neq 0$ and divide by x^3y to find that

$4y = x$. Substitute for x in the last equation.

$$-2x - x + 60 = 0$$

$$x = 20, y = 20/4 = 5$$

$$f(20, 5) = 4,000,000.$$

The maximum utility is 4,000,000, obtained by purchasing 20 units of x and 5 units of y .

- 26.** Maximize $f(x, y) = x^3y^4$ subject to $3x + 3y = 42$ or $x + y = 14$.

$$1. \quad g(x, y) = x + y - 14$$

$$2. \quad F(x, y) = x^3y^4 - \lambda(x + y - 14)$$

$$3. \quad F_x(x, y, \lambda) = 3x^2y^4 - \lambda$$

$$F_y(x, y, \lambda) = 4x^3y^3 - \lambda$$

$$F_\lambda(x, y, \lambda) = -x - y + 14$$

$$4. \quad 3x^2y^4 - \lambda = 0$$

$$4x^3y^3 - \lambda = 0$$

$$-x - y + 14 = 0$$

5. Solve the first two equations for λ and eliminate λ .

$$3x^2y^4 = \lambda$$

$$4x^3y^3 = \lambda$$

$$3x^2y^4 = 4x^3y^3$$

If either x or y equals 0, the utility will have a minimum value of 0. So we can assume $xy \neq 0$ and divide by x^2y^3 to find that $3y = 4x$ or $y = (4/3)x$. Substitute for y in the last equation.

$$-x - \frac{4}{3}x + 14 = 0$$

$$7x = 42$$

$$x = 6, y = \left(\frac{4}{3}\right)6 = 8$$

$$f(6, 8) = 884,736.$$

The maximum utility is 884,736, obtained by purchasing 6 units of x and 8 units of y .

- 27.** Let x be the width and y be the length of a field such that the cost in dollars to enclose the field is

$$6x + 6y + 4x + 4y = 1200$$

$$10x + 10y = 1200.$$

The area is

$$f(x, y) = xy.$$

$$1. \quad g(x, y) = 10x + 10y - 1200$$

$$2. \quad F(x, y) = xy - \lambda(10x + 10y - 1200)$$

$$3. \quad F_x(x, y, \lambda) = y - 10\lambda$$

$$F_y(x, y, \lambda) = x - 10\lambda$$

$$F_\lambda(x, y, \lambda) = -(10x + 10y - 1200)$$

$$4. \quad y - 10\lambda = 0$$

$$x - 10\lambda = 0$$

$$10x + 10y - 1200 = 0$$

$$5. \quad 10\lambda = y \text{ and } 10\lambda = x$$

$$y = x$$

Substituting into the third equation gives

$$10x + 10x - 1200 = 0$$

$$20x - 1200 = 0$$

$$x = 60$$

$$y = 60.$$

These dimensions, 60 feet by 60 feet, will maximize the area.

- 28.** Let x be the length of the fence opposite the building and y be the length of each end. The area is then xy and the total cost is $25x + 15(2y)$.

Restate the problem as follows.

Maximize xy , subject to $25x + 30y = 2400$.

$$1. \quad g(x, y) = 25x + 30y - 2400$$

$$2. \quad F(x, y, \lambda) = xy - \lambda(25x + 30y - 2400)$$

$$3. \quad F_x(x, y, \lambda) = y - 25\lambda$$

$$F_y(x, y, \lambda) = x - 30\lambda$$

$$F_\lambda(x, y, \lambda) = -(25x + 30y - 2400)$$

$$4. \quad y - 25\lambda = 0 \quad (1)$$

$$x - 30\lambda = 0 \quad (2)$$

$$25x + 30y - 2400 = 0 \quad (3)$$

5. Equations (1) and (2) give $\lambda = \frac{y}{25}$ and

$$\lambda = \frac{x}{30}. \text{ Thus,}$$

$$\frac{y}{25} = \frac{x}{30}$$

$$y = \frac{5}{6}x.$$

Substituting $y = \frac{5}{6}x$ into equation (3) gives

$$25x + 30\left(\frac{5}{6}x\right) - 2400 = 0$$

$$50x - 2400 = 0$$

$$x = 48.$$

$$\text{So } y = \frac{5}{6}x = 40.$$

The dimensions are 48 ft (opposite the building) by 40 ft (the ends).

- 29.** Maximize $C(x, y) = 2x^2 + 6y^2 + 4xy + 10$, subject to $x + y = 10$.

$$1. \quad g(x, y) = x + y - 10$$

$$2. \quad F(x, y) = 2x^2 + 6y^2 + 4xy + 10 - \lambda(x + y - 10)$$

$$3. \quad F_x(x, y, \lambda) = 4x + 4y - \lambda \\ F_y(x, y, \lambda) = 12y + 4x - \lambda \\ F_\lambda(x, y, \lambda) = -(x + y - 10)$$

$$4. \quad 4x + 4y - \lambda = 0 \\ 12y + 4x - \lambda = 0 \\ x + y - 10 = 0$$

$$5. \quad \lambda = 4x + 4y \text{ and } \lambda = 12y + 4x.$$

$$4x + 4y = 12y + 4x$$

$$8y = 0$$

$$y = 0$$

Since $x + y = 10$, $x = 10$.

10 large kits and no small kits will maximize the cost.

- 30.** Maximize $P(x, y) = -x^2 - y^2 + 4x + 8y$, subject to $x + y = 6$.

$$1. \quad g(x, y) = x + y - 6$$

$$2. \quad F(x, y, \lambda) = -x^2 - y^2 + 4x + 8y - \lambda(x + y - 6)$$

$$3. \quad F_x(x, y, \lambda) = -2x + 4 - \lambda \\ F_y(x, y, \lambda) = -2y + 8 - \lambda \\ F_\lambda(x, y, \lambda) = -(x + y - 6)$$

$$4. \quad -2x + 4 - \lambda = 0 \\ -2y + 8 - \lambda = 0 \\ x + y - 6 = 0$$

$$5. \quad \lambda = -(2x - 4) \text{ and } \lambda = -(2y - 8)$$

$$2x - 4 = 2y - 8$$

$$y = x + 2$$

Substituting into $x + y - 6 = 0$, we have

$$x + (x + 2) - 6 = 0 \text{ so } x = 2 \text{ and } y = 4.$$

- 31.** Maximize $f(x, y) = 3x^{1/3}y^{2/3}$, subject to $80x + 150y = 40,000$.

$$1. \quad g(x, y) = 80x + 150y - 40,000$$

$$2. \quad F(x, y) = 3x^{1/3}y^{2/3} - \lambda(80x + 150y - 40,000)$$

$$3, 4. \quad F_x(x, y, \lambda) = x^{-2/3}y^{2/3} - 80\lambda = 0$$

$$F_y(x, y, \lambda) = 2x^{1/3}y^{-1/3} - 150\lambda = 0$$

$$F_\lambda(x, y, \lambda) = -(80x + 150y - 40,000) = 0$$

$$5. \quad \frac{x^{-2/3}y^{2/3}}{80} = \frac{2x^{1/3}y^{-1/3}}{150} \\ \frac{15y}{16} = x$$

Substitute into the third equation.

$$80\left(\frac{15y}{16}\right) + 150y - 40,000 = 0$$

$$y = 178(\text{rounded})$$

$$x = \frac{15(178)}{16} \\ \approx 167$$

Use about 167 units of labor and 178 units of capital to maximize production.

- 32.** Maximize $f(x, y) = 12x^{3/4}y^{1/4}$, subject to $100x + 180y = 25,200$.

$$1. \quad g(x, y) = 100x + 180y - 25,200$$

$$2. \quad F(x, y, \lambda) = 12x^{3/4}y^{1/4} - \lambda(100x + 180y - 25,200)$$

$$\begin{aligned}
 3. \quad F_x(x, y, \lambda) &= \frac{3}{4}(12x^{-1/4}y^{1/4}) - 100\lambda \\
 &= \frac{9y^{1/4}}{x^{1/4}} - 100\lambda \\
 F_y(x, y, \lambda) &= \frac{1}{4}(12x^{3/4}y^{-3/4}) - 180\lambda \\
 &= \frac{3x^{3/4}}{y^{3/4}} - 180\lambda \\
 F_\lambda(x, y, \lambda) &= -(100x + 180y - 25,200)
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \frac{9y^{1/4}}{x^{1/4}} - 100\lambda &= 0 \\
 \frac{3x^{3/4}}{y^{3/4}} - 180\lambda &= 0 \\
 100x + 180y - 25,200 &= 0
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \lambda &= \frac{9y^{1/4}}{100x^{1/4}} \quad \text{and} \quad \lambda = \frac{3x^{3/4}}{180y^{3/4}} \\
 &= \frac{x^{3/4}}{60y^{3/4}} \\
 \frac{9y^{1/4}}{100x^{1/4}} &= \frac{x^{3/4}}{60y^{3/4}} \\
 100x &= 540y \\
 x &= \frac{27y}{5}
 \end{aligned}$$

Substitute into

$$\begin{aligned}
 100x + 180y - 25,200 &= 0. \\
 100\left(\frac{27y}{5}\right) + 180y &= 25,200 \\
 540y + 180y &= 25,200 \\
 720y &= 25,200 \\
 y &= 35 \\
 x &= \frac{27(35)}{5} = 189
 \end{aligned}$$

Production will be maximized with 189 units of labor and 35 units of capital.

- 33.** Let x and y be the dimensions of the field such that $2x + 2y = 500$, and the area is $f(x, y) = xy$.

$$\begin{aligned}
 1. \quad g(x, y) &= 2x + 2y - 500 \\
 2. \quad F(x, y) &= xy - \lambda(2x + 2y - 500) \\
 3. \quad F_x(x, y, \lambda) &= y - 2\lambda \\
 F_y(x, y, \lambda) &= x - 2\lambda \\
 F_\lambda(x, y, \lambda) &= -(2x + 2y - 500)
 \end{aligned}$$

$$\begin{aligned}
 4. \quad y - 2\lambda &= 0 \\
 x - 2\lambda &= 0 \\
 2x + 2y - 500 &= 0
 \end{aligned}$$

$$\begin{aligned}
 5. \quad 2\lambda &= y \quad \text{and} \quad 2\lambda = x, \text{ so } x = y. \\
 2x + 2x - 500 &= 0 \\
 4x - 500 &= 0 \\
 x &= 125 \\
 \text{Thus,} \quad y &= 125.
 \end{aligned}$$

Dimensions of 125 m by 125 m will maximize the area.

- 34.** If x and y are the dimensions of the field, we must maximize $f(x, y) = xy$ subject to $x + 2y = 600$.

$$\begin{aligned}
 1. \quad g(x, y) &= x + 2y - 600 \\
 2. \quad F(x, y, \lambda) &= xy - \lambda(x + 2y - 600) \\
 3. \quad F_x(x, y, \lambda) &= y - \lambda \\
 F_y(x, y, \lambda) &= x - 2\lambda \\
 F_\lambda(x, y, \lambda) &= -(x + 2y - 600) \\
 4. \quad x - \lambda &= 0 \\
 x - 2\lambda &= 0 \\
 x + 2y - 600 &= 0
 \end{aligned}$$

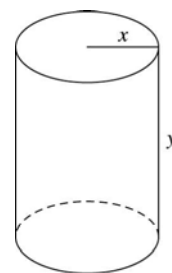
$$\begin{aligned}
 5. \quad \lambda &= y \quad \text{and} \quad \lambda = \frac{x}{2} \\
 y &= \frac{x}{2}
 \end{aligned}$$

Substituting into $x + 2y - 600 = 0$, we have

$$\begin{aligned}
 x + 2\left(\frac{x}{2}\right) - 600 &= 0, \text{ so } x = 300 \text{ and} \\
 y &= 150.
 \end{aligned}$$

The largest area is $(300)(150) = 45,000 \text{ m}^2$.

- 35.** Let x be the radius r of the circular base and y the height h of the can, such that the volume is $\pi x^2 y = 250\pi$.



The surface area is

$$f(x, y) = 2\pi xy + 2\pi x^2.$$

1. $g(x, y) = \pi x^2 y + 250\pi$
2. $F(x, y) = 2\pi xy + 2\pi x^2$
 $\quad - \lambda(\pi x^2 y - 250\pi)$
3. $F_x(x, y, \lambda) = 2\pi y + 4\pi x - \lambda(2\pi xy)$
 $F_y(x, y, \lambda) = 2\pi x - \lambda(\pi x^2)$
 $F_\lambda(x, y, \lambda) = -(\pi x^2 y - 250\pi)$
4. $2\pi y + 4\pi x - \lambda(2\pi xy) = 0$
 $2\pi x - \lambda\pi x^2 = 0$
 $\pi x^2 y - 250\pi = 0$

Simplifying these equations gives

$$\begin{aligned} y + 2x - \lambda xy &= 0 \\ 2x - \lambda x^2 &= 0 \\ x^2 y - 250 &= 0. \end{aligned}$$

5. From the second equation,

$$\begin{aligned} x(2 - \lambda x) &= 0 \\ x = 0 \quad \text{or} \quad \lambda &= \frac{2}{x}. \end{aligned}$$

If $x = 0$, the volume will be 0, which is not possible.

Substituting $x = \frac{2}{\lambda}$ into the first equation gives

$$\begin{aligned} y + 2\left(\frac{2}{\lambda}\right) - \lambda\left(\frac{2}{\lambda}\right)y &= 0 \\ y + \frac{4}{\lambda} - 2y &= 0 \\ \frac{4}{\lambda} &= y \\ \lambda &= \frac{4}{y}. \end{aligned}$$

Since $\lambda = \frac{2}{x}$, $y = 2x$.

Substituting into third equation gives

$$\begin{aligned} x^2(2x) - 250 &= 0 \\ 2x^3 - 250 &= 0 \\ x &= 5 \\ y &= 10. \end{aligned}$$

Since $g(1, 250) = 0$ and

$$f(1, 250) = 502\pi > f(5, 10) = 150\pi,$$

a can with radius of 5 inches and height of 10 inches will have a minimum surface area.

36. Let x be the radius of the can and y be the height.

Minimize surface area $f(x, y) = 2\pi xy + 2\pi x^2$, subject to the constraint that $\pi x^2 y = 25$.

1. $g(x, y) = \pi x^2 y - 25$
2. $F(x, y, \lambda) = 2\pi xy + 2\pi x^2 - \lambda(\pi x^2 y - 25)$
3. $F_x(x, y, \lambda) = 2\pi y + 4\pi x - 2\lambda\pi xy$
 $F_y(x, y, \lambda) = 2\pi x - \lambda\pi x^2$
 $F_\lambda(x, y, \lambda) = -(\pi x^2 y - 25)$
4. $2\pi y + 4\pi x - 2\lambda\pi xy = 0$
 $2\pi x - \lambda\pi x^2 = 0$
 $\pi x^2 y - 25 = 0$

$$5. \quad \lambda = \frac{2x + y}{xy} \quad \text{and} \quad \lambda = \frac{2}{x}$$

$$\frac{2x + y}{xy} = \frac{2}{x}$$

$$2x^2 + xy = 2xy$$

$$2x^2 - xy = 0$$

$$x = 0 \quad \text{or} \quad y = 2x$$

$x = 0$ is impossible.

Substituting $y = 2x$ into $\pi x^2 y - 25 = 0$, we

have $\pi x^2(2x) - 25 = 0$, so $x = \sqrt[3]{\frac{25}{2\pi}}$

≈ 1.585 inches and $y = 2\sqrt[3]{\frac{25}{2\pi}} \approx 3.169$ inches.

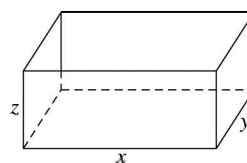
The can with minimum surface area will have a radius of approximately 1.585 inches and a height of approximately 3.169 inches.

37. Let x , y , and z be the dimensions of the box such that the surface area is

$$xy + 2yz + 2xz = 500$$

and the volume is

$$f(x, y, z) = xyz.$$



1. $g(x, y, z) - 500 = 0$
2. $F(x, y, z)$
 $= xyz - \lambda(xy + 2yz + 2xz - 500)$

3,4.

$$F_x(x, y, z, \lambda) = yz - \lambda(y + 2z) = 0 \quad (1)$$

$$F_y(x, y, z, \lambda) = xz - \lambda(x + 2z) = 0 \quad (2)$$

$$F_z(x, y, z, \lambda) = xy - \lambda(2y + 2x) = 0 \quad (3)$$

$$F_\lambda(x, y, z, \lambda) = -(xy + 2xz + 2yz - 500) = 0 \quad (4)$$

Multiplying equation (1) by x , equation (2) by y , and equation (3) by z gives

$$xyz - \lambda x(y + 2z) = 0$$

$$xyz - \lambda y(x + 2z) = 0$$

$$xyz - \lambda z(2y + 2x) = 0.$$

5. Subtracting the first equation from the second equation gives

$$\lambda x(y + 2z) - \lambda y(x + 2z) = 0$$

$$2\lambda xz - 2\lambda yz = 0$$

$$\lambda z(x - y) = 0,$$

$$\text{so} \quad x = y.$$

Subtracting the third equation from the second equation gives

$$\lambda z(2y + 2x) - \lambda y(x + 2z) = 0$$

$$2\lambda xz - \lambda xy = 0$$

$$\lambda x(2z - y) = 0,$$

$$\text{so} \quad z = \frac{y}{2}.$$

Substituting into the fourth equation gives

$$y^2 + 2y\left(\frac{y}{2}\right) + 2y\left(\frac{y}{2}\right) - 500 = 0$$

$$3y^2 = 500$$

$$y = \sqrt{\frac{500}{3}}$$

$$\approx 12.9099$$

$$x \approx 12.9099$$

$$z \approx \frac{12.9099}{2}$$

$$\approx 6.4549.$$

The dimensions are 12.91 m by 12.91 m by 6.455 m.

38. If the box is x by x by y , we must minimize surface area $f(x, y) = 2x^2 + 4xy$, subject to $x^2y = 185$.

$$1. \quad g(x, y) = x^2y - 185$$

$$2. \quad F(x, y, \lambda) = 2x^2 + 4xy - \lambda(x^2y - 185)$$

$$3. \quad F_x(x, y, \lambda) = 4x + 4y - 2\lambda xy$$

$$F_y(x, y, \lambda) = 4x - \lambda x^2$$

$$F_\lambda(x, y, \lambda) = -(x^2y - 185)$$

$$4. \quad 4x + 4y - 2\lambda xy = 0$$

$$4x - \lambda x^2 = 0$$

$$x^2y - 185 = 0$$

$$5. \quad \lambda = \frac{2x + 2y}{xy} \text{ and } \lambda = \frac{4}{x}$$

$$\frac{2x + 2y}{xy} = \frac{4}{x}$$

$$2x^2 + 2xy = 4xy$$

$$2x^2 - 2xy = 0$$

$$2x(x - y) = 0$$

$$x = 0 \text{ or } y = x$$

$x = 0$ is impossible.

Substituting $y = x$ into $x^2y - 185 = 0$, we have

$$y = x = \sqrt[3]{185} \approx 5.698.$$

The dimensions are 5.698 inches by 5.698 inches by 5.698 inches.

39. Let x , y , and z be the dimensions of the box. The surface area is

$$2xy + 2xz + 2yz.$$

We must minimize

$$f(x, y, z) = 2xy + 2xz + 2yz$$

subject to $xyz = 125$.

$$1. \quad g(x, y, z) = xyz - 125$$

$$2. \quad F(x, y, z) = 2xy + 2xz + 2yz - \lambda(xyz - 125)$$

$$3. \quad F_x(x, y, z, \lambda) = 2y + 2z - \lambda yz$$

$$F_y(x, y, z, \lambda) = 2x + 2z - \lambda xz$$

$$F_z(x, y, z, \lambda) = 2x + 2y - \lambda xy$$

$$F_\lambda(x, y, z, \lambda) = -(xyz - 125)$$

$$\begin{aligned}
 4. \quad & 2y + 2z - \lambda yz = 0 \quad (1) \\
 & 2x + 2z - \lambda xz = 0 \quad (2) \\
 & 2x + 2y - \lambda xy = 0 \quad (3) \\
 & xyz - 125 = 0 \quad (4)
 \end{aligned}$$

5. Equations (1) and (2) give

$$\frac{2y + 2z}{yz} = \lambda \quad \text{and} \quad \frac{2x + 2z}{xz} = \lambda.$$

Thus,

$$\frac{2y + 2z}{yz} = \frac{2x + 2z}{xz}$$

$$2xyz + 2xz^2 = 2xyz + 2yz^2$$

$$2xz^2 - 2yz^2 = 0$$

$$2z^2 = 0 \quad \text{or} \quad x - y = 0$$

$$z = 0 \quad (\text{impossible}) \quad \text{or} \quad x = y.$$

Equations (2) and (3) give

$$\frac{2x + 2z}{xz} = \lambda \quad \text{and} \quad \frac{2x + 2y}{xy} = \lambda.$$

$$\frac{2x + 2z}{xz} = \frac{2x + 2y}{xy}$$

Thus,

$$2x^2y + 2xyz = 2x^2z + 2xyz$$

$$2x^2y - 2x^2z = 0$$

$$2x^2(y - z) = 0$$

$$2x^2 = 0 \quad \text{or} \quad y - z = 0$$

$$x = 0 \quad (\text{impossible}) \quad \text{or} \quad y = z.$$

Therefore, $x = y = z$. Substituting into equation (4) gives

$$x^3 - 125 = 0$$

$$x^3 = 125$$

$$x = 5.$$

Thus,

$$y = 5 \text{ and } z = 5.$$

The dimensions that will minimize the surface area are 5 m by 5 m by 5 m.

40. Let the dimensions of the bottom be x by y , and let the height be z . We must minimize $f(x, y, z) = xy + 2xz + 2yz$ subject to $xyz = 32$.

$$1. \quad g(x, y, z) = xyz - 32$$

$$\begin{aligned}
 2. \quad & F(x, y, z, \lambda) \\
 & = xy + 2xz + 2yz - \lambda(xyz - 32)
 \end{aligned}$$

$$\begin{aligned}
 3. \quad & F_x(x, y, z, \lambda) = y + 2z - \lambda yz \\
 & F_y(x, y, z, \lambda) = x + 2z - \lambda xz \\
 & F_z(x, y, z, \lambda) = 2x + 2y - \lambda xy \\
 & F_\lambda(x, y, z, \lambda) = -(xyz - 32)
 \end{aligned}$$

$$\begin{aligned}
 4. \quad & y + 2z - \lambda yz = 0 \\
 & x + 2z - \lambda xz = 0 \\
 & 2x + 2y - \lambda xy = 0 \\
 & xyz - 32 = 0
 \end{aligned}$$

$$5. \quad \lambda = \frac{y + 2z}{yz}$$

$$\lambda = \frac{x + 2z}{xz}$$

$$\lambda = \frac{2x + 2y}{xy}$$

$$xyz = 32$$

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz}$$

$$xyz + 2xz^2 = xyz + 2yz^2$$

$$2z^2(x - y) = 0$$

$$z^2 = 0 \quad \text{or} \quad x - y = 0$$

$$z = 0 \quad (\text{impossible}) \quad \text{or} \quad x = y$$

$$\frac{x + 2z}{xz} = \frac{2x + 2y}{xy}$$

$$x^2y + 2xyz = 2x^2z + 2xyz$$

$$x^2(y - 2z) = 0$$

$$x^2 = 0 \quad \text{or} \quad y - 2z = 0$$

$$x = 0 \quad (\text{impossible}) \quad \text{or} \quad y = 2z$$

Since $x = y$ and $y = 2z$ and since $xyz = 32$, we have

$$(2z)(2z)z = 32$$

$$z^3 = 8$$

$$z = 2.$$

If $z = 2$, $y = 4$ and $x = 4$.

The dimensions are 4 feet by 4 feet for the base and 2 feet for the height.

41. (a) The surface area of the box is

$$SA = xy + 2xz + 2yz.$$

Let the sides with area xz be those made from the free material (along with the bottom). This gives the constraint $2xz + xy = 4$. The volume of the box is xyz , thus it will take $\frac{400}{xyz}$ trips to transport the material at a cost of $\frac{400}{xyz}(.10) = \frac{40}{xyz}$.

The total cost also includes the cost of the material for the ends of the box: $(2yz)(20) = 40yz$.

Thus, the total cost is

$$f(x, y, z) = \frac{40}{xyz} + 40yz.$$

- (b) Using a spreadsheet, $x = 2$ yards, $y = 1$ yard, $z = \frac{1}{2}$ yard.

42. (a)
$$\begin{aligned} P(r, s, t) &= rs(1-t) + (1-r)st \\ &\quad + r(1-s)t + rst \\ g(r, s, t) &= r + s + t - \alpha \\ F(r, s, t, \lambda) &= rs(1-t) + (1-r)st \\ &\quad + r(1-s)t + rst \\ &\quad - \lambda(r + s + t - \alpha) \end{aligned}$$

(b)

3.
$$\begin{aligned} F_r(r, s, t, \lambda) &= s(1-t) - st + (1-s)t + st - \lambda \\ &= s + t - 2st - \lambda \\ F_s(r, s, t, \lambda) &= r(1-t) + (1-r)t - rt + rt - \lambda \\ &= r + t - 2rt - \lambda \\ F_t(r, s, t, \lambda) &= -rs + (1-r)s + r(1-s) + rs - \lambda \\ &= r + s - 2rs - \lambda \\ F_\lambda(r, s, t, \lambda) &= -(r + s + t - \alpha) \end{aligned}$$

4.
$$\begin{aligned} s + t - 2st - \lambda &= 0 \\ r + t - 2rt - \lambda &= 0 \\ r + s - 2rs - \lambda &= 0 \\ r + s + t - \alpha &= 0 \end{aligned}$$

5.
$$\begin{aligned} -\lambda &= 2st - s - t \\ -\lambda &= 2rt - r - t \\ -\lambda &= 2rs - r - s \\ r + s + t &= \alpha \\ 2st - s - t &= 2rt - r - t \\ s(2t - 1) &= r(2t - 1) \\ (s - r)(2t - 1) &= 0 \\ s = r \quad \text{or} \quad t &= \frac{1}{2} \\ 2rt - r - t &= 2rs - r - s \\ t(2r - 1) &= s(2r - 1) \\ (t - s)(2r - 1) &= 0 \end{aligned}$$

$$t = s \quad \text{or} \quad r = \frac{1}{2}$$

$$2st - s - t = 2rs - r - s$$

$$t(2s - 1) = r(2s - 1)$$

$$(t - r)(2s - 1) = 0$$

$$t = r \quad \text{or} \quad s = \frac{1}{2}$$

Since r, s and t are probabilities, $0 \leq r, s, t \leq 1$. also, $r + s + t = \alpha = 0.75$. If $t = \frac{1}{2}$, then either $t = s = \frac{1}{2}$ or $r = \frac{1}{2}$, both of which are impossible (the third value would have to be -0.25 to get a sum of 0.75).

Thus, $r = s = t = 0.25$.

- (c) Now we have $r + s + t = \alpha = 3$. If $t = \frac{1}{2}$, then either $t = s = \frac{1}{2}$ or $r = \frac{1}{2}$, both of which are impossible (the third value would have to be 2 to get a sum of 3).

Thus, $r = s = t = 1.0$.

9.5 Total Differentials and Approximations

Your Turn 1

$$f(x, y) = 3x^2y^4 + 6\sqrt{x^2 - 7y^2}$$

$$f_x(x, y) = 6xy^4 + \frac{6x}{\sqrt{x^2 - 7y^2}}$$

$$f_y(x, y) = 12x^2y^3 - \frac{42y}{\sqrt{x^2 - 7y^2}}$$

$$\begin{aligned} \text{(a)} \quad dz &= f_x(x, y)dx + f_y(x, y)dy \\ &= \left(6xy^4 + \frac{6x}{\sqrt{x^2 - 7y^2}} \right) dx \\ &\quad + \left(12x^2y^3 - \frac{42y}{\sqrt{x^2 - 7y^2}} \right) dy \end{aligned}$$

$$\text{(b)} \quad \text{Evaluate } dz \text{ for } x = 4, y = 1, dx = 0.02, dy = -0.03.$$

$$\begin{aligned} dz &= \left(6(4)(1)^4 + \frac{6(4)}{\sqrt{(4)^2 - 7(1)^2}} \right) (0.02) \\ &\quad + \left(12(4)^2(1)^3 - \frac{42(1)}{\sqrt{(4)^2 - 7(1)^2}} \right) (-0.03) \\ &= \left(24 + \frac{24}{3} \right) (0.02) + \left(192 - \frac{42}{3} \right) (-0.03) \\ &= (32)(0.02) - 178(0.03) \\ &= -4.7 \end{aligned}$$

Your Turn 2

As in Example 2, we let $f(x, y) = \sqrt{x^2 + y^2}$. As we found in Example 2,

$$dz = \left(\frac{x}{\sqrt{x^2 + y^2}} \right) dx + \left(\frac{y}{\sqrt{x^2 + y^2}} \right) dy$$

To approximate $\sqrt{5.03^2 + 11.99^2}$ we let $x = 5$, $y = 12$, $dx = 0.03$, $dy = -0.01$.

Then $f(x + dx, y + dy) \approx f(x, y) + dz$

$$\begin{aligned} &\sqrt{5.03^2 + 11.99^2} \\ &\approx \sqrt{5^2 + 12^2} + \left(\frac{5}{\sqrt{5^2 + 12^2}} \right) (0.03) \\ &\quad + \left(\frac{12}{\sqrt{5^2 + 12^2}} \right) (-0.01) \\ &= 13 + \frac{0.15}{13} - \frac{0.12}{13} \\ &\approx 13.0023 \end{aligned}$$

Your Turn 3

As calculated in Example 3, $\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$. Thus the maximum percent error in the volume if the errors in the radius and length are at most 4% and 2% is $\frac{dV}{V} = 2(0.04) + (0.02) = 0.10$ or 10%.

9.5 Exercises

- $z = f(x, y) = 2x^2 + 4xy + y^2$
 $x = 5, dx = 0.03, y = -1, dy = -0.02$
 $f_x(x, y) = 4x + 4y$
 $f_y(x, y) = 4x + 2y$
 $dz = (4x + 4y)dx + (4x + 2y)dy$
 $= [4(5) + 4(-1)](0.03)$
 $\quad + [4(5) + 2(-1)](-0.02)$
 $= 0.48 - 0.36 = 0.12$
- $z = f(x, y) = 5x^3 + 2xy^2 - 4y$
 $x = 1, dx = 0.01, y = 3, dy = 0.02$
 $f_x(x, y) = 15x^2 + 2y^2$
 $f_y(x, y) = 4xy - 4$
 $dz = (15x^2 + 2y^2)dx + (4xy - 4)dy$
 $= [15(1)^2 + 2(3)^2](0.01)$
 $\quad + [4(1)(3) - 4](0.02)$
 $= 0.33 + 0.16 = 0.49$

$$3. \quad z = \frac{y^2 + 3x}{y^2 - x}, \quad x = 4, y = -4,$$

$$dx = 0.01, dy = 0.03$$

$$\begin{aligned} dz &= \frac{(y^2 - x) \cdot 3 - (y^2 + 3x) \cdot (-1)}{(y^2 - x)^2} dx \\ &\quad + \frac{(y^2 - x) \cdot 2y - (y^2 + 3x) \cdot 2y}{(y^2 - x)^2} dy \\ &= \frac{4y^2}{(y^2 - x)^2} dx - \frac{8xy}{(y^2 - x)^2} dy \\ &= \frac{4(-4)^2}{[(-4)^2 - 4]^2} (0.01) - \frac{8(4)(-4)}{[(-4)^2 - 4]^2} (0.03) \\ &\approx 0.0311 \end{aligned}$$

$$4. \quad z = \ln(x^2 + y^2) \\ x = 2, y = 3, dx = 0.02, dy = -0.03$$

$$dz = \frac{2x dx}{x^2 + y^2} + \frac{2y dy}{x^2 + y^2}$$

Substitute the given information.

$$\begin{aligned} dz &= \frac{2(2)(0.02)}{2^2 + 3^2} + \frac{2(3)(-0.03)}{2^2 + 3^2} \\ &= -0.00769 \end{aligned}$$

$$5. \quad w = \frac{5x^2 + y^2}{z + 1}$$

$$x = -2, y = 1, z = 1$$

$$dx = 0.02, dy = -0.03, dz = 0.02$$

$$\begin{aligned} f_x(x, y) &= \frac{(z + 1)10x - (5x^2 + y^2)(0)}{(z + 1)^2} \\ &= \frac{10x}{z + 1} \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \frac{(z + 1)(2y) - (5x^2 + y^2)(0)}{(z + 1)^2} \\ &= \frac{2y}{z + 1} \end{aligned}$$

$$\begin{aligned} f_z(x, y) &= \frac{(z + 1)(0) - (5x^2 + y^2)(1)}{(z + 1)^2} \\ &= \frac{-5x^2 - y^2}{(z + 1)^2} \end{aligned}$$

$$dw = \frac{10x}{z + 1} dx + \frac{2y}{z + 1} dy + \frac{-5x^2 - y^2}{(z + 1)^2} dz$$

Substitute the given values.

$$\begin{aligned} dw &= \frac{-20}{2} (0.02) + \frac{2}{2} (-0.03) \\ &\quad + \frac{[-5(4) - 1](0.02)}{(2)^2} \\ &= -0.2 - 0.03 - \frac{21}{4} (0.02) \\ &= -0.335 \end{aligned}$$

$$6. \quad w = x \ln(yz) - y \ln\left(\frac{x}{z}\right)$$

$$= x(\ln y + \ln z) - y(\ln x - \ln z),$$

$$x = 2, y = 1, z = 4, dx = 0.03, dy = 0.02,$$

$$dz = -0.01$$

$$\begin{aligned} dw &= \left(\ln y + \ln z - \frac{y}{x} \right) dx \\ &\quad + \left(\frac{x}{y} - \ln x + \ln z \right) dy \\ &\quad + \left(\frac{x}{z} + \frac{y}{z} \right) dz \\ &= \left(\ln 1 + \ln 4 - \frac{1}{2} \right) (0.03) \\ &\quad + \left(\frac{2}{1} - \ln 2 + \ln 4 \right) (0.02) \\ &\quad + \left(\frac{2}{4} + \frac{1}{4} \right) (-0.01) \\ &\approx 0.0730 \end{aligned}$$

$$7. \quad \text{Let } z = f(x, y) = \sqrt{x^2 + y^2}.$$

Then

$$\begin{aligned} dz &= f_x(x, y) dx + f_y(x, y) dy \\ &= \frac{1}{2} (x^2 + y^2)^{-1/2} (2x) dx \\ &\quad + \frac{1}{2} (x^2 + y^2)^{-1/2} (2y) dy \\ &= \frac{xdx + ydy}{\sqrt{x^2 + y^2}}. \end{aligned}$$

To approximate $\sqrt{8.05^2 + 5.97^2}$, we let $x = 8$, $dx = 0.05$, $y = 6$ and $dy = -0.03$.

$$\begin{aligned} dz &= \frac{8(0.05) + 6(-0.03)}{\sqrt{8^2 + 6^2}} \\ &= \frac{4}{5} (0.05) + \frac{3}{5} (-0.03) \\ &= 0.04 - 0.018 = 0.022 \end{aligned}$$

$$\begin{aligned}
 f(8.05, 5.97) &= f(8, 6) + \Delta z \\
 &\approx f(8, 6) + dz \\
 &= \sqrt{8^2 + 6^2} + 0.222 \\
 &= 10.022
 \end{aligned}$$

Thus, $\sqrt{8.05^2 + 5.97^2} \approx 10.022$.

Using a calculator, $\sqrt{8.05^2 + 5.97^2} \approx 10.0221$.

The absolute value of the difference of the two results is $|10.022 - 10.0221| = 0.0001$.

8. Let $z = f(x, y) = \sqrt{x^2 + y^2}$.

Then

$$\begin{aligned}
 dz &= f_x(x, y)dx + f_y(x, y)dy \\
 &= \frac{1}{2}(x^2 + y^2)^{-1/2}(2x)dx \\
 &\quad + \frac{1}{2}(x^2 + y^2)^{-1/2}(2y)dy \\
 &= \frac{xdx + ydy}{\sqrt{x^2 + y^2}}.
 \end{aligned}$$

To approximate $\sqrt{4.96^2 + 12.06^2}$, we let $x = 5$, $dx = -0.04$, $y = 12$, and $dy = 0.06$.

$$\begin{aligned}
 dz &= \frac{5(-0.04) + 12(0.06)}{\sqrt{5^2 + 12^2}} \\
 &= \frac{5}{13}(-0.04) + \frac{12}{13}(0.06) \\
 &= 0.04
 \end{aligned}$$

$$\begin{aligned}
 f(4.96, 12.06) &= f(5, 12) + \Delta z \\
 &\approx f(5, 12) + dz \\
 &= \sqrt{5^2 + 12^2} + 0.04 \\
 &= 13.04
 \end{aligned}$$

Thus, $\sqrt{4.96^2 + 12.06^2} \approx 13.04$.

Using a calculator, $\sqrt{4.96^2 + 12.06^2} \approx 13.0401$.

The absolute value of the difference of the two results is $|13.04 - 13.0401| = 0.0001$.

9. Let $z = f(x, y) = (x^2 + y^2)^{1/3}$.

Then

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

$$\begin{aligned}
 dz &= \frac{1}{3}(x^2 + y^2)^{-2/3}(2x)dx \\
 &\quad + \frac{1}{3}(x^2 + y^2)^{-2/3}(2y)dy \\
 &= \frac{2x}{3(x^2 + y^2)^{2/3}}dx + \frac{2y}{3(x^2 + y^2)^{2/3}}dy
 \end{aligned}$$

To approximate $(1.92^2 + 2.1^2)^{1/3}$, we let

$x = 2$, $dx = -0.08$, $y = 2$, and $dy = 0.1$.

$$\begin{aligned}
 dz &= \frac{2(2)}{3[(2)^2 + (2)^2]^{2/3}}(-0.08) \\
 &\quad + \frac{2(2)}{3[(2)^2 + (2)^2]^{2/3}}(0.1) \\
 &= \frac{4}{12}(-0.08) + \frac{4}{12}(0.1) \\
 &= 0.00\bar{6}
 \end{aligned}$$

$$\begin{aligned}
 f(1.92, 2.1) &= f(2, 2) + \Delta z \\
 &\approx f(2, 2) + dz \\
 &= 2 + 0.00\bar{6} \\
 f(1.92, 2.1) &\approx 2.0067
 \end{aligned}$$

Using a calculator, $(1.92^2 + 2.1^2)^{1/3} \approx 2.0080$.

The absolute value of the difference of the two results is $|2.0067 - 2.0080| = 0.0013$.

10. Let $z = f(x, y) = (x^2 - y^2)^{1/3}$.

Then

$$\begin{aligned}
 dz &= f_x(x, y)dx + f_y(x, y)dy \\
 &= \frac{1}{3}(x^2 - y^2)^{-2/3}(2x)dx \\
 &\quad + \frac{1}{3}(x^2 - y^2)^{-2/3}(-2y)dy \\
 &= \frac{2x}{3(x^2 - y^2)^{2/3}}dx - \frac{2y}{3(x^2 - y^2)^{2/3}}dy
 \end{aligned}$$

To approximate $(2.93^2 - 0.94^2)^{1/3}$, we let

$x = 3$, $dx = -0.07$, $y = 1$, and $dy = -0.06$.

$$\begin{aligned}
 dz &= \frac{2(3)}{3[(3)^2 - (1)^2]^{2/3}}(-0.07) \\
 &\quad - \frac{2(1)}{3[(3)^2 - (1)^2]^{2/3}}(-0.06) \\
 &= \frac{1}{2}(-0.07) - \frac{1}{6}(-0.06) \\
 &= -0.025
 \end{aligned}$$

$$\begin{aligned}
 f(2.93, 0.94) &= f(3, 1) + \Delta z \\
 &\approx f(3, 1) + dz \\
 &= 2 + (-0.025) \\
 f(2.93, 0.94) &\approx 1.975
 \end{aligned}$$

Using a calculator, $(2.93^2 - 0.94^2)^{1/3} \approx 1.9748$.

The absolute value of the difference of the two results is $|1.975 - 1.9748| = 0.0002$.

11. Let $z = f(x, y) = xe^y$.

Then

$$\begin{aligned}
 dz &= f_x(x, y)dx + f_y(x, y)dy \\
 &= e^y dx + xe^y dy.
 \end{aligned}$$

To approximate $1.03e^{0.04}$, we let $x = 1$, $dx = 0.03$, $y = 0$, and $dy = 0.04$.

$$\begin{aligned}
 dz &= e^0(0.03) + 1 \cdot e^0(0.04) \\
 &= 0.07 \\
 f(1.03, 0.04) &= f(1, 0) + \Delta z \\
 &\approx f(1, 0) + dz \\
 &= 1 \cdot e^0 + 0.07 \\
 &= 1.07
 \end{aligned}$$

Thus, $1.03e^{0.04} \approx 1.07$.

Using a calculator, $1.03e^{0.04} \approx 1.0720$.

The absolute value of the difference of the two results is $|1.07 - 1.0720| = 0.0020$.

12. Let $z = f(x, y) = xe^y$.

Then

$$\begin{aligned}
 dz &= f_x(x, y)dx + f_y(x, y)dy \\
 &= e^y dx + xe^y dy.
 \end{aligned}$$

To approximate $0.98e^{-0.04}$, we let $x = 1$, $dx = -0.02$, $y = 0$, and $dy = -0.04$.

$$\begin{aligned}
 dz &= e^0(-0.02) + 1 \cdot e^0(-0.04) \\
 &= -0.06 \\
 f(0.98, -0.04) &= f(1, 0) + \Delta z \\
 &\approx f(1, 0) + dz \\
 &= 1 \cdot e^0 - 0.06 \\
 &= 0.94
 \end{aligned}$$

Thus, $0.98e^{-0.04} \approx 0.94$.

Using a calculator, $0.98e^{-0.04} \approx 0.9416$. The absolute value of the difference of the two results is $|0.94 - 0.9416| = 0.0016$.

13. Let $z = f(x, y) = x \ln y$.

Then

$$\begin{aligned}
 dz &= f_x(x, y)dx + f_y(x, y)dy \\
 &= \ln y dx + \frac{x}{y} dy
 \end{aligned}$$

To approximate $0.99 \ln 0.98$, we let $x = 1$, $dx = -0.01$, $y = 1$, and $dy = -0.02$.

$$\begin{aligned}
 dz &= \ln(1) \cdot (-0.01) + \frac{1}{1}(-0.02) \\
 &= -0.02
 \end{aligned}$$

$$\begin{aligned}
 f(0.99, 0.98) &= f(1, 1) + \Delta z \\
 &\approx f(1, 1) + dz \\
 &= 1 \cdot \ln(1) - 0.02 \\
 &\approx -0.02
 \end{aligned}$$

Thus, $0.99 \ln 0.98 \approx -0.02$.

Using a calculator, $0.99 \ln 0.98 \approx -0.0200$.

The absolute value of the difference of the two results is $|-0.02 - (-0.0200)| = 0$.

14. Let $z = f(x, y) = x \ln y$.

Then

$$\begin{aligned}
 dz &= f_x(x, y)dx + f_y(x, y)dy \\
 &= \ln y dx + \frac{x}{y} dy.
 \end{aligned}$$

To approximate $2.03 \ln 1.02$, we let $x = 2$, $dx = 0.03$, $y = 1$, and $dy = 0.02$.

$$\begin{aligned}
 dz &= \ln(1) \cdot (0.03) + \frac{2}{1}(0.02) \\
 &= 0.04
 \end{aligned}$$

$$\begin{aligned}
 f(2.03, 1.02) &= f(2, 1) + \Delta z \\
 &\approx f(2, 1) + dz \\
 &= 2 \cdot \ln(1) + 0.04 \\
 &= 0.04
 \end{aligned}$$

Thus, $2.03 \ln 1.02 \approx 0.04$.

Using a calculator, $2.03 \ln 1.02 \approx 0.0402$.

The absolute value of the difference of the two results is $|0.04 - 0.0402| = 0.0002$.

15. The volume of the can is

$$V = \pi r^2 h,$$

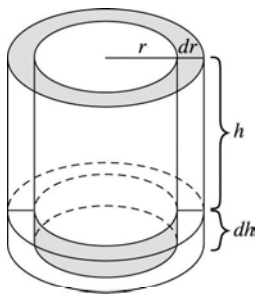
With

$$r = 2.5 \text{ cm}, h = 14 \text{ cm}, dr = 0.08, dh = 0.16.$$

$$\begin{aligned} dV &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi(2.5)(14)(0.08) + \pi(2.5)^2(0.16) \\ &\approx 20.73 \end{aligned}$$

Approximately 20.73 cm^3 of aluminum are needed.

16. Let r be the radius inside the tumbler and h be the height inside.



$$V = \pi r^2 h, r = 1.5, h = 9, dr = dh = 0.2$$

$$\begin{aligned} dV &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi(1.5)(9)(0.2) + \pi(1.5)^2(0.2) \\ &\approx 18.4 \end{aligned}$$

Approximately 18.4 cm^3 of material is needed.

17. The volume of the box is

$$V = LWH$$

with $L = 10$, $W = 9$, and $H = 18$.

Since 0.1 inch is applied to each side and each dimension has a side at each end,

$$\begin{aligned} dL &= dW = dH = 2(0.1) = 0.2 \\ dV &= WH dL + LH dW + LW dH. \end{aligned}$$

Substitute.

$$\begin{aligned} dV &= (9)(18)(0.2) + (10)(18)(0.2) \\ &\quad + (10)(9)(0.2) \\ &= 86.4 \end{aligned}$$

Approximately 86.4 in^3 are needed.

18. $M(x, y) = 45x^2 + 40y^2 - 20xy + 50$
 $x = 8, dx = 8.25 - 8 = 0.25, y = 14,$
 $dy = 13.75 - 14 = -0.25$

$$M_x(x, y) = 90x - 20y$$

$$M_y(x, y) = 80y - 20x$$

$$\begin{aligned} dM &= (90x - 20y)dx \\ &\quad + (80y - 20x)dy \\ &= [90(8) - 20(14)](0.25) \\ &\quad + [80(14) - 20(8)](-0.25) \\ &= 110 - 240 = -130 \end{aligned}$$

The change in cost will be a decrease of \$130.

19. $z = x^{0.65}y^{0.35}$
 $x = 50, y = 29,$
 $dx = 52 - 50 = 2$
 $dy = 27 - 29 = -2$

$$\begin{aligned} f_x(x, y) &= y^{0.35}(0.65)(x^{-0.35}) \\ &= 0.65 \left(\frac{y}{x} \right)^{0.35} \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= (x^{0.65})(0.35)(y^{-0.65}) \\ &= 0.35 \left(\frac{x}{y} \right)^{0.65} \end{aligned}$$

$$dz = 0.65 \left(\frac{y}{x} \right)^{0.35} dx + 0.35 \left(\frac{x}{y} \right)^{0.65} dy$$

Substitute.

$$\begin{aligned} dz &= 0.65 \left(\frac{29}{50} \right)^{0.35} (2) + 0.35 \left(\frac{50}{29} \right)^{0.65} (-2) \\ &= 0.07694 \text{ unit} \end{aligned}$$

20. $z = x^{0.8}y^{0.2}, x = 20, y = 18,$
 $dx = 21 - 20 = 1, dy = 16 - 18 = -2$

$$\begin{aligned} dz &= 0.8x^{-0.2}y^{0.2}dx + 0.2x^{0.8}y^{-0.8}dy \\ &= 0.8 \left(\frac{y^{0.2}}{x^{0.2}} \right) dx + 0.2 \left(\frac{x^{0.8}}{y^{0.8}} \right) dy \\ &= 0.8 \left(\frac{x}{y} \right)^{0.2} dx + 0.2 \left(\frac{x}{y} \right)^{0.8} dy \\ &= 0.8 \left(\frac{18}{20} \right)^{0.2} (1) + 0.2 \left(\frac{20}{18} \right)^{0.8} (-2) \\ &\approx 0.348 \end{aligned}$$

The change in production is 0.348 unit.

21. The volume of the bone is

$$V = \pi r^2 h,$$

$$\begin{aligned} \text{with } h = 7, r = 1.4, dr = 0.09, dh = 2(0.09) \\ = 0.18 \end{aligned}$$

$$\begin{aligned} dV &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi(1.4)(7)(0.09) + \pi(1.4)^2(0.18) \\ &= 6.65 \end{aligned}$$

6.65 cm³ of preservative are used.

22. Assume that blood vessels are cylindrical.

$$\begin{aligned} V &= \pi r^2 h, r = 0.8, h = 7.9, dr = dh \\ &= \pm 0.15 \end{aligned}$$

$$\begin{aligned} dV &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi(0.8)(7.9)(\pm 0.15) \\ &\quad + \pi(0.8)^2(\pm 0.15) \\ &\approx \pm 6.258 \text{ cm}^3 \end{aligned}$$

The maximum possible error is 6.26 cm³.

23. $C = \frac{b}{a-v} = b(a-v)^{-1}$

$$a = 160,$$

$$b = 200, v = 125$$

$$da = 145 - 160 = -15$$

$$db = 190 - 200 = -10$$

$$dv = 130 - 125 = 5$$

$$\begin{aligned} dC &= -b(a-v)^{-2}da \\ &\quad + \frac{1}{a-v}db + b(a-v)^{-2}dv \\ &= \frac{-b}{(a-v)^2}da + \frac{1}{a-v}db + \frac{b}{(a-v)^2}dv \\ &= \frac{-200}{(160-125)^2}(-15) + \frac{1}{160-125}(-10) \\ &\quad + \frac{200}{(160-125)^2}(5) \\ &\approx 2.98 \text{ liters} \end{aligned}$$

24. Using the partial derivatives $H_T(m, T, A)$ and $H_A(m, T, A)$ found in Exercise 54 of Section 2, we have

$$\begin{aligned} H_T(25, 36, 12) &= \frac{15.2(25^{0.67})}{10.23 \ln 25 - 10.74} \\ &= 5.92 \end{aligned}$$

and

$$H_A(25, 36, 12) = -5.92.$$

$$H_m(m, T, A)$$

$$\begin{aligned} &\left[\frac{(15.2)(0.67)m^{-0.33}(10.23 \ln m - 10.74)}{(10.23 \ln m - 10.74)^2} \right. \\ &\quad \left. - 15.2m^{0.67} \left(\frac{10.23}{m} \right) \right] \end{aligned}$$

$$\begin{aligned} H_M(25, 36, 12) &= \frac{24(78.1156 - 53.7525)}{492.3561} \\ &= 1.19 \end{aligned}$$

The estimated change is

$$(1)(1.19) + (0.5)(5.92) + (-2)(-5.92) = 15.99$$

The approximate change in H is 16.0 W.

$$\begin{aligned} 25. \quad C(t, g) &= 0.6(0.96)^{(210t/1500)-1} \\ &\quad + \frac{gt}{126t - 900} \left[1 - (0.96)^{(210t/1500)-1} \right] \end{aligned}$$

(a)

$$\begin{aligned} C(180, 8) &= 0.6(0.96)^{(210(180)/1500)-1} \\ &\quad + \frac{(8)(180)}{126(180) - 900} \left[1 - (0.96)^{(210(180)/1500)-1} \right] \\ &\approx 0.2649 \end{aligned}$$

(b) $C_t(t, g)$

$$\begin{aligned} &= 0.6(\ln 0.96) \left(\frac{210}{1500} \right) (0.96)^{(210t/1500)-1} \\ &\quad + \frac{g(126t - 900) - 126(gt)}{(126t - 900)^2} \\ &\quad \times [1 - (0.96)^{(210t/1500)-1}] \\ &\quad - \frac{gt}{126t - 900} (\ln 0.96) \left(\frac{210}{1500} \right) (0.96)^{(210t/1500)-1} \end{aligned}$$

$$C_g(t, g)$$

$$= \frac{t}{126t - 900} [1 - (0.96)^{(210t/1500)-1}]$$

$$C(180 - 10, 8 + 1)$$

$$\approx C(180, 8) + C_t(180, 8) \cdot (-10)$$

$$+ C_g(180, 8) \cdot (1)$$

$$\approx 0.2649 + (-0.00115)(-10) + 0.00519(1)$$

$$\approx 0.2816$$

$$C(170, 9) \approx 0.2817$$

The approximation is very good.

$$26. \quad V = \frac{h\pi}{3} (r_1^2 + r_1 r_2 + r_2^2)$$

$$(a) \quad V = \frac{40\pi}{3} (5^2 + 5 \cdot 3 + 3^2) \approx 2052.51$$

The volume is about 2052.5 cm³.

(b)

$$\begin{aligned} dV &= V_h dh + V_{r_1} dr_1 + V_{r_2} dr_2 \\ &= \frac{\pi}{3} (r_1^2 + r_1 r_2 + r_2^2) dh + \frac{h\pi}{3} (2r_1 + r_2) dr_1 \\ &\quad + \frac{h\pi}{3} (r_1 + 2r_2) dr_2 \end{aligned}$$

$$dh = 42 - 40 = 2$$

$$dr_1 = 5.1 - 5 = 0.1$$

$$dr_2 = 2.9 - 3 = -0.1$$

$$\begin{aligned} dV(40, 5, 3) &= \frac{\pi}{3} (5^2 + 5 \cdot 3 + 3^2)(2) \\ &\quad + \frac{40\pi}{3} (2 \cdot 5 + 3)(0.1) \\ &\quad + \frac{40\pi}{3} (5 + 2 \cdot 3)(-0.1) \\ &\approx 111.00 \end{aligned}$$

$$\begin{aligned} V(42, 5.1, 2.9) &\approx V(40, 5, 3) + dV \\ &\approx 2052.51 + 111.00 \\ &\approx 2163.51 \end{aligned}$$

Using differentials, the volume is about 2163.5 cm³.

$$\begin{aligned} V(42, 5.1, 2.9) &= \frac{42\pi}{3} (5.1^2 + 5.1 \cdot 2.9 + 2.9^2) \\ &\approx 2164.37 \end{aligned}$$

The volume found by using the original formula is about 2164.37 cm³.

$$27. \quad P(A, B, D) = \frac{1}{1 + e^{3.68 - 0.016A - 0.77B - 0.12D}}$$

(a) Since bird pecking is present, $B = 1$.

$$\begin{aligned} P(150, 1, 20) &= \frac{1}{1 + e^{3.68 - 0.016(150) - 0.77(1) - 0.12(20)}} \\ &= \frac{1}{1 + e^{-1.89}} \approx 0.8688 \end{aligned}$$

The probability is about 87%.

(b) Since bird pecking is not present, $B = 0$.

$$\begin{aligned} P(150, 0, 20) &= \frac{1}{1 + e^{3.68 - 0.016(150) - 0.77(0) - 0.12(20)}} \\ &= \frac{1}{1 + e^{-1.12}} \approx 0.7540 \end{aligned}$$

The probability is about 75%.

(c) Let $B = 0$. To simplify the notation, let $X = 3.68 - 0.016A - 0.12D$. Then

$$\begin{aligned} P(A, 0, D) &= \frac{1}{1 + e^{3.68 - 0.016A - 0.12D}} \\ &= \frac{1}{1 + e^X}. \end{aligned}$$

Some other values that we will need are

$$dA = 160 - 150 = 10$$

$$dD = 25 - 20 = 5$$

$$\begin{aligned} X(150, 20) &= 3.68 - 0.016(150) - 0.12(20) \\ &= -1.12 \end{aligned}$$

$$X_A = \frac{\partial X}{\partial A} = -0.016$$

$$X_D = \frac{\partial X}{\partial D} = -0.12.$$

$$P_A(A, 0, D) = \frac{X_A e^X}{(1 + e^X)^2} = \frac{0.016e^X}{(1 + e^X)^2}$$

$$P_D(A, 0, D) = \frac{X_D e^X}{(1 + e^X)^2} = \frac{0.12e^X}{(1 + e^X)^2}$$

$$\begin{aligned} dP &= P_A(A, 0, D) dA + P_D(A, 0, D) dD \\ &= \frac{0.016e^X}{(1 + e^X)^2} dA + \frac{0.12e^X}{(1 + e^X)^2} dD \end{aligned}$$

Substituting the given and calculated values,

$$\begin{aligned} dP &= \frac{0.016e^{-1.12}}{(1 + e^{-1.12})^2} (10) + \frac{0.12e^{-1.12}}{(1 + e^{-1.12})^2} (5) \\ &= (0.016 \cdot 10 + 0.12 \cdot 5) \frac{e^{-1.12}}{(1 + e^{-1.12})^2} \\ &\approx 0.76 \cdot 0.1855 \approx 0.14. \end{aligned}$$

Therefore,

$$\begin{aligned} P(160, 0, 25) &= P(150, 0, 20) + \Delta P \\ &\approx P(150, 0, 20) + dP \\ &= 0.75 + 0.14 = 0.89. \end{aligned}$$

The probability is about 89%.

Using a calculator, $P(160, 0, 25) \approx 0.8676$, or about 87%.

$$28. \quad t(x, y, p, C) = \frac{\sqrt{x^2 + (y - p)^2}}{331.45 + 0.6C}$$

$$\begin{aligned} (a) \quad t(5, -2, 20, 20) &= \frac{\sqrt{5^2 + (-2 - 20)^2}}{331.45 + 0.6(20)} \\ &= \frac{\sqrt{509}}{343.45} \approx 0.06569 \end{aligned}$$

$$\begin{aligned} t(5, -2, 10, 20) &= \frac{\sqrt{5^2 + (-2 - 10)^2}}{331.45 + 0.6(20)} \\ &= \frac{\sqrt{169}}{343.45} \approx 0.0379 \end{aligned}$$

In a close race, this difference could certainly affect the outcome.

- (b) Since the starter remains stationary, $dx = dy = 0$, so $t_x(x, y, p, C)$ and $t_y(x, y, p, C)$ do not need to be computed.

$$\begin{aligned} t_p(x, y, p, C) &= \frac{-2(y - p)}{2(331.45 + 0.6C)\sqrt{x^2 + (y - p)^2}} \\ &= \frac{p - y}{(331.45 + 0.6C)\sqrt{x^2 + (y - p)^2}} \\ t_C(x, y, p, C) &= -\frac{0.6\sqrt{x^2 + (y - p)^2}}{(331.45 + 0.6C)^2} \end{aligned}$$

$$\begin{aligned} dx &= 0, dy = 0, dp = 0.5, dC = -5 \\ dt &= t_x(5, -2, 20, 20) \cdot 0 + t_y(5, -2, 20, 20) \cdot 0 \\ &\quad + t_p(5, -2, 20, 20) \cdot 0.5 \\ &\quad + t_C(5, -2, 20, 20) \cdot (-5) \\ &= \frac{20 - (-2)}{(331.45 + 0.6(20))\sqrt{5^2 + (-2 - 20)^2}} \cdot 0.5 \\ &\quad - \frac{0.6\sqrt{5^2 + (-2 - 20)^2}}{(331.45 + 0.6(20))^2} \cdot (-5) \\ &\approx 0.001993 \text{ sec} \end{aligned}$$

This is the approximate change in the time when the swimmer stands 0.5 m farther from the starter and the temperature decreases by 5°C .

29. The area is $A = \frac{1}{2}bh$ with $b = 15.8$ cm, $h = 37.5$ cm, $db = 1.1$ cm, and $dh = 0.8$ cm.

$$\begin{aligned} dA &= \frac{1}{2}b dh + \frac{1}{2}h db \\ &= \frac{1}{2}(15.8)(0.8) + \frac{1}{2}(37.5)(1.1) \\ &= 26.945 \end{aligned}$$

The maximum possible error is 26.945 cm^2 .

30. $V = \frac{1}{3}\pi r^2 h$, $r = 3.2$, $h = 9.3$,
 $dr = dh = \pm 0.1$

$$\begin{aligned} dV &= \frac{2}{3}\pi r h dr + \frac{1}{3}\pi r^2 dh \\ &= \frac{2}{3}\pi(3.2)(9.3)(\pm 0.1) + \frac{1}{3}\pi(3.2)^2(\pm 0.1) \\ &\approx \pm 7.305 \end{aligned}$$

The maximum possible error is 7.305 cm^3 .

31. Let $z = f(L, W, H) = LWH$

Then

$$\begin{aligned} dz &= f_L(L, W, H)dL + f_W(L, W, H)dW \\ &\quad + f_H(L, W, H)dH \\ &= WHdL + LHdW + LWdH. \end{aligned}$$

A maximum 1% error in each measurement means that the maximum values of dL , dW , and dH are given by $dL = 0.01L$, $dW = 0.01W$, and $dH = 0.01H$. Therefore,

$$\begin{aligned} dz &= WH(0.01L) + LH(0.01W) + LW(0.01H) \\ &= 0.01LWH + 0.01LWH + 0.01LWH \\ &= 0.03LWH. \end{aligned}$$

Thus, an estimate of the maximum error in calculating the volume is 3%.

32. Let $z = f(r, h) = \frac{1}{3}\pi r^2 h$.

Then $dz = f_r(r, h)dr + f_h(r, h)dh$

$$dz = \frac{2}{3}\pi r h dr + \frac{1}{3}\pi r^2 dh.$$

Since there is a maximum error of $a\%$ in measuring the radius, the maximum value of dr is $\frac{a}{100}r$. Similarly, the maximum value of dh is $\frac{b}{100}h$. Therefore, the maximum value of dz is

$$\begin{aligned} dz &= \frac{2}{3}\pi r h \left(\frac{a}{100}r \right) + \frac{1}{3}\pi r^2 \left(\frac{b}{100}h \right) \\ &= \frac{2a}{100} \left(\frac{1}{3}\pi r^2 h \right) + \frac{b}{100} \left(\frac{1}{3}\pi r^2 h \right) \\ &= \left(\frac{2a + b}{100} \right) \left(\frac{1}{3}\pi r^2 h \right). \end{aligned}$$

The maximum percent error in calculating the volume is $(2a + b)\%$.

Since the volume of a cylinder is $\pi r^2 h$, the maximum percent error in calculating the volume of the cylinder is the same.

33. The volume of a cone is $V = \frac{\pi}{3}r^2h$.

$$\begin{aligned}\frac{dV}{V} &= \frac{\frac{2\pi}{3}rh}{\frac{\pi}{3}r^2h}dr + \frac{\frac{\pi}{3}r^2}{\frac{\pi}{3}r^2h}dh \\ &= \frac{2}{r}dr + \frac{1}{h}dh\end{aligned}$$

When $r = 1$ and $h = 4$, a 1% change in radius changes the volume by 2%, and a 1% change in height changes the volume by $\frac{1}{4}\%$. So the change produced by changing the radius is 8 times the change produced by changing the height.

34. The volume of a cylinder is $V = \pi r^2h$.

$$\begin{aligned}\frac{dV}{V} &= \frac{2\pi rh}{\pi r^2h}dr + \frac{\pi r^2}{\pi r^2h}dh \\ &= \frac{2}{r}dr + \frac{1}{h}dh\end{aligned}$$

We need to use the same units for both dimensions; in inches, $r = 0.5$ and $h = (20)(12) = 240$.

When r changes by a , the relative volume change is $\frac{2}{0.5}a = 4a$. When h changes by a , the relative volume change is $\frac{1}{240}a$. So the radius has a greater effect by a factor of $\frac{4a}{(1/240)a} = 960$.

9.6 Double Integrals

Your Turn 1

$$\begin{aligned}&\int_1^3 (6x^2y^2 + 4xy + 8x^3 + 10y^4 + 3) dy \\ &= (2x^2y^3 + 2xy^2 + 8x^3y + 2y^5 + 3y) \Big|_{y=1}^{y=3} \\ &= (2x^2(3)^3 + 2x(3)^2 + 8x^3(3) + 2(3)^5 + 3(3)) \\ &\quad - (2x^2(1)^3 + 2x(1)^2 + 8x^3(1) + 2(1)^5 + 3(1)) \\ &= 52x^2 + 16x + 16x^3 + 484 + 6 \\ &= 16x^3 + 52x^2 + 16x + 490\end{aligned}$$

Your Turn 2

$$\int_0^2 \left[\int_1^3 (6x^2y^2 + 4xy + 8x^3 + 10y^4 + 3) dy \right] dx$$

Use the result from Your Turn 1 to evaluate the inner integral.

$$\begin{aligned}&\int_0^2 (16x^3 + 52x^2 + 16x + 490) dx \\ &= \left(4x^4 + \frac{52}{3}x^3 + 8x^2 + 490x \right) \Big|_0^2 \\ &= 4(16) + \frac{52}{3}(8) + 8(4) + 490(2) \\ &= \frac{3644}{3}\end{aligned}$$

Integrating in the other order:

$$\begin{aligned}&\int_1^3 \left[\int_0^2 (6x^2y^2 + 4xy + 8x^3 + 10y^4 + 3) dx \right] dy \\ &= \int_1^3 \left[(2x^3y^2 + 2x^2y + 2x^4 + 10xy^4 + 3x) \Big|_{x=0}^{x=2} \right] dy \\ &= \int_1^3 (16y^2 + 8y + 32 + 20y^4 + 6) dy \\ &= \int_1^3 (20y^4 + 16y^2 + 8y + 38) dy \\ &= \left(4y^5 + \frac{16}{3}y^3 + 4y^2 + 38y \right) \Big|_1^3 \\ &= (972 + 144 + 36 + 114) - \left(4 + \frac{16}{3} + 4 + 38 \right) \\ &= \frac{3644}{3}\end{aligned}$$

Your Turn 3

Let the region R be defined by $0 \leq x \leq 5$ and $1 \leq y \leq 6$.

$$\begin{aligned}&\iint_R \frac{1}{\sqrt{x+y+3}} dx dy \\ &= \int_1^6 \int_0^5 \frac{1}{\sqrt{x+y+3}} dx dy \\ &= \int_1^6 \left(2\sqrt{x+y+3} \right) \Big|_{x=0}^{x=5} dy \\ &= 2 \int_1^6 (\sqrt{y+8} - \sqrt{y+3}) dy\end{aligned}$$

$$\begin{aligned}
&= 2 \frac{2}{3} [(y+8)^{3/2} - (y+3)^{3/2}] \Big|_1^6 \\
&= \frac{4}{3} [(14^{3/2} - 9^{3/2}) - (9^{3/2} - 4^{3/2})] \\
&= \frac{4}{3} (14\sqrt{14} - 46) \\
&= \frac{56\sqrt{14} - 184}{3}
\end{aligned}$$

Your Turn 4

The function $4 - x^3 - y^3$ is positive over the region $0 \leq x \leq 1$ and $0 \leq y \leq 1$, so the volume under the surface $z = 4 - x^3 - y^3$ over this region is

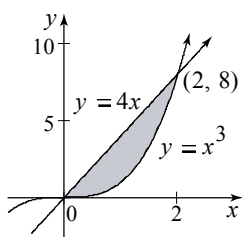
$$\begin{aligned}
&\iint_R (4 - x^3 - y^3) dx dy \\
&= \int_0^1 \int_0^1 (4 - x^3 - y^3) dx dy \\
&= \int_0^1 \int_0^1 (4 - x^3 - y^3) dx dy \\
&= \int_0^1 \left(4x - \frac{x^4}{4} - xy^3 \right) \Big|_{x=0}^{x=1} dy \\
&= \int_0^1 \left(\frac{15}{4} - y^3 \right) dy \\
&= \left(\frac{15}{4}y - \frac{y^4}{4} \right) \Big|_0^1 \\
&= \frac{15}{4} - \frac{1}{4} = \frac{7}{2}
\end{aligned}$$

Your Turn 5

Find $\iint_R (x^3 + 4y) dy dx$ over the region bounded by

$$y = 4x \text{ and } y = x^3 \text{ for } 0 \leq x \leq 2.$$

The region is shown in the figure below.



Note that throughout the region, $4x \geq x^3$.

$$\begin{aligned}
&\iint_R (x^3 + 4y) dy dx \\
&= \int_0^2 \int_{x^3}^{4x} (x^3 + 4y) dy dx \\
&= \int_0^2 (x^3 y + 2y^2) \Big|_{y=x^3}^{y=4x} dx \\
&= \int_0^2 [(4x^4 + 32x^2) - (x^6 + 2x^6)] dx \\
&= \int_0^2 (4x^4 + 32x^2 - 3x^6) dx \\
&= \left(\frac{4}{5}x^5 + \frac{32}{3}x^3 - \frac{3}{7}x^7 \right) \Big|_0^2 \\
&= \frac{128}{5} + \frac{256}{3} - \frac{384}{7} \\
&= \frac{5888}{105}
\end{aligned}$$

9.6 Exercises

- $$\begin{aligned}
1. \quad \int_0^5 (x^4 y + y) dx &= \left(\frac{x^5 y}{5} - xy \right) \Big|_0^5 \\
&= (625y + 5y) - 0 = 630y
\end{aligned}$$
- $$\begin{aligned}
2. \quad \int_1^2 (xy^3 - x) dy &= \left(\frac{xy^4}{4} + xy \right) \Big|_1^2 \\
&= (4x - 2x) - \left(\frac{x}{4} - x \right) \\
&= \frac{11}{4}x
\end{aligned}$$
- $$\begin{aligned}
3. \quad \int_4^5 x \sqrt{x^2 + 3y} dy &= \int_4^5 x(x^2 + 3y)^{1/2} dy \\
&= \frac{2x}{9} [(x^2 + 3y)^{3/2}] \Big|_4^5 \\
&= \frac{2x}{9} [(x^2 + 15)^{3/2} - (x^2 + 12)^{3/2}]
\end{aligned}$$

$$4. \int_3^6 x\sqrt{x^2 + 3y} \, dx$$

Let $u = x^2 + 3y$. Then $du = 2x \, dx$

When $x = 6, u = 36 + 3y$.

When $x = 3, u = 9 + 3y$.

$$\begin{aligned} &= \frac{1}{2} \int_{9+3y}^{36+3y} u^{1/2} \, du \\ &= \frac{1}{2} \cdot \frac{2}{3} (u^{3/2}) \Big|_{9+3y}^{36+3y} \\ &= \frac{1}{3} [(36 + 3y)^{3/2} - (9 + 3y)^{3/2}] \end{aligned}$$

$$\begin{aligned} 5. \int_4^9 \frac{3 + 5y}{\sqrt{x}} \, dx &= (3 + 5y) \int_4^9 x^{-1/2} \, dx \\ &= (3 + 5y) 2x^{1/2} \Big|_4^9 \\ &= (3 + 5y) 2[\sqrt{9} - \sqrt{4}] \\ &= 6 + 10y \end{aligned}$$

$$\begin{aligned} 6. \int_2^7 \frac{3 + 5y}{\sqrt{x}} \, dy &= \left(\frac{3y}{\sqrt{x}} + \frac{5y^2}{2\sqrt{x}} \right) \Big|_2^7 \\ &= \frac{1}{\sqrt{x}} \left[\left(21 + \frac{245}{2} \right) - \left(6 + \frac{20}{2} \right) \right] \\ &= \frac{255}{2\sqrt{x}} \end{aligned}$$

$$\begin{aligned} 7. \int_2^6 e^{2x+3y} \, dx &= \frac{1}{2} e^{2x+3y} \Big|_2^6 \\ &= \frac{1}{2} (e^{12+3y} - e^{4+3y}) \end{aligned}$$

$$\begin{aligned} 8. \int_{-1}^1 e^{2x+3y} \, dy &= \frac{1}{3} e^{2x+3y} \Big|_{-1}^1 \\ &= \frac{1}{3} (e^{2x+3} - e^{2x-3}) \end{aligned}$$

$$9. \int_0^3 ye^{4x+y^2} \, dx$$

Let $u = 4x + y^2$; then $du = 4y \, dx$.

If $y = 0$ then $u = 4x$.

If $y = 3$ then $u = 4x + 9$.

$$\begin{aligned} \int_{4x}^{4x+9} e^u \cdot \frac{1}{2} \, du &= \frac{1}{2} e^u \Big|_{4x}^{4x+9} \\ &= \frac{1}{2} (e^{4x+9} - e^{4x}) \end{aligned}$$

$$10. \int_1^5 ye^{4x+y^2} \, dx$$

Let $u = 4x + y^2$; then $du = 4 \, dx$.

If $x = 1$ then $u = 4 + y^2$.

If $x = 5$ then $u = 20 + y^2$.

$$\begin{aligned} \int_{4+y^2}^{20+y^2} ye^u \cdot \frac{1}{4} \, du &= \frac{1}{4} ye^u \Big|_{4+y^2}^{20+y^2} \\ &= \frac{1}{4} y (e^{20+y^2} - e^{4+y^2}) \end{aligned}$$

$$11. \int_1^2 \int_0^5 (x^4 y + y) \, dx \, dy$$

From Exercise 1

$$\int_0^5 (x^4 y + y) \, dx = 630y.$$

Therefore,

$$\begin{aligned} \int_1^2 \left[\int_0^5 (x^4 y + y) \, dx \right] dy &= \int_1^2 630y \, dy \\ &= 315y^2 \Big|_1^2 \\ &= 315(4 - 1) = 945. \end{aligned}$$

$$12. \int_0^3 \int_1^2 (xy^3 - x) \, dy \, dx$$

From Exercise 2.

$$\int_1^2 (xy^3 - x) \, dy = \frac{11}{4} x.$$

Therefore,

$$\begin{aligned} \int_0^3 \int_1^2 [(xy^3 - x) \, dy] \, dx &= \int_0^3 \frac{11}{4} x \, dx \\ &= \frac{11}{8} x^2 \Big|_0^3 \\ &= \frac{99}{8}. \end{aligned}$$

$$13. \int_0^1 \left[\int_3^6 x\sqrt{x^2 + 3y} dx \right] dy$$

From Exercise 4,

$$\begin{aligned} & \int_3^6 x\sqrt{x^2 + 3y} dx \\ &= \frac{1}{3}[(36 + 3y)^{3/2} - (9 + 3y)^{3/2}] \\ & \int_0^1 \left[\int_3^6 x\sqrt{x^2 + 3y} dx \right] dy \\ &= \int_0^1 \frac{1}{3}[(36 + 3y)^{3/2} - (9 + 3y)^{3/2}] dy \end{aligned}$$

Let $u = 36 + 3y$. Then $du = 3dy$.

When $y = 0$, $u = 36$.

When $y = 1$, $u = 39$.

Let $z = 9 + 3y$. Then $dz = 3dy$.

When $y = 0$, $z = 9$.

When $y = 1$, $z = 12$.

$$\begin{aligned} & \frac{1}{9} \left[\int_{36}^{39} u^{3/2} du - \int_9^{12} z^{3/2} dz \right] \\ &= \frac{1}{9} \cdot \frac{2}{5} [(39)^{5/2} - (36)^{5/2} \\ & \quad - (12)^{5/2} + (9)^{5/2}] \\ &= \frac{2}{45} [(39)^{5/2} - (12)^{5/2} - 6^5 + 3^5] \\ &= \frac{2}{45} (39^{5/2} - 12^{5/2} - 7533) \end{aligned}$$

14. (See Exercise 3.)

$$\begin{aligned} & \int_0^3 \left[\int_4^5 x\sqrt{x^2 + 3y} dy \right] dx \\ &= \int_0^3 \frac{2x}{9} [(x^2 + 15)^{3/2} - (x^2 + 12)^{3/2}] dx \\ &= \frac{2}{45} [(x^2 + 15)^{5/2} - (x^2 + 12)^{5/2}] \Big|_0^3 \\ &= \frac{2}{45} (24^{5/2} - 21^{5/2} - 15^{5/2} + 12^{5/2}) \end{aligned}$$

$$15. \int_1^2 \left[\int_4^9 \frac{3 + 5y}{\sqrt{x}} dx \right] dy$$

From Exercise 5,

$$\begin{aligned} & \int_4^9 \frac{3 + 5y}{\sqrt{x}} dx = 6 + 10y. \\ & \int_1^2 \left[\int_4^9 \frac{3 + 5y}{\sqrt{x}} dx \right] dy \\ &= \int_1^2 (6 + 10y) dy \\ &= 6y \Big|_1^2 + 5y^2 \Big|_1^2 \\ &= 6(2 - 1) + 5(4 - 1) \\ &= 6 + 15 \\ &= 21 \end{aligned}$$

16. (See Exercise 6.)

$$\begin{aligned} & \int_{16}^{25} \left[\int_2^7 \frac{3 + 5y}{\sqrt{x}} dy \right] dx \\ &= \int_{16}^{25} \frac{255}{2\sqrt{x}} dx \\ &= \int_{16}^{25} \frac{255}{2} x^{-1/2} dx \\ &= 255x^{1/2} \Big|_{16}^{25} = 255(5 - 4) \\ &= 255 \end{aligned}$$

$$\begin{aligned} 17. \int_1^3 \int_1^3 \frac{dy dx}{xy} &= \int_1^3 \left[\int_1^3 \frac{1}{xy} dy \right] dx \\ &= \int_1^3 \left(\frac{1}{x} \ln |y| \right) \Big|_1^3 dx \\ &= \int_1^3 \frac{\ln 3}{x} dx \\ &= (\ln 3) \ln |x| \Big|_1^3 \\ &= (\ln 3)(\ln 3 - 0) \\ &= (\ln 3)^2 \end{aligned}$$

$$\begin{aligned}
 18. \quad \int_1^5 \int_2^4 \frac{dx \, dy}{y} &= \int_1^5 \left[\int_2^4 \frac{1}{y} dx \right] dy \\
 &= \int_1^5 \left(\frac{1}{y} x \right) \Big|_2^4 dy \\
 &= \int_1^5 \left[\frac{1}{y} (4 - 2) \right] dy \\
 &= \int_1^5 \frac{2}{y} dy \\
 &= 2 \ln |y| \Big|_1^5 \\
 &= 2 \ln 5
 \end{aligned}$$

$$\begin{aligned}
 19. \quad \int_2^4 \int_3^5 \left(\frac{x}{y} + \frac{y}{3} \right) dx \, dy &= \int_2^4 \left(\frac{x^2}{2y} + \frac{yx}{3} \right) \Big|_3^5 dy \\
 &= \int_2^4 \left[\frac{25}{2y} + \frac{5y}{3} - \left(\frac{9}{2y} + \frac{3y}{3} \right) \right] dy \\
 &= \int_2^4 \left(\frac{16}{2y} + \frac{2y}{3} \right) dy \\
 &= \left(8 \ln |y| + \frac{y^2}{3} \right) \Big|_2^4 \\
 &= 8 (\ln 4 - \ln 2) + \frac{16}{3} - \frac{4}{3} \\
 &= 8 \ln \frac{4}{2} + \frac{12}{3} \\
 &= 8 \ln 2 + 4
 \end{aligned}$$

$$\begin{aligned}
 20. \quad \int_3^4 \int_1^2 \left(\frac{6x}{5} + \frac{y}{x} \right) dx \, dy &= \int_3^4 \left(\frac{3x^2}{5} + y \ln x \right) \Big|_1^2 dy \\
 &= \int_3^4 \left(\frac{12}{5} + y \ln 2 - \frac{3}{5} \right) dy
 \end{aligned}$$

$$\begin{aligned}
 &= \int_3^4 \left(\frac{9}{5} + y \ln 2 \right) dy \\
 &= \left(\frac{9y}{5} + \frac{y^2}{2} \ln 2 \right) \Big|_3^4 \\
 &= \frac{36}{5} + 8 \ln 2 - \frac{27}{5} - \frac{9}{2} \ln 2 \\
 &= \frac{9}{5} + \frac{7}{2} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \iint_R (3x^2 + 4y) dx \, dy; \\
 0 \leq x \leq 3, 1 \leq y \leq 4
 \end{aligned}$$

$$\begin{aligned}
 \iint_R (3x^2 + 4y) dx \, dy &= \int_1^4 \int_0^3 (3x^2 + 4y) dx \, dy \\
 &= \int_1^4 (x^3 + 4xy) \Big|_0^3 dy \\
 &= \int_1^4 (27 + 12y) dy \\
 &= (27y + 6y^2) \Big|_1^4 \\
 &= (108 + 96) - (27 + 6) = 171
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \iint_R (x^2 + 4y^3) dy \, dx; \\
 1 \leq x \leq 2, 0 \leq y \leq 3
 \end{aligned}$$

$$\begin{aligned}
 \iint_R (x^2 + 4y^3) dy \, dx &= \int_1^2 \int_0^3 (x^2 + 4y^3) dy \, dx \\
 &= \int_1^2 (x^2 y + y^4) \Big|_0^3 dx \\
 &= \int_1^2 (3x^2 + 81) dx \\
 &= (x^3 + 81x) \Big|_1^2 \\
 &= (8 + 162) - (1 + 81) = 88
 \end{aligned}$$

$$23. \iint_R \sqrt{x+y} \, dy \, dx; 1 \leq x \leq 3, 0 \leq y \leq 1$$

$$\begin{aligned} & \iint_R \sqrt{x+y} \, dy \, dx \\ &= \int_1^3 \int_0^1 (x+y)^{1/2} \, dy \, dx \\ &= \int_1^3 \left[\frac{2}{3} (x+y)^{3/2} \right]_0^1 \, dx \\ &= \int_1^3 \frac{2}{3} [(x+1)^{3/2} - x^{3/2}] \, dx \\ &= \frac{2}{3} \cdot \frac{2}{5} [(x+1)^{5/2} - x^{5/2}]_1^3 \\ &= \frac{4}{15} (4^{5/2} - 3^{5/2} - 2^{5/2} + 1^{5/2}) \\ &= \frac{4}{15} (32 - 3^{5/2} - 2^{5/2} + 1) \\ &= \frac{4}{15} (33 - 3^{5/2} - 2^{5/2}) \end{aligned}$$

$$24. \iint_R x^2 \sqrt{x^3 + 2y} \, dx \, dy; \\ 0 \leq x \leq 2, 0 \leq y \leq 3$$

$$\begin{aligned} & \iint_R x^2 \sqrt{x^3 + 2y} \, dx \, dy \\ &= \int_0^3 \int_0^2 x^2 (x^3 + 2y)^{1/2} \, dx \, dy \\ &= \int_0^3 \frac{2}{9} (x^3 + 2y)^{3/2} \Big|_0^2 \, dy \\ &= \int_0^3 \frac{2}{9} [(8 + 2y)^{3/2} - (2y)^{3/2}] \, dy \\ &= \frac{2}{45} [(8 + 2y)^{5/2} - (2y)^{5/2}]_0^3 \\ &= \frac{2}{45} (14^{5/2} - 6^{5/2} - 8^{5/2}) \end{aligned}$$

$$25. \iint_R \frac{3}{(x+y)^2} \, dy \, dx; 2 \leq x \leq 4, 1 \leq y \leq 6$$

$$\begin{aligned} & \iint_R \frac{3}{(x+y)^2} \, dy \, dx \\ &= -3 \int_2^4 \int_1^6 (x+y)^{-2} \, dy \, dx \\ &= -3 \int_2^4 (x+y)^{-1} \Big|_1^6 \, dx \\ &= -3 \int_2^4 \left(\frac{1}{x+6} - \frac{1}{x+1} \right) \, dx \\ &= -3 (\ln|x+6| - \ln|x+1|) \Big|_2^4 \\ &= -3 \left(\ln \left| \frac{x+6}{x+1} \right| \right) \Big|_2^4 \\ &= -3 \left(\ln 2 - \ln \frac{8}{3} \right) \\ &= -3 \ln \frac{2}{8/3} \\ &= -3 \ln \frac{3}{4} \text{ or } 3 \ln \frac{4}{3} \end{aligned}$$

$$26. \iint_R \frac{y}{\sqrt{2x+5y^2}} \, dx \, dy; \\ 0 \leq x \leq 2, 1 \leq y \leq 3$$

$$\begin{aligned} & \iint_R \frac{y}{\sqrt{2x+5y^2}} \, dx \, dy \\ &= \int_1^3 \int_0^2 y(2x+5y^2)^{-1/2} \, dx \, dy \\ &= \int_1^3 y \cdot \frac{1}{2} \cdot 2(2x+5y^2)^{1/2} \Big|_0^2 \, dy \\ &= \int_1^3 [y(4+5y^2)^{1/2} - y(5y^2)^{1/2}] \, dy \\ &= \left[\frac{1}{10} \cdot \frac{2}{3} (4+5y^2)^{3/2} - \frac{1}{10} \cdot \frac{2}{3} (5y^2)^{3/2} \right]_1^3 \\ &= \frac{1}{15} [(49^{3/2} - 45^{3/2}) - (9^{3/2} - 5^{3/2})] \\ &= \frac{1}{15} (343 - 135\sqrt{5} - 27 + 5\sqrt{5}) \\ &= \frac{1}{15} (316 - 130\sqrt{5}) \end{aligned}$$

27. $\iint_R ye^{(x+y^2)} dx dy; 2 \leq x \leq 3, 0 \leq y \leq 2$

$$\begin{aligned} & \iint_R ye^{(x+y^2)} dx dy \\ &= \int_0^2 \int_2^3 ye^{x+y^2} dx dy \\ &= \int_0^2 ye^{x+y^2} \Big|_2^3 dy \\ &= \int_0^2 (ye^{3+y^2} - ye^{2+y^2}) dy \\ &= e^3 \int_0^2 ye^{y^2} dy - e^2 \int_0^2 ye^{y^2} dy \\ &= \frac{e^3}{2} (e^{y^2}) \Big|_0^2 - \frac{e^2}{2} (e^{y^2}) \Big|_0^2 \\ &= \frac{e^3}{2} (e^4 - e^0) - \frac{e^2}{2} (e^4 - e^0) \\ &= \frac{1}{2} (e^7 - e^6 - e^3 + e^2) \end{aligned}$$

28. $\iint_R x^2 e^{x^3+2y} dx dy; 1 \leq x \leq 2, 1 \leq y \leq 3$

$$\begin{aligned} & \iint_R x^2 e^{x^3+2y} dx dy \\ &= \int_1^3 \int_1^2 x^2 e^{x^3+2y} dx dy \\ &= \int_1^3 \frac{1}{3} e^{x^3+2y} \Big|_1^2 dy \\ &= \int_1^3 \frac{1}{3} (e^{8+2y} - e^{1+2y}) dy \\ &= \frac{1}{6} (e^{8+2y} - e^{1+2y}) \Big|_1^3 \\ &= \frac{1}{6} (e^{14} - e^7 - e^{10} + e^3) \end{aligned}$$

29. $z = 8x + 4y + 10; -1 \leq x \leq 1, 0 \leq y \leq 3$

$$\begin{aligned} V &= \int_{-1}^1 \int_0^3 (8x + 4y + 10) dy dx \\ &= \int_{-1}^1 (8xy + 2y^2 + 10y) \Big|_0^3 dx \\ &= \int_{-1}^1 (24x + 18 + 30 - 0) dx \\ &= \int_{-1}^1 (24x + 48) dx \\ &= (12x^2 + 48x) \Big|_{-1}^1 \\ &= (12 + 48) - (12 - 48) = 96 \end{aligned}$$

30. $z = 3x + 10y + 20; 0 \leq x \leq 3, -2 \leq y \leq 1$

$$\begin{aligned} V &= \int_0^3 \int_{-2}^1 (3x + 10y + 20) dy dx \\ &= \int_0^3 (3xy + 5y^2 + 20y) \Big|_{-2}^1 dx \\ &= \int_0^3 [(3x + 5 + 20) - (-6x + 20 - 40)] dx \\ &= \int_0^3 (9x + 45) dx \\ &= \left(\frac{9}{2} x^2 + 45x \right) \Big|_0^3 \\ &= \frac{81}{2} + 135 - 0 = \frac{351}{2} \end{aligned}$$

31. $z = x^2; 0 \leq x \leq 2, 0 \leq y \leq 5$

$$\begin{aligned} V &= \int_0^2 \int_0^5 x^2 dy dx \\ &= \int_0^2 x^2 y \Big|_0^5 dx \\ &= \int_0^2 5x^2 dx \\ &= \frac{5}{3} x^3 \Big|_0^2 \\ &= \frac{40}{3} \end{aligned}$$

32. $z = \sqrt{y}; 0 \leq x \leq 4, 0 \leq y \leq 9$

$$\begin{aligned} V &= \int_0^4 \int_0^9 y^{1/2} dy dx \\ &= \int_0^4 \left. \frac{2}{3} y^{3/2} \right|_0^9 dx = \int_0^4 \frac{2}{3} (27) dx \\ &= \int_0^4 18 dx = 18x \Big|_0^4 \\ &= 72 \end{aligned}$$

$$\begin{aligned} &= \int_0^1 \frac{1}{3} [y(16 + y^2)^{3/2} - y^4] dy \\ &= \frac{1}{3} \left[\frac{1}{5} (16 + y^2)^{5/2} - \frac{y^5}{5} \right] \Big|_0^1 \\ &= \frac{1}{15} (17^{5/2} - 1 - 16^{5/2}) \\ &= \frac{1}{15} (17^{5/2} - 1 - 1024) \\ &= \frac{1}{15} (17^{5/2} - 1025) \end{aligned}$$

33. $z = x\sqrt{x^2 + y}; 0 \leq x \leq 1, 0 \leq y \leq 1$

$$V = \int_0^1 \int_0^1 x\sqrt{x^2 + y} dx dy$$

Let $u = x^2 + y$. Then $du = 2x dx$.

When $x = 0, u = y$.

When $x = 1, u = 1 + y$.

$$\begin{aligned} &= \int_0^1 \left[\int_y^{1+y} u^{1/2} du \right] dy \\ &= \int_0^1 \frac{1}{2} \left(\frac{2}{3} u^{3/2} \right) \Big|_y^{1+y} dy \\ &= \int_0^1 \frac{1}{3} [(1+y)^{3/2} - y^{3/2}] dy \\ &= \frac{1}{3} \cdot \frac{2}{5} [(1+y)^{5/2} - y^{5/2}] \Big|_0^1 \\ &= \frac{2}{15} (2^{5/2} - 1 - 1) \\ &= \frac{2}{15} (2^{5/2} - 2) \end{aligned}$$

34. $z = yx\sqrt{x^2 + y^2}; 0 \leq x \leq 4, 0 \leq y \leq 1$

$$\begin{aligned} V &= \int_0^1 \int_0^4 yx(x^2 + y^2)^{1/2} dx dy \\ &= \int_0^1 \frac{y}{3} (x^2 + y^2)^{3/2} \Big|_0^4 dy \\ &= \int_0^1 \frac{y}{3} [(16 + y^2)^{3/2} - (y^2)^{3/2}] dy \end{aligned}$$

35. $z = \frac{xy}{(x^2 + y^2)^2}; 1 \leq x \leq 2, 1 \leq y \leq 4$

$$\begin{aligned} V &= \int_1^2 \int_1^4 \frac{xy}{(x^2 + y^2)^2} dy dx \\ &= \int_1^2 \left[\int_1^4 xy(x^2 + y^2)^{-2} dy \right] dx \\ &= \int_1^2 \left[\int_1^4 \frac{1}{2} x(x^2 + y^2)^{-2} (2y) dy \right] dx \\ &= \int_1^2 \left[-\frac{1}{2} x(x^2 + y^2)^{-1} \right] \Big|_1^4 dx \\ &= \int_1^2 \left[-\frac{1}{2} x(x^2 + 16)^{-1} + \frac{1}{2} x(x^2 + 1)^{-1} \right] dx \\ &= -\frac{1}{2} \int_1^2 \frac{1}{2} (x^2 + 16)^{-1} (2x) dx \\ &\quad + \frac{1}{2} \int_1^2 \frac{1}{2} (x^2 + 1)^{-1} (2x) dx \\ &= -\frac{1}{2} \cdot \frac{1}{2} \ln |x^2 + 16| \Big|_1^2 \\ &\quad + \frac{1}{2} \cdot \frac{1}{2} \ln |x^2 + 1| \Big|_1^2 \\ &= -\frac{1}{4} \cdot \ln 20 + \frac{1}{4} \ln 17 \\ &\quad + \frac{1}{4} \ln 5 - \frac{1}{4} \ln 2 \\ &= \frac{1}{4} (-\ln 20 + \ln 17 + \ln 5 - \ln 2) \\ &= \frac{1}{4} \ln \frac{(17)(5)}{(20)(2)} \\ &= \frac{1}{4} \ln \frac{17}{8} \end{aligned}$$

36. $z = e^{x+y}; 0 \leq x \leq 1, 0 \leq y \leq 1$

$$\begin{aligned} V &= \int_0^1 \int_0^1 e^{x+y} dx dy \\ &= \int_0^1 e^{x+y} \Big|_0^1 dy \\ &= \int_0^1 (e^{1+y} - e^y) dy \\ &= (e^{1+y} - e^y) \Big|_0^1 \\ &= e^2 - e - e + 1 \\ &= e^2 - 2e + 1 \end{aligned}$$

37. $\iint_R x e^{xy} dx dy; 0 \leq x \leq 2; 0 \leq y \leq 1$

$$\begin{aligned} &\iint_R x e^{xy} dx dy \\ &= \int_0^2 \int_0^1 x e^{xy} dy dx \\ &= \int_0^2 \frac{x}{x} e^{xy} \Big|_0^1 dx \\ &= \int_0^2 (e^x - e^0) dx \\ &= (e^x - x) \Big|_0^2 \\ &= e^2 - 2 - e^0 + 0 \\ &= e^2 - 3 \end{aligned}$$

38. $\iint_R 2x^3 e^{x^2 y} dx dy; 0 \leq x \leq 1, 0 \leq y \leq 1$

$$\begin{aligned} &\iint_R 2x^3 e^{x^2 y} dx dy \\ &= \int_0^1 \int_0^1 2x^3 e^{x^2 y} dy dx \\ &= \int_0^1 \left(2x^3 \cdot \frac{1}{x^2} e^{x^2 y} \right) \Big|_0^1 dx \\ &= \int_0^1 (2x e^{x^2} - 2x e^0) dx \\ &= (e^{x^2} - x^2) \Big|_0^1 \\ &= (e^1 - 1) - (e^0 - 0) \\ &= e - 2 \end{aligned}$$

39. $\int_2^4 \int_2^{x^2} (x^2 + y^2) dy dx$

$$\begin{aligned} &= \int_2^4 \left(x^2 y + \frac{y^3}{3} \right) \Big|_2^{x^2} dx \\ &= \int_2^4 \left(x^4 + \frac{x^6}{3} - 2x^2 - \frac{8}{3} \right) dx \\ &= \left(\frac{x^5}{5} + \frac{x^7}{21} - \frac{2}{3} x^3 - \frac{8}{3} x \right) \Big|_2^4 \\ &= \frac{1024}{5} + \frac{16,384}{21} - \frac{2}{3}(64) - \frac{8}{3}(4) \\ &\quad - \left(\frac{32}{5} + \frac{128}{21} - \frac{16}{3} - \frac{16}{3} \right) \\ &= \frac{1024}{5} - \frac{32}{5} + \frac{16,384 - 128}{21} \\ &\quad - \frac{128}{3} - \frac{32}{3} - \left(\frac{-32}{3} \right) \\ &= \frac{992}{5} + \frac{16,256}{21} - \frac{128}{3} \\ &= \frac{20,832}{105} + \frac{81,280}{105} - \frac{4480}{105} \\ &= \frac{97,632}{105} \end{aligned}$$

40. $\int_0^2 \int_0^{3y} (x^2 + y) dx dy$

$$\begin{aligned} &= \int_0^2 \left(\frac{x^3}{3} + xy \right) \Big|_0^{3y} dy \\ &= \int_0^2 (9y^3 + 3y^2) dy \\ &= \left(\frac{9}{4} y^4 + y^3 \right) \Big|_0^2 \\ &= 36 + 8 - 0 = 44 \end{aligned}$$

$$\begin{aligned}
41. \quad & \int_0^4 \int_0^x \sqrt{xy} \, dy \, dx \\
&= \int_0^4 \int_0^x (xy)^{1/2} \, dy \, dx \\
&= \int_0^4 \left[\frac{2(xy)^{3/2}}{3x} \right]_0^x \, dx \\
&= \frac{2}{3} \int_0^4 \left[\frac{(\sqrt{x^2})^3}{x} - \frac{0}{x} \right] \, dx \\
&= \frac{2}{3} \int_0^4 x^2 \, dx = \frac{2}{3} \cdot \frac{x^3}{3} \bigg|_0^4 = \frac{2}{9} (64) \\
&= \frac{128}{9}
\end{aligned}$$

$$\begin{aligned}
42. \quad & \int_1^4 \int_0^x \sqrt{x+y} \, dy \, dx \\
&= \int_1^4 \int_0^x (x+y)^{1/2} \, dy \, dx \\
&= \int_1^4 \frac{2}{3} (x+y)^{3/2} \bigg|_0^x \, dx \\
&= \int_1^4 \frac{2}{3} [(2x)^{3/2} - x^{3/2}] \, dx \\
&= \frac{2}{3} \left[\frac{1}{5} (2x)^{5/2} - \frac{2}{5} x^{5/2} \right] \bigg|_1^4 \\
&= \frac{2}{15} (8^{5/2} - 2(4)^{5/2} - 2^{5/2} + 2) \\
&= \frac{2}{15} (8^{5/2} - 64 - 2^{5/2} + 2) \\
&= \frac{2}{15} (8^{5/2} - 62 - 2^{5/2})
\end{aligned}$$

$$\begin{aligned}
43. \quad & \int_2^6 \int_{2y}^{4y} \frac{1}{x} \, dx \, dy \\
&= \int_2^6 (\ln |x|) \bigg|_{2y}^{4y} \, dy \\
&= \int_2^6 (\ln |4y| - \ln |2y|) \, dy \\
&= \int_2^6 \ln \left| \frac{4y}{2y} \right| \, dy \\
&= \int_2^6 \ln 2 \, dy \\
&= (\ln 2)y \bigg|_2^6 \\
&= (\ln 2)(6 - 2) = 4 \ln 2
\end{aligned}$$

Note: We can write $4 \ln 2$ as $\ln 2^4$, or $\ln 16$.

$$\begin{aligned}
44. \quad & \int_1^4 \int_x^{x^2} \frac{1}{y} \, dy \, dx \\
&= \int_1^4 \ln y \bigg|_x^{x^2} \, dx \\
&= \int_1^4 (\ln x^2 - \ln x) \, dx \\
&= \int_1^4 (2 \ln x - \ln x) \, dx \\
&= \int_1^4 \ln x \, dx \\
&= (x \ln x - \int dx) \bigg|_1^4 \\
&\quad \text{Integration by parts} \\
&\quad u = \ln x, dv = dx \\
&\quad du = \frac{1}{x} dx, v = x \\
&= (x \ln x - x) \bigg|_1^4 \\
&= 4 \ln 4 - 4 + 1 \\
&= 4 \ln 4 - 3
\end{aligned}$$

Note: We can write $\ln y$ instead of $\ln |y|$ since x is in $[1, 4]$ and y is in $[x, x^2]$, so $y > 0$.

$$\begin{aligned}
 45. \quad & \int_0^4 \int_1^{e^x} \frac{x}{y} dy dx \\
 &= \int_0^4 (x \ln |y|) \Big|_1^{e^x} dx \\
 &= \int_0^4 (x \ln e^x - x \ln 1) dx \\
 &= \int_0^4 x^2 dx = \frac{x^3}{3} \Big|_0^4 = \frac{64}{3}
 \end{aligned}$$

$$\begin{aligned}
 46. \quad & \int_0^1 \int_{2x}^{4x} e^{x+y} dy dx \\
 &= \int_0^1 e^{x+y} \Big|_{2x}^{4x} dx \\
 &= \int_0^1 (e^{5x} - e^{3x}) dx \\
 &= \left(\frac{1}{5} e^{5x} - \frac{1}{3} e^{3x} \right) \Big|_0^1 \\
 &= \frac{1}{5} e^5 - \frac{1}{3} e^3 - \frac{1}{5} + \frac{1}{3} \\
 &= \frac{e^5}{5} - \frac{e^3}{3} + \frac{2}{15}
 \end{aligned}$$

$$\begin{aligned}
 47. \quad & \iint_R (5x + 8y) dy dx; 1 \leq x \leq 3, \\
 & \quad 0 \leq y \leq x - 1 \\
 & \iint_R (5x + 8y) dy dx \\
 &= \int_1^3 \int_0^{x-1} (5x + 8y) dy dx \\
 &= \int_1^3 (5xy + 4y^2) \Big|_0^{x-1} dx \\
 &= \int_1^3 [5x(x-1) + 4(x-1)^2 - 0] dx \\
 &= \int_1^3 (9x^2 - 13x + 4) dx \\
 &= \left(3x^3 - \frac{13}{2}x^2 + 4x \right) \Big|_1^3 \\
 &= \left(81 - \frac{117}{2} + 12 \right) - \left(3 - \frac{13}{2} + 4 \right) \\
 &= 34
 \end{aligned}$$

$$\begin{aligned}
 48. \quad & \iint_R (2x + 6y) dy dx; \\
 & \quad 2 \leq x \leq 4, 2 \leq y \leq 3x \\
 & \iint_R (2x + 6y) dy dx \\
 &= \int_2^4 \int_2^{3x} (2x + 6y) dy dx \\
 &= \int_2^4 [2xy + 3y^2] \Big|_2^{3x} dx \\
 &= \int_2^4 [2x(3x) + 3(3x)^2 - (4x + 12)] dx \\
 &= \int_2^4 [33x^2 - 4x - 12] dx \\
 &= (11x^3 - 2x^2 - 12x) \Big|_2^4 \\
 &= (704 - 32 - 48) - (88 - 8 - 24) \\
 &= 568
 \end{aligned}$$

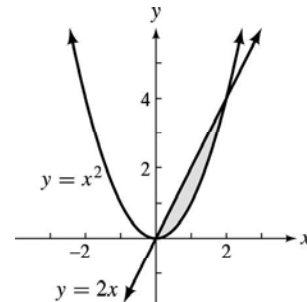
$$\begin{aligned}
 49. \quad & \iint_R (4 - 4x^2) dy dx; 0 \leq x \leq 1, \\
 & \quad 0 \leq y \leq 2 - 2x \\
 & \iint_R (4 - 4x^2) dy dx \\
 &= \int_0^1 \int_0^{2-2x} 4(1 - x^2) dy dx \\
 &= \int_0^1 [4(1 - x^2)y] \Big|_0^{2(1-x)} dx \\
 &= \int_0^1 4(1 - x^2)(2)(1 - x) dx \\
 &= 8 \int_0^1 (1 - x - x^2 + x^3) dx \\
 &= 8 \left(x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^1 \\
 &= 8 \left(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \right) \\
 &= 8 \left(\frac{1}{2} - \frac{1}{12} \right) \\
 &= 8 \cdot \frac{5}{12} = \frac{10}{3}
 \end{aligned}$$

$$\begin{aligned}
 50. \quad & \iint_R \frac{dy \, dx}{x}; 1 \leq x \leq 2, 0 \leq y \leq x-1 \\
 & \iint_R \frac{dy \, dx}{x} \\
 &= \int_1^2 \int_0^{x-1} \frac{dy \, dx}{x} \\
 &= \int_1^2 \left. \frac{y}{x} \right|_0^{x-1} dx \\
 &= \int_1^2 \frac{x-1}{x} dx \\
 &= \int_1^2 \left[1 - \frac{1}{x} \right] dx \\
 &= (x - \ln x) \Big|_1^2 \\
 &= 2 - \ln 2 - 1 \\
 &= 1 - \ln 2
 \end{aligned}$$

$$\begin{aligned}
 51. \quad & \iint_R e^{x/y^2} dx \, dy; 1 \leq y \leq 2, 0 \leq x \leq y^2 \\
 & \iint_R e^{x/y^2} dx \, dy \\
 &= \int_1^2 \int_0^{y^2} e^{x/y^2} dx \, dy \\
 &= \int_1^2 \left[y^2 e^{x/y^2} \right]_0^{y^2} dy \\
 &= \int_1^2 (y^2 e^{y^2/y^2} - y^2 e^0) dy \\
 &= \int_1^2 (ey^2 - y^2) dy \\
 &= (e-1) \frac{y^3}{3} \Big|_1^2 \\
 &= (e-1) \left(\frac{8}{3} - \frac{1}{3} \right) \\
 &= \frac{7(e-1)}{3}
 \end{aligned}$$

$$\begin{aligned}
 52. \quad & \iint_R (x^2 - y) dy \, dx; -1 \leq x \leq 1, \\
 & -x^2 \leq y \leq x^2 \\
 & \iint_R (x^2 - y) dy \, dx \\
 &= \int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y) dy \, dx \\
 &= \int_{-1}^1 \left[x^2 y - \frac{y^2}{2} \right]_{-x^2}^{x^2} dx \\
 &= \int_{-1}^1 \left[x^4 - \frac{x^4}{2} + x^4 + \frac{x^4}{2} \right] dx \\
 &= \int_{-1}^1 2x^4 dx = \frac{2x^5}{5} \Big|_{-1}^1 \\
 &= \frac{2}{5} + \frac{2}{5} = \frac{4}{5}
 \end{aligned}$$

$$53. \quad \iint_R x^3 y \, dy \, dx; R \text{ bounded by } y = x^2, y = 2x$$



The points of intersection can be determined by solving the following system for x .

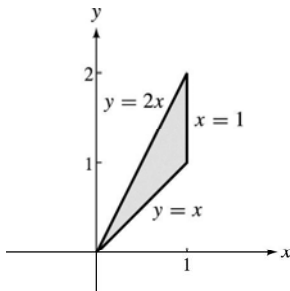
$$\begin{aligned}
 y &= x^2 \\
 y &= 2x \\
 x^2 &= 2x \\
 x(x-2) &= 0 \\
 x &= 0 \quad \text{or} \quad x = 2
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \iint_R x^3 y \, dy \, dx \\
 &= \int_0^2 \int_{x^2}^{2x} x^3 y \, dy \, dx = \int_0^2 \left[x^3 \frac{y^2}{2} \right]_{x^2}^{2x} dx
 \end{aligned}$$

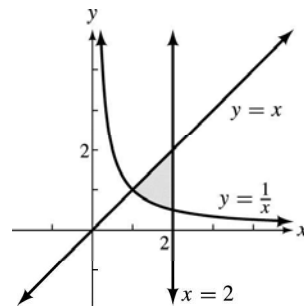
$$\begin{aligned}
&= \int_0^2 \left[x^3 \frac{(4x^2)}{2} - x^3 \frac{(x^4)}{2} \right] dx \\
&= \int_0^2 \left(2x^5 - \frac{x^7}{2} \right) dx \\
&= \left(\frac{1}{3}x^6 - \frac{1}{16}x^8 \right) \Big|_0^2 \\
&= \frac{1}{3} \cdot 2^6 - \frac{1}{16} \cdot 2^8 \\
&= \frac{64}{3} - 16 \\
&= \frac{16}{3}.
\end{aligned}$$

54. $\iint_R x^2 y^2 dy dx$; R bounded by $y = x$, $y = 2x$, and $x = 1$



$$\begin{aligned}
&\int_0^1 \int_x^{2x} x^2 y^2 dy dx \\
&= \int_0^1 \left. \frac{x^2 y^3}{3} \right|_x^{2x} dx \\
&= \int_0^1 \left(\frac{8x^5}{3} - \frac{x^5}{3} \right) dx \\
&= \int_0^1 \frac{7x^5}{3} dx \\
&= \left. \frac{7x^6}{18} \right|_0^1 \\
&= \frac{7}{18}
\end{aligned}$$

55. $\iint_R \frac{dy dx}{y}$; R bounded by $y = x$, $y = \frac{1}{x}$, $x = 2$.



The graphs of $y = x$ and $y = \frac{1}{x}$ intersect at (1, 1).

$$\begin{aligned}
\int_1^2 \int_{1/x}^x \frac{dy}{y} dx &= \int_1^2 \ln y \Big|_{1/x}^x dx \\
&= \int_1^2 \left(\ln x - \ln \frac{1}{x} \right) dx \\
&= \int_1^2 2 \ln x dx \\
&= 2(x \ln x - x) \Big|_1^2 \\
&= 2[(2 \ln 2 - 2) - (\ln 1 - 1)] \\
&= 4 \ln 2 - 2
\end{aligned}$$

56. $\iint_R e^{2y/x} dy dx$; R bounded by $y = x^2$, $y = 0$, $x = 2$.

The region lies between the curve $y = x^2$ and the x -axis between $x = 0$ and $x = 2$.

$$\begin{aligned}
&\iint_R e^{2y/x} dy dx \\
&= \int_0^2 \int_0^{x^2} e^{2y/x} dy dx \\
&= \int_0^2 \left(\frac{x}{2} e^{2y/x} \right) \Big|_{y=0}^{y=x^2} dx \\
&= \int_0^2 \left(\frac{x}{2} e^{2x} - \frac{x}{2} \right) dx \\
&= \frac{1}{2} \int_0^2 x e^{2x} dx - \frac{1}{2} \int_0^2 x dx
\end{aligned}$$

For the first integral, use integration by parts with $u = x$ and $dv = e^{2x} dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$.

$$\begin{aligned}\frac{1}{2} \int_0^2 x e^{2x} dx &= \frac{1}{2} \left(\frac{1}{2} x e^{2x} \Big|_0^2 - \frac{1}{2} \int_0^2 e^{2x} dx \right) \\ &= \frac{1}{4} (2e^4) - \frac{1}{4} \left(\frac{1}{2} e^{2x} \right) \Big|_0^2 \\ &= \frac{1}{2} e^4 - \frac{1}{8} (e^4 - 1)\end{aligned}$$

Evaluate the second integral.

$$\begin{aligned}-\frac{1}{2} \int_0^2 x dx &= -\frac{1}{2} \left(\frac{1}{2} x^2 \right) \Big|_0^2 \\ &= -1\end{aligned}$$

Combine.

$$\begin{aligned}\frac{1}{2} e^4 - \frac{1}{8} (e^4 - 1) - 1 &= \frac{3}{8} e^4 - \frac{7}{8} \\ &= \frac{3e^4 - 7}{8}\end{aligned}$$

$$57. \int_0^{\ln 2} \int_{e^y}^2 \frac{1}{\ln x} dx dy$$

Changing the order of integration,

$$\begin{aligned}\int_0^{\ln 2} \int_{e^y}^2 \frac{1}{\ln x} dx dy &= \int_1^2 \int_0^{\ln x} \frac{1}{\ln x} dy dx \\ &= \int_1^2 \left[\frac{1}{\ln x} y \Big|_0^{\ln x} \right] dx \\ &= \int_1^2 (1 - 0) dx \\ &= x \Big|_1^2 \\ &= 2 - 1 = 1\end{aligned}$$

$$58. \int_0^2 \int_{y/2}^1 e^{x^2} dx dy$$

Change the order of integration.

$$\begin{aligned}\int_0^2 \int_{y/2}^1 e^{x^2} dx dy &= \int_0^1 \int_0^{2x} e^{x^2} dy dx \\ &= \int_0^1 e^{x^2} y \Big|_0^{2x} dx \\ &= \int_0^1 (2xe^{x^2} - 0) dx \\ &= e^{x^2} \Big|_0^1 \\ &= e^1 - e^0 \\ &= e - 1\end{aligned}$$

$$61. f(x, y) = 6xy + 2x; 2 \leq x \leq 5, 1 \leq y \leq 3$$

The area of region R is

$$A = (5 - 2)(3 - 1) = 6.$$

The average value of f over R is

$$\begin{aligned}\frac{1}{A} \iint_R f(x, y) dy dx &= \frac{1}{6} \int_2^5 \int_1^3 (6xy + 2x) dy dx \\ &= \frac{1}{6} \int_2^5 (3xy^2 + 2xy) \Big|_1^3 dx \\ &= \frac{1}{6} \int_2^5 [(27x + 6x) - (3x + 2x)] dx \\ &= \frac{1}{6} \int_2^5 28x dx \\ &= \frac{1}{6} 14x^2 \Big|_2^5 \\ &= \frac{7}{3} (25 - 4) = 49.\end{aligned}$$

$$62. f(x, y) = x^2 + y^2; 0 \leq x \leq 2, 0 \leq y \leq 3$$

The area of region R is

$$A = (2 - 0)(3 - 0) = 6.$$

The average value of

$$f(x, y) = x^2 + y^2$$

over R is

$$\begin{aligned}
& \frac{1}{A} \iint_R (x^2 + y^2) dy dx \\
&= \frac{1}{6} \int_0^2 \int_0^3 (x^2 + y^2) dy dx = \frac{1}{6} \int_0^2 \left(x^2 y + \frac{y^3}{3} \right) \bigg|_0^3 dx \\
&= \frac{1}{6} \int_0^2 (3x^2 + 9) dx = \frac{1}{6} (x^3 + 9x) \bigg|_0^2 \\
&= \frac{1}{6} (8 + 18 - 0) = \frac{1}{6} \cdot 26 = \frac{13}{3}.
\end{aligned}$$

63. $f(x, y) = e^{-5y+3x}$; $0 \leq x \leq 2, 0 \leq y \leq 2$

The area of region R is

$$(2 - 0)(2 - 0) = 4.$$

The average value of f over R is

$$\begin{aligned}
\frac{1}{4} \int_0^2 \int_0^2 e^{-5y+3x} dy dx &= \frac{1}{4} \int_0^2 -\frac{1}{5} e^{-5y+3x} \bigg|_0^2 dx = \frac{1}{4} \int_0^2 -\frac{1}{5} [e^{3x-10} - e^{3x}] dx \\
&= -\frac{1}{20} \left[\frac{1}{3} e^{3x-10} - \frac{1}{3} e^{3x} \right] \bigg|_0^2 = -\frac{1}{60} [e^{-4} - e^6 - e^{-10} + 1] = \frac{e^6 + e^{-10} - e^{-4} - 1}{60}.
\end{aligned}$$

64. $f(x, y) = e^{2x+y}$; $1 \leq x \leq 2, 2 \leq y \leq 3$

The area of region R is

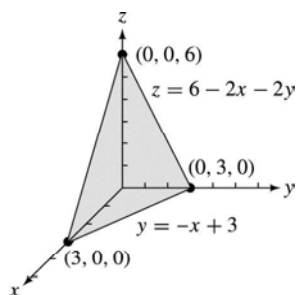
$$A = (2 - 1)(3 - 2) = 1.$$

The average value of f over R is

$$\begin{aligned}
& \frac{1}{A} \iint_R e^{2x+y} dy dx \\
&= \iint_R e^{2x+y} dy dx = \int_1^2 \int_2^3 e^{2x+y} dy dx \\
&= \int_1^2 (e^{2x+y}) \bigg|_2^3 dx = \int_1^2 (e^{2x+3} - e^{2x+2}) dx \\
&= \frac{1}{2} (e^{2x+3} - e^{2x+2}) \bigg|_1^2 = \frac{1}{2} (e^{4+3} - e^{2+3} - e^{4+2} + e^4) \\
&= \frac{e^7 - e^6 - e^5 + e^4}{2}.
\end{aligned}$$

65. The plane that intersects the axes has the equation

$$z = 6 - 2x - 2y.$$



$$\begin{aligned} V &= \iint_R f(x, y) \, dA \\ &= \int_0^3 \int_0^{-x+3} (6 - 2x - 2y) \, dy \, dx \\ &= \int_0^3 (6y - 2xy - y^2) \Big|_0^{-x+3} \, dx \\ &= \int_0^3 [-6x + 18 - 2x(-x + 3) - (3 - x)^2] \, dx \\ &= \int_0^3 (-6x + 18 + 2x^2 - 6x - 9 + 6x - x^2) \, dx \\ &= \int_0^3 (x^2 - 6x + 9) \, dx = \left(\frac{x^3}{3} - 3x^2 + 9x \right) \Big|_0^3 \\ &= (9 - 27 + 27) - 0 = 9 \end{aligned}$$

The volume is 9 in^3 .

66. $C(x, y) = \frac{1}{9}x^2 + 2x + y^2 + 5y + 100$

The area of region R is

$$A(80 - 40)(70 - 30) = 1600.$$

The average cost is

$$\begin{aligned} &\frac{1}{A} \iint_R C(x, y) \, dy \, dx \\ &= \frac{1}{1600} \int_{30}^{70} \int_{40}^{80} \left(\frac{1}{9}x^2 + 2x + y^2 + 5y + 100 \right) \, dy \, dx \\ &= \frac{1}{1600} \int_{30}^{70} \left(\frac{1}{27}x^3 + x^2 + xy^2 + 5xy + 100x \right) \Big|_{40}^{80} \, dx \\ &= \int_{30}^{70} \left[\left(\frac{320}{27} + 4 + \frac{1}{20}y^2 + \frac{1}{4}y + 5 \right) - \left(\frac{40}{27} + 1 + \frac{1}{40}y^2 + \frac{1}{8}y + \frac{5}{2} \right) \right] \, dy \end{aligned}$$

$$\begin{aligned}
&= \int_{30}^{70} \left(\frac{1}{40} y^2 + \frac{1}{8} y + \frac{857}{54} \right) dy \\
&= \left(\frac{1}{120} y^3 + \frac{1}{16} y^2 + \frac{857}{54} y \right) \Big|_{30}^{70} \\
&= \left(\frac{8575}{3} + \frac{1225}{4} + \frac{29,995}{27} \right) - \left(225 + \frac{225}{4} + \frac{4285}{9} \right) \\
&= \frac{94,990}{27} \\
&\approx 3518.
\end{aligned}$$

The average cost is about \$3518.

67. $P(x, y) = 500x^{0.2}y^{0.8}, 10 \leq x \leq 50,$
 $20 \leq y \leq 40$

$$A = 40 \cdot 20 = 800$$

Average production:

$$\begin{aligned}
&\frac{1}{800} \int_{10}^{50} \int_{20}^{40} 500x^{0.2}y^{0.8} dy dx \\
&= \frac{5}{8} \int_{10}^{50} \frac{x^{0.2}y^{1.8}}{1.8} \Big|_{20}^{40} dx = \frac{25}{72} \int_{10}^{50} x^{0.2}(40^{1.8} - 20^{1.8}) dx \\
&= \frac{25(40^{1.8} - 20^{1.8})}{72} \cdot \frac{x^{1.2}}{1.2} \Big|_{10}^{50} = \frac{125}{432} (40^{1.8} - 20^{1.8})(50^{1.2} - 10^{1.2}) \\
&\approx 14,753 \text{ units}
\end{aligned}$$

68. $P(x, y) = -(x - 100)^2 - (y - 50)^2 + 2000$

$$\begin{aligned}
\text{Area} &= (150 - 100)(80 - 40) = (50)(40) \\
&= 2000
\end{aligned}$$

The average weekly profit is

$$\begin{aligned}
&\frac{1}{2000} \iint_R [-(x - 100)^2 - (y - 50)^2 + 2000] dy dx \\
&= \frac{1}{2000} \int_{100}^{150} \int_{40}^{80} [-(x - 100)^2 - (y - 50)^2 + 2000] dy dx \\
&= \frac{1}{2000} \int_{100}^{150} \left[-(x - 100)^2 y - \frac{(y - 50)^3}{3} + 2000y \right] \Big|_{40}^{80} dx \\
&= \frac{1}{2000} \int_{100}^{150} \left[-(x - 100)^2(80 - 40) - \frac{(80 - 50)^3}{3} + \frac{(40 - 50)^3}{3} + 2000(80 - 40) \right] dx \\
&= \frac{1}{2000} \int_{100}^{150} \left[-40(x - 100)^2 - \frac{30^3}{3} + \frac{(-10)^3}{3} + 2000(40) \right] dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2000} \int_{100}^{150} \left[-40(x-100)^2 - \frac{28,000}{3} + 80,000 \right] dx \\
&= \frac{1}{2000} \cdot \left[\frac{-40(x-100)^3}{3} - \frac{28,000}{3}x + 80,000x \right] \Big|_{100}^{150} \\
&= \frac{1}{2000} \left[-\frac{40}{3}(150-100)^3 + \frac{40}{3}(100-100)^3 - \frac{28,000}{3}(150-100) + 80,000(150-100) \right] \\
&= \frac{1}{2000} \left[-\frac{40}{3}(50)^3 + \frac{40}{3} \cdot 0 - \frac{28,000}{3}(50) + 80,000(50) \right] \\
&= \frac{1}{2000} \left(\frac{-5,000,000 - 1,400,000 + 12,000,000}{3} \right) \\
&= \frac{1}{2000} \left(\frac{5,600,000}{3} \right) = \$933.33.
\end{aligned}$$

69. $R = q_1 p_1 + q_2 p_2$ where

$$q_1 = 300 - 2p_1,$$

$$q_2 = 500 - 1.2p_2, \quad 25 \leq p_1 \leq 50,$$

$$\text{and } 50 \leq p_2 \leq 75.$$

$$A = 25 \cdot 25 = 625$$

$$R = (300 - 2p_1)p_1 + (500 - 1.2p_2)p_2$$

$$R = 300p_1 - 2p_1^2 + 500p_2 - 1.2p_2^2$$

Average Revenue:

$$\begin{aligned}
&\frac{1}{625} \int_{25}^{50} \int_{50}^{75} (300p_1 - 2p_1^2 + 500p_2 - 1.2p_2^2) dp_2 dp_1 \\
&= \frac{1}{625} \int_{25}^{50} (300p_1 p_2 - 2p_1^2 p_2 + 250p_2^2 - 0.4p_2^3) \Big|_{50}^{75} dp_1 \\
&= \frac{1}{625} \int_{25}^{50} \left[22,500p_1 - 150p_1^2 + 1,406,250 - 168,750 \right] dp_1 \\
&= \frac{1}{625} \int_{25}^{50} (662,500 + 7500p_1 - 50p_1^2) dp_1 \\
&= \frac{1}{625} \left(662,500p_1 + 3750p_1^2 - \frac{50p_1^3}{3} \right) \Big|_{25}^{50} \\
&= \frac{1}{625} \left(33,125,000 + 9,375,000 - \frac{6,250,000}{3} - 16,562,500 - 2,343,750 + \frac{781,250}{3} \right) \\
&\approx \$34,833
\end{aligned}$$

70. $T(x, y) = x^4 + 16y^4 - 32xy + 40$

Area = $(4 - 0)(2 - 0) = 8$

The average time is

$$\begin{aligned} & \frac{1}{8} \iint_R (x^4 + 16y^4 - 32xy + 40) dy dx \\ &= \frac{1}{8} \int_0^4 \int_0^2 (x^4 + 16y^4 - 32xy + 40) dy dx \\ &= \frac{1}{8} \int_0^4 \left[x^4 y + \frac{16y^5}{5} - \frac{32xy^2}{2} + 40y \right]_0^2 dx \\ &= \frac{1}{8} \int_0^4 \left[x^4(2 - 0) + \frac{16(2 - 0)^5}{5} \right. \\ &\quad \left. - \frac{32x(2 - 0)^2}{2} + 40(2 - 0) \right] dx \\ &= \frac{1}{8} \int_0^4 \left(2x^4 + \frac{512}{5} - 64x + 80 \right) dx \\ &= \frac{1}{8} \left[\frac{2x^5}{5} + \frac{512}{5}x - \frac{64x^2}{2} + 80x \right]_0^4 \\ &= \frac{1}{8} \left[\frac{2(4 - 0)^5}{5} + \frac{512}{5}(4 - 0) \right. \\ &\quad \left. - \frac{64(4 - 0)^2}{2} + 80(4 - 0) \right] \\ &= \frac{1}{8} \left(\frac{2048}{5} + \frac{2048}{5} - 512 + 320 \right) \\ &= 78.4 \text{ hours} \end{aligned}$$

71. $P(x, y) = 36xy - x^3 - 8y^3$

Areas = $(8 - 0)(4 - 0)$
 $= 32$

The average profit is

$$\begin{aligned} & \frac{1}{32} \iint_R (36xy - x^3 - 8y^3) dy dx \\ &= \frac{1}{32} \int_0^8 \int_0^4 (36xy - x^3 - 8y^3) dy dx \\ &= \frac{1}{32} \int_0^8 \left[\frac{36xy^2}{2} - x^3 y - \frac{8y^4}{4} \right]_0^4 dx \\ &= \frac{1}{32} \int_0^8 \left[\frac{36x(4 - 0)^2}{2} - x^3(4 - 0) - \frac{8(4 - 0)^4}{4} \right] dx \\ &= \frac{1}{32} \int_0^8 (288x - 4x^3 - 512) dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{32} \left[\frac{288x^2}{2} - \frac{4x^4}{4} - 512x \right]_0^8 \\ &= \frac{1}{32} \left[\frac{288(8 - 0)^2}{2} - \frac{4(8 - 0)^4}{4} - 512(8 - 0) \right] \\ &= \frac{1}{32} (9216 - 4096 - 4096) \\ &= \$32,000 \end{aligned}$$

Chapter 9 Review Exercises

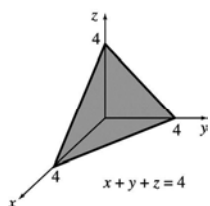
- True
- True
- True
- True
- False: $f(x + h, y) = 3(x + h)^2 + 2(x + h)y + y^2$
- False: (a, b) could be a saddle point.
- False: No; near a saddle point the function takes on values both larger and smaller than its value at the saddle point.
- True
- False: We need to test values of the function at nearby points that satisfy the constraints to tell if the point found represents a maximum or minimum.
- False: When dx and dy are interchanged, the limits on the first integral must be exchanged with the limits on the second integral.
- True
- False: The two integrals are over different regions, and neither region is a simple region of the sort that we deal with in this chapter.
- $f(x, y) = -4x^2 + 6xy - 3$
 $f(-1, 2) = -4(-1)^2 + 6(-1)(2) - 3 = -19$
 $f(6, -3) = -4(6)^2 + 6(6)(-3) - 3$
 $= -4(36) + (-108) - 3$
 $= -255$

18. $f(x, y) = 2x^2y^2 - 7x + 4y$
 $f(-1, 2) = 2(1)(4) - 7(-1) + 4(2) = 23$
 $f(6, -3) = 2(36)(9) - 7(6) + 4(-3) = 594$

19. $f(x, y) = \frac{x - 2y}{x + 5y}$
 $f(-1, 2) = \frac{(-1) - 2(2)}{(-1) + 5(2)} = \frac{-5}{9} = -\frac{5}{9}$
 $f(6, -3) = \frac{(6) - 2(-3)}{(6) + 5(-3)} = \frac{12}{-9} = -\frac{4}{3}$

20. $f(x, y) = \frac{\sqrt{x^2 + y^2}}{x - y}$
 $f(-1, 2) = \frac{\sqrt{1 + 4}}{-1 - 2} = -\frac{\sqrt{5}}{3}$
 $f(6, -3) = \frac{\sqrt{36 + 9}}{6 + 3} = \frac{\sqrt{45}}{9} = \frac{\sqrt{5}}{3}$

21. The plane $x + y + z = 4$ intersects the axes at $(4, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 4)$.



22. $x + 2y + 6z = 6$
 x -intercept: $y = 0, z = 0$
 $x = 6$

y -intercept: $x = 0, z = 0$

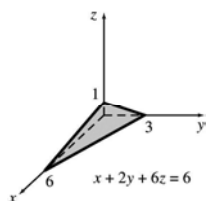
$2y = 6$

$y = 3$

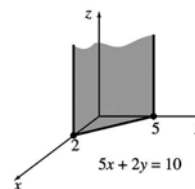
z -intercept: $x = 0, y = 0$

$6z = 6$

$z = 1$



23. The plane $5x + 2y = 10$ intersects the x - and y -axes at $(2, 0, 0)$ and $(0, 5, 0)$. Note that there is no z -intercept since $x = y = 0$ is not a solution of the equation of the plane.



24. $4x + 3z = 12$
 No y -intercept
 x -intercept: $y = 0, z = 0$

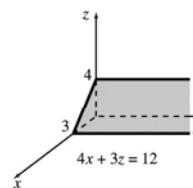
$4x = 12$

$x = 3$

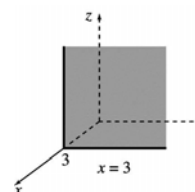
z -intercept: $x = 0, y = 0$

$3z = 12$

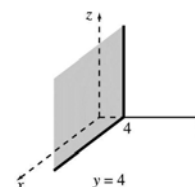
$z = 4$



25. $x = 3$
 The plane is parallel to the yz -plane. It intersects the x -axis at $(3, 0, 0)$.



26. $y = 4$
 No x -intercept, no z -intercept
 The graph is a plane parallel to the xz -plane.



$$27. \quad z = f(x, y) = 3x^3 + 4x^2y - 2y^2$$

$$(a) \quad \frac{\partial z}{\partial x} = 9x^2 + 8xy$$

$$(b) \quad \frac{\partial z}{\partial y} = 4x^2 - 4y$$

$$\left(\frac{\partial z}{\partial y} \right) (-1, 4) = 4(-1)^2 - 4(4) = -12$$

$$(c) \quad f_{xy}(x, y) = 8x$$

$$f_{xy}(x, y)(2, -1) = 8(2) = 16$$

$$28. \quad z = f(x, y) = \frac{x + y^2}{x - y^2}$$

$$(a) \quad \frac{\partial z}{\partial y} = \frac{(x - y^2) \cdot 2y - (x + y^2)(-2y)}{(x - y^2)^2}$$

$$= \frac{4xy}{(x - y^2)^2}$$

$$(b) \quad \frac{\partial z}{\partial x} = \frac{(x - y^2) \cdot 1 - (x + y^2) \cdot 1}{(x - y^2)^2}$$

$$= \frac{-2y^2}{(x - y^2)^2}$$

$$= -2y^2(x - y^2)^{-2}$$

$$\left(\frac{\partial z}{\partial x} \right) (0, 2) = \frac{-8}{(-4)^2} = -\frac{1}{2}$$

$$(c) \quad f_{xx}(x, y) = 4y^2(x - y^2)^{-3}$$

$$= \frac{4y^2}{(x - y^2)^3}$$

$$f_{xx}(-1, 0) = \frac{0}{1} = 0$$

$$29. \quad f(x, y) = 6x^2y^3 - 4y$$

$$f_x(x, y) = 12xy^3$$

$$f_y(x, y) = 18x^2y^2 - 4$$

$$30. \quad f(x, y) = 5x^4y^3 - 6x^5y$$

$$f_x(x, y) = 20x^3y^3 - 30x^4y$$

$$f_y(x, y) = 15x^4y^2 - 6x^5$$

$$31. \quad f(x, y) = \sqrt{4x^2 + y^2}$$

$$f_x(x, y) = \frac{1}{2}(4x^2 + y^2)^{-1/2}(8x)$$

$$= \frac{4x}{(4x^2 + y^2)^{-1/2}}$$

$$f_y(x, y) = \frac{1}{2}(4x^2 + y^2)^{-1/2}(2y)$$

$$= \frac{y}{(4x^2 + y^2)^{1/2}}$$

$$32. \quad f(x, y) = \frac{2x + 5y^2}{3x^2 + y^2}$$

$$f_x(x, y) = \frac{(3x^2 + y^2) \cdot 2 - (2x + 5y^2) \cdot 6x}{(3x^2 + y^2)^2}$$

$$= \frac{2y^2 - 6x^2 - 30xy^2}{(3x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(3x^2 + y^2) \cdot 10y - (2x + 5y^2) \cdot 2y}{(3x^2 + y^2)^2}$$

$$= \frac{30x^2y - 4xy}{(3x^2 + y^2)^2}$$

$$33. \quad f(x, y) = x^3e^{3y}$$

$$f_x(x, y) = 3x^2e^{3y}$$

$$f_y(x, y) = 3x^3e^{3y}$$

$$34. \quad f(x, y) = (y - 2)^2e^{x+2y}$$

$$f_x(x, y) = (y - 2)^2e^{x+2y}$$

$$f_y(x, y) = e^{x+2y} \cdot 2(y - 2) + (y - 2)^2 \cdot 2e^{x+2y}$$

$$= 2(y - 2)[1 + (y - 2)]e^{x+2y}$$

$$= 2(y - 2)(y - 1)e^{x+2y}$$

$$35. \quad f(x, y) = \ln |2x^2 + y^2|$$

$$f_x(x, y) = \frac{1}{2x^2 + y^2} \cdot 4x$$

$$= \frac{4x}{2x^2 + y^2}$$

$$f_y(x, y) = \frac{1}{2x^2 + y^2} \cdot 2y$$

$$= \frac{2y}{2x^2 + y^2}$$

$$\begin{aligned}
 36. \quad f(x, y) &= \ln |2 - x^2 y^3| \\
 f_x(x, y) &= \frac{1}{2 - x^2 y^3} \cdot (-2xy^3) \\
 &= \frac{-2xy^3}{2 - x^2 y^3} \\
 f_y(x, y) &= \frac{1}{2 - x^2 y^3} \cdot (-3x^2 y^2) \\
 &= \frac{-3x^2 y^2}{2 - x^2 y^3}
 \end{aligned}$$

$$\begin{aligned}
 37. \quad f(x, y) &= 5x^3 y - 6xy^2 \\
 f_x(x, y) &= 15x^2 y - 6y^2 \\
 f_{xx}(x, y) &= 30xy \\
 f_{xy}(x, y) &= 15x^2 - 12y
 \end{aligned}$$

$$\begin{aligned}
 38. \quad f(x, y) &= -3x^2 y^3 + x^3 y \\
 f_x(x, y) &= -6xy^3 + 3x^2 y \\
 f_{xx}(x, y) &= -6y^3 + 6xy \\
 f_{xy}(x, y) &= -18xy^2 + 3x^2
 \end{aligned}$$

$$\begin{aligned}
 39. \quad f(x, y) &= \frac{3x}{2x - y} \\
 f_x(x, y) &= \frac{(2x - y) \cdot 3 - 3x \cdot 2}{(2x - y)^2} \\
 &= \frac{-3y}{(2x - y)^2} \\
 f_{xx}(x, y) &= \frac{(2x - y)^2 \cdot 0 - (-3y) \cdot 2(2x - y) \cdot 2}{(2x - y)^4} \\
 &= \frac{12y}{(2x - y)^3} \\
 f_{xy}(x, y) &= \frac{\left[(2x - y)^2 \cdot (-3) - (-3y) \cdot 2(2x - y) \cdot (-1) \right]}{(2x - y)^4} \\
 &= \frac{-6x - 3y}{(2x - y)^3}
 \end{aligned}$$

$$\begin{aligned}
 40. \quad f(x, y) &= \frac{3x + y}{x - 1} \\
 f_x(x, y) &= \frac{(x - 1) \cdot 3 - (3x + y) \cdot 1}{(x - 1)^2} \\
 &= \frac{-3 - y}{(x - 1)^2} = (-3 - y)(x - 1)^{-2} \\
 f_{xx}(x, y) &= -2(-3 - y)(x - 1)^{-3} \\
 &= \frac{2(3 + y)}{(x - 1)^3} \\
 f_{xy}(x, y) &= \frac{-1}{(x - 1)^2}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad f(x, y) &= 4x^2 e^{2y} \\
 f_x(x, y) &= 8xe^{2y} \\
 f_{xx}(x, y) &= 8e^{2y} \\
 f_{xy}(x, y) &= 16xe^{2y}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad f(x, y) &= ye^{x^2} \\
 f_x(x, y) &= 2xye^{x^2} \\
 f_{xx}(x, y) &= 2xy \cdot 2xe^{x^2} + e^{x^2} \cdot 2y \\
 &= 2ye^{x^2}(2x^2 + 1) \\
 f_{xy}(x, y) &= 2xe^{x^2}
 \end{aligned}$$

$$\begin{aligned}
 43. \quad f(x, y) &= \ln |2 - x^2 y| \\
 f_x(x, y) &= \frac{1}{2 - x^2 y} \cdot (-2xy) \\
 &= \frac{2xy}{x^2 y - 2} \\
 f_{xx}(x, y) &= \frac{(x^2 y - 2)2y - 2xy(2xy)}{(x^2 y - 2)^2} \\
 &= \frac{2y[(x^2 y - 2) - 2x^2 y]}{(x^2 y - 2)^2} \\
 &= \frac{2y(-x^2 y - 2)}{(x^2 y - 2)^2} \\
 &= \frac{-2x^2 y^2 - 4y}{(2 - x^2 y)^2}
 \end{aligned}$$

$$\begin{aligned}
 f_{xy}(x, y) &= \frac{2x(x^2y - 2) - x^2(2xy)}{(x^2y - 2)^2} \\
 &= \frac{2x[(x^2y - 2) - x^2y]}{(x^2y - 2)^2} \\
 &= \frac{2x(-2)}{(x^2y - 2)^2} \\
 &= \frac{-4x}{(2 - x^2y)^2}
 \end{aligned}$$

44. $f(x, y) = \ln |1 + 3xy^2|$

$$\begin{aligned}
 f_x(x, y) &= \frac{1}{1 + 3xy^2} \cdot 3y^2 \\
 &= \frac{3y^2}{1 + 3xy^2} \\
 &= 3y^2(1 + 3xy^2)^{-1} \\
 f_{xx}(x, y) &= 3y^2 \cdot (-3y^2)(1 + 3xy^2)^{-2} \\
 &= \frac{-9y^4}{(1 + 3xy^2)^2} \\
 f_{xy}(x, y) &= \frac{(1 + 3xy^2) \cdot 6y - 3y^2(6xy)}{(1 + 3xy^2)^2} \\
 &= \frac{6y}{(1 + 3xy^2)^2}
 \end{aligned}$$

45. $z = 2x^2 - 3y^2 + 12y$

$$\begin{aligned}
 z_x(x, y) &= 4x \\
 z_y(x, y) &= -6y + 12
 \end{aligned}$$

If $z_x(x, y) = 0$, $x = 0$. If $z_y(x, y) = 0$, $y = 2$.

Therefore, $(0, 2)$ is a critical point.

$$\begin{aligned}
 z_{xx}(x, y) &= 4 \\
 z_{yy}(x, y) &= -6 \\
 z_{xy}(x, y) &= 0
 \end{aligned}$$

$$D = 4(-6) - 0^2 = -24 < 0$$

There is a saddle point at $(0, 2)$.

46. $z = x^2 + y^2 + 9x - 8y + 1$
 $z_x(x, y) = 2x + 9$, $z_y(x, y) = 2y - 8$

$$\begin{aligned}
 2x + 9 &= 0 \\
 x &= -\frac{9}{2} \\
 2y - 8 &= 0 \\
 y &= 4
 \end{aligned}$$

$$z_{xx}(x, y) = 2, z_{yy}(x, y) = 2, z_{xy}(x, y) = 0$$

$$D = 2(2) - (0)^2 = 4 > 0 \text{ and } z_{xx}(x, y) > 0.$$

Relative minimum at $\left(-\frac{9}{2}, 4\right)$

47. $f(x, y) = x^2 + 3xy - 7x + 5y^2 - 16y$
 $f_x(x, y) = 2x + 3y - 7$
 $f_y(x, y) = 3x + 10y - 16$

Solve the system $f_x(x, y) = 0$, $f_y(x, y) = 0$.

$$\begin{aligned}
 2x + 3y - 7 &= 0 \\
 3x + 10y - 16 &= 0 \\
 -6x - 9y + 21 &= 0 \\
 \frac{6x + 20y - 32}{11y - 11} &= 0 \\
 y &= 1
 \end{aligned}$$

$$\begin{aligned}
 2x + 3(1) - 7 &= 0 \\
 2x &= 4 \\
 x &= 2
 \end{aligned}$$

Therefore, $(2, 1)$ is a critical point.

$$\begin{aligned}
 f_{xx}(x, y) &= 2 \\
 f_{yy}(x, y) &= 10 \\
 f_{xy}(x, y) &= 3
 \end{aligned}$$

$$D = 2 \cdot 10 - 3^2 = 11 > 0$$

Since $f_{xx} = 2 > 0$, there is a relative minimum at $(2, 1)$.

48. $z = x^3 - 8y^2 + 6xy + 4$
 $z_x(x, y) = 3x^2 + 6y, z_y(x, y) = -16y + 6x$

$$\begin{aligned} 3x^2 + 6y &= 0 \\ x^2 + 2y &= 0 \\ y &= -\frac{x^2}{2} \\ -16y + 6x &= 0 \\ -8y + 3x &= 0 \end{aligned}$$

Substituting, we have

$$\begin{aligned} -8\left(-\frac{x^2}{2}\right) + 3x &= 0 \\ 4x^2 + 3x &= 0 \\ x(4x + 3) &= 0 \\ x = 0 \text{ or } x &= -\frac{3}{4} \\ y = 0 \text{ or } y &= -\frac{9}{32} \end{aligned}$$

$$z_{xx}(x, y) = 6x, z_{yy}(x, y) = -16, z_{xy}(x, y) = 6$$

$$D = 6x(-16) - (6)^2 = -96x - 36$$

$$\text{At } (0, 0), D = -36 < 0.$$

Saddle point at $(0, 0)$.

$$\begin{aligned} \text{At } \left(-\frac{3}{4}, -\frac{9}{32}\right), D &= 36 > 0 \text{ and } z_{xx}(x, y) \\ &= -\frac{9}{2} < 0. \end{aligned}$$

Relative maximum at $\left(-\frac{3}{4}, -\frac{9}{32}\right)$

49. $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 2xy - 5x - 7y + 10$

$$\begin{aligned} z_x(x, y) &= x + 2y - 5 \\ z_y(x, y) &= y + 2x - 7 \end{aligned}$$

Setting $z_x = z_y = 0$ and solving yields

$$\begin{aligned} x + 2y &= 5 \\ 2x + y &= 7 \\ -2x - 4y &= -10 \\ \frac{2x + y}{-3y} &= \frac{7}{-3} \\ -3y &= -3 \\ y &= 1, x = 3. \end{aligned}$$

$$z_{xx}(x, y) = 1, z_{yy}(x, y) = 1, z_{xy}(x, y) = 2$$

For $(3, 1)$,

$$D = 1 \cdot 1 - 4 = -3 < 0.$$

Therefore, z has a saddle point at $(3, 1)$.

50. $f(x, y) = 2x^2 + 4xy + 4y^2 - 3x + 5y - 15$

$$f_x(x, y) = 4x + 4y - 3$$

$$f_y(x, y) = 4x + 8y + 5$$

Solve the system $f_x(x, y) = 0, f_y(x, y) = 0$.

$$4x + 4y - 3 = 0$$

$$4x + 8y + 5 = 0$$

$$-4x - 4y + 3 = 0$$

$$\frac{4x + 8y + 5 = 0}{4y + 8 = 0}$$

$$4y + 8 = 0$$

$$y = -2$$

$$4x + 4(-2) - 3 = 0$$

$$4x = 11$$

$$x = \frac{11}{4}$$

Therefore, $\left(\frac{11}{4}, -2\right)$ is a critical point.

$$f_{xx}(x, y) = 4$$

$$f_{yy}(x, y) = 8$$

$$f_{xy}(x, y) = 4$$

$$D = 4 \cdot 8 - 4^2 = 16 > 0$$

Since $f_{xx} = 4 > 0$, there is a relative minimum at $\left(\frac{11}{4}, -2\right)$.

51. $z = x^3 + y^2 + 2xy - 4x - 3y - 2$

$$z_x(x, y) = 3x^2 + 2y - 4$$

$$z_y(x, y) = 2y + 2x - 3$$

Setting $z_x(x, y) = z_y(x, y) = 0$ yields

$$3x^2 + 2y - 4 = 0 \quad (1)$$

$$2y + 2x - 3 = 0 \quad (2)$$

Solving for $2y$ in equation (2) gives

$$2y = -2x + 3.$$

Substitute into equation (1).

$$3x^2 + (-2x) + 3 - 4 = 0$$

$$3x^2 - 2x - 1 = 0$$

$$(3x + 1)(x - 1) = 0$$

$$x = -\frac{1}{3} \quad \text{or} \quad x = 1$$

$$y = \frac{11}{6} \quad \text{or} \quad y = \frac{1}{2}$$

$$z_{xx}(x, y) = 6x, \quad z_{yy}(x, y) = 2, \quad z_{xy}(x, y) = 2$$

For $\left(-\frac{1}{3}, \frac{11}{6}\right)$,

$$\begin{aligned} D &= 6\left(-\frac{1}{3}\right)(2) - 4 \\ &= -4 - 4 = -8 < 0, \end{aligned}$$

so z has a saddle point at $\left(-\frac{1}{3}, \frac{11}{6}\right)$.

$$D = 6(1)(2) - 4 = 8 > 0.$$

$z_{xx}\left(1, \frac{1}{2}\right) = 6 > 0$, so z has a relative minimum at $\left(1, \frac{1}{2}\right)$.

$$\begin{aligned} 52. \quad f(x, y) &= 7x^2 + y^2 - 3x + 6y - 5xy \\ f_x(x, y) &= 14x - 3 - 5y \\ f_y(x, y) &= 2y + 6 - 5x \end{aligned}$$

$$\begin{aligned} 14x - 5y - 3 &= 0 \\ -5x + 2y + 6 &= 0 \end{aligned}$$

$$\begin{aligned} 28x - 10y - 6 &= 0 \\ -25x + 10y + 30 &= 0 \\ \hline 3x &+ 24 = 0 \\ x &= -8 \end{aligned}$$

$$\begin{aligned} -5(-8) + 2y + 6 &= 0 \\ 2y &= -46 \\ y &= -23 \end{aligned}$$

$$\begin{aligned} f_{xx}(x, y) &= 14, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -5 \\ D &= 14(2) - (-5)^2 = 3 > 0 \text{ and } f_{xx}(x, y) > 0. \end{aligned}$$

Relative minimum at $(-8, -23)$

$$54. \quad f(x, y) = x^2y; \quad x + y = 4$$

$$1. \quad g(x) = x + y - 4$$

$$2. \quad F(x, y, \lambda) = x^2y - \lambda(x + y - 4)$$

$$3. \quad F_x(x, y, \lambda) = 2xy - \lambda$$

$$F_y(x, y, \lambda) = x^2 - \lambda$$

$$F_\lambda(x, y, \lambda) = -(x + y - 4)$$

$$4. \quad 2xy - \lambda = 0 \quad (1)$$

$$x^2 - \lambda = 0 \quad (2)$$

$$x + y - 4 = 0 \quad (3)$$

$$5. \quad \lambda = 2xy$$

$$\lambda = x^2$$

$$2xy = x^2$$

$$2xy - x^2 = 0$$

$$x(2y - x) = 0$$

$$x = 0 \quad \text{or} \quad 2y = x$$

Substituting into equation (3) gives

$$y = 4 \quad \text{or} \quad y = \frac{4}{3}.$$

$$\text{If } y = \frac{4}{3}, x = \frac{8}{3}.$$

The critical points are $(0, 4)$ and $\left(\frac{8}{3}, \frac{4}{3}\right)$.

$$\begin{aligned} f(0, 4) &= 0 \\ f\left(\frac{8}{3}, \frac{4}{3}\right) &= \frac{64}{9} \cdot \frac{4}{3} \\ &= \frac{256}{27} \end{aligned}$$

Therefore, f has a minimum of 0 at $(0, 4)$ and a maximum of $\frac{256}{27}$ at $\left(\frac{8}{3}, \frac{4}{3}\right)$.

$$55. \quad f(x, y) = x^2 + y^2; \quad x = y - 6.$$

$$1. \quad g(x, y) = x - y + 6$$

$$2. \quad F(x, y, \lambda) = x^2 + y^2 - \lambda(x - y + 6)$$

$$3. \quad F_x(x, y, \lambda) = 2x - \lambda$$

$$F_y(x, y, \lambda) = 2y + \lambda$$

$$F_\lambda(x, y, \lambda) = -(x - y + 6)$$

4. $2x - \lambda = 0$ (1)
 $2y + \lambda = 0$ (2)
 $x - y + 6 = 0$ (3)
5. Equations (1) and (2) give $\lambda = 2x$, and $\lambda = -2y$. Thus,

$$2x = -2y$$

$$x = -y.$$

Substituting into equation (3),

$$(-y) - y + 6 = 0$$

$$y = 3.$$

So $x = -3.$

And $f(-3, 3) = (-3)^2 + (3)^2 = 18.$

$$F_{xx}(x, y, \lambda) = 2$$

$$F_{yy}(x, y, \lambda) = 2$$

$$F_{xy}(x, y, \lambda) = 0$$

$$D = 2 \cdot 2 - 0^2 = 4 > 0$$

Since $F_{xx} > 0$, there is a relative minimum of 18 at $(-3, 3)$.

- 56.** Let x and y be the numbers such that $x + y = 80$ and $f(x, y) = x^2y$.

- $g(x) = x + y - 80$
- $F(x, y, \lambda) = x^2y - \lambda(x + y - 80)$
- $F_x(x, y, \lambda) = 2xy - \lambda$
 $F_y(x, y, \lambda) = x^2 - \lambda$
 $F_\lambda(x, y, \lambda) = -(x + y - 80)$
- $2xy - \lambda = 0$ (1)
 $x^2 - \lambda = 0$ (2)
 $x + y - 80 = 0$ (3)
- $\lambda = 2xy$
 $\lambda = x^2$
 $2xy = x^2$

$$2xy - x^2 = 0$$

$$x(2y - x) = 0$$

$$x = 0 \text{ or } x = 2y$$

Substituting into equation (3) gives

$$y = 80 \text{ or } 2y + y - 80 = 0$$

$$3y = 80$$

$$y = \frac{80}{3}$$

and $x = \frac{160}{3}.$

$$f(0, 80) = 0 \cdot 80^2 = 0$$

$$f\left(\frac{160}{3}, \frac{80}{3}\right) = \frac{(160)^2 (80)}{9 \cdot 3}$$

$$= \frac{2,048,000}{27} > f(0, 80)$$

f has a maximum at $\left(\frac{160}{3}, \frac{80}{3}\right).$

Therefore, if $x = \frac{160}{3}$ and $y = \frac{80}{3}$, then x^2y is maximized.

- 57.** Maximize $f(x, y) = xy^2$, subject to $x + y = 75$.

- $g(x, y) = x + y - 75$
- $F(x, y, \lambda) = xy^2 - \lambda(x + y - 75)$
- $F_x(x, y, \lambda) = y^2 - \lambda$
 $F_y(x, y, \lambda) = 2xy - \lambda$
 $F_\lambda(x, y, \lambda) = -(x + y - 75)$

- $y^2 - \lambda = 0$ (1)
 $2xy - \lambda = 0$ (2)
 $x + y - 75 = 0$ (3)

5. Equations (1) and (2) give $\lambda = y^2$ and $\lambda = 2xy$. Thus,

$$y^2 = 2xy$$

$$y(y - 2x) = 0$$

$$y = 0 \text{ or } y = 2x$$

Substituting $y = 0$ into equation (3),

$$x + (0) - 75 = 0$$

$$x = 75.$$

Substituting $y = 2x$ into equation (3),

$$x + (2x) - 75 = 0$$

$$x = 25.$$

So $y = 2x = 50.$

Thus,

$$f(75, 0) = 75(0)^2 = 0, \text{ and}$$

$$f(25, 50) = 25(50)^2 = 62,500.$$

Since $f(25, 50) > f(75, 0)$, $x = 25$ and $y = 50$ will maximize $f(x, y) = xy^2$.

58. No, a maximum does not exist without the requirement that x and y are positive. Consider maximizing x^2y with $x + y = 80$. If $x < 0$, then $y = 80 - x > 80$. The values of y and x^2y will increase as the value of $|x|$ increases, so there is no maximum. A similar situation occurs by xy^2 when $x + y = 50$, by taking $y < 0$ and considering what happens to the values of x and xy^2 as $|y|$ increases.

59. $z = f(x, y) = 6x^2 - 7y^2 + 4xy$
 $x = 3, y = -1, dx = 0.03, dy = 0.01$

$$f_x(x, y) = 12x + 4y$$

$$f_y(x, y) = -14y + 4x$$

$$\begin{aligned} dz &= (12x + 4y)dx + (-14y + 4x)dy \\ &= [12(3) + 4(-1)](0.03) \\ &\quad + [-14(-1) + 4(3)](0.01) \\ &= 0.96 + 0.26 = 1.22 \end{aligned}$$

60. $z = f(x, y) = \frac{x + 5y}{x - 2y}$
 $x = 1, y = -2, dx = -0.04, dy = 0.02$

$$f_x(x, y) = \frac{(x - 2y)(1) - (x + 5y)(1)}{(x - 2y)^2}$$

$$= \frac{-7y}{(x - 2y)^2}$$

$$f_y(x, y) = \frac{(x - 2y)(5) - (x + 5y)(-2)}{(x - 2y)^2}$$

$$= \frac{7x}{(x - 2y)^2}$$

$$dz = \frac{-7y}{(x - 2y)^2}dx + \frac{7x}{(x - 2y)^2}dy$$

$$= \frac{-7(-2)}{[1 - 2(-2)]^2}(-0.04)$$

$$+ \frac{7(1)}{[1 - 2(-2)]^2}(0.02)$$

$$= -0.0224 + 0.0056 = -0.0168$$

61. Let $z = f(x, y) = \sqrt{x^2 + y^2}$.

Then

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

$$\begin{aligned} dz &= \frac{1}{2}(x^2 + y^2)^{-1/2}(2x)dx \\ &\quad + \frac{1}{2}(x^2 + y^2)^{-1/2}(2y)dy \\ &= \frac{x}{\sqrt{x^2 + y^2}}dx + \frac{y}{\sqrt{x^2 + y^2}}dy \end{aligned}$$

To approximate $\sqrt{5.1^2 + 12.05^2}$, we let $x = 5$, $dx = 0.1$, $y = 12$, and $dy = 0.05$.

Then,

$$\begin{aligned} dz &= \frac{5}{\sqrt{5^2 + 12^2}}(0.1) + \frac{12}{\sqrt{5^2 + 12^2}}(0.05) \\ &= \frac{5}{13}(0.1) + \frac{12}{13}(0.05) \approx 0.0846. \end{aligned}$$

Therefore,

$$\begin{aligned} f(5.1, 12.05) &= f(5, 12) + \Delta z \\ &\approx f(5, 12) + dz \\ &= \sqrt{5^2 + 12^2} + 0.0846 \\ f(5.1, 12.05) &\approx 13.0846 \end{aligned}$$

Using a calculator, $\sqrt{5.1^2 + 12.05^2} \approx 13.0848$.

The absolute value of the difference of the two results is $|13.0846 - 13.0848| = 0.0002$.

62. Let $z = f(x, y) = \sqrt{x}e^y$.

Then

$$\begin{aligned} dz &= f_x(x, y)dx + f_y(x, y)dy \\ &= \frac{1}{2}x^{-1/2}e^ydx + x^{1/2}e^ydy \\ &= \frac{e^y}{2\sqrt{x}}dx + \sqrt{x}e^ydy. \end{aligned}$$

To approximate $\sqrt{4.06}e^{0.04}$, let $x = 4$, $dx = 0.06$, $y = 0$, and $dy = 0.04$.

Therefore,

$$\begin{aligned} dz &= \frac{e^0}{2\sqrt{4}}(0.06) + \sqrt{4}e^0(0.04) \\ dz &= \frac{1}{4}(0.06) + 2(0.04) \\ dz &= 0.095. \end{aligned}$$

$$\begin{aligned}
 f(4.06, 0.04) &= f(4, 0) + \Delta z \\
 &\approx f(4, 0) + dz \\
 &= \sqrt{4}e^0 + 0.095 \\
 f(4.06, 0.04) &\approx 2.095
 \end{aligned}$$

Using a calculator, $\sqrt{4.06}e^{0.04} \approx 2.0972$.

The absolute value of the difference of the two results is $|2.095 - 2.0972| = 0.0022$.

$$\begin{aligned}
 63. \quad \int_1^4 \frac{4y-3}{\sqrt{x}} dx &= (4y-3)(2\sqrt{x}) \Big|_1^4 \\
 &= (4y-3)(2 \cdot 2 - 2 \cdot 1) \\
 &= 8y - 6
 \end{aligned}$$

$$\begin{aligned}
 64. \quad \int_1^5 e^{3x+5y} dx &= \frac{1}{3} e^{3x+5y} \Big|_1^5 \\
 &= \frac{1}{3} (e^{15+5y} - e^{3+5y})
 \end{aligned}$$

$$\begin{aligned}
 65. \quad \int_0^5 \frac{6x}{\sqrt{4x^2 + 2y^2}} dx & \\
 \text{Let } u = 4x^2 + 2y^2; \text{ then } du &= 8x dx. \\
 \text{When } x = 0, u = 2y^2. & \\
 \text{When } x = 5, u = 100 + 2y^2. & \\
 &= \frac{3}{4} \int_{2y^2}^{100+2y^2} u^{-1/2} du \\
 &= \frac{3}{4} (2u^{1/2}) \Big|_{2y^2}^{100+2y^2} \\
 &= \frac{3}{4} \cdot 2[(100 + 2y^2)^{1/2} - (2y^2)^{1/2}] \\
 &= \frac{3}{2} [(100 + 2y^2)^{1/2} - (2y^2)^{1/2}]
 \end{aligned}$$

$$\begin{aligned}
 66. \quad \int_1^3 y^2(7x + 11y^3)^{-1/2} dy & \\
 &= \frac{2}{33} (7x + 11y^3)^{1/2} \Big|_1^3 \\
 &= \frac{2}{33} [(7x + 297)^{1/2} - (7x + 11)^{1/2}]
 \end{aligned}$$

$$\begin{aligned}
 67. \quad \int_0^2 \left[\int_0^4 (x^2 y^2 + 5x) dx \right] dy & \\
 &= \int_0^2 \left(\frac{1}{3} x^3 y^2 + \frac{5}{2} x^2 \right) \Big|_0^4 dy \\
 &= \int_0^2 \left(\frac{64}{3} y^2 + 40 \right) dy \\
 &= \left(\frac{64y^3}{9} + 40y \right) \Big|_0^2 \\
 &= \frac{64}{9} (8) + 40(2) \\
 &= \frac{512}{9} + \frac{720}{9} \\
 &= \frac{1232}{9}
 \end{aligned}$$

$$\begin{aligned}
 68. \quad \int_0^3 \int_0^5 (2x + 6y + y^2) dy dx & \\
 &= \int_0^3 \left(2xy + 3y^2 + \frac{1}{3} y^3 \right) \Big|_0^5 dx \\
 &= \int_0^3 \left[\left(10x + 75 + \frac{125}{3} \right) - 0 \right] dx \\
 &= \int_0^3 \left(10x + \frac{350}{3} \right) dx \\
 &= \left(5x^2 + \frac{350}{3} x \right) \Big|_0^3 \\
 &= (45 + 350) - 0 = 395
 \end{aligned}$$

$$\begin{aligned}
 69. \quad \int_3^4 \left[\int_2^5 \sqrt{6x + 3y} dx \right] dy & \\
 &= \int_3^4 \frac{1}{9} (6x + 3y)^{3/2} \Big|_2^5 dy \\
 &= \int_3^4 \frac{1}{9} [(30 + 3y)^{3/2} - (12 + 3y)^{3/2}] dy \\
 &= \frac{1}{3} \cdot \frac{1}{9} \cdot \frac{2}{5} [(30 + 3y)^{5/2} - (12 + 3y)^{5/2}] \Big|_3^4 \\
 &= \frac{2}{135} [(42)^{5/2} - (24)^{5/2} - (39)^{5/2} + (21)^{5/2}]
 \end{aligned}$$

$$\begin{aligned}
 70. \quad & \int_3^5 e^{2x-7y} dx \\
 &= \frac{1}{2} e^{2x-7y} \Big|_3^5 \\
 &= \frac{1}{2} (e^{10-7y} - e^{6-7y}) \\
 & \int_1^2 \left[\int_3^5 (e^{2x-7y}) dx \right] dy \\
 &= \int_1^2 \frac{1}{2} (e^{10-7y} - e^{6-7y}) dy \\
 &= -\frac{1}{14} (e^{10-7y} - e^{6-7y}) \Big|_1^2 \\
 &= -\frac{1}{14} (e^{-4} - e^{-8} - e^3 + e^{-1}) \\
 &= \frac{e^3 + e^{-8} - e^{-4} - e^{-1}}{14}
 \end{aligned}$$

$$\begin{aligned}
 71. \quad & \int_2^4 \int_2^4 \frac{dx dy}{y} = \int_2^4 \left(\frac{1}{y} x \right) \Big|_2^4 dy \\
 &= \int_2^4 \left[\frac{1}{y} (4 - 2) \right] dy \\
 &= 2 \ln |y| \Big|_2^4 \\
 &= 2 \ln \left| \frac{4}{2} \right| \\
 &= 2 \ln 2 \text{ or } \ln 4
 \end{aligned}$$

$$\begin{aligned}
 72. \quad & \int_1^2 \int_1^2 \frac{dx dy}{x} = \int_1^2 \ln x \Big|_1^2 dy \\
 &= \int_1^2 \ln 2 dy \\
 &= y \ln 2 \Big|_1^2 \\
 &= 2 \ln 2 - \ln 2 \\
 &= \ln 2
 \end{aligned}$$

$$\begin{aligned}
 73. \quad & \iint_R (x^2 + 2y^2) dx dy; 0 \leq x \leq 5, 0 \leq y \leq 2 \\
 & \iint_R (x^2 + 2y^2) dx dy \\
 &= \int_0^2 \int_0^5 (x^2 + 2y^2) dx dy \\
 &= \int_0^2 \left(\frac{1}{3} x^3 + 2xy^2 \right) \Big|_0^5 dy \\
 &= \int_0^2 \left[\left(\frac{125}{3} + 10y^2 \right) - 0 \right] dy \\
 &= \int_0^2 \left(\frac{125}{3} + 10y^2 \right) dy \\
 &= \left(\frac{125}{3} y + \frac{10}{3} y^3 \right) \Big|_0^2 \\
 &= \frac{250}{3} + \frac{80}{3} = 110
 \end{aligned}$$

$$\begin{aligned}
 74. \quad & \iint_R \sqrt{2x+y} dx dy; 1 \leq x \leq 3, 2 \leq y \leq 5 \\
 & \iint_R \sqrt{2x+y} dx dy \\
 &= \int_1^3 \int_2^5 (2x+y)^{1/2} dy dx \\
 &= \int_1^3 \frac{2}{3} (2x+y)^{3/2} \Big|_2^5 dx \\
 &= \int_2^3 \frac{2}{3} [(2x+5)^{3/2} - (2x+2)^{3/2}] dx \\
 &= \frac{2}{15} [(2x+5)^{5/2} - (2x+2)^{5/2}] \Big|_1^3 \\
 &= \frac{2}{15} (11^{5/2} - 8^{5/2} - 7^{5/2} + 4^{5/2}) \\
 &= \frac{2}{15} (11^{5/2} - 8^{5/2} - 7^{5/2} + 32)
 \end{aligned}$$

$$75. \iint_R \sqrt{y+x} \, dx \, dy; 0 \leq x \leq 7, 1 \leq y \leq 9$$

$$\begin{aligned} & \iint_R \sqrt{y+x} \, dx \, dy \\ &= \int_0^7 \int_1^9 \sqrt{y+x} \, dy \, dx \\ &= \int_0^7 \left[\frac{2}{3} (y+x)^{3/2} \right]_1^9 dx \\ &= \int_0^7 \frac{2}{3} [(9+x)^{3/2} - (1+x)^{3/2}] dx \\ &= \frac{2}{3} \cdot \frac{2}{5} [(9+x)^{5/2} - (1+x)^{5/2}] \Big|_0^7 \\ &= \frac{4}{15} [(16)^{5/2} - (8)^{5/2} - (9)^{5/2} + (1)^{5/2}] \\ &= \frac{4}{15} [4^5 - (2\sqrt{2})^5 - 3^5 + 1] \end{aligned}$$

$$\begin{aligned} &= \frac{4}{15} (1024 - 32(4\sqrt{2}) - 243 + 1) \\ &= \frac{4}{15} (782 - 128\sqrt{2}) \\ &= \frac{4}{15} (782 - 8^{5/2}) \end{aligned}$$

$$76. \iint_R ye^{y^2+x} \, dx \, dy; 0 \leq x \leq 1, 0 \leq y \leq 1$$

$$\begin{aligned} & \iint_R ye^{y^2+x} \, dx \, dy \\ &= \int_0^1 \int_0^1 ye^{y^2+x} \, dx \, dy \\ &= \int_0^1 \frac{1}{2} e^{y^2+x} \Big|_0^1 dx \\ &= \int_0^1 \frac{1}{2} [e^{1+x} - e^x] dx \\ &= \frac{1}{2} [e^{1+x} - e^x] \Big|_0^1 \\ &= \frac{1}{2} [e^2 - e - e + 1] \\ &= \frac{e^2 - 2e + 1}{2} \end{aligned}$$

$$77. z = x + 8y + 4; 0 \leq x \leq 3, 1 \leq y \leq 2$$

$$\begin{aligned} V &= \int_0^3 \int_1^2 (x + 8y + 4) \, dy \, dx \\ &= \int_0^3 (xy + 4y^2 + 4y) \Big|_1^2 dx \\ &= \int_0^3 [(2x + 16 + 8) - (x + 4 + 4)] dx \\ &= \int_0^3 (x + 16) dx \\ &= \left(\frac{1}{2} x^2 + 16x \right) \Big|_0^3 \\ &= \left(\frac{9}{2} + 48 \right) - 0 = \frac{105}{2} \end{aligned}$$

$$78. z = x^2 + y^2; 3 \leq x \leq 5, 2 \leq y \leq 4$$

$$\begin{aligned} V &= \iint_R (x^2 + y^2) \, dy \, dx \\ &= \int_3^5 \int_2^4 (x^2 + y^2) \, dy \, dx \\ &= \int_3^5 \left[x^2 y + \frac{y^3}{3} \right]_2^4 dx \\ &= \int_3^5 \left[4x^2 + \frac{64}{3} - 2x^2 - \frac{8}{3} \right] dx \\ &= \int_3^5 \left(2x^2 + \frac{56}{3} \right) dx = \left(\frac{2x^3}{3} + \frac{56x}{3} \right) \Big|_3^5 \\ &= \frac{250}{3} + \frac{280}{3} - 18 - 56 \\ &= \frac{308}{3} \end{aligned}$$

$$\begin{aligned} 79. & \int_0^1 \int_0^{2x} xy \, dy \, dx \\ &= \int_0^1 \left(\frac{xy^2}{2} \right) \Big|_0^{2x} dx \\ &= \int_0^1 \frac{x}{2} (4x^2 - 0) dx \\ &= \int_0^1 2x^3 dx \\ &= \left(\frac{1}{2} x^4 \right) \Big|_0^1 = \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
 80. \quad & \int_1^2 \int_2^{2x^2} y \, dy \, dx \\
 &= \int_1^2 \left. \frac{1}{2} y^2 \right|_2^{2x^2} dx \\
 &= \int_1^2 (2x^4 - 2) \, dx \\
 &= \left(\frac{2}{5} x^5 - 2x \right) \Big|_1^2 \\
 &= \left(\frac{64}{5} - 4 \right) - \left(\frac{2}{5} - 2 \right) \\
 &= \frac{52}{5}
 \end{aligned}$$

$$\begin{aligned}
 81. \quad & \int_0^1 \int_{x^2}^x x^3 y \, dy \, dx \\
 & \int_0^1 \left(\frac{x^3}{2} y^2 \right) \Big|_{x^2}^x dx \\
 &= \int_0^1 \frac{x^3}{2} (x^2 - x^4) \, dx \\
 &= \frac{1}{2} \int_0^1 (x^5 - x^7) \, dx \\
 &= \frac{1}{2} \left(\frac{x^6}{6} - \frac{x^8}{8} \right) \Big|_0^1 \\
 &= \frac{1}{2} \left(\frac{1}{6} - \frac{1}{8} \right) = \frac{1}{2} \cdot \frac{1}{24} = \frac{1}{48}
 \end{aligned}$$

$$\begin{aligned}
 82. \quad & \int_0^1 \int_y^{\sqrt{y}} x \, dx \, dy = \int_0^1 \left. \frac{x^2}{2} \right|_y^{\sqrt{y}} dy \\
 &= \int_0^1 \frac{1}{2} (y - y^2) \, dy \\
 &= \frac{1}{2} \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1 \\
 &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{12}
 \end{aligned}$$

$$83. \quad \int_0^2 \int_{x/2}^1 \frac{1}{y^2 + 1} \, dy \, dx$$

Change the order of integration.

$$\begin{aligned}
 & \int_0^2 \int_{x/2}^1 \frac{1}{y^2 + 1} \, dy \, dx \\
 &= \int_0^1 \int_0^{2y} \frac{1}{y^2 + 1} \, dx \, dy \\
 &= \int_0^1 \frac{x}{y^2 + 1} \Big|_0^{2y} dy \\
 &= \int_0^1 \left[\frac{1}{y^2 + 1} (2y) - \frac{1}{y^2 + 1} (0) \right] dy \\
 &= \int_0^1 \frac{2y}{y^2 + 1} \, dy \\
 &= \ln(y^2 + 1) \Big|_0^1 \\
 &= \ln 2 - \ln 1 \\
 &= \ln 2 - 0 = \ln 2
 \end{aligned}$$

$$84. \quad \int_0^8 \int_{x/2}^4 \sqrt{y^2 + 4} \, dy \, dx$$

Change the order of integration.

$$\begin{aligned}
 & \int_0^8 \int_{x/2}^4 \sqrt{y^2 + 4} \, dy \, dx \\
 &= \int_0^4 \int_0^{2y} \sqrt{y^2 + 4} \, dx \, dy \\
 &= \int_0^4 (y^2 + 4)^{1/2} x \Big|_0^{2y} dy \\
 &= \int_0^4 [(y^2 + 4)^{1/2} (2y)] - (y^2 + 4)(0) \, dy \\
 &= \int_0^4 (y^2 + 4)^{1/2} (2y) \, dy \\
 &= \frac{(y^2 + 4)^{3/2}}{\frac{3}{2}} \Big|_0^4 \\
 &= \frac{2}{3} (4^2 + 4)^{3/2} - \frac{2}{3} (0^2 + 4)^{3/2} \\
 &= \frac{2}{3} (20^{3/2} - 4^{3/2}) \\
 &= \frac{2}{3} (20\sqrt{20} - 8) \\
 &= \frac{2}{3} (40\sqrt{5} - 8) \\
 &= \frac{16}{3} (5\sqrt{5} - 1)
 \end{aligned}$$

85. $\iint_R (2x + 3y) \, dx \, dy; 0 \leq y \leq 1,$
 $y \leq x \leq 2 - y$

$$\begin{aligned} & \int_0^1 \int_y^{2-y} (2x + 3y) \, dx \, dy \\ &= \int_0^1 (x^2 + 3xy) \Big|_y^{2-y} \, dy \\ &= \int_0^1 [(2-y)^2 - y^2 + 3y(2-y-y)] \, dy \\ &= \int_0^1 (4 - 4y + y^2 - y^2 + 6y - 6y^2) \, dy \\ &= \int_0^1 (4 + 2y - 6y^2) \, dy \\ &= (4y + y^2 - 2y^3) \Big|_0^1 \\ &= 4 + 1 - 2 = 3 \end{aligned}$$

86. $\iint_R (2 - x^2 - y^2) \, dy \, dx; 0 \leq x \leq 1,$
 $x^2 \leq y \leq x$

$$\begin{aligned} & \int_0^1 \int_{x^2}^x (2 - x^2 - y^2) \, dy \, dx \\ &= \int_0^1 \left(2y - x^2 y - \frac{y^3}{3} \right) \Big|_{x^2}^x \, dx \\ &= \int_0^1 \left(2x - x^3 - \frac{x^3}{3} - 2x^2 + x^4 + \frac{x^6}{3} \right) \, dx \\ &= \int_0^1 \left(2x - 2x^2 - \frac{4x^3}{3} + x^4 + \frac{x^6}{3} \right) \, dx \\ &= \left(x^2 - \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + \frac{x^7}{21} \right) \Big|_0^1 \\ &= 1 - \frac{2}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{21} = \frac{26}{105} \end{aligned}$$

87. $C(x, y) = 4x^2 + 5y^2 - 4xy + \sqrt{x}$

(a) $C(10, 5)$
 $= 4(10)^2 + 5(5)^2 - 4(10)(5) + \sqrt{10}$
 $= 400 + 125 - 200 + \sqrt{10}$
 $= 325 + \sqrt{10} \approx 328.16$

The cost is about \$328.16.

(b) $C(15, 10)$
 $= 4(15)^2 + 5(10)^2 - 4(15)(10) + \sqrt{15}$
 $= 900 + 500 - 600 + \sqrt{15}$
 $= 800 + \sqrt{15} \approx 803.87$

The cost is about \$803.87.

(c) $C(20, 20)$
 $= 4(20)^2 + 5(20)^2 - 4(20)(20) + \sqrt{20}$
 $= 1600 + 2000 - 1600 + \sqrt{20}$
 $= 2000 + \sqrt{20} \approx 2004.47$

The cost is about \$2004.47.

88. $c(x, y) = 2x + y^2 + 4xy + 25$

(a) $c_x = 2 + 4y$
 $c_x(640, 6) = 2 + 4(6)$
 $= 26$

For an additional 1 MB of memory, the approximate change in cost is \$26.

(b) $c_y = 2y + 4x$
 $c_y(640, 6) = 2(6) + 4(640)$
 $= 2572$

For an additional hour of labor, the approximate change in cost is \$2572.

89. $z = x^{0.7} y^{0.3}$

(a) The marginal productivity of labor is

$$\frac{\partial z}{\partial x} = 0.7x^{0.7-1}y^{0.3} = \frac{0.7y^{0.3}}{x^{0.3}}$$

(b) The marginal productivity of capital is

$$\frac{\partial z}{\partial y} = 0.3x^{0.7}y^{0.3-1} = \frac{0.3x^{0.7}}{y^{0.7}}$$

90. (a) Minimize $c(x, y)$

$$\begin{aligned} &= x^2 + 5y^2 + 4xy - 70x \\ &\quad - 164y + 1800 \end{aligned}$$

$$c_x = 2x + 4y - 70$$

$$c_y = 10y + 4x - 164$$

$$2x + 4y - 70 = 0$$

$$4x + 10y - 164 = 0$$

$$\begin{aligned}
 -4x - 8y + 140 &= 0 \\
 4x + 10y - 164 &= 0 \\
 \hline
 2y - 24 &= 0 \\
 y &= 12 \\
 4x + 10(12) - 164 &= 0 \\
 4x &= 44 \\
 x &= 11
 \end{aligned}$$

Extremum at (11,12)

$$c_{xx} = 2, c_{yy} = 10, c_{xy} = 4$$

For (11, 12),

$$\begin{aligned}
 D &= (2)(10) - 16 = 4 > 0 \text{ and} \\
 c_{xx}(11, 12) &= 2 > 0.
 \end{aligned}$$

There is a relative minimum at (11, 12).

$$\begin{aligned}
 \text{(b) } c(11, 12) &= (11)^2 + 5(12)^2 + 4(11)(12) \\
 &\quad - 70(11) - 164(12) + 1800 \\
 &= 121 + 720 + 528 - 770 \\
 &\quad - 1968 + 1800 \\
 &= \$431
 \end{aligned}$$

91. Maximize $f(x, y) = xy^3$ subject to $2x + 4y = 80$.

- $g(x, y) = 2x + 4y - 80$
- $F(x, y) = xy^3 - \lambda(2x + 4y - 80)$
- $F_x(x, y, \lambda) = y^3 - 2\lambda$
 $F_y(x, y, \lambda) = 3xy^2 - 4\lambda$
 $F_\lambda(x, y, \lambda) = -2x - 4y + 80$
- $y^3 - 2\lambda = 0$
 $3xy^2 - 4\lambda = 0$
 $-2x - 4y + 80 = 0$
- Use the first and second equations to express 4λ and eliminate 4λ .

$$\begin{aligned}
 2y^3 &= 4\lambda \\
 3xy^2 &= 4\lambda \\
 2y^3 &= 3xy^2
 \end{aligned}$$

If y equals 0, the utility will have a minimum value of 0. So we can assume $y \neq 0$ and divide by y^2 to find that

$2y = 3x$. Substitute $6x$ for $4y$ in the last equation.

$$\begin{aligned}
 -2x - 6x + 80 &= 0 \\
 x = 10, y &= \frac{3}{2}x = 15. \\
 f(10, 15) &= 33,750.
 \end{aligned}$$

The maximum utility is 33,750, obtained by purchasing 10 units of x and 15 units of y .

92. Maximize $f(x, y) = x^5y^2$ subject to $10x + 6y = 42$ or $5x + 3y = 21$.

- $g(x, y) = 5x + 3y - 21$
- $F(x, y) = x^5y^2 - \lambda(5x + 3y - 21)$
- $F_x(x, y, \lambda) = 5x^4y^2 - 5\lambda$
 $F_y(x, y, \lambda) = 2x^5y - 3\lambda$
 $F_\lambda(x, y, \lambda) = -5x - 3y + 21$
- $5x^4y^2 - 5\lambda = 0$
 $2x^5y - 3\lambda = 0$
 $-5x - 3y + 21 = 0$
- Solve the first two equations for λ and eliminate λ .

$$\begin{aligned}
 x^4y^2 &= \lambda \\
 \frac{2}{3}x^5y &= \lambda \\
 x^4y^2 &= \frac{2}{3}x^5y
 \end{aligned}$$

If either x or y equals 0, the utility will have a minimum value of 0. So we can assume $xy \neq 0$ and divide by x^4y to find that $y = (2/3)x$. Substitute for y in the last equation.

$$\begin{aligned}
 -5x - 3\left(\frac{2}{3}x\right) + 21 &= 0 \\
 7x &= 21 \\
 x = 3, y &= \frac{2}{3}(3) = 2. \\
 f(3, 2) &= 972.
 \end{aligned}$$

The maximum utility is 972, obtained by purchasing 3 units of x and 2 units of y .

93. $C(x, y) = 100 \ln(x^2 + y) + e^{xy/20}$
 $x = 15, y = 9, dx = 1, dy = -1$

$$\begin{aligned} dC &= \left(\frac{200x}{x^2 + y} + \frac{y}{20} e^{xy/20} \right) dx \\ &\quad + \left(\frac{100}{x^2 + y} + \frac{x}{20} e^{xy/20} \right) dy \\ dC(15, 9) &= \left(\frac{200(15)}{15^2 + 9} + \frac{9}{20} e^{(15)(9)/20} \right) (1) \\ &\quad + \left(\frac{100}{15^2 + 9} + \frac{15}{20} e^{(9)(15)/20} \right) (-1) \\ &= \frac{1450}{117} - \frac{3}{10} e^{27/4} \\ &= -243.82 \end{aligned}$$

Costs decrease by \$243.82.

94. $V = \frac{1}{3} \pi r^2 h$
 $r = 2 \text{ cm}, h = 8 \text{ cm}$
 $dr = 0.21 \text{ cm}, dh = 0.21 \text{ cm}$

$$\begin{aligned} dV &= \frac{\pi}{3} (2rh \, dr + r^2 \, dh) \\ &= \frac{\pi}{3} [2(2)(8)(0.21) + 4(0.21)] \\ &= \frac{\pi}{3} (6.72 + 0.84) \\ &= \frac{\pi}{3} (7.56) \\ &\approx 7.92 \text{ cm}^3 \end{aligned}$$

95. $V = \frac{4}{3} \pi r^3, r = 2 \text{ ft},$

$$dr = 1 \text{ in} = \frac{1}{12} \text{ ft}$$

$$dV = 4\pi r^2 dr = 4\pi(2)^2 \left(\frac{1}{12} \right) \approx 4.19 \text{ ft}^3$$

96. $V = \frac{1}{3} \pi r^2 h$
 $r = 2.9 \text{ cm}, h = 11.4 \text{ cm}$
 $dr = dh = 0.2 \text{ cm}$

$$\begin{aligned} dV &= \frac{\pi}{3} (2rh \, dr + r^2 \, dh) \\ &= \frac{\pi}{3} [2(2.9)(11.4)(0.2) + (2.9)^2(0.2)] \\ &= \frac{\pi}{3} (13.224 + 1.682) \\ &= \frac{\pi}{3} (14.906) \\ &\approx 15.6 \text{ cm}^3 \end{aligned}$$

97. $P(x, y) = 0.01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600)$
with $x + y = 280$.

(a) $y = 280 - x$

$$\begin{aligned} P(x) &= 0.01[-x^2 + 3x(280 - x) + 160x \\ &\quad - 5(280 - x)^2 + 200(280 - x) \\ &\quad + 2600] \\ &= 0.01(-x^2 + 840x - 3x^2 + 160x \\ &\quad - 392,000 + 2800x - 5x^2 \\ &\quad + 56,000 - 200x + 2600) \end{aligned}$$

$$\begin{aligned} P(x) &= 0.01(-9x^2 + 3600x - 333,400) \\ P'(x) &= 0.01(-18x + 3600) \\ 0.01(-18x + 3600) &= 0 \\ -18x &= -3600 \\ x &= 200 \end{aligned}$$

If $x < 200, P'(x) > 0$, and if $x > 200, P'(x) < 0$.

Therefore, P is maximum when $x = 200$.
If $x = 200, y = 80$.

$$\begin{aligned} P(200, 80) &= 0.01[-200^2 + 3(200)(80) + 160(200) \\ &\quad - 5(80)^2 + 200(80) + 2600] \\ &= 0.01(26,600) = 266 \end{aligned}$$

Thus, \$200 spent on fertilizer and \$80 spent on seed will produce a maximum profit of \$266 per acre.

$$(b) \quad P(x, y) = 0.01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600)$$

$$P_x(x, y) = 0.01(-2x + 3y + 160)$$

$$P_y(x, y) = 0.01(3x - 10y + 200)$$

$$0.01(-2x + 3y + 160) = 0$$

$$0.01(3x - 10y + 200) = 0$$

These equations simplify to

$$-2x + 3y = -160$$

$$3x - 10y = -200.$$

Solve this system.

$$-6x + 9y = -480$$

$$\underline{6x - 20y = -400}$$

$$-11y = -880$$

$$y = 80$$

If $y = 80$,

$$3x - 10(80) = -200$$

$$3x = 600$$

$$x = 200.$$

$$P_{xx}(x, y) = 0.01(-2) = -0.02$$

$$P_{yy}(x, y) = 0.01(-10) = -0.1$$

$$P_{xy}(x, y) = 0$$

For $(200, 80)$, $D = (-0.02)(-0.1) - 0^2 = 0.002 > 0$, and $P_{xx} < 0$, so there is a relative maximum at $(200, 80)$.

$P(200, 80) = 266$, as in part (a) Thus, \$200 spent on fertilizer and \$80 spent on seed will produce a maximum profit of \$ 266 per acre.

(c) Maximize $P(x, y)$

$$= 0.01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600)$$

subject to $x + y = 280$.

$$1. \quad g(x, y) = x + y - 280$$

$$2. \quad F(x, y, \lambda) = 0.01(-x^2 + 3xy + 160x - 5y^2 + 200y + 2600) - \lambda(x + y - 280)$$

$$3. \quad \begin{aligned} F_x &= 0.01(-2x + 3y + 160) - \lambda \\ F_y &= 0.01(3x - 10y + 200) - \lambda \\ F_\lambda &= -(x + y - 280) \end{aligned}$$

$$4. \quad 0.01(-2x + 3y + 160) - \lambda = 0 \quad (1)$$

$$0.01(3x - 10y + 200) - \lambda = 0 \quad (2)$$

$$x + y - 280 = 0 \quad (3)$$

5. Equations (1) and (2) give

$$\begin{aligned} 0.01(-2x + 3y + 160) &= 0.01(3x - 10y + 200) \\ -2x + 3y + 160 &= 3x - 10y + 200 \\ -5x + 13y &= 40. \end{aligned}$$

Multiplying equation (3) by 5 gives

$$5x + 5y - 1400 = 0.$$

$$-5x + 13y = 40$$

$$\underline{5x + 5y = 1400}$$

$$18y = 1440$$

$$y = 80$$

If $y = 80$,

$$5x + 5(80) = 1400$$

$$5x = 1000$$

$$x = 200.$$

Thus, $P(200, 80)$ is a maximum. As before, $P(200, 80) = 266$.

Thus, \$200 spent on fertilizer and \$80 spent on seed will produce a maximum profit of \$266 per acre.

98. Assume that blood vessels are cylindrical.

$$V = \pi r^2 h$$

$$r = 0.7, h = 2.7$$

$$dr = dh = \pm 0.1$$

$$dV = 2\pi r h dr + \pi r^2 dh$$

$$= 2\pi(0.7)(2.7)(\pm 0.1) + \pi(0.7)^2(\pm 0.1)$$

$$\approx \pm 1.341$$

The possible error is 1.341 cm^3 .

$$99. \quad T(A, W, S) = -18.37 - 0.09A + 0.34W + 0.25S$$

$$\begin{aligned} (a) \quad T(65, 85, 180) &= -18.37 - 0.09(65) \\ &\quad + 0.34(85) + 0.25(180) \\ &= 49.68 \end{aligned}$$

The total body water is 49.68 liters.

(b) $T_A(A, W, S) = -0.09$

The approximate change in total body water if age is increased by 1 yr and mass and height are held constant is -0.09 liter.

$$T_W(A, W, S) = 0.34$$

The approximate change in total body water if mass is increased by 1 kg and height are held constant is 0.34 liter.

$$T_S(A, W, S) = 0.25$$

The approximate change in total body water if height is increased by 1 cm and age and mass are held constant is 0.25 liter.

100. $L(w, t) = (0.00082t + 0.0955)e^{(\ln w + 10.49)/2.842}$

(a) $L(450, 4)$

$$= [0.00082(4) + 0.0955] e^{(\ln(450) + 10.49)/2.842} \approx 33.982$$

The length is about 33.98 cm.

(b) $L_w(w, t) = (0.00082t + 0.0955)e^{(\ln w + 10.49)/2.842} \cdot \frac{1}{2.842w}$

$$L_w(450, 7) \approx 0.02723$$

The approximate change in the length of a trout if its mass increases from 450 to 451 g while age is held constant at 7 yr is 0.027 cm.

$$L_t(w, t) = 0.00082e^{(\ln w + 10.49)/2.842}$$

$$L_t(450, 7) \approx 0.2821$$

The approximate change in the length of a trout if its age increases from 7 to 8 yr while mass is held constant at 450 g is 0.28 cm.

101. (a) $f(60, 1900) \approx 50$

In 1900, 50% of those born 60 years earlier are still alive.

(b) $f(70, 2000) \approx 75$

In 2000, 75% of those born 70 years earlier are still alive.

(c) $f_x(60, 1900) \approx -1.25$

In 1900, the percent of those born 60 years earlier who are still alive was dropping at a rate of 1.25 percent per additional year of life.

(d) $f_x(70, 2000) \approx -2$

In 2000, the percent of those born 70 years earlier who are still alive was dropping at a rate of 2 percent per additional year of life.

102. $f(a, b) = \frac{1}{4}b\sqrt{4a^2 - b^2}$

(a) $f(3, 2) = \frac{1}{4}(2)\sqrt{4(3)^2 - 2^2}$

$$= \frac{1}{2}\sqrt{32} = 2\sqrt{2} \approx 2.828$$

The area of the bottom of the planter is approximately 2.828 ft².

$$\begin{aligned}
 \text{(b)} \quad A &= \frac{1}{4}b\sqrt{4a^2 - b^2} \\
 dA &= \frac{1}{4}b \cdot \frac{1}{2}(4a^2 - b^2)^{-1/2}(8a)da \\
 &\quad + \left[\frac{1}{4}b \cdot \frac{1}{2}(4a^2 - b^2)^{-1/2}(-2b) \right. \\
 &\quad \left. + \frac{1}{4}(4a^2 - b^2)^{1/2} \right] db \\
 dA &= \frac{ab}{\sqrt{4a^2 - b^2}} da \\
 &\quad + \frac{1}{4} \left(\frac{-b^2}{\sqrt{4a^2 - b^2}} + \sqrt{4a^2 - b^2} \right) db
 \end{aligned}$$

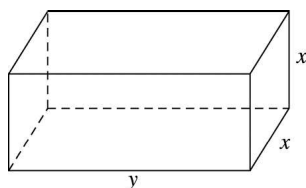
If $a = 3$, $b = 2$, $da = 0$, and $db = 0.5$,

$$dA = \frac{1}{4} \left(\frac{-2^2}{\sqrt{4(3)^2 - 2^2}} + \sqrt{4(3)^2 - 2^2} \right) (0.5)$$

$$dA \approx 0.6187.$$

The approximate effect on the area is an increase of 0.6187 ft^2 .

- 103.** Let x be the length of each of the square faces of the box and y be the length of the box.



Since the volume must be 125, the constraint is $125 = x^2y$.

$f(x, y) = 2x^2 + 4xy$ is the surface area of the box.

1. $g(x) = x^2y - 125$
2. $F(x, y, \lambda) = 2x^2 + 4xy - \lambda(x^2y - 125)$
3. $F_x(x, y, \lambda) = 4x + 4y - 2xy\lambda$
 $F_y(x, y, \lambda) = 4x - x^2\lambda$
 $F_\lambda(x, y, \lambda) = -(x^2y - 125)$
4. $4x + 4y - 2xy\lambda = 0 \quad (1)$
 $4x - x^2\lambda = 0 \quad (2)$
 $x^2y - 125 = 0 \quad (3)$

5. Factoring equation (2) gives

$$x(4 - x\lambda) = 0$$

$$x = 0 \quad \text{or} \quad 4 - x\lambda = 0.$$

Since $x = 0$ is not a solution of equation (3), then

$$4 - x\lambda = 0$$

$$\lambda = \frac{4}{x}.$$

Substituting into equation (1) gives

$$4x + 4y - 2xy \left(\frac{4}{x} \right) = 0$$

$$\text{or} \quad 4x + 4y - 8y = 0$$

$$x = y.$$

Substituting $x = y$ into equation (3) gives

$$x^2y - 125 = 0$$

$$y^3 = 125$$

$$y = 5.$$

Therefore, $x = y = 5$. The dimensions are 5 inches by 5 inches by 5 inches.

- 104.** Maximize $f(x, y) = xy$, subject to $2x + y = 400$.

1. $g(x, y) = 2x + y - 400$
2. $F(x, y, \lambda) = xy - \lambda(2x + y - 400)$
3. $F_x = y - 2\lambda$
 $F_y = x - \lambda$
 $F_\lambda = -(2x + y - 400)$
4. $y - 2\lambda = 0$
 $x - \lambda = 0$
 $2x + y - 400 = 0$
5. $\lambda = \frac{y}{2}, \lambda = x$
 $\frac{y}{2} = x$
 $y = 2x$

Substituting into $2x + y - 400$, we have

$$2x + 2x - 400 = 0,$$

$$\text{so} \quad x = 100, y = 200.$$

Dimensions are 100 feet by 200 feet for maximum area of $20,000 \text{ ft}^2$.

Extended Application: Using Multivariable Fitting to Create a Response Surface Design

1. The general cubic function of two variables has the form

$$\begin{aligned} G(x, y) = & Ax^3 + By^3 + Cx^2y + Dxy^2 \\ & + Ex^2 + Fy^2 + Gxy + Hx \\ & + Ty + J \end{aligned}$$

which has 10 terms.

2. The maximum appears to be close to orange = 56, banana = 48.

$$\begin{aligned} 3. \quad G(x, y) = & -0.00202x^2 - 0.00163y^2 \\ & + 0.000194xy + 0.21380x \\ & + 0.14768y - 2.36204 \\ G_x(x, y) = & -0.00404x + 0.000194y \\ & + 0.21380 \\ G_y(x, y) = & -0.00326y + 0.000194x \\ & + 0.14768 \\ -0.00404x + 0.000194y = & -0.21380 \\ -0.00326y + 0.000194x = & -0.14768 \end{aligned}$$

The solution to the system is $x \approx 55.254$,
 $y \approx 48.589$.

$$\begin{aligned} 4. \quad G_{xx}(x, y) = & -0.00404 \\ G_{yy}(x, y) = & -0.00326 \\ G_{xy}(x, y) = & 0.000194 \end{aligned}$$

$$\begin{aligned} D = & (-0.00404)(-0.00326) - (0.000194)^2 \\ \approx & 0.0000131 \end{aligned}$$

Since $D > 0$ and $G_{xx} = -0.00404 < 0$,
 $G(55.254, 48.589)$ is a relative maximum of
 $G(x, y)$.

5. The test results include random errors or “noise,” which may explain the 7.2 rating. Also, the function that best fits all of the data points is not necessarily above all of the points.
6. The two flavor surfaces have saddle points near the middle of the domain. For overall flavor, the maxima are at the edges of the domain, either 400 minutes at about 130°C or 20 minutes at about 160°C.
7. With this more stringent requirement, the allowable regions in the three contour plots do not overlap. The blanching efficiency would need to be allowed to be less than 93%.
8. The lower area of overlap suggests a processing temperature between 145°C and 155°C with treatment times from 50 to 70 seconds. This region represents processing nuts at higher temperatures but for a shorter time.

DIFFERENTIAL EQUATIONS

10.1 Solutions of Elementary and Separable Differential Equations

Your Turn 1

Find all solutions of $\frac{dy}{dx} = 12x^5 + \sqrt{x} + e^{5x}$.

$$\begin{aligned} y &= \int (12x^5 + \sqrt{x} + e^{5x}) dx \\ &= 2x^6 + \frac{2}{3}x^{3/2} + \frac{1}{5}e^{5x} + C \end{aligned}$$

Your Turn 2

Find the particular solution of $\frac{dy}{dx} - 12x^3 = 6x^2$,

$$y(2) = 60.$$

First solve for dy/dx .

$$\begin{aligned} \frac{dy}{dx} &= 12x^3 + 6x^2 \\ y &= 3x^4 + 2x^3 + C \end{aligned}$$

Use the initial condition to find the value of C .

$$\begin{aligned} y(2) &= 60 \\ 60 &= 3(2)^4 + 2(2)^3 + C \\ 60 &= 48 + 16 + C \\ 60 &= 64 + C \\ -4 &= C \end{aligned}$$

The particular solution is $y = 3x^4 + 2x^3 - 4$.

Your Turn 3

Find the general solution of $\frac{dy}{dx} = \frac{x^2 + 1}{xy^2}$.

Separate the variables.

$$\begin{aligned} y^2 dy &= \frac{x^2 + 1}{x} dx \\ y^2 dy &= \left(x + \frac{1}{x} \right) dx \end{aligned}$$

Now integrate.

$$\begin{aligned} y^2 dy &= \left(x + \frac{1}{x} \right) dx \\ \int y^2 dy &= \int \left(x + \frac{1}{x} \right) dx \\ \int y^2 dy &= \int x dx + \int \frac{1}{x} dx \\ \frac{1}{3}y^3 &= \frac{1}{2}x^2 + \ln|x| + C \\ y^3 &= \frac{3}{2}x^2 + 3\ln|x| + C \\ y &= \left(\frac{3}{2}x^2 + 3\ln|x| + C \right)^{1/3} \end{aligned}$$

Your Turn 4

Find the goat population in 5 years if the reserve can support 6000 goats, the growth rate is 15%, and there are currently 1200 goats in the area.

The general solution will be $y = N - Me^{-kt}$ as in Example 5, where N is the maximum population, k is the growth rate constant, and M is a constant to be determined using the initial population. For this problem, $N = 6000$ and $k = 20 = 15\% = 0.15$. Solve for M .

$$\begin{aligned} 1200 &= 6000 - Me^{(-0.15)(0)} \\ M &= 6000 - 1200 \\ M &= 4800 \end{aligned}$$

The model is $y = 6000 - 4800e^{-0.15t}$.

Now find $y(5)$.

$$\begin{aligned} y(5) &= 1200 - 4800e^{-0.15(5)} \\ &= 6000 - 4800e^{-0.75} \approx 3733 \end{aligned}$$

The goat population in 5 years will be 3733.

10.1 Exercises

$$\begin{aligned} 1. \quad \frac{dy}{dx} &= -4x + 6x^2 \\ y &= \int (-4x + 6x^2) dx \\ &= -2x^2 + 2x^3 + C \end{aligned}$$

$$\begin{aligned}
 2. \quad \frac{dy}{dx} &= 4e^{-3x} \\
 y &= \int 4e^{-3x} dx \\
 &= \frac{-4e^{-3x}}{3} + C
 \end{aligned}$$

$$3. \quad 4x^3 - 2\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$.

$$\begin{aligned}
 \frac{dy}{dx} &= 2x^3 \\
 y &= 2 \int x^3 dx \\
 &= 2 \left(\frac{x^4}{4} \right) + C \\
 &= \frac{x^4}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 4. \quad 3x^2 - 3\frac{dy}{dx} &= 2 \\
 \frac{dy}{dx} &= \frac{3x^2 - 2}{3} \\
 &= x^2 - \frac{2}{3} \\
 y &= \int \left(x^2 - \frac{2}{3} \right) dx \\
 &= \frac{x^3}{3} - \frac{2x}{3} + C
 \end{aligned}$$

$$5. \quad y \frac{dy}{dx} = x^2$$

Separate the variables and take antiderivatives.

$$\begin{aligned}
 \int y dy &= \int x^2 dx \\
 \frac{y^2}{2} &= \frac{x^3}{3} + K \\
 y^2 &= \frac{2}{3}x^3 + 2K \\
 y^2 &= \frac{2}{3}x^3 + C
 \end{aligned}$$

$$\begin{aligned}
 6. \quad y \frac{dy}{dx} &= x^2 - x \\
 y dy &= (x^2 - x) dx \\
 \int y dy &= \int (x^2 - x) dx \\
 \frac{y^2}{2} &= \frac{x^3}{3} - \frac{1}{2}x^2 + C \\
 y^2 &= \frac{2x^3}{3} - x^2 + C
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \frac{dy}{dx} &= 2xy \\
 \int \frac{dy}{y} &= \int 2x dx \\
 \ln |y| &= \frac{2x^2}{2} + C \\
 \ln |y| &= x^2 + C \\
 e^{\ln |y|} &= e^{x^2 + C} \\
 y &= \pm e^{x^2} + C \\
 y &= \pm e^{x^2} \cdot e^C \\
 y &= ke^{x^2}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \frac{dy}{dx} &= x^2 y \\
 \frac{1}{y} dy &= x^2 dx \\
 \int \frac{1}{y} dy &= \int x^2 dx \\
 \ln |y| &= \frac{x^3}{3} + C \\
 |y| &= e^{x^3/3 + C} = e^{x^3/3} \cdot e^C \\
 y &= \pm e^C e^{x^3/3} \\
 y &= Me^{x^3/3}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \frac{dy}{dx} &= 3x^2y - 2xy \\
 \frac{dy}{dx} &= y(3x^2 - 2x) \\
 \int \frac{dy}{y} &= \int (3x^2 - 2x) dx \\
 \ln |y| &= \frac{3x^3}{3} - \frac{2x^2}{2} + C \\
 e^{\ln |y|} &= e^{x^3 - x^2 + C} \\
 y &= \pm \left(e^{x^3 - x^2} \right) e^C \\
 y &= ke^{x^3 - x^2}
 \end{aligned}$$

$$\begin{aligned}
 10. \quad (y^2 - y) \frac{dy}{dx} &= x \\
 (y^2 - y) dy &= x dx \\
 \int (y^2 - y) dy &= \int x dx \\
 \frac{y^3}{3} - \frac{y^2}{2} &= \frac{x^2}{2} + C \\
 2y^3 - 3y^2 &= 3x^2 + C
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \frac{dy}{dx} &= \frac{y}{x}, x > 0 \\
 \int \frac{dy}{y} &= \int \frac{dx}{x} \\
 \ln |y| &= \ln x + C_1 \\
 e^{\ln |y|} &= e^{\ln x + C_1} \\
 y &= \pm e^{\ln x} \cdot e^{C_1} \\
 y &= Ce^{\ln x} \\
 y &= Cx
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \frac{dy}{dx} &= \frac{y}{x^2} \\
 \frac{1}{y} dy &= \frac{1}{x^2} dx \\
 \int \frac{1}{y} dy &= \int x^{-2} dx \\
 \ln |y| &= \frac{x^{-1}}{-1} + C_1 \\
 |y| &= e^{-1/x + C_1} \\
 &= e^{-1/x} \cdot e^{C_1} \\
 y &= \pm e^{C_1} e^{-1/x} \\
 y &= Ce^{-1/x}
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \frac{dy}{dx} &= \frac{y^2 + 6}{2y} \\
 \frac{2y}{y^2 + 6} dy &= dx \\
 \int \frac{2y}{y^2 + 6} dy &= \int dx \\
 \ln |y^2 + 6| &= x + C
 \end{aligned}$$

Since $y^2 + 6$ is always greater than 0 we can write this as $\ln(y^2 + 6) = x + C$.

$$\begin{aligned}
 14. \quad \frac{dy}{dx} &= \frac{e^{y^2}}{y} \\
 \frac{y}{e^{y^2}} dy &= dx \\
 \int \frac{y}{e^{y^2}} dy &= \int dx \\
 \int -2ye^{-y^2} dy &= \int -2dx \\
 e^{-y^2} &= -2x + C
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \frac{dy}{dx} &= y^2 e^{2x} \\
 \int y^{-2} dy &= \int e^{2x} dx \\
 -y^{-1} &= \frac{1}{2} e^{2x} + C \\
 -\frac{1}{y} &= \frac{1}{2} e^{2x} + C \\
 y &= \frac{-1}{\frac{1}{2} e^{2x} + C}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \frac{dy}{dx} &= \frac{e^x}{e^y} \\
 e^y dy &= e^x dx \\
 \int e^y dy &= \int e^x dx \\
 e^y &= e^x + C \\
 y &= \ln(e^x + C)
 \end{aligned}$$

17. $\frac{dy}{dx} + 3x^2 = 2x$

$$\frac{dy}{dx} = 2x - 3x^2$$

$$y = \frac{2x^2}{2} - \frac{3x^3}{3} + C$$

$$y = x^2 - x^3 + C$$

Since $y = 5$ when $x = 0$,

$$5 = 0 - 0 + C$$

$$C = 5.$$

Thus,

$$y = x^2 - x^3 + 5.$$

18. $\frac{dy}{dx} = 4x^3 - 3x^2 + x$; $y = 0$ when $x = 1$.

$$\begin{aligned} y &= \int (4x^3 - 3x^2 + x) dx \\ &= x^4 - x^3 + \frac{x^2}{2} + C \end{aligned}$$

Substitute.

$$0 = 1 - 1 + \frac{1}{2} + C$$

$$-\frac{1}{2} = C$$

$$y = x^4 - x^3 + \frac{x^2}{2} - \frac{1}{2}$$

19. $2\frac{dy}{dx} = 4xe^{-x}$
 $\frac{dy}{dx} = 2xe^{-x}$

Use the table of integrals or integrate by parts.

$$y = 2(-x - 1)e^{-x} + C$$

Since $y = 42$ when $x = 0$,

$$42 = 2(0 - 1)(1) + C$$

$$42 = -2 + C$$

$$C = 44$$

Thus,

$$y = -2xe^{-x} - 2e^{-x} + 44.$$

20. $x\frac{dy}{dx} = x^2e^{3x}$; $y = \frac{8}{9}$ when $x = 0$.

$$\frac{dy}{dx} = xe^{3x}$$

$$y = \int xe^{3x} dx$$

Let $u = x$ $dv = e^{3x} dx$

$$du = dx \quad v = \frac{e^{3x}}{3}.$$

$$y = \frac{x}{3}e^{3x} - \int \frac{e^{3x}}{3} dx$$

$$y = \frac{x}{3}e^{3x} - \frac{e^{3x}}{9} + C$$

$$\frac{8}{9} = 0 - \frac{1}{9} + C$$

$$C = 1$$

$$y = \frac{x}{3}e^{3x} - \frac{e^{3x}}{9} + 1$$

21. $\frac{dy}{dx} = \frac{x^3}{y}$; $y = 5$ when $x = 0$.

$$\int y dy = \int x^3 dx$$

$$\frac{y^2}{2} = \frac{x^4}{4} + C$$

$$y^2 = \frac{1}{2}x^4 + 2C$$

$$y^2 = \frac{1}{2}x^4 + k$$

Since $y = 5$ when $x = 0$,

$$25 = 0 + k$$

$$k = 25.$$

So $y^2 = \frac{1}{2}x^4 + 25$.

22. $x^2\frac{dy}{dx} - y\sqrt{x} = 0$; $y = e^{-2}$ when $x = 1$.

$$\frac{1}{y} dy = \frac{\sqrt{x}}{x^2} dx$$

$$\int \frac{1}{y} dy = \int x^{-3/2} dx$$

$$\ln|y| = -2x^{-1/2} + C$$

$$|y| = e^{-2x^{-1/2}} + C$$

$$\begin{aligned}
 y &= Me^{-2x^{-1/2}} \\
 e^{-2} &= Me^{-2} \\
 M &= 1 \\
 y &= e^{-2x^{-1/2}}
 \end{aligned}$$

23. $(2x + 3)y = \frac{dy}{dx}; y = 1$ when $x = 0$.

$$\int (2x + 3)dx = \int \frac{dy}{y}$$

$$\frac{2x^2}{2} + 3x + C = \ln|y|$$

$$e^{x^2+3x+C} = e^{\ln|y|}$$

$$y = (e^{x^2+3x})(\pm e^C)$$

$$y = ke^{x^2+3x}$$

Since $y = 1$ when $x = 0$.

$$1 = ke^{0+0}$$

$$k = 1.$$

So $y = e^{x^2+3x}$.

24. $\frac{dy}{dx} = \frac{x^2 + 5}{2y - 1}; y = 11$ when $x = 0$.

$$(2y - 1)dy = (x^2 + 5)dx$$

$$\int (2y - 1)dy = \int (x^2 + 5)dx$$

$$y^2 - y = \frac{x^3}{3} + 5x + C$$

$$121 - 11 = C$$

$$C = 110$$

$$y^2 - y = \frac{x^3}{3} + 5x + 110$$

25. $\frac{dy}{dx} = \frac{2x + 1}{y - 3}; y = 4$ when $x = 0$.

$$\int (y - 3)dy = \int (2x + 1)dx$$

$$\frac{y^2}{2} - 3y = \frac{2x^2}{2} + x + C$$

Since $y = 4$ when $x = 0$,

$$\frac{16}{2} - 12 = 0 + 0 + C$$

$$C = -4.$$

So,

$$\frac{y^2}{2} - 3y = x^2 + x - 4.$$

26. $x^2 \frac{dy}{dx} = y; y = -1$ when $x = 1$.

$$\frac{1}{y}dy = \frac{1}{x^2}dx$$

See Exercise 12.

$$y = Me^{-1/x}$$

$$-1 = Me^{-1}$$

$$M = -e$$

$$y = -e^{1-1/x} = -e^{(-1/x)+1}$$

27. $\frac{dy}{dx} = \frac{y^2}{x}; y = 3$ when $x = e$.

$$\int y^{-2}dy = \int \frac{dx}{x}$$

$$-y^{-1} = \ln|x| + C$$

$$-\frac{1}{y} = \ln|x| + C$$

$$y = \frac{-1}{\ln|x| + C}$$

Since $y = 3$ when $x = e$,

$$3 = \frac{-1}{\ln e + C}$$

$$3 = \frac{-1}{1 + C}$$

$$3 + 3C = -1$$

$$3C = -4$$

$$C = -\frac{4}{3}.$$

$$\text{So } y = \frac{-1}{\ln|x| - \frac{4}{3}} = \frac{-3}{3\ln|x| - 4}.$$

28. $\frac{dy}{dx} = x^{1/2}y^2$; $y = 9$ when $x = 4$.

$$\begin{aligned}\frac{1}{y^2} dy &= x^{1/2} dx \\ \int y^{-2} dy &= \int x^{1/2} dx \\ -y^{-1} &= \frac{2}{3} x^{3/2} + C \\ -\frac{1}{9} &= \frac{2}{3} (4)^{3/2} + C \\ C &= \frac{49}{9} \\ -\frac{1}{y} &= \frac{2}{3} x^{3/2} - \frac{49}{9} \\ \frac{-1}{y} &= \frac{6x^{3/2} - 49}{9} \\ y &= \frac{-9}{6x^{3/2} - 49}\end{aligned}$$

29. $\frac{dy}{dx} = (y-1)^2 e^{x-1}$; $y = 2$ when $x = 1$.

$$\begin{aligned}\frac{dy}{(y-1)^2} &= e^{x-1} dx \\ \int (y-1)^{-2} dy &= \int e^{x-1} dx \\ \frac{(y-1)^{-1}}{-1} &= e^{x-1} + C \\ -\frac{1}{y-1} &= e^{x-1} + C \\ -(y-1) &= \frac{1}{e^{x-1} + C} \\ -y + 1 &= \frac{1}{e^{x-1} + C} \\ 1 - \frac{1}{e^{x-1} + C} &= y \\ y &= \frac{e^{x-1} + C}{e^{x-1} + C} - \frac{1}{e^{x-1} + C} \\ y &= \frac{e^{x-1} + C - 1}{e^{x-1} + C}\end{aligned}$$

$y = 2$, when $x = 1$.

$$\begin{aligned}2 &= \frac{e^0 + C - 1}{e^0 + C} \\ 2 &= \frac{C}{1 + C} \\ 2 + 2C &= C \\ C &= -2 \\ y &= \frac{e^{x-1} - 3}{e^{x-1} - 2}.\end{aligned}$$

30. $\frac{dy}{dx} = (x+2)^2 e^y$; $y = 0$ when $x = 1$.

$$\begin{aligned}e^{-y} dy &= (x+2)^2 dx \\ \int e^{-y} dy &= \int (x+2)^2 dx \\ -e^{-y} &= \frac{(x+2)^3}{3} + C \\ -1 &= 9 + C \\ C &= -10 \\ -e^{-y} &= \frac{(x+2)^3}{3} - 10 \\ e^{-y} &= 10 - \frac{(x+2)^3}{3} \\ -y &= \ln \left[10 - \frac{(x+2)^3}{3} \right] \\ y &= -\ln \left[10 - \frac{(x+2)^3}{3} \right]\end{aligned}$$

31. $\frac{dy}{dx} = \frac{k}{N}(N-y)y$

(a) $\frac{N dy}{(N-y)y} = k dx$

Since $\frac{1}{y} + \frac{1}{N-y} = \frac{N}{(N-y)y}$,

$$\begin{aligned}\int \frac{dy}{y} + \int \frac{dy}{N-y} &= k dx \\ \ln \left| \frac{y}{N-y} \right| &= kx + C \\ \frac{y}{N-y} &= Ce^{kx}.\end{aligned}$$

For $0 < y < N$, $Ce^{kx} > 0$.

For $0 < N < y$, $Ce^{kx} < 0$.

Solve for y .

$$y = \frac{Ce^{kx}N}{1 + Ce^{kx}} = \frac{N}{1 + C^{-1}e^{-kx}}$$

Let $b = C^{-1} > 0$ for $0 < y < N$.

$$y = \frac{N}{1 + be^{-kx}}$$

Let $-b = C^{-1} < 0$ for $0 < N < y$.

$$y = \frac{N}{1 - be^{-kx}}$$

(b) For $0 < y < N$; $t = 0$, $y = y_0$.

$$y_0 = \frac{N}{1 + be^0} = \frac{N}{1 + b}$$

Solve for b .

$$b = \frac{N - y_0}{y_0}$$

(c) For $0 < N < y$; $t = 0$, $y = y_0$.

$$y_0 = \frac{N}{1 - be^0} = \frac{N}{1 - b}$$

Solve for b .

$$b = \frac{y_0 - N}{y_0}$$

$$32. \quad \frac{dz}{dx} = k(1 - z)z, 0 < z < 1$$

Note $1 - z > 0$ also.

$$\frac{1}{(1 - z)z} dz = k dx$$

Observe that

$$\begin{aligned} \frac{1}{1 - z} + \frac{1}{z} &= \frac{1}{(1 - z)z} \\ \left[\frac{1}{1 - z} + \frac{1}{z} \right] dz &= dz = k dx \\ \int \left[\frac{1}{1 - z} + \frac{1}{z} \right] dz &= \int k dx \\ -\ln(1 - z) + \ln z &= kx + C \\ \ln \left(\frac{z}{1 - z} \right) &= kx + C \\ \frac{z}{1 - z} &= e^{kx+C} \end{aligned}$$

Solve for z .

$$\begin{aligned} z &= e^{kx+C} - ze^{kx+C} \\ &= \frac{e^{kx+C}}{1 + e^{kx+C}} \\ &= \frac{1}{1 + e^{-kx-C}} \end{aligned}$$

$$\text{Letting } b = e^{-C}, z = \frac{1}{1 + be^{-kx}}.$$

Let x_0 be the time when $z = 1/2$, then

$$1 + be^{-kx_0} = 2 \text{ and } b = e^{kx_0}.$$

33. (a) $0 < y_0 < N$ implies that $y_0 > 0$, $N > 0$, and $N - y_0 > 0$.

Therefore,

$$b = \frac{N - y_0}{y_0} > 0.$$

Also, $e^{-kx} > 0$ for all x , which implies that $1 + be^{-kx} > 1$.

$$(1) \quad y(x) = \frac{N}{1 + be^{-kx}} < N \text{ since}$$

$$1 + be^{-kx} > 1.$$

$$(2) \quad y(x) = \frac{N}{1 + be^{-kx}} > 0 \text{ since } N > 0$$

$$\text{and } 1 + be^{-kx} > 0.$$

Combining statements (1) and (2), we have

$$\begin{aligned} 0 < \frac{N}{1 + be^{-kx}} &= y(x) \\ &= \frac{N}{1 + be^{-kx}} < N \end{aligned}$$

or $0 < y(x) < N$ for all x .

$$(b) \quad \lim_{x \rightarrow \infty} \frac{N}{1 + be^{-kx}} = \frac{N}{1 + b(0)} = N$$

$$\lim_{x \rightarrow -\infty} \frac{N}{1 + be^{-kx}} = 0$$

Note that as $x \rightarrow -\infty$, $1 + be^{-kx}$ becomes infinitely large.

Therefore, the horizontal asymptotes are $y = N$ and $y = 0$.

$$\begin{aligned} (c) \quad y'(x) &= \frac{(1 + be^{-kx})(0) - N(-kbe^{-kx})}{(1 + be^{-kx})^2} \\ &= \frac{Nkbe^{-kx}}{(1 + be^{-kx})^2} > 0 \text{ for all } x. \end{aligned}$$

Therefore, $y(x)$ is an increasing function.

- (d) To find $y''(x)$, apply the quotient rule to find the derivation of $y'(x)$. The numerator of $y''(x)$, is

$$\begin{aligned} y''(x) &= (1 + be^{-kx})^2 (-Nk^2be^{-kx}) \\ &\quad - Nkbe^{-kx}[-2kbe^{-kx}(1 + be^{-kx})] \\ &= -Nk^2be^{-kx}(1 - be^{-kx})(1 + be^{-kx}), \end{aligned}$$

and the denominator is

$$[(1 + be^{-kx})^2]^2 = (1 + be^{-kx})^4.$$

Thus,

$$y''(x) = \frac{-Nk^2be^{-kx}(1 - be^{-kx})}{(1 + be^{-kx})^3}.$$

$y''(x) = 0$ when

$$\begin{aligned} k - kbe^{-kx} &= 0 \\ be^{-kx} &= 1 \\ e^{-kx} &= \frac{1}{b} \\ -kx &= \ln\left(\frac{1}{b}\right) \\ x &= \frac{\ln\left(\frac{1}{b}\right)}{k} \\ &= \frac{\ln\left(\frac{1}{b}\right)^{-1}}{k} = \frac{\ln b}{k}. \end{aligned}$$

When $x = \frac{\ln b}{k}$,

$$\begin{aligned} y &= \frac{N}{1 + be^{-k\left(\frac{\ln b}{k}\right)}} = \frac{N}{1 + be^{(-\ln b)}} \\ &= \frac{N}{1 + be^{\ln(1/b)}} = \frac{N}{1 + b\left(\frac{1}{b}\right)} = \frac{N}{2}. \end{aligned}$$

Therefore, $\left(\frac{\ln b}{k}, \frac{N}{2}\right)$ is a point of inflection.

- (e) To locate the maximum of $\frac{dy}{dx}$, we must consider, from part (d),

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{-Nkbe^{-kx}(k - kbe^{-kx})}{(1 + be^{-kx})^3}.$$

Since $y''(x) > 0$ for $x < \frac{\ln b}{k}$ and

$$y''(x) < 0 \text{ for } x > \frac{\ln b}{k},$$

we know that $x = \frac{\ln b}{k}$ locates a relative maximum of $\frac{dy}{dx}$.

34. $y = \frac{N}{1 - be^{-kx}}$ for all $x \neq \frac{\ln b}{k}$, where $0 < N < y_0$ and

$$b = \frac{y_0 - N}{y_0}.$$

- (a) Since $0 < N < y_0$, $0 < y_0 - N$ and $0 < y_0$, so $b > 0$. Also $y_0 - N < y_0$, so

$$\frac{y_0 - N}{y_0} < 1 \text{ and } b < 1.$$

- (b) $\lim_{x \rightarrow \infty} e^{-kx} = 0$ (assuming $k > 0$), so

$$\lim_{x \rightarrow \infty} \frac{N}{1 - be^{-kx}} = \frac{N}{1 - 0} = N.$$

Thus, $y = N$ is a horizontal asymptote to the curve as $x \rightarrow \infty$.

$$\lim_{x \rightarrow -\infty} e^{-kx} = \infty, \text{ so}$$

$$\lim_{x \rightarrow -\infty} \frac{N}{1 - be^{-kx}} = 0.$$

Thus, $y = 0$ is a horizontal asymptote to the curve as $x \rightarrow -\infty$.

- (c) The graph has a vertical asymptote when

$$1 - be^{-kx} = 0$$

$$1 = be^{-kx}$$

$$\frac{1}{b} = e^{-kx}$$

$$b^{-1} = e^{-kx}$$

$$\ln b^{-1} = \ln e^{-kx}$$

$$-\ln b = -kx$$

$$\frac{\ln b}{k} = x.$$

$$(d) \quad y = N(1 - be^{-kx})^{-1}, x \neq \frac{\ln b}{k}$$

We assume that $k > 0$.

$$\begin{aligned} \frac{dy}{dx} &= -N(1 - be^{-kx})^{-2}(-be^{-kx} \cdot (-k)) \\ &= \frac{-kbNe^{-kx}}{(1 - be^{-kx})^2} \end{aligned}$$

$$e^{-kx} > 0 \text{ for all } x, (1 - be^{-kx})^2 > 0 \text{ for all } x \neq \frac{\ln b}{k}, k > 0, b > 0, \text{ and } N > 0.$$

Therefore, $\frac{dy}{dx} < 0$, and y is decreasing on $\left(\frac{\ln b}{k}, \infty\right)$ and on $\left(-\infty, \frac{\ln b}{k}\right)$.

(e) To find $\frac{d^2y}{dx^2}$, see Exercise 33(d) and just change the sign of b. Thus,

$$\frac{d^2y}{dx^2} = \frac{k^2bNe^{-kx}(1 + be^{-kx})}{(1 - be^{-kx})^3}.$$

$e^{-kx} > 0$ and $b > 0$, so $1 + be^{-kx} > 0$ for all x . $k^2 > 0$ and $N > 0$, so the numerator is always positive.

$(1 - be^{-kx})^3 > 0$, if and only if

$$\begin{aligned} 1 - be^{-kx} &> 0 \\ 1 &> be^{-kx} \\ b^{-1} &> e^{-kx} \\ \ln b^{-1} &> \ln e^{-kx} \\ -\ln b &> -kx \\ \frac{\ln b}{k} &> x. \end{aligned}$$

Thus, y is concave upward on $\left(\frac{\ln b}{k}, \infty\right)$ and concave downward on $\left(-\infty, \frac{\ln b}{k}\right)$.

$$\begin{aligned} 35. \quad \frac{dy}{dx} &= \frac{100}{32 - 4x} \\ y &= 100 \left(-\frac{1}{4} \right) \ln |32 - 4x| + C \\ y &= -25 \ln |32 - 4x| + C \\ \text{Now, } y &= 1000 \text{ when } x = 0. \end{aligned}$$

$$\begin{aligned} 1000 &= -25 \ln |32| + C \\ C &= 1000 + 25 \ln 32 \\ C &\approx 1086.64 \end{aligned}$$

Thus,

$$y = -25 \ln |32 - 4x| + 1086.64.$$

(a) Let $x = 3$.

$$\begin{aligned} y &= -25 \ln |32 - 12| + 1086.64 \\ &\approx \$1011.75 \end{aligned}$$

(b) Let $x = 5$.

$$\begin{aligned} y &= -25 \ln |32 - 20| + 1086.64 \\ &\approx \$1024.52 \end{aligned}$$

(c) Advertising expenditures can never reach \$8000. If $x = 8$, the denominator becomes zero.

36. Let $y = \text{sales}$, $t = \text{time}$, and $k = -0.15$.

$$(a) \quad \frac{dy}{dt} = -0.15y$$

(b) From Example 5, $y = Me^{-0.15t}$.

(c) If $t = 0$, $y = 0.25M$.

$$\begin{aligned} 0.25M &= Me^{-0.15t} \\ 0.25 &= e^{-0.15t} \\ -0.15t &= \ln 0.25 \\ t &= \frac{\ln 0.25}{-0.15} \approx 9.2 \end{aligned}$$

Sales will become 25% of their original value in approximately 9.2 yr.

$$37. \quad \frac{dy}{dt} = -0.05y$$

See Example 5.

$$\begin{aligned} \int \frac{dy}{y} &= \int -0.05 \, dt \\ \ln |y| &= -0.05t + C \\ e^{\ln |y|} &= e^{-0.05t + C} \\ e^{\ln |y|} &= e^{-0.05t} \cdot e^C \\ |y| &= e^{-0.05t} \cdot e^C \\ y &= Me^{-0.05t} \end{aligned}$$

Let $y = 1$ when $t = 0$.

Solve M :

$$\begin{aligned} 1 &= Me^0 \\ M &= 1. \end{aligned}$$

So $y = e^{-0.05t}$.

If $y = 0.50$,

$$0.5 = e^{-0.05t}$$

$$t = \frac{-\ln 0.5}{0.05} \approx 13.9$$

It will take about 13.9 years for \$1 to lose half its value.

38. $E = -\frac{p}{q} \cdot \frac{dq}{dp}$

If $E = \frac{4p^2}{q^2}$,

$$\frac{4p^2}{q^2} = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$4p dp = -q dq.$$

$$\int 4p dp = -\int q dq$$

$$2p^2 = -\frac{1}{2}q^2 + C_1$$

Multiplying by 2, we have

$$4p^2 = -q^2 + 2C_2$$

$$q^2 = -4p^2 + 2C_2.$$

Let $C = 2C_2$. Then

$$q^2 = -4p^2 + C$$

$$q = \pm\sqrt{-4p^2 + C}.$$

Since q cannot be negative,

$$q = \sqrt{-4p^2 + C}.$$

39. $E = -\frac{p}{q} \cdot \frac{dq}{dp}$ with $p > 0$ and $q > 0$

If $E = 2$,

$$2 = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$\frac{2}{p} dp = -\frac{1}{q} dq$$

$$\int \frac{2}{p} dp = -\int \frac{1}{q} dq$$

$$2 \ln p = -\ln q + K$$

$$\ln p^2 + \ln q = K$$

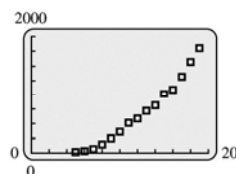
$$\ln(p^2 q) = K$$

$$p^2 q = e^K$$

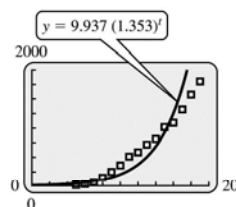
$$p^2 q = C$$

$$q = \frac{C}{p^2}.$$

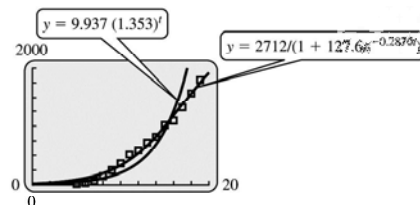
40. (a)



(b) $y = 9.937(1.353)^t$



(c) $y = \frac{2712}{1 + 127.6e^{-0.2870t}}$



(d) As t gets very large, the value of the function in (c) approaches 2712, so 2712 million is the limiting number of internet users predicted by the model.

41. $\frac{dA}{dt} = Ai$

$$\frac{dA}{A} = i dt$$

$$\int \frac{dA}{A} = \int i dt$$

$$\ln A = it + C$$

$$e^{\ln A} = e^{it+C}$$

$$A = Me^{it}$$

When $t = 0$, $A = 5000$. Therefore, $M = 5000$.

Find i so that $A = 20,000$

when $t = 24$.

$$20,000 = 5000e^{24i}$$

$$4 = e^{24i}$$

$$\ln 4 = 24i$$

$$i = \frac{\ln 4}{24}$$

$$= \frac{2\ln 2}{24}$$

$$= \frac{\ln 2}{12}$$

The answer is d.

$$42. \quad \frac{dy}{dt} = -0.03y$$

$$\int \frac{dy}{y} = -0.03y \int dt$$

$$\ln |y| = -0.03t + C$$

$$e^{\ln |y|} = e^{-0.03t+C}$$

$$y = Me^{-0.03t}$$

Since $y = 6$ when $t = 0$,

$$6 = Me^0$$

$$M = 6.$$

So $y = 6e^{-0.03t}$.

If $t = 10$,

$$y = 6e^{-0.03(10)} \\ \approx 4.4$$

After 10 minutes, about 4.4 cc of dye will be present.

$$43. \quad (a) \quad \frac{dI}{dW} = 0.088(2.4 - I)$$

Separate the variables and take anti-derivatives.

$$\int \frac{dI}{2.4 - I} = \int 0.088 dW$$

$$-\ln |2.4 - I| = 0.088W + k$$

Solve for I .

$$\ln |2.4 - I| = -0.088W - k$$

$$|2.4 - I| = e^{-0.088W-k} = e^{-k}e^{-0.088W}$$

$$I - 2.4 = Ce^{-0.088W}, \text{ where } C = \pm e^{-k}.$$

$$I = 2.4 + Ce^{-0.088W}$$

Since $I(0) = 1$, then

$$1 = 2.4 + Ce^0$$

$$C = 1 - 2.4 = -1.4.$$

Therefore, $I = 2.4 - 1.4e^{-0.088W}$.

(b) Note that as W gets larger and larger $e^{-0.088W}$ approaches 0, so

$$\lim_{W \rightarrow \infty} I = \lim_{W \rightarrow \infty} (2.4 - 1.4e^{-0.088W}) \\ = 2.4 - 1.4(0) = 2.4,$$

so I approaches 2.4.

$$44. \quad \frac{dy}{dx} = 0.02(4000 - y)$$

$$\int \frac{dy}{4000 - y} = \int 0.02 dx$$

$$-\ln |4000 - y| = 0.02x + C$$

$$\ln |4000 - y| = -0.02x + C$$

$$|4000 - y| = e^{-0.02x+C}$$

$$|4000 - y| = e^{-0.02x} \cdot e^C$$

$$4000 - y = Me^{-0.02x}$$

$$y = 4000 - Me^{-0.02x}$$

Since $y = 320$ when $x = 0$,

$$320 = 4000 - Me^0$$

$$M = 3680.$$

So $y = 4000 - 3680e^{-0.02x}$.

If $x = 10$,

$$y = 4000 - 3680e^{-0.02(10)} \\ \approx 987.$$

At the end of 10 years, there will be about 987 fish.

$$45. \quad (a) \quad \frac{dw}{dt} = k(C - 17.5w)$$

C being constant implies that the calorie intake per day is constant.

(b) pounds/day = k (calories/day)

$$\frac{\text{pounds/day}}{\text{calories/day}} = k$$

The units of k are pounds/calorie.

(c) Since 3500 calories is equivalent to 1 pound, $k = \frac{1}{3500}$ and

$$\frac{dw}{dt} = \frac{1}{3500}(C - 17.5w).$$

(d) $\frac{dw}{dt} = \frac{1}{3500}(C - 17.5w)$; $w = w_0$ when $t = 0$.

$$\begin{aligned}\frac{3500}{C - 17.5w} dw &= dt \\ \frac{3500}{-17.5} \int \frac{-17.5}{C - 17.5w} dw &= \int dt \\ -200 \ln|C - 17.5w| &= t + k \\ \ln|C - 17.5w| &= 0.005t - 0.005k \\ |C - 17.5w| &= e^{-0.005t - 0.005k} \\ |C - 17.5w| &= e^{-0.005t} \cdot e^{-0.005k} \\ C - 17.5w &= e^{-0.005M} e^{-0.005t} \\ -17.5w &= -C + e^{-0.005M} e^{-0.005t} \\ w &= \frac{C}{17.5} - \frac{e^{-0.005M}}{17.5} e^{-0.005t}\end{aligned}$$

(e) Since $w = w_0$ when $t = 0$,

$$\begin{aligned}w_0 &= \frac{C}{17.5} - \frac{e^{-0.005M}}{17.5} \quad (1) \\ w_0 - \frac{C}{17.5} &= -\frac{e^{-0.005M}}{17.5} \\ \frac{e^{-0.005M}}{17.5} &= \frac{C}{17.5} - w_0.\end{aligned}$$

Therefore,

$$\begin{aligned}w &= \frac{C}{17.5} - \left(\frac{C}{17.5} - w_0 \right) e^{-0.005t} \\ w &= \frac{C}{17.5} + \left(w_0 - \frac{C}{17.5} \right) e^{-0.005t}.\end{aligned}$$

46. (a) $\frac{dw}{dt} = \frac{1}{3500}(2500 - 17.5w)$; $w_0 = 180$
when $t = 0$.

$$\begin{aligned}\frac{3500}{2500 - 17.5w} dw &= dt \\ \int \frac{3500}{2500 - 17.5w} dw &= \int dt \\ \frac{3500}{-17.5} \int \frac{-17.5}{2500 - 17.5w} dw &= \int dt \\ -200 \ln|2500 - 17.5w| &= t + C_1 \\ \ln|2500 - 17.5w| &= -0.005t + C_2 \\ |2500 - 17.5w| &= e^{-0.005t + C_2} \\ |2500 - 17.5w| &= e^{C_2} e^{-0.005t} \\ 2500 - 17.5w &= \pm e^{C_2} e^{-0.005t} \\ -17.5w &= -2500 + C_3 e^{-0.005t} \\ w &= 143 + C_4 e^{-0.005t}\end{aligned}$$

Since $w = 180$ when $t = 0$,

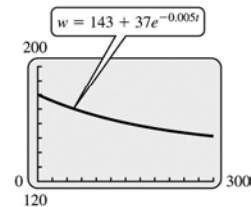
$$180 = 143 + C_4(1)$$

$$C_4 = 37$$

$$w = 143 + 37e^{-0.005t}.$$

$$\begin{aligned}\text{(b)} \quad \lim_{t \rightarrow \infty} w &= \lim_{t \rightarrow \infty} (143 + 37e^{-0.005t}) \\ &= 143 + 37(0) \\ &= 143\end{aligned}$$

The asymptote is $w = 143$.

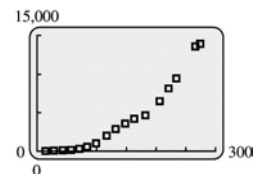


According to the model, the value 143 will never be attained.

$$\begin{aligned}\text{(c)} \quad 145 &= 143 + 37e^{-0.005t} \\ 2 &= 37e^{-0.005t} \\ e^{-0.005t} &= \frac{2}{37} \\ -0.005t &= \ln\left(\frac{2}{37}\right) \\ t &= \frac{\ln\left(\frac{2}{37}\right)}{-0.005} \\ t &\approx 583.55\end{aligned}$$

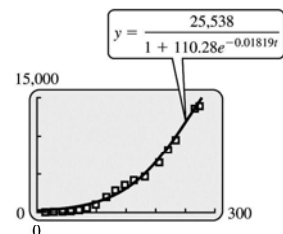
About 584 days (about 19.5 months) would be required to reach a weight of 145.

47. (a)



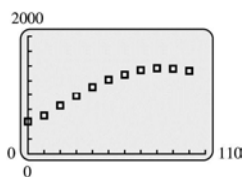
$$\text{(b)} \quad y = \frac{25,538}{1 + 110.28e^{-0.01819t}}$$

- (c)

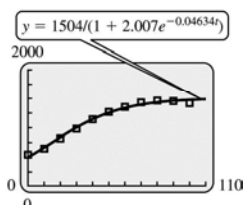


- (d) As t gets very large, the value of the function in (b) approaches 25,538, so this is the limiting number of deaths predicted by the model.

48. (a)

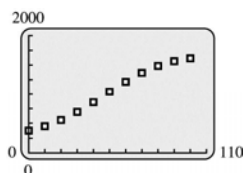


(b)
$$y = \frac{1504}{1 + 2.007e^{-0.04634t}}$$

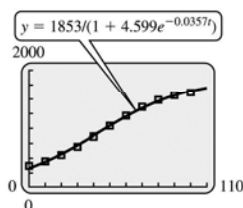


(c) According to the model, the limiting size of the Chinese population is 1504 million.

(d)



$$y = \frac{1853}{1 + 4.599e^{-0.0357t}}$$



According to the model, the limiting size of the Indian population is 1853 million.

49.
$$\frac{dy}{dt} = ky$$

First separate the variables and integrate.

$$\begin{aligned} \frac{dy}{y} &= k dt \\ \int \frac{dy}{y} &= \int k dt \\ \ln|y| &= kt + C. \end{aligned}$$

Solve for y .

$$\begin{aligned} |y| &= e^{kt+C_1} = e^{C_1}e^{kt} \\ y &= Ce^{kt}, \text{ where } C = \pm e^{C_1}. \\ y(0) &= 35.6, \text{ so } 35.6 = Ce^0 = C, \text{ and} \\ y &= 35.6e^{kt}. \end{aligned}$$

Since $y(50) = 102.6$, then $102.6 = 35.6e^{50k}$.

Solve for k .

$$\begin{aligned} e^{50k} &= \frac{102.6}{35.6} \\ 50k &= \ln\left(\frac{102.6}{35.6}\right) \\ k &= \frac{\ln\left(\frac{102.6}{35.6}\right)}{50} \approx 0.02117, \\ \text{so } y &= 35.6e^{0.02117t}. \end{aligned}$$

50.
$$\frac{dy}{dt} = ky$$

First separate the variables and integrate.

$$\begin{aligned} \frac{dy}{y} &= k dt \\ \int \frac{dy}{y} &= \int k dt \\ \ln|y| &= kt + C_1 \end{aligned}$$

Solve for y .

$$\begin{aligned} |y| &= e^{kt+C_1} = e^{C_1}e^{kt} \\ y &= Ce^{kt}, \text{ where } C = \pm e^{C_1}. \\ y(0) &= 10.7 \text{ so } 10.7 = Ce^0 = C, \text{ and} \\ y &= 10.7e^{kt}. \end{aligned}$$

Solve for k . Since $y(50) = 33.4$, then

$$\begin{aligned} 33.4 &= 10.7e^{50k} \\ e^{50k} &= \frac{33.4}{10.7} \\ 50k &= \ln\left(\frac{33.4}{10.7}\right) \\ k &= \frac{\ln\left(\frac{33.4}{10.7}\right)}{50} \approx 0.02277, \\ \text{so } y &= 10.7e^{0.02277t} \end{aligned}$$

51. (a) $\frac{dy}{dt} = ky$

$$\int \frac{dy}{y} = \int k dt$$

$$\ln|y| = kt + C$$

$$e^{\ln|y|} = e^{kt+C}$$

$$y = \pm(e^{kt})(e^C)$$

$$y = Me^{kt}$$

If $y = 1$ when $t = 0$ and $y = 5$ when $t = 2$, we have the system of equations

$$1 = Me^{k(0)}$$

$$5 = Me^{2k}$$

$$1 = M(1)$$

$$M = 1$$

Substitute.

$$5 = (1)e^{2k}$$

$$e^{2k} = 5$$

$$2k \ln e = \ln 5$$

$$k = \frac{\ln 5}{2}$$

$$\approx 0.8$$

(b) If $k = 0.8$ and $M = 1$,

$$y = e^{0.8t}$$

When $t = 3$,

$$y = e^{0.8(3)}$$

$$= e^{2.4}$$

$$\approx 11.$$

(c) When $t = 5$,

$$y = e^{0.8(5)}$$

$$= e^4$$

$$\approx 55.$$

(d) When $t = 10$,

$$y = e^{0.8(10)}$$

$$= e^8$$

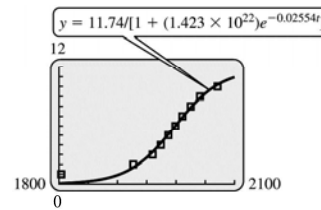
$$\approx 3000.$$

52. (a) A calculator with a logistic regression function determined that

$$y = \frac{11.74}{1 + (1.423 \times 10^{22})e^{-0.02554t}}$$

best fits the data.

- (b) From the graph, the function from part (a) seems to fit the data from 1927 on very well. For the year 1804, the function does not fit the data very well.

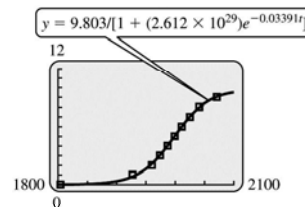


- (c) After subtracting 0.99 from the y -values in the list, a calculator with a logistic regression function determines that

$$y = \frac{9.803}{1 + (2.612 \times 10^{29})e^{-0.03391t}}$$

best fits the data.

- (d)



From the graph, the function in part (c) does seem to fit the data better than the graph found in part (b).

- (e) As x gets larger and larger $e^{-0.03391x}$ approaches 0 so that y approaches

$$\frac{9.803}{1 + 0} = 9.803.$$

If you add back the 0.99 that was subtracted from the y -values, the result is approximately 10.79 billion.

- (f) For the function found in part (c), as x gets smaller and approaches negative infinity, the denominator of this logistic function approaches infinity so that y approaches 0. After adding back the 0.99 that was subtracted earlier, this would imply that the limiting value for the world population as you go further and further back in time is 0.99 billion. This does not seem reasonable though because the world population was not always more than 990 million.

53. $\frac{dy}{dx} = 7.5e^{-0.3y}$, $y = 0$ when $x = 0$.

$$\begin{aligned} e^{0.3y} dy &= 7.5 dx \\ \int e^{0.3y} dy &= \int 7.5 dx \\ \frac{e^{0.3y}}{0.3} &= 7.5x + C \\ e^{0.3y} &= 2.25x + C \\ 1 &= 0 + C = C \\ e^{0.3y} &= 2.25x + 1 \\ 0.3y &= \ln(2.25x + 1) \\ y &= \frac{\ln(2.25x + 1)}{0.3} \end{aligned}$$

When $x = 8$,

$$y = \frac{\ln[2.25(8) + 1]}{0.3} \approx 10 \text{ items.}$$

54. (a) $\frac{dy}{dt} = -0.03y$

(b) $\int \frac{dy}{y} = -\int 0.03 dt$
 $\ln|y| = -0.03t + C$
 $e^{\ln|y|} = e^{-0.03t + C}$
 $y = \pm e^{-0.03t} \cdot e^C$
 $y = Me^{-0.03t}$

(c) Since $y = 75$ when $t = 0$.

$$\begin{aligned} 75 &= Me^0 \\ M &= 75. \end{aligned}$$

So $y = 75e^{-0.03t}$.

(d) At $t = 10$,

$$y = 75e^{-0.03(10)} \approx 56.$$

After 10 months, about 56 g are left.

55. Let $t = 0$ be the time it started snowing. If h is the height of the snow and if the rate of snowfall is constant, $\frac{dh}{dt} = k_1$, where k_1 is a constant.

$$\frac{dh}{dt} = k_1 \text{ and } h = 0 \text{ when } t = 0.$$

$$dh = k_1 dt$$

$$\int dh = \int k_1 dt$$

$$h = k_1 t + C_1$$

Since $h = 0$ and $t = 0$, $0 = k_1(0) + C_1$.

Thus, $C_1 = 0$ and $h = k_1 t$.

Since the snowplow removes a constant volume of snow per hour and the volume is proportional to the height of the snow, the rate of travel of the snowplow is inversely proportional to the height of the snow.

$$\frac{dx}{dt} = \frac{k^2}{h}, \text{ where } k_2 \text{ is a constant.}$$

When $t = T$, $x = 0$.

When $t = T + 1$, $x = 2$.

When $t = T + 2$, $x = 3$.

Since $\frac{dy}{dt} = \frac{k^2}{h}$ and $h = k_1 t$,

$$\frac{dy}{dt} = \frac{k_2}{k_1 t}$$

$$\frac{dx}{dt} = \frac{k_2}{k_1} \cdot \frac{1}{t}.$$

Let $k_3 = \frac{k_2}{k_1}$. Then

$$\frac{dx}{dt} = k_3 \frac{1}{t}$$

$$dx = k_3 \frac{1}{t} dt$$

$$\begin{aligned} \int dx &= \int k_3 \frac{1}{t} dt \\ x &= k_3 \ln t + C_2. \end{aligned}$$

Since $x = 0$, when $t = T$,

$$0 = k_3 \ln T + C_2$$

$$C_2 = -k_3 \ln T.$$

Thus,

$$x = k_3 \ln t - k_3 \ln T$$

$$x = k_3 (\ln t - \ln T)$$

$$x = k_3 \ln \left(\frac{t}{T} \right).$$

Since $x = 2$, when $t = T + 1$,

$$2 = k_3 \ln \left(\frac{T+1}{T} \right). \quad (I)$$

Since $x = 3$ when $t = T + 2$,

$$3 = k_3 \ln \left(\frac{T+2}{T} \right). \quad (2)$$

We want to solve for T , so we divide equation (1) by equation (2).

$$\begin{aligned} \frac{2}{3} &= \frac{k_3 \ln \left(\frac{T+1}{T} \right)}{k_3 \ln \left(\frac{T+2}{T} \right)} \\ \frac{2}{3} &= \frac{\ln(T+1) - \ln T}{\ln(T+2) - \ln T} \end{aligned}$$

$$\begin{aligned} 2 \ln(T+2) - 2 \ln T &= 3 \ln(T+1) - 3 \ln T \\ \ln(T+2)^2 - \ln T^2 - \ln(T+1)^3 + \ln T^3 &= 0 \end{aligned}$$

$$\ln \frac{(T+2)^2 T^3}{T^2 (T+1)^3} = 0$$

$$\frac{T(T+2)^2}{(T+1)^3} = 1$$

$$T(T^2 + 4T + 4) = T^3 + 3T^2 + 3T + 1$$

$$T^3 + 4T^2 + 4T = T^3 + 3T^2 + 3T + 1$$

$$T^2 + T - 1 = 0$$

$$T = \frac{-1 \pm \sqrt{1+4}}{2}$$

$T = \frac{-1-\sqrt{5}}{2}$ is negative and is not a possible solution.

Thus, $T = \frac{-1+\sqrt{5}}{2} \approx 0.618$ hr.

0.618 hr $\approx$ 37 min and 5 sec

Now, 37 min and 5 sec before 8:00 A.M. is 7:22:55 A.M.

Thus, it started snowing at 7:22:55 A.M.

57. If $T = Ce^{-kt} + T_M$,

$$\begin{aligned} \lim_{t \rightarrow \infty} T &= \lim_{t \rightarrow \infty} (Ce^{-kt} + T_M) \\ &= \lim_{t \rightarrow \infty} (Ce^{-kt}) + T_M \\ &= T_M \end{aligned}$$

(The exponential term has limit 0 since $k > 0$.)

58. Use the formula from Exercise 57:

$$T = Ce^{-kt} + T_M.$$

(a) We know that $T_M = 68$, $T(0) = 98.6$ and $T(1) = 90$.

$$98.6 = (C)(1) + 68$$

$$C = 30.6$$

$$T = 30.6e^{-kt} + 68$$

$$90 = 30.6e^{-k(1)} + 68$$

$$30.6e^{-kt} = 22$$

$$e^{-k} = \frac{22}{30.6}$$

$$-k = \ln \left(\frac{22}{30.6} \right)$$

$$k = -\ln \left(\frac{22}{30.6} \right)$$

$$\approx 0.33$$

Therefore, $T = 30.6e^{-0.33t} + 68$.

When $t = 2$, $T = 30.6e^{-0.33(2)} + 68$
 $= 83.8$.

After two hours the temperature of the body will be 83.8°F.

$$(b) \quad 75 = 30.6e^{-0.33t} + 68$$

$$7 = 30.6e^{-0.33t}$$

$$e^{-0.33t} = \frac{7}{30.6}$$

$$-0.33t = \ln \left(\frac{7}{30.6} \right)$$

$$t = \frac{\ln \left(\frac{7}{30.6} \right)}{-0.33}$$

$$\approx 4.5$$

The temperature of the body will be 75°F in approximately 4.5 hours.

$$(c) \quad 68.01 = 30.6e^{-0.33t} + 68$$

$$0.01 = 30.6e^{-0.33t}$$

$$e^{-0.33t} = \frac{0.01}{30.6}$$

$$-0.33t = \ln \left(\frac{0.01}{30.6} \right)$$

$$t = \frac{\ln \left(\frac{0.01}{30.6} \right)}{-0.33}$$

$$\approx 24.3$$

The temperature of the body will be within 0.01° of the surrounding air in approximately 24.3 hours.

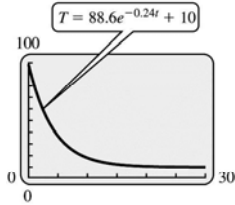
59. Use the formula from Exercise 57:

$$T = Ce^{-kt} + T_M.$$

- (a) In this problem, $T_0 = 10$, $C = 88.6$, and $k = 0.24$.

Therefore, $T = 88.6e^{-0.24t} + 10$.

- (b)



- (c) The graph shows the most rapid decrease in the first few hours which is just after death.

- (d) If $t = 4$,

$$T = 88.6e^{-0.24(4)} + 10$$

$$T \approx 43.9.$$

The temperature of the body will be 43.9°F after 4 hours.

- (e) $40 = 88.6e^{-0.24t} + 10$

$$88.6e^{-0.24t} = 30$$

$$e^{-0.24t} = \frac{30}{88.6}$$

$$-0.24t = \ln\left(\frac{30}{88.6}\right)$$

$$t = \frac{\ln\left(\frac{30}{88.6}\right)}{-0.24}$$

$$t \approx 4.5$$

The body will reach a temperature of 40°F in 4.5 hours.

10.2 Linear First-Order Differential Equations

Your Turn 1

Give the general solution of

$$x \frac{dy}{dx} - y - x^2 e^x = 0, x > 0.$$

Step 1: Put the equation in the required form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = xe^x$$

Step 2: Find the integrating factor $I(x)$.

$$\begin{aligned} I(x) &= e^{\int \left(-\frac{1}{x}\right) dx} \\ &= e^{-\ln x} \\ &= \frac{1}{x} \end{aligned}$$

Step 3: Multiply each term in the equation by the integrating factor x^{-1} .

$$(x^{-1}) \frac{dy}{dx} - (x^{-2})y = e^x$$

Step 4: Write the left side as the derivative of a product with respect to x .

$$D_x(x^{-1}y) = e^x$$

Step 5: Integrate on both sides.

$$\begin{aligned} \int D_x(x^{-1}y) dx &= \int e^x dx \\ x^{-1}y &= e^x + C \end{aligned}$$

Step 6: Multiply both sides by x to solve for y .

$$y = xe^x + Cx$$

Your Turn 2

Solve the initial value problem

$$\frac{dy}{dx} + 2xy - xe^{-x^2} = 0, y(0) = 3.$$

$$\frac{dy}{dx} + (2x)y = xe^{-x^2}$$

The integrating factor is

$$\begin{aligned} I(x) &= e^{\int 2x dx} \\ &= e^{x^2} \end{aligned}$$

Multiply all terms by the integrating factor, express the left side as the derivative of a product, and integrate on both sides.

$$\begin{aligned} e^{x^2} \frac{dy}{dx} + 2xe^{x^2} y &= x \\ D_x(e^{x^2} y) &= x \\ e^{x^2} y &= \int x dx \\ e^{x^2} y &= \frac{1}{2} x^2 + C \\ y &= \frac{\frac{1}{2} x^2 + C}{e^{x^2}} \\ &= \frac{x^2 + C}{2e^{x^2}} \end{aligned}$$

To find the particular solution, substitute 0 for x and 3 for y .

$$\begin{aligned} 3 &= \frac{(0)^2 + C}{2(1)} \\ C &= 6 \end{aligned}$$

The particular solution is $y = \frac{x^2 + 6}{2e^{x^2}}$.

10.2 Exercises

1. $\frac{dy}{dx} + 3y = 6$

$$I(x) = e^{\int 3 dx} = e^{3x}$$

Multiply each term by e^{3x} .

$$\begin{aligned} e^{3x} \frac{dy}{dx} + 3e^{3x} y &= 6e^{3x} \\ D_x(e^{3x} y) &= 6e^{3x} \end{aligned}$$

Integrate both sides.

$$\begin{aligned} e^{3x} y &= \int 6e^{3x} dx \\ &= 2e^{3x} + C \\ y &= 2 + Ce^{-3x} \end{aligned}$$

2. $\frac{dy}{dx} + 5y = 12$

$$\begin{aligned} I(x) &= e^{\int 5 dx} = e^{5x} \\ e^{5x} \frac{dy}{dx} + 5e^{5x} y &= 12e^{5x} \\ D_x(e^{5x} y) &= 12e^{5x} \\ e^{5x} y &= \int 12e^{5x} dx \\ &= \frac{12}{5} e^{5x} + C \\ y &= \frac{12}{5} + Ce^{-5x} \end{aligned}$$

3. $\frac{dy}{dx} + 2xy = 4x$

$$\begin{aligned} I(x) &= e^{\int 2x dx} = e^{x^2} \\ e^{x^2} \frac{dy}{dx} + 2xe^{x^2} y &= 4xe^{x^2} \\ D_x(e^{x^2} y) &= 4xe^{x^2} \\ e^{x^2} y &= \int 4xe^{x^2} dx \\ &= 2e^{x^2} + C \\ y &= 2 + Ce^{-x^2} \end{aligned}$$

4. $\frac{dy}{dx} + 4xy = 4x$

$$\begin{aligned} I(x) &= e^{\int 4x dx} = e^{2x^2} \\ e^{2x^2} \frac{dy}{dx} + 4xe^{2x^2} y &= 4xe^{2x^2} \\ D_x(e^{2x^2} y) &= 4xe^{2x^2} \\ e^{2x^2} y &= \int 4xe^{2x^2} dx \\ &= e^{2x^2} + C \\ y &= 1 + Ce^{-2x^2} \end{aligned}$$

5. $x \frac{dy}{dx} - y - x = 0; x > 0$

$$\begin{aligned} \frac{dy}{dx} - \frac{1}{x} y &= 1 \\ I(x) &= e^{-\int 1/x dx} \\ &= e^{-\ln x} = \frac{1}{x} \end{aligned}$$

$$\frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x}$$

$$D_x \left(\frac{1}{x} y \right) = \frac{1}{x}$$

$$\frac{y}{x} = \int \frac{1}{x} dx$$

$$\frac{y}{x} = \ln x + C$$

$$y = x \ln x + Cx$$

$$6. \quad x \frac{dy}{dx} + 2xy - x^2 = 0; x > 0$$

$$\frac{dy}{dx} + 2y = x$$

$$I(x) = e^{\int 2 dx} = e^{2x}$$

$$e^{2x} \frac{dy}{dx} + 2e^{2x} y = xe^{2x}$$

$$D_x(e^{2x} y) = xe^{2x}$$

$$e^{2x} y = \int xe^{2x} dx$$

Integration by parts:

$$\text{Let } u = x \quad dv = e^{2x} dx$$

$$du = dx \quad v = \frac{e^{2x}}{2}$$

$$e^{2x} y = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2} dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$$y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$$

$$7. \quad 2 \frac{dy}{dx} - 2xy - x = 0$$

$$\frac{dy}{dx} - xy = \frac{x}{2}$$

$$I(x) = e^{-\int x dx} = e^{-x^2/2}$$

$$e^{-x^2/2} \frac{dy}{dx} - xe^{-x^2/2} y = \frac{x}{2} e^{-x^2/2}$$

$$D_x(e^{-x^2/2} y) = \frac{x}{2} e^{-x^2/2}$$

$$e^{-x^2/2} y = \int \frac{x}{2} e^{-x^2/2} dx$$

$$= \frac{-1}{2} e^{-x^2/2} + C$$

$$y = -\frac{1}{2} + Ce^{x^2/2}$$

$$8. \quad 3 \frac{dy}{dx} + 6xy + x = 0$$

$$\frac{dy}{dx} + 2xy = -\frac{x}{3}$$

$$I(x) = e^{\int 2x dx} = e^{x^2}$$

$$e^{x^2} \frac{dy}{dx} + 2xe^{x^2} y = -\frac{x}{3} e^{x^2}$$

$$D_x(e^{x^2} y) = -\frac{x}{3} e^{x^2}$$

$$e^{x^2} y = \int -\frac{x}{3} e^{x^2} dx$$

$$= -\frac{1}{6} e^{x^2} + C$$

$$y = -\frac{1}{6} + Ce^{-x^2}$$

$$9. \quad x \frac{dy}{dx} + 2y = x^2 + 6x; x > 0$$

$$\frac{dy}{dx} + \frac{2}{x} y = x + 6$$

$$I(x) = e^{\int 2/x dx} = e^{2 \ln x} = x^2$$

$$x^2 \frac{dy}{dx} + 2xy = x^3 + 6x^2$$

$$D_x(x^2 y) = x^3 + 6x^2$$

$$x^2 y = \int (x^3 + 6x^2) dx$$

$$= \frac{x^4}{4} + 2x^3 + C$$

$$y = \frac{x^2}{4} + 2x + \frac{C}{x^2}$$

$$10. \quad x^3 \frac{dy}{dx} - x^2 y = x^3 - 4x^3; x > 0$$

$$\frac{dy}{dx} - \frac{1}{x} y = x - 4$$

$$I(x) = e^{\int -1/x dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x} \cdot \frac{dy}{dx} - \frac{1}{x^2} y = 1 - \frac{4}{x}$$

$$D_x \left(\frac{y}{x} \right) = 1 - \frac{4}{x}$$

$$\frac{y}{x} = \int \left(1 - \frac{4}{x} \right) dx$$

$$= x - 4 \ln x + C$$

$$y = x^2 - 4x \ln x + Cx$$

11. $y - x \frac{dy}{dx} = x^3; x > 0$

$$\frac{dy}{dx} - \frac{y}{x} = -x^2$$

$$I(x) = e^{-\int 1/x \, dx} = e^{-\ln x} = x^{-1}$$

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = -x$$

$$D_x \left(\frac{1}{x} y \right) = -x$$

$$\frac{y}{x} = \int -x \, dx$$

$$= \frac{-x^2}{2} + C$$

$$y = \frac{-x^3}{2} + Cx$$

12. $2xy + x^3 = x \frac{dy}{dx}$

$$x \frac{dy}{dx} - 2xy = x^3$$

$$\frac{dy}{dx} - 2y = x^2$$

$$I(x) = e^{\int -2 \, dx} = e^{-2x}$$

$$e^{-2x} \frac{dy}{dx} - 2e^{-2x} y = x^2 e^{-2x}$$

$$D_x(e^{-2x} y) = x^2 e^{-2x}$$

$$e^{-2x} y = \int x^2 e^{-2x} \, dx$$

Integration by parts:

Let $u = x^2 \quad dv = e^{-2x} \, dx$

$$du = 2x \, dx \quad v = \frac{e^{-2x}}{-2}$$

$$e^{-2x} y = \frac{-x^2 e^{-2x}}{2} + \int x e^{-2x} \, dx$$

Let $u = x \quad dv = e^{-2x} \, dx$

$$du = dx \quad v = \frac{e^{-2x}}{-2}$$

$$e^{-2x} y = \frac{-x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} + \int \frac{e^{-2x}}{2} \, dx$$

$$= \frac{-x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

$$y = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4} + C e^{2x}$$

13. $\frac{dy}{dx} + y = 4e^x; y = 50 \text{ when } x = 0.$

$$I(x) = e^{\int 1 \, dx} = e^x$$

$$e^x \frac{dy}{dx} + y e^x = 4e^{2x}$$

$$D_x(e^x y) = 4e^{2x}$$

$$e^x y = \int 4e^{2x} \, dx$$

$$= 2e^{2x} + C$$

$$y = 2e^x + C e^{-x}$$

Since $y = 50$ when $x = 0$,

$$50 = 2e^0 + C e^0$$

$$50 = 2 + C$$

$$C = 48.$$

Therefore,

$$y = 2e^x + 48e^{-x}.$$

14. $\frac{dy}{dx} + 4y = 9e^{5x}; y = 25 \text{ when } x = 0.$

$$I(x) = e^{\int 4 \, dx} = e^{4x}$$

$$e^{4x} \frac{dy}{dx} + 4e^{4x} y = 9e^{5x} e^{4x}$$

$$D_x(e^{4x} y) = 9e^{9x}$$

$$e^{4x} y = \int 9e^{9x} \, dx + C$$

$$= e^{9x} + C$$

$$y = e^{5x} + C e^{-4x}$$

$$25 = 1 + C$$

$$C = 24$$

$$y = e^{5x} + 24e^{-4x}$$

15. $\frac{dy}{dx} - 2xy - 4x = 0; y = 20 \text{ when } x = 1.$

$$\frac{dy}{dx} - 2xy - 4x = 0$$

$$\frac{dy}{dx} - 2xy = 4x$$

$$I(x) = e^{-\int 2x \, dx} = e^{-x^2}$$

$$e^{-x^2} \frac{dy}{dx} - 2x e^{-x^2} y = 4x e^{-x^2}$$

$$\begin{aligned}
 D_x(e^{-x^2}y) &= 4xe^{-x^2} \\
 e^{-x^2}y &= \int 4xe^{-x^2} dx \\
 e^{-x^2}y &= -2e^{-x^2} + C \\
 y &= -2 + Ce^{x^2}
 \end{aligned}$$

Since $y = 20$ when $x = 1$,

$$\begin{aligned}
 20 &= -2 + Ce^1 \\
 22 &= Ce \\
 C &= \frac{22}{e}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 y &= -2 + \frac{22}{e}(e^{x^2}) \\
 &= -2 + 22e^{(x^2-1)}.
 \end{aligned}$$

16. $x \frac{dy}{dx} - 3y + 2 = 0$; $y = 8$ when $x = 1$.

$$\begin{aligned}
 \frac{dy}{dx} - \frac{3}{x}y &= -\frac{2}{x} \\
 I(x) &= e^{\int -3/x dx} = e^{-3 \ln x} \\
 &= e^{\ln x^{-3}} = x^{-3} \\
 \frac{1}{x^3} \frac{dy}{dx} - \frac{3}{x^4}y &= -\frac{2}{x^4} \\
 D_x\left(\frac{1}{x^3}y\right) &= -\frac{2}{x^4} \\
 \frac{1}{x^3}y &= \int -2x^{-4} dx \\
 &= \frac{2x^{-3}}{3} + C \\
 y &= \frac{2}{3} + Cx^3 \\
 8 &= \frac{2}{3} + C \\
 C &= \frac{22}{3} \\
 y &= \frac{2}{3} + \frac{22x^3}{3}
 \end{aligned}$$

17. $x \frac{dy}{dx} + 5y = x^2$; $y = 12$ when $x = 2$.

$$\begin{aligned}
 x \frac{dy}{dx} + 5y &= x^2 \\
 \frac{dy}{dx} + \frac{5}{x}y &= x \\
 I(x) &= e^{\int 5/x dx} = e^{5 \ln x} = x^5 \\
 x^5 \frac{dy}{dx} + 5x^4y &= x^6 \\
 D_x(x^5y) &= x^6 \\
 x^5y &= \int x^6 dx \\
 x^5y &= \frac{x^7}{7} + C \\
 y &= \frac{x^2}{7} + \frac{C}{x^5}
 \end{aligned}$$

Since $y = 12$, when $x = 2$,

$$\begin{aligned}
 12 &= \frac{4}{7} + \frac{C}{32} \\
 \frac{80}{7} &= \frac{C}{32} \\
 C &= \frac{2560}{7}.
 \end{aligned}$$

Therefore,

$$y = \frac{x^2}{7} + \frac{2560}{7x^5}.$$

18. $2 \frac{dy}{dx} - 4xy = 5x$; $y = 10$ when $x = 1$.

$$\begin{aligned}
 \frac{dy}{dx} - 2xy &= \frac{5x}{2} \\
 I(x) &= e^{\int -2x dx} = e^{-x^2} \\
 e^{-x^2} \frac{dy}{dx} - 2xe^{-x^2}y &= \frac{5x}{2}e^{-x^2} \\
 D_x\left(e^{-x^2}y\right) &= \frac{5x}{2}e^{-x^2} \\
 e^{-x^2}y &= \int \frac{5x}{2}e^{-x^2} dx \\
 &= -\frac{5}{4}e^{-x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 y &= -\frac{5}{4} + Ce^{x^2} \\
 10 &= -\frac{5}{4} + Ce \\
 C &= \frac{45}{4e} \\
 y &= -\frac{5}{4} + \frac{45}{4e}e^{x^2} \\
 \text{or } &-\frac{5}{4} + \frac{45}{4}e^{x^2-1}
 \end{aligned}$$

19. $x \frac{dy}{dx} + (1+x)y = 3$; $y = 50$ when $x = 4$

$$\frac{dy}{dx} + \left(\frac{1+x}{x} \right) y = \frac{3}{x}$$

$$\begin{aligned}
 I(x) &= e^{\int (1+x) dx/x} \\
 &= e^{\int (1/x) dx + dx} \\
 &= e^{(\ln x) + x} \\
 &= e^{\ln x} \cdot e^x \\
 &= xe^x
 \end{aligned}$$

$$xe^x \frac{dy}{dx} + (1+x)e^x y = 3e^x$$

$$\begin{aligned}
 D_x(xe^x y) &= 3e^x \\
 xe^x y &= \int 3e^x dx \\
 xe^x y &= 3e^x + C \\
 y &= \frac{3}{x} + \frac{C}{xe^x}
 \end{aligned}$$

Since $y = 50$ when $x = 4$,

$$\begin{aligned}
 50 &= \frac{3}{4} + \frac{C}{4e^4} \\
 \frac{197}{4} &= \frac{C}{4e^4} \\
 C &= 197e^4.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 y &= \frac{3}{x} + \frac{197e^4}{xe^x} \\
 &= \frac{3}{x} + \frac{197}{x}e^{4-x} \\
 &= \frac{3 + 197e^{4-x}}{x}.
 \end{aligned}$$

20. $\frac{dy}{dx} + 3x^2y - 2xe^{-x^3}$; $y = 100$ when $x = 0$.

$$\begin{aligned}
 I(x) &= e^{\int 3x^2 dx} = e^{x^3} \\
 e^{x^3} \frac{dy}{dx} + 3x^2 e^{x^3} y &= 2xe^{x^3} \cdot e^{-x^3} \\
 D_x(e^{x^3} y) &= 2x \\
 e^{x^3} y &= \int 2x dx \\
 &= x^2 + C \\
 y &= x^2 e^{-x^3} + Ce^{-x^3} \\
 100 &= 0 + C \\
 y &= x^2 e^{-x^3} + 100e^{-x^3}
 \end{aligned}$$

21. $\frac{dy}{dx} = cy - py^2$

(a) Let $y = \frac{1}{z}$ and $\frac{dy}{dx} = -\frac{z'}{z^2}$.

$$-\frac{z'}{z^2} = c\left(\frac{1}{z}\right) - p\left(\frac{1}{z^2}\right)$$

$$z' = -cz + p$$

$$z' + cz = p$$

$$I(x) = e^{\int c dx} = e^{cx}$$

$$D_x(e^{cx} \cdot z) = \int pe^{cx} dx$$

$$e^{cx} \cdot z = \frac{p}{c} e^{cx} + K$$

$$z = \frac{p}{c} + Ke^{-cx}$$

$$= \frac{p + Kce^{-cx}}{c}$$

Therefore,

$$y = \frac{c}{p + Kce^{-cx}}.$$

(b) Let $z(0) = \frac{1}{y_0}$.

$$\frac{1}{y_0} = \frac{p + Kce^0}{c} = \frac{p + Kc}{c}$$

$$\frac{c}{y_0} = p + Kc$$

$$Kc = \frac{c}{y_0} - p = \frac{c - py_0}{y_0}$$

$$K = \frac{c - py_0}{cy_0}$$

$$y = \frac{c}{p + \left(\frac{c - py_0}{cy_0}\right)ce^{-cx}}$$

From part (a)

$$= \frac{cy_0}{py_0 + (c - py_0)e^{-cx}}$$

$$\begin{aligned} \text{(c)} \quad \lim_{x \rightarrow \infty} y &= \lim_{x \rightarrow \infty} \left(\frac{cy_0}{py_0 + (c - py_0)e^{-cx}} \right) \\ &= \frac{cy_0}{py_0 - 0} \\ &= \frac{c}{p} \end{aligned}$$

22.

$$\frac{dG}{dt} = a - KG$$

$$dG = (a - KG)dt$$

$$\begin{aligned} \frac{1}{a - KG} dG &= dt \\ \int \frac{1}{a - KG} dG &= \int dt \\ -\frac{1}{K} \int \frac{-K}{a - KG} dG &= \int dt \\ -\frac{1}{K} \ln|a - KG| &= t + C_1 \\ \ln|a - KG| &= -Kt + C_2 \\ |a - KG| &= e^{-Kt + C_2} \\ &= e^{C_2} e^{-Kt} \\ a - KG &= ke^{-Kt} \\ -KG &= -a + ke^{-Kt} \end{aligned}$$

Dividing by $-K$, we get

$$G = \frac{a}{K} + Ce^{-Kt} \text{ where } C = \frac{k}{-K}.$$

$$\begin{aligned} \text{23. (a)} \quad \frac{dC}{dt} &= -kC + D(t) \\ \frac{dC}{dt} + kC &= D(t) \\ I &= e^{\int k dt} = e^{kt} \end{aligned}$$

$$\begin{aligned} e^{kt} \frac{dC}{dt} + e^{kt} kC &= e^{kt} D(t) \\ Dt(e^{kt} C) &= e^{kt} D(t) \\ e^{kt} C &= \int e^{kt} D(t) dt \\ &= \int_0^t e^{ky} D(y) dy \\ C(t) &= e^{-kt} \int_0^t e^{ky} D(y) dy + C_2 \end{aligned}$$

If $C(0) = 0$,

$$\begin{aligned} C(0) &= e^0 \int_0^0 e^{ky} D(y) dy + C_2 \\ 0 &= 0 + C_2 \end{aligned}$$

Therefore,

$$C(t) = e^{-kt} \int_0^t e^{ky} D(y) dy.$$

(b) Let $D(y) = D$, a constant.

$$\begin{aligned} C(t) &= e^{-kt} \int_0^t e^{ky} D dy \\ &= De^{-kt} \int_0^t e^{ky} dy \\ &= De^{-kt} \left(\frac{1}{k} \right) (e^{kt} - e^{k(0)}) \\ C(t) &= \frac{D(1 - e^{-kt})}{k} \end{aligned}$$

24.

$$\begin{aligned} \frac{dN}{dt} &= rN - \frac{\alpha r(\alpha + b + v)}{\beta \left[\alpha - r \left(1 + \frac{v}{b + \gamma} \right) \right]} \\ \frac{dN}{dt} - rN &= -\frac{\alpha r(\alpha + b + v)}{\beta \left[\alpha - r \left(1 + \frac{v}{b + \gamma} \right) \right]} \end{aligned}$$

Integrating factor: $I = e^{\int -r dt} = e^{-rt}$.

$$\begin{aligned} e^{-rt} \frac{dN}{dt} - e^{-rt} rN &= e^{-rt} \left(-\frac{\alpha r(\alpha + b + v)}{\beta \left[\alpha - r \left(1 + \frac{v}{b + \gamma} \right) \right]} \right) \\ D_t(Ne^{-rt}) &= -\frac{\alpha r(\alpha + b + v)}{\beta \left[\alpha - r \left(1 + \frac{v}{b + \gamma} \right) \right]} e^{-rt} \end{aligned}$$

$$\begin{aligned}
 Ne^{-rt} &= -\frac{\alpha r(\alpha + b + v)}{\beta \left[\alpha - r \left(1 + \frac{v}{b+\gamma} \right) \right]} \int e^{-rt} dt \\
 &= -\frac{\alpha r(\alpha + b + v)}{\beta \left[\alpha - r \left(1 + \frac{v}{b+\gamma} \right) \right]} \left(-\frac{1}{r} \right) e^{-rt} + C \\
 &= \frac{\alpha(\alpha + b + v)}{\beta \left[\alpha - r \left(1 + \frac{v}{b+\gamma} \right) \right]} e^{-rt} + C \\
 N(t) &= \frac{\alpha + b + v}{\beta} \cdot \frac{\alpha}{\left[\alpha - r \left(1 + \frac{v}{b+\gamma} \right) \right]} + Ce^{rt}
 \end{aligned}$$

Use the initial condition $N(0) = (\alpha + b + \gamma)/\beta$.

$$\begin{aligned}
 N(0) &= \frac{\alpha + b + v}{\beta} \cdot \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} + Ce^{r(0)} \\
 \frac{\alpha + b + v}{\beta} &= \frac{\alpha + b + v}{\beta} \cdot \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} + C \\
 C &= \frac{\alpha + b + v}{\beta} - \frac{\alpha + b + v}{\beta} \cdot \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} \\
 &= \frac{\alpha + b + v}{\beta} \left(1 - \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} \right)
 \end{aligned}$$

Replace this in the anti-derivative previously found.

$$\begin{aligned}
 N(t) &= \frac{\alpha + b + v}{\beta} \cdot \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} \\
 &\quad + \frac{\alpha + b + v}{\beta} \left(1 - \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} \right) e^{rt}
 \end{aligned}$$

Now use the substitution

$$\begin{aligned}
 R &= \alpha - r \left(1 + \frac{v}{b+\gamma} \right) \\
 N(t) &= \frac{\alpha + b + v}{\beta} \cdot \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} \\
 &\quad + \frac{\alpha + b + v}{\beta} \cdot \left(1 - \frac{\alpha}{\alpha - r \left(1 + \frac{v}{b+\gamma} \right)} \right) e^{rt} \\
 &= \frac{\alpha + b + v}{\beta} \cdot \frac{\alpha}{R} + \frac{\alpha + b + v}{\beta} \left(1 - \frac{\alpha}{R} \right) e^{rt} \\
 &= \frac{\alpha + b + v}{\beta R} [\alpha + (R - \alpha)e^{rt}] \\
 &= \frac{\alpha + b + v}{\beta R} [(R - \alpha)e^{rt} + \alpha]
 \end{aligned}$$

25. $\frac{dy}{dt} = 0.02y + e^t$; $y = 10,000$ when $t = 0$.

$$\frac{dy}{dt} - 0.02y = e^t$$

$$I(t) = e^{\int -0.02 dt} = e^{-0.02t}$$

$$e^{-0.02t} \frac{dy}{dt} - 0.02e^{-0.02t} y = e^{-0.02t} \cdot e^t$$

$$D_t(e^{-0.02t} y) = e^{0.98t}$$

$$e^{-0.02t} y = \int e^{0.98t} dt$$

$$= \frac{e^{0.98t}}{0.98} + C$$

$$y = \frac{e^t}{0.98} + Ce^{0.02t}$$

$$10,000 = \frac{1}{0.98} + C$$

$$C \approx 9999$$

$$y \approx \frac{e^t}{0.98} + 9999e^{0.02t}$$

$$= 1.02e^t + 9999e^{0.02t}$$

26. $f(t) = e^{-t}$

$$\frac{dy}{dt} = ky + f(t), k = 0.02$$

$$\frac{dy}{dt} = 0.02y + e^{-t}$$

$$\frac{dy}{dt} - 0.02y = e^{-t}$$

$$I(t) = e^{\int -0.02 dt} = e^{-0.02t}$$

$$D_t(y \cdot e^{-0.02t}) = e^{-0.02t} e^{-t}$$

$$ye^{-0.02t} = \int e^{-1.02t} dt$$

$$= -0.98e^{-1.02t} + C$$

$$y = -0.98e^{-t} + Ce^{-0.02t}$$

Since $y = 10,000$ when $t = 0$,

$$10,000 = -0.98e^0 + Ce^0 = -0.98 + C$$

$$C = 10,000.98.$$

Therefore,

$$y = -0.98e^{-t} + 10,000.98e^{0.02t}.$$

27. $\frac{dy}{dt} = 0.02y - t$; $y = 10,000$ when $t = 0$.

$$\frac{dy}{dt} - 0.02y = -t$$

$$I(t) = e^{\int -0.02} = e^{-0.02t}$$

$$e^{-0.02t} \frac{dy}{dt} - 0.02e^{-0.02t} y = -te^{-0.02t}$$

$$D_t(e^{-0.02t} y) = -te^{-0.02t}$$

$$e^{-0.02t} y = \int -te^{-0.02t} dt$$

Integration by parts:

Let $u = -t$ $dv = e^{-0.02t} dt$

$$du = -dt \quad v = \frac{e^{-0.02t}}{-0.02}$$

$$e^{-0.02t} = \frac{te^{-0.02t}}{0.02} - \int \frac{e^{-0.02t}}{-0.02} dt$$

$$e^{-0.02t} y = \frac{te^{-0.02t}}{0.02} + \frac{e^{-0.02t}}{0.0004} + C$$

$$y = 50t + 2500 + Ce^{0.02t}$$

$$10,000 = 2500 + C$$

$$C = 7500$$

$$y = 50t + 2500 + 7500e^{0.02t}$$

28. $f(t) = t$

$$\frac{dy}{dt} = ky + f(t)$$

$$\frac{dy}{dt} = ky + t$$

$$\frac{dy}{dt} - ky = t$$

$$I(t) = e^{-\int kt} = e^{-kt}$$

$$e^{-kt} \frac{dy}{dt} - ke^{-kt} y = te^{-kt}$$

$$D_t(e^{-kt} y) = te^{-kt}$$

$$e^{-kt} y = \int te^{-kt} dt$$

Use the table or integration by parts.

$$e^{-kt} y = -\left(\frac{kt + 1}{k^2}\right)e^{-kt} + C$$

$$y = -\left(\frac{kt + 1}{k^2}\right) + Ce^{kt}$$

At $k = 0.02$, $y = 10,000$, $t = 0$,

$$10,000 = -\left(\frac{0 + 1}{(0.02)^2}\right) + Ce^0$$

$$= -\left(\frac{1}{0.0004}\right) + C(1)$$

$$12,500 = C.$$

Therefore,

$$y = -\left(\frac{0.02t + 1}{0.0004}\right) + 12,500e^{0.02t}$$

$$= -50t - 2500 + 12,500e^{0.02t}.$$

29. $\frac{dT}{dt} = -k(T - T_M)$

$$\frac{dT}{dt} = -kT + kT_M$$

$$\frac{dT}{dt} + kT = kT_M$$

$$I(t) = e^{\int kdt} = e^{kt}$$

Multiply both sides by e^{kt} .

$$e^{kt} \frac{dT}{dt} + ke^{kt} T = kT_M e^{kt}$$

$$D_t(Te^{kt}) = kT_M e^{kt}$$

$$Te^{kt} = \int kT_M e^{kt} dt$$

$$Te^{kt} = T_M e^{kt} + C$$

$$T = T_M + Ce^{-kt}$$

$$T = Ce^{-kt} + T_M$$

10.3 Euler's Method

Your Turn 1

Use Euler's method to approximate the solution of $dy/dx - x^2 y^2 = 1$, $y(0) = 2$, for the interval $[0, 1]$ with $h = 0.2$.

First rewrite the equation with the derivative on the left.

$$\frac{dy}{dx} = 1 + x^2 y^2$$

$$g(x, y) = 1 + x^2 y^2$$

The following table shows the calculations, including the value of $g(x_i, y_i)$ used in computing y_{i+1} .

i	x_i	y_i	$g(x_i, y_i)$
0	0	2	1
1	0.2	2.2	1.1936
2	0.4	2.43872	1.95157684
3	0.6	2.82903537	3.88123880
4	0.8	3.60528313	9.31876252
5	1.0	5.46903563	

10.3 Exercises

Note: In each step of the calculation shown in this section, all digits should be kept in your calculator as you proceed through Euler's method. Do not round intermediate results.

1. $\frac{dy}{dx} = x^2 + y^2$; $y(0) = 2$, $h = 0.1$. Find $y(0.5)$.

$$g(x, y) = x^2 + y^2$$

$$x_0 = 0; y_0 = 2$$

$$g(x_0, y_0) = 0 + 4 = 4$$

$$x_1 = 0.1; y_1 = 2 + 4(0.1) = 2.4$$

$$g(x_1, y_1) = (0.1)^2 + (2.4)^2 = 5.77$$

$$x_2 = 0.2; y_2 = 2.4 + 5.77(0.1) = 2.977$$

$$g(x_2, y_2) = (0.2)^2 + (2.977)^2 \approx 8.903$$

$$x_3 = 0.3; y_3 = 2.977 + 8.903(0.1) \approx 3.867$$

$$g(x_3, y_3) = (0.3)^2 + (3.867)^2 \approx 15.046$$

$$x_4 = 0.4; y_4 = 3.867 + 15.046(0.1) \approx 5.372$$

$$g(x_4, y_4) = (0.4)^2 + (5.372)^2 \approx 29.016$$

$$x_5 = 0.5; y_5 = 5.372 + 29.016(0.1) \approx 8.273$$

These results are tabulated as follows.

x_i	y_i
0	2
0.1	2.4
0.2	2.977
0.3	3.867
0.4	5.372
0.5	8.273

$$y(0.5) \approx 8.273.$$

2. $\frac{dy}{dx} = xy + 4$; $y(0) = 0$, $h = 0.1$. Find $y(0.5)$.

$$g(x, y) = xy + 4$$

$$x_0 = 0; y_0 = 0$$

$$g(x_0, y_0) = 0 + 4 = 4$$

$$x_1 = 0.1; y_1 = 0 + 4(0.1) = 0.4$$

$$g(x_1, y_1) = (0.1)(0.4) + 4 = 4.04$$

$$x_2 = 0.2; y_2 = 0.4 + 4.04(0.1) = 0.804$$

$$g(x_2, y_2) = (0.2)(0.804) + 4 \approx 4.161$$

$$x_3 = 0.3; y_3 = 0.804 + 4.161(0.1) \approx 1.220$$

$$g(x_3, y_3) = (0.3)(1.220) + 4 \approx 4.366$$

$$x_4 = 0.4; y_4 = 1.220 + 4.366(0.1) \approx 1.657$$

$$g(x_4, y_4) = (0.4)(1.657) + 4 \approx 4.663$$

$$x_5 = 0.5; y_5 = 1.657 + 4.663(0.1) \approx 2.123$$

These results are tabulated as follows.

x_i	y_i
0	0
0.1	0.4
0.2	0.804
0.3	1.220
0.4	1.657
0.5	2.123

$$y(0.5) \approx 2.123$$

Use Euler's method as outlined as in the solutions for Exercises 1 and 2 in the following exercises. The results are tabulated.

3. $\frac{dy}{dx} = 1 + y$; $y(0) = 2$, $h = 0.1$; find $y(0.6)$.

x_i	y_i
0	2
0.1	2.3
0.2	2.63
0.3	2.993
0.4	3.3923
0.5	3.8315
0.6	4.31465

$$y(0.6) \approx 4.315$$

4. $\frac{dy}{dx} = x + y^2; y(0) = 0, h = 0.1$; find $y(0.6)$.

x_i	y_i
0	0
0.1	0.0
0.2	0.01
0.3	0.030
0.4	0.0601
0.5	0.100
0.6	0.151

$$y(0.6) \approx 0.151.$$

5. $\frac{dy}{dx} = x + \sqrt{y}; y(0) = 1, h = 0.1$; find $y(0.4)$.

x_i	y_i
0	1
0.1	1.1
0.2	1.215
0.3	1.345
0.4	1.491

$$y(0.4) \approx 1.491$$

6. $\frac{dy}{dx} = 1 + \frac{y}{x}; y(1) = 0, h = 0.1$; find $y(1.4)$.

x_i	y_i
1	0
1.1	0.1
1.2	0.209
1.3	0.327
1.4	0.452

$$y(1.4) \approx 0.452.$$

7. $\frac{dy}{dx} = 2x\sqrt{1 + y^2}; y(1) = 0, h = 0.1$;
find $y(1.5)$.

x_i	y_i
1	2
1.1	2.447
1.2	3.029
1.3	3.794
1.4	4.815
1.5	6.191

$$y(1.5) \approx 6.191.$$

8. $\frac{dy}{dx} = e^{-y} + e^x; y(1) = 1; h = 0.1$; find $y(1.5)$.

x_i	y_i
1	1
1.1	1.309
1.2	1.636
1.3	1.988
1.4	2.368
1.5	2.783

$$y(1.5) \approx 2.783.$$

9. $\frac{dy}{dx} = -4 + x; y(0) = 1, h = 0.1$; find $y(0.4)$.

x_i	y_i
0	1
0.1	0.6
0.2	0.21
0.3	-0.17
0.4	-0.540

$$y(0.4) \approx -0.540$$

Exact solution:

$$\begin{aligned}\frac{dy}{dx} &= -4 + x \\ y &= -4x + \frac{x^2}{2} + C\end{aligned}$$

At $y(0) = 1$,

$$\begin{aligned}1 &= -4(0) + \frac{0}{2} + C \\ C &= 1\end{aligned}$$

Therefore,

$$\begin{aligned}y &= -4x + \frac{x^2}{2} + 1 \\ y(0.4) &= -4(0.4) + \frac{(0.4)^2}{2} + 1 \\ &= -0.520.\end{aligned}$$

10. $\frac{dy}{dx} = 4x + 3; y(1) = 0, h = 0.1$; find $y(1.5)$.

x_i	y_i
1	0
1.1	0.7
1.2	1.44
1.3	2.22
1.4	3.04
1.5	3.9

$$y(1.5) \approx 3.900.$$

Exact solution:

$$\begin{aligned}\frac{dy}{dx} &= 4x + 3 \\ y &= 2x^2 + 3x + C\end{aligned}$$

At $y(1) = 0$,

$$\begin{aligned}0 &= 2(1)^2 + 3(1) + C \\ C &= -5.\end{aligned}$$

Therefore,

$$\begin{aligned}y &= 2x^2 + 3x - 5 \\ y(1.5) &= 2(1.5)^2 + 3(1.5) - 5 \\ &= 4.000.\end{aligned}$$

11. $\frac{dy}{dx} = x^3$; $y(0) = 4$, $h = 0.1$, find $y(0.4)$.

x_i	y_i
0	4
0.1	4
0.2	4.0001
0.3	4.0009
0.4	4.0036
0.5	4.010

$$y(0.5) \approx 4.010$$

Exact solution:

$$\begin{aligned}\frac{dy}{dx} &= x^3 \\ y &= \frac{x^4}{4} + C\end{aligned}$$

At $y(0) = 4$,

$$\begin{aligned}4 &= \frac{0}{4} + C \\ C &= 4.\end{aligned}$$

Therefore,

$$\begin{aligned}y &= \frac{x^4}{4} + 4 \\ y(0.5) &= \frac{(0.5)^4}{4} + 4 \approx 4.016.\end{aligned}$$

12. $\frac{dy}{dx} = \frac{3}{x}$; $y(1) = 2$, $h = 0.1$; find $y(1.4)$.

x_i	y_i
1	2
1.1	2.3
1.2	2.573
1.3	2.823
1.4	3.053

$$y(1.4) \approx 3.053.$$

Exact solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{3}{x} \\ y &= 3\ln x + C\end{aligned}$$

At $y(1) = 2$,

$$2 = 3\ln 1 + C = C.$$

Therefore,

$$\begin{aligned}y &= 3\ln x + 2 \\ y(1.4) &= 3\ln 1.4 + 2 \approx 3.009.\end{aligned}$$

13. $\frac{dy}{dx} = 2xy$; $y(1) = 1$, $h = 0.1$, find $y(1.6)$.

$$g(x, y) = 2xy$$

x_i	y_i
1	1
1.1	1.2
1.2	1.464
1.3	1.815
1.4	2.287
1.5	2.927
1.6	3.806

$$y(1.6) \approx 3.806$$

Exact solution:

$$\begin{aligned}\frac{dy}{y} &= 2x \, dx \\ \int \frac{dy}{y} &= \int 2x \, dx \\ \ln |y| &= x^2 + C \\ |y| &= e^{x^2 + C} \\ y &= ke^{x^2}\end{aligned}$$

At $y(1) = 1$,

$$1 = ke^1 = ke$$

$$k = \frac{1}{e}.$$

Therefore,

$$y = \frac{1}{e} \left(e^{x^2} \right)$$

$$= e^{x^2-1}$$

$$y(1.6) = e^{(1.6)^2-1}$$

$$= 4.759.$$

14. $\frac{dy}{dx} = x^2y$; $y(0) = 1$, $h = 0.1$; find $y(0.6)$.

x_i	y_i
0	1
0.1	1
0.2	1.001
0.3	1.005
0.4	1.014
0.5	1.030
0.6	1.056

$y(0.6) \approx 1.056$.

Exact solution:

$$\frac{dy}{y} = x^2 dx$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{x^3}{3} + C$$

At $y(0) = 1$,

$$\ln 1 = 0 + C$$

$$0 = C.$$

Therefore,

$$\ln y = \frac{x^3}{3}$$

$$y = y(x) = e^{x^3/3}.$$

$$y(0.6) = e^{(0.6)^3/3} \approx 1.075.$$

15. $\frac{dy}{dx} = ye^x$; $y(0) = 2$, $h = 0.1$, find $y(0.4)$.

x_i	y_i
0	2
0.1	2.2
0.2	2.443
0.3	2.742
0.4	3.112

So, $y(0.4) \approx 3.112$.

Exact solution:

$$\frac{dy}{y} = e^x dx$$

$$\int \frac{dy}{y} = \int e^x + c$$

$$\ln |y| = e^x + c$$

$$|y| = e^{e^x+c} = e^c e^{e^x}$$

$$y = ke^x, \text{ where } k = \pm e^c.$$

At $y(0) = 2$, $2 = ke^{e^0} = ke$, so $k = \frac{2}{e}$.

Therefore,

$$y = \frac{2}{e} e^{e^x} = 2e^{e^x-1},$$

so

$$y(0.4) = 2e^{e^{0.4}-1} \approx 3.271.$$

16. $\frac{dy}{dx} = \frac{2x}{y}$; $y(0) = 3$, $h = 0.1$; find $y(0.6)$.

x_i	y_i
0	3
0.1	3
0.2	3.007
0.3	3.020
0.4	3.040
0.5	3.066
0.6	3.099

Therefore, $y(0.6) \approx 3.099$.

Exact solution:

$$y dy = 2x dx$$

$$\int y dy = \int 2x dx$$

$$y \frac{dy}{dx} = x$$

$$\frac{y^2}{2} = x^2 + C$$

At $y(0) = 3$,

$$\frac{9}{2} = 0 + C = C.$$

$$\frac{y^2}{2} = x^2 + \frac{9}{2}$$

$$y^2 = 2x^2 + 9$$

$$y = \sqrt{2x^2 + 9}, \text{ assuming } y > 0.$$

$$y(0.6) = \sqrt{2(0.6)^2 + 9} \approx 3.118.$$

17. $\frac{dy}{dx} + y = 2e^x$; $y(0) = 100$, $h = 0.1$.
Find $y(0.3)$.

x_i	y_i
0	100
0.1	90.2
0.2	81.401
0.3	73.505

$$y(0.3) \approx 73.505.$$

Exact solution:

$$I(x) = e^{\int dx} = e^x$$

$$e^x \frac{dy}{dx} + e^x y = 2e^x e^x$$

$$D_x(e^x y) = 2e^{2x}$$

$$e^x y = \int 2e^{2x} dx + C$$

$$= e^{2x} + C$$

$$y = e^x + Ce^{-x}$$

$$100 = 1 + C$$

$$C = 99$$

$$y = e^x + 99e^{-x}$$

$$y(0.3) = e^{0.3} + 99e^{-0.3} \approx 74.691$$

18. $\frac{dy}{dx} - 2y = e^{2x}$; $y(0) = 10$, $h = 0.1$. Find $y(0.4)$.

x_i	y_i
0	10
0.1	12.1
0.2	14.642
0.3	17.720
0.4	21.446

$$y(0.4) \approx 21.446.$$

Exact solution:

$$I(x) = e^{\int -2dx} = e^{-2x}$$

$$e^{-2x} \frac{dy}{dx} - 2e^{-2x} y = e^{-2x} e^{2x}$$

$$D_x(e^{-2x} y) = 1$$

$$e^{-2x} y = \int dx + C$$

$$= x + C$$

$$y = (x + C)e^{2x}$$

$$10 = C$$

$$y = (x + 10)e^{2x}$$

$$y(0.4) = (0.4 + 10)e^{2(0.4)} \approx 23.146$$

19. $\frac{dy}{dx} = \sqrt[3]{x}$, $y(0) = 0$

Using the program for Euler's method in the Graphing Calculator Manual, the following values are obtained:

x_i	y_i	$y'(x_i)$	$y_i - y'(x_i)$
0	0	0	0
0.2	0	0.08772053	-0.08772053
0.4	0.11696071	0.22104189	-0.10408118
0.6	0.26432197	0.37954470	-0.11522273
0.8	0.43300850	0.55699066	-0.12398216
1.0	0.61867206	0.75000000	-0.13132794

20. $\frac{dy}{dx} = y$, $y(0) = 1$

Using the program for Euler's method in the Graphing Calculator Manual, the following values are obtained:

x_i	y_i	$y'(x_i)$	$y_i - y'(x_i)$
0	1	1	0
0.2	1.2	1.2214028	-0.0214028
0.4	1.44	1.4918247	-0.0518247
0.6	1.728	1.8221188	-0.0941188
0.8	2.0736	2.2255409	-0.1519409
1.0	2.48832	2.7182818	-0.2299618

21. $\frac{dy}{dx} = 4 - y$, $y(0) = 0$

Using the program for Euler's method in the Graphing Calculator Manual, the following values are obtained:

x_i	y_i	$y(x_i)$	$y_i - y(x_i)$
0	0	0	0
0.2	0.8	0.725077	0.07492
0.4	1.44	1.3187198	0.12128
0.6	1.952	1.8047535	0.14725
0.8	2.3616	2.2026841	0.15892
1.0	2.68928	2.5284822	0.16080

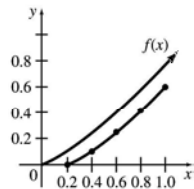
22. $\frac{dy}{dx} = x - 2xy$, $y(0) = 1$

Using the program in the Graphing Calculator Manual, the following values are obtained:

x_i	y_i	$y(x_i)$	$y_i - y(x_i)$
0	1	1	0
0.2	1	0.9803947	0.01961
0.4	0.96	0.9260719	0.03393
0.6	0.8864	0.8488382	0.03756
0.8	0.793664	0.7636462	0.03002
1.0	0.699692	0.6839397	0.01575

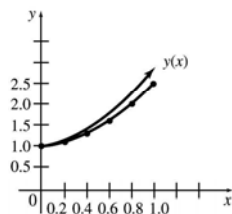
23. $\frac{dy}{dx} = \sqrt[3]{x}$; $y(0) = 0$

$y = \frac{3}{4}x^{4/3}$ See Exercise 19.



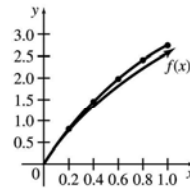
24. $\frac{dy}{dx} = y$; $y(0) = 1$

$y = e^x$ See Exercise 20.



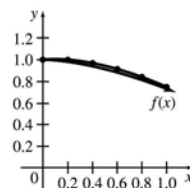
25. $\frac{dy}{dx} = 4 - y$; $y(0) = 0$

$y = 4(1 - e^{-x})$ See Exercise 21.



26. $y' = x - 2xy$; $y(0) = 1$

$y = \frac{1}{2}(1 + e^{-x^2})$ See Exercise 22.



27. $\frac{dy}{dx} = y^2$; $y(0) = 1$

(a)

x_i	y_i
0	1
0.2	1.2
0.4	1.488
0.6	1.9308288
0.8	2.676448771
1.0	4.109124376

Thus, $y(1.0) \approx 4.109$.

(b) $\frac{dy}{dx} = y^2$; $y = 1$ when $x = 0$

$$\frac{dy}{dx} = y^2$$

$$\frac{1}{y^2} dy = dx$$

$$\int \frac{1}{y^2} dy = \int dx$$

$$-\frac{1}{y} = x + C$$

When $x = 0$, $y = 1$.

$$-\frac{1}{1} = 0 + C$$

$$C = -1$$

$$-\frac{1}{y} = x - 1$$

$$-1 = (x - 1)y$$

$$y = \frac{-1}{x - 1}$$

$$y = \frac{1}{1 - x}$$

As x approaches 1 from the left,
 y approaches ∞ .

28. (a) $\frac{dy}{dt} = ky(125 - y)$; $k = 0.002$

$$\frac{dy}{dt} = 0.002y(125 - y)$$

$$\frac{dy}{dt} = 0.25y - 0.002y^2$$

(b) If $t_0 = 0$ corresponds to 2005, then
 $t_4 = 5$ corresponds to 2010.

$$y_0 = 20; g(t, y) = 0.25y - 0.002y^2$$

$$g(t_0, y_0) = 0.25(20) - 0.002(20)^2$$

$$= 4.2$$

$$t_1 = 1; y_1 = 20 + (4.2)(1) = 24.2$$

$$g(t_1, y_1) = 0.25(24.2) - 0.002(24.2)^2$$

$$= 4.87872$$

$$t_2 = 2; y_2 = 24.2 + (4.87872)(1)$$

$$= 29.0787$$

$$g(t_2, y_2) = 0.25(29.0787) - 0.002(29.0787)^2$$

$$= 5.57857$$

$$t_3 = 3; y_3 = 29.07870 + (5.57857)(1)$$

$$= 34.6573$$

$$g(t_3, y_3) = 0.25(34.6573) - 0.002(34.6573)^2$$

$$= 6.26206$$

$$t_4 = 4; y_4 = 34.6573 + (6.26206)(1)$$

$$= 40.9193$$

$$g(t_4, y_4) = 0.25(40.9193) - 0.002(40.9193)^2$$

$$= 6.88105$$

$$t_5 = 5; y_5 = 40.9193 + (6.88105)(1)$$

$$= 47.8004$$

About 48 firms are bankrupt in 2010.

29. Let y = the number of algae (in thousands) at time.

$$y \leq 500; y(0) = 5$$

(a) $\frac{dy}{dt} = 0.01y(500 - y)$

$$= 5y - 0.01y^2$$

(b) Find $y(2)$; $h = 0.5$.

x_i	y_i
0	5
0.5	17.375
1	59.303
1.5	189.976
2	484.462

Therefore, $y(2) \approx 484$, so about 484,000
algae are present when $t = 2$.

30. $\frac{dy}{dt} = 0.02(100 - y^{1/2}) = 2 - 0.02\sqrt{y}$

$$t_0 = 0; y_0 = 10; h = 0.5$$

$$g(t, y) = 2 - 0.02\sqrt{y}$$

$$g(t_0, y_0) = 2 - 0.02\sqrt{10} = 1.9368$$

$$t_1 = 0.5; y_1 = 10 + 1.9368(0.5)$$

$$= 10.9684$$

$$g(t_1, y_1) = 2 - 0.02\sqrt{10.9684}$$

$$= 1.9338$$

$$t_2 = 1; y_2 = 10.9684 + 1.9338(0.5)$$

$$= 11.9353$$

$$g(t_2, y_2) = 2 - 0.02\sqrt{11.9353}$$

$$= 1.9309$$

$$t_3 = 1.5; y_3 = 11.9353 + 1.9309(0.5)$$

$$= 12.9008$$

$$g(t_3, y_3) = 2 - 0.02\sqrt{12.9008}$$

$$= 1.9282$$

$$t_4 = 2; y_4 = 12.9008 + 1.9282(0.5)$$

$$= 13.8649$$

$$g(t_4, y_4) = 2 - 0.02\sqrt{13.8649}$$

$$= 1.9255$$

$$t_5 = 2.5; y_5 = 13.8649 + 1.9255(0.5)$$

$$= 14.8277$$

$$g(t_5, y_5) = 2 - 0.02\sqrt{14.8277}$$

$$= 1.9230$$

$$t_6 = 3, y_6 = 14.8277 + 1.9230(0.5) \\ = 15.7892$$

$$g(t_6, y_6) = 2 - 0.02\sqrt{15.7892} \\ = 1.9205$$

$$t_7 = 3.5, y_7 = 15.7892 + 1.9205(0.5) \\ = 16.7495$$

$$g(t_7, y_7) = 2 - 0.02\sqrt{16.7495} \\ = 1.9181$$

$$t_8 = 4, y_8 = 16.7495 + 1.9181(0.5) \\ = 17.7086$$

$$g(t_8, y_8) = 2 - 0.02\sqrt{17.7086} \\ = 1.9158$$

$$t_9 = 4.5, y_9 = 17.7086 + 1.9158(0.5) \\ = 18.6665$$

$$g(t_9, y_9) = 2 - 0.02\sqrt{18.6665} \\ = 1.9136$$

$$t_{10} = 5, y_{10} = 18.6665 + 1.9136(0.5) \\ = 19.6233$$

There will be about 20 species.

31. $\frac{dy}{dt} = 0.05y - 0.1y^{1/2}$; $y(0) = 60, h = 1$;
find $y(6)$.

x_i	y_i
0	60
1	62.22541
2	64.54785
3	66.97182
4	69.50205
5	72.14347
6	74.90127

Therefore, $y(6) \approx 75$, so about 75 insects are present after 6 weeks.

32. $\frac{dy}{dt} = -y + 0.02y^2 + 0.003y^3$; for $[0, 4]$
 $h = 1, t_0 = 0, y_0 = 15$
- $$g(t, y) = -y + 0.02y^2 + 0.003y^3$$
- $$g(t_0, y_0) = -15 + 0.02(15)^2 + 0.003(15)^3 \\ = -0.375$$

$$t_1 = 1; y_1 = 15 + (-0.375)(1) \\ = 14.625$$

$$g(t_1, y_1) = -14.625 + 0.02(14.625)^2 \\ + 0.003(14.625)^3 \\ = -0.963$$

$$t_2 = 2; y_2 = 14.625 + (-0.963)(1) \\ = 13.662$$

$$g(t_2, y_2) = -13.662 + 0.02(13.662)^2 \\ + 0.003(13.662)^2 \\ = -2.279$$

$$t_3 = 3; y_3 = 13.662 + (-2.279)(1) \\ = 11.383$$

$$g(t_3, y_3) = -11.383 + 0.02(11.383)^2 \\ + 0.003(11.383)^3 \\ = -4.367$$

$$t_4 = 4; y_4 = 11.383 + (-4.367)(1) \\ = 7.016 \text{ thousand}$$

There will be about 7000 whales.

33. $\frac{dW}{dt} = -0.01189W + 0.92389W^{0.016}$
 $W(0) = 3.65; h = 1$; find $W(5)$

Using the program for Euler's method in *The Graphing Calculator Manual*, the following values are obtained:

t_i	W_i
0	3.6500000
1	4.5498301
2	5.4422927
3	6.3268604
4	7.2032009
5	8.0710987

The weight of a goat at 5 weeks is about 8.07 kg.

34. (a) Using Method 2 described after Example 1 of the text, store $\frac{dy}{dt}$ for the function variable Y_1 with $k = 0.5$ and $m = 4$. That is, Y_1 should equal $0.5(P - P^2)^{1.5}$. Store 5 for H (use keystrokes $5 \rightarrow H$), to $-h = -5$ for $T(-5 \rightarrow T)$, and $p_0 = 0.1$ for $P(0.1 \rightarrow P)$. Next enter the keystrokes $T + H \rightarrow T: P + Y_1H \rightarrow P$. Each time the ENTER key is pressed, the subsequent

values for t_i will be stored into T and the corresponding values for p_i will appear on the screen. This summarized in the table below.

t_i	p_i
0	0.1
5	0.1675
10	0.297678
15	0.536660
20	0.846644
25	0.963605
30	0.980024

- (b) By continuing to press the ENTER key, it appears that the values for p_i are approaching 1.

35.

$$\frac{dN}{dt} = 0.02(500 - N)N^{1/2}; N(0) = 2, h = 0.5;$$

Find $N(3)$.

t_i	N_i
0	2
0.5	9.043
1	23.806
1.5	47.041
2	78.108
2.5	115.394
3	156.709

Therefore, $N(3) \approx 157$, so about 157 people have heard the rumor.

10.4 Applications of Differential Equations

Your Turn 1

For this problem, $r = 0.05$ and $D = 1200$ so the differential equation is

$$\frac{dA}{dt} = 0.05A + 1200$$

Separate variables and integrate on both sides.

$$\begin{aligned} \frac{dA}{0.05A + 1200} &= dt \\ \frac{1}{0.05} \ln(0.05A + 1200) &= t + C \\ \ln(0.05A + 1200) &= 0.05t + K \end{aligned}$$

$$0.05A + 1200 = e^{0.05t+K}$$

$$0.05A = -1200 + Me^{0.05t}$$

$$A = -24,000 + \frac{M}{0.05}e^{0.05t}$$

Use the fact that $A(0) = 6000$.

$$6000 = -24,000 + \frac{M}{0.05}e^{0.05(0)}$$

$$30,000(0.05) = M$$

$$1500 = M$$

$$A = -24,000 + \frac{1500}{0.05}e^{0.05t}$$

$$A = -24,000 + 30,000e^{0.05t}$$

Find A when $t = 21$.

$$A = -24,000 + 30,000e^{0.05(21)}$$

$$A \approx 61,729.53$$

When Michael is 21 the account will be worth \$61,729.53.

Your Turn 2

Solve the equation given in Example 2 with $p = 4$, $q = 1$, $r = 3$, and $s = 5$, given that there was a time when $x = 1$ and $y = 1$.

$$\frac{dy}{dx} = \frac{y(4-x)}{x(5y-3)}$$

$$\frac{5y-3}{y} dy = \frac{4-x}{x} dx$$

$$\int \frac{5y-3}{y} dy = \int \frac{4-x}{x} dx$$

$$5y - 3 \ln y = 4 \ln x - x + C$$

Use the condition $y = 1$ when $x = 1$ to find C .

$$5(1) - 3 \ln(1) = 4 \ln(1) - 1 + C$$

$$6 = C$$

The equation relating x and y is

$$x + 5y - 4 \ln x - 3 \ln y = 6.$$

Your Turn 3

Using the notation of Example 3,

$$\begin{aligned} b &= \frac{N - y_0}{y_0} \\ &= \frac{50,000 - 80}{80} \\ &= 624. \end{aligned}$$

Thus the formula for y is

$$y = \frac{50,000}{1 + 624e^{-kt}}.$$

Use the fact that $y(15) = 640$ to find k .

$$\begin{aligned} 640 &= \frac{50,000}{1 + 624e^{-k(15)}} \\ 640 + 399,360e^{-15k} &= 50,000 \\ e^{-15k} &= \frac{49,360}{399,360} \\ k &= \frac{1}{15} \ln \left(\frac{399,360}{49,360} \right) \\ k &\approx 0.13938 \end{aligned}$$

Therefore, $y = \frac{50,000}{1 + 624e^{-0.13938t}}.$

When $t = 25$,

$$\begin{aligned} y &= \frac{50,000}{1 + 624e^{-0.13938(25)}} \\ &= 2483. \end{aligned}$$

About 2483 people are infected 25 days into the epidemic.

Your Turn 4

Suppose that a tank initially contains 500 liters of a solution of water and 5 kg of salt, and that pure water flows in at a rate of 6 L/min and solution flows out at a rate of 4 L/min. These last two numbers indicate that the volume of solution increases at the rate of 2 L/min, so at time t the volume $V(t) = 500 + 2t$. How many kilograms of salt remain after 20 minutes? We will assume that the tank is large enough so that it does not overflow during the 20 minutes of interest. Following the argument in Example 4 we can write the differential equation for this process as

$$\begin{aligned} \frac{dy}{dt} &= -\frac{4y}{V(t)} \\ \frac{dy}{dt} &= -\frac{4y}{500 + 2t} \end{aligned}$$

with initial condition $y(0) = 5$, where y is the amount of salt in the tank in kilograms.

Separate the variables and integrate on both sides.

$$\begin{aligned} \frac{dy}{dt} &= -\frac{4y}{500 + 2t} \\ \frac{1}{y} dy &= -\frac{4}{500 + 2t} dt \\ \int \frac{1}{y} dy &= -\int \frac{4}{500 + 2t} dt \\ \int \frac{1}{y} dy &= -2 \int \frac{1}{500 + 2t} 2dt \\ \ln y &= -2 \ln(500 + 2t) + C \end{aligned}$$

Use the initial condition $y(0) = 5$.

$$\begin{aligned} \ln 5 &= -2 \ln[500 + 2(0)] + C \\ C &= \ln 5 + 2 \ln 500 \\ C &= \ln(5 \cdot 500^2) \\ \ln y &= -2 \ln(500 + 2t) + \ln(5 \cdot 500^2) \\ y &= \frac{5 \cdot 500^2}{(500 + 2t)^2} \end{aligned}$$

When $t = 20$,

$$\begin{aligned} y &= \frac{5 \cdot 500^2}{(500 + 40)^2} \\ &\approx 4.29. \end{aligned}$$

After twenty minutes, the tank contains 540 L of solution and 4.29 kg of salt.

10.4 Exercises

1. $\frac{dA}{dt} = rA + D$; $r = 0.06$, $D = \$5000$

$$\begin{aligned} \frac{dA}{dt} &= 0.06A + 3000 \\ \int \frac{dA}{0.06A + 3000} &= \int dt \\ \ln|0.06A + 3000| &= 0.06t + C \\ |0.06A + 3000| &= e^{0.06t+C} \\ 0.06A + 3000 &= Me^{0.06t} \\ A &= \frac{M}{0.06} e^{0.06t} - \frac{3000}{0.06} \\ A &= 16.66667Me^{0.06t} - 50,000 \\ t = 0, A &= 5000 \end{aligned}$$

$$\begin{aligned} 5000 &= 16.66667Me^0 - 50,000 \\ M &= 3300 \\ A &= 16.66667(3300)e^{0.06t} - 50,000 \\ &= 55,000e^{0.06t} - 50,000 \end{aligned}$$

When $t = 10$ yr,

$$A = 55,000e^{0.06(10)} - 50,000 \\ = \$50,216.53.$$

$$2. \quad A = \frac{-3000 + 3300e^{0.06t}}{0.06}$$

$$30,000 = \frac{-3000 + 3300e^{0.06t}}{0.06}$$

$$\frac{16}{11} = e^{0.06t}$$

$$\ln\left(\frac{16}{11}\right) = 0.06t$$

$$t = \frac{1}{0.06} \ln\left(\frac{16}{11}\right)$$

$$\approx 6.2 \text{ yr}$$

$$3. \quad \frac{dA}{dt} = rA + D; r = 0.01; D = \$50,000; \\ t = 0, A = 0$$

$$\frac{dA}{dt} = 0.1A + 50,000$$

$$\frac{dA}{0.1A + 50,000} = dt$$

$$\frac{1}{0.1} \ln|0.1A + 50,000| = t + C$$

$$\ln|0.1A + 50,000| = 0.1t + C$$

$$0.1A + 50,000 = e^{0.1t+C}$$

$$= Me^{0.1t}$$

$$A = \frac{M}{0.1}e^{0.1t} - \frac{50,000}{0.1}$$

When $t = 0$, $A = 0$.

$$0 = \frac{M}{0.1}(1) - \frac{50,000}{0.1}$$

$$M = 50,000$$

$$A = \frac{50,000}{0.1}e^{0.1t} - \frac{50,000}{0.1}$$

$$= 500,000(e^{0.1t} - 1)$$

Find t when $A = \$500,000$.

$$500,000 = 500,000(e^{0.1t} - 1)$$

$$e^{0.1t} = 2$$

$$t = \frac{1}{0.1} \ln 2$$

$$\approx 6.9 \text{ yr}$$

It will take about 6.9 years to accumulate \$500,000.

$$4. \quad \frac{dA}{dt} = 0.1A + D$$

$$\frac{1}{0.1A + D} dA = dt$$

$$\frac{\ln(0.1A + D)}{0.1} = t + C$$

$$A = \frac{D}{0.1}(-1 + e^{0.1t})$$

When $t = 3$, $A = 500,000$.

$$500,000 = \frac{D}{0.1}(-1 + e^{0.1(3)})$$

$$= \frac{D}{0.1}(-1 + e^{0.3})$$

$$D = \frac{500,000(0.1)}{-1 + e^{0.3}} \approx \$142,914.80$$

$$5. \quad (a) \quad \frac{dA}{dt} = rA + D; r = 0.06; D = \$1200;$$

$$A(0) = 8000$$

$$\frac{dA}{dt} = 0.06A - 1200$$

(b) To solve for A , first separate the variables.

$$\frac{dA}{0.06A - 1200} = dt$$

Integrate.

$$\int \frac{dA}{0.06A - 1200} = \int dt$$

$$\frac{1}{0.06} \ln|0.06A - 1200| = t + k$$

Solve for A . $\ln|0.06A - 1200| = 0.06t + C$, where $C = 0.06k$.

$$|0.06A - 1200| = e^{0.06t+C} = e^C e^{0.06t}$$

$$0.06A - 1200 = Me^{0.06t}, \text{ where } M = \pm e^C$$

$$0.06A = 1200 + Me^{0.06t}, \text{ so}$$

$$A = 20,000 + C_1 e^{0.06t},$$

$$\text{Where } C_1 = \frac{M}{0.06}.$$

Solve for C_1 . Since $A(0) = 8000$, then

$$8000 = 20,000 + C_1 e^0, \text{ so}$$

$$C_1 = -12,000. \text{ Therefore,}$$

$$A = 20,000 - 12,000e^{0.06t}$$

Since $A(2) = 20,000 - 12,000e^{0.06(2)}$

≈ 6470.04 , the account will have

\$6470.04 left after two years.

(c) If $A = 0$, then

$$20,000 - 12,000e^{0.06t} = 0.$$

$$12,000e^{0.06t} = 20,000$$

$$e^{0.06t} = \frac{5}{3}$$

$$0.06t = \ln\left(\frac{5}{3}\right), \text{ so}$$

$$t = \frac{1}{0.06} \ln\left(\frac{5}{3}\right) \approx 8.51.$$

Therefore, the account will be completely depleted in 8.51 years.

7. (a)

$$\frac{dy}{dt} = 4y - 2xy$$

$$= y(4 - 2x)$$

$$\frac{dx}{dt} = -3x + 2xy$$

$$= x(-3 + 2y)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y(4 - 2x)}{x(-3 + 2y)}$$

$$\int \left(\frac{2y - 3}{y} \right) dy = \int \left(\frac{4 - 2x}{x} \right) dx$$

$$\int \left(2 - \frac{3}{y} \right) dy = \int \left(\frac{4}{x} - 2 \right) dx$$

$$2y - 3 \ln y = 4 \ln x - 2x + C$$

When $x = 1$, $y = 1$,

$$2 - 3 \ln 1 = 4 \ln 1 - 2 + C$$

$$4 = C.$$

Therefore, $2y - 3 \ln y - 4 \ln x + 2x = 4$ is an equation relating x and y in this case.

(b) $0 = 4y - 2xy = -y(-4 + 2x)$

$$0 = -3x + 2xy = x(-3 + 2y)$$

$$0 = -y(-4 + 2x) \text{ and } 0 = x(-3 + 2y)$$

$$y = 0 \text{ and } x = 0 \text{ or}$$

$$-4 + 2x = 0 \quad -3 + 2y = 0$$

$$x = 2 \text{ and } y = \frac{3}{2}$$

8. $\frac{dx}{dt} = -4x + 4xy, \frac{dy}{dt} = -3y + 2xy$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{-3y + 2xy}{-4x + 4xy}$$

$$= \frac{y(-3 + 2x)}{x(-4 + 4y)}$$

$$\frac{-4 + 4y}{y} dy = \frac{-3 + 2x}{x} dx$$

$$\int \left(-\frac{4}{y} + 4 \right) dy = \int \left(-\frac{3}{x} + 2 \right) dx$$

$$-4 \ln y + 4y = -3 \ln x + 2x + C$$

(a) When $x = 5$, $y = 1$. so

$$4 = -3 \ln 5 + 10 + C$$

$$-6 + 3 \ln 5 = C$$

$$3 \ln x - 4 \ln y - 2x + 4y = 3 \ln 5 - 6$$

(b) $-4x + 4xy = 0$

$$x(-4 + 4y) = 0$$

$$-3y + 2xy = 0$$

$$y(-3 + 2x) = 0$$

The solution $x = 0$, $y = 0$ may be feasible. If not, $x = \frac{3}{2}$ and $y = 1$.

(c) When $y > 1$, both populations increase.

When $y < 1$, the x population increases while the y population decreases.

9. (a) Let y = the number of individuals infected. The differential equation is

$$\frac{dy}{dt} = a(N - y)y.$$

The solution is Equation 12 in Example 4, which is

$$y = \frac{N}{1 + (N - 1)e^{-aNt}}, \text{ where } a = \frac{k}{N}.$$

The number of individuals uninfected at time t is

$$\begin{aligned} y &= N - \frac{N}{1 + (N - 1)e^{-aNt}} \\ &= \frac{N + N(N - 1)e^{-aNt} - N}{1 + (N - 1)e^{-aNt}} \\ &= \frac{N(N - 1)}{N - 1 + e^{aNt}}. \end{aligned}$$

Now substitute $N = 5000$ and $a = 0.00005$.

$$y = \frac{5000(5000 - 1)}{5000 - 1 + e^{(0.00005)(5000)t}}$$

$$= \frac{24,995,000}{4999 + e^{0.25t}}$$

(b) $t = 30$

$$y = \frac{24,995,000}{4999 + e^{0.25(30)}} = 3672$$

(c) $t = 50$

$$\frac{24,995,000}{4999 + e^{0.25(50)}} = 91$$

(d) From Example 4,

$$t_m = \frac{\ln(N - 1)}{aN}$$

$$= \frac{\ln(5000 - 1)}{(0.00005)(5000)}$$

$$= 34.$$

The maximum infection rate will occur on the 34th day.

10. Let N = population size, y = number infected.

$$(a) \quad N - y = N - \frac{N}{1 + (N - 1)e^{-aNt}}$$

Example 4

$$= \frac{N + N(N - 1)e^{-aNt} - N}{1 + (N - 1)e^{-aNt}}$$

$$= \frac{N(N - 1)e^{-aNt}}{1 + (N - 1)e^{-aNt}}$$

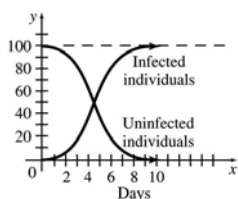
$$= \frac{N(N - 1)}{e^{aNt} + (N - 1)}$$

$$= \frac{N(N - 1)}{N - 1 + e^{aNt}}$$

$$= \frac{N(N - 1)}{N - 1 + e^{kt}} \text{ with } k = aN$$

(b) $N = 100, a = 0.01$

$$y = \frac{100}{1 + 99e^{-t}} \text{ and } 100 - y = \frac{9900}{99 + e^t}$$



$$(c) \quad y = \frac{100}{1 + 99e^{-t}} = 100(1 + 99e^{-t})^{-1}$$

$$\frac{dy}{dt} = -100(1 + 99e^{-t})^{-2}(-99e^{-t})$$

$$= 9900 \cdot \frac{e^{-t}}{(1 + 99e^{-t})}$$

$$\frac{d^2y}{dt^2} = 9900$$

$$= \frac{(1 + 99e^{-t}) \cdot (-e^{-t}) - e^{-t} \cdot 2(1 + 99e^{-t}) \cdot (-99e^{-t})}{(1 + 99e^{-t})^4}$$

$$= \frac{9900(-e^{-t})(1 - 99e^{-t})}{(1 + 99e^{-t})^3}$$

Since $\frac{-9900e^{-t}}{(1 + 99e^{-t})^3} < 0$ for all t , $\frac{d^2y}{dt^2}$ changes sign.

Thus, $1 - 99e^{-t}$ changes sign:

$$1 - 99e^{-t} > 0$$

$$\frac{1}{99} > \frac{1}{e^t}$$

$$e^t > 99$$

$$t > \ln 99.$$

Therefore, y is concave upward if $t > \ln 99$ and concave downward if $t < \ln 99$.

If $t = \ln 99$,

$$y = \frac{100}{1 + 99\left(\frac{1}{99}\right)} = 50.$$

Therefore, $(\ln 99, 50) \approx (4.6 \text{ days}, 50 \text{ people})$ is a point of inflection.

The second derivative of $100 - y$ merely has the opposite sign as $\frac{d^2y}{dt^2}$ above, so $100 - y$ has the same point of inflection as y .

(d) At the point of inflection, the number of infected people is the same as the number of uninfected people.

$$(e) \quad \lim_{t \rightarrow \infty} y = \lim_{t \rightarrow \infty} \frac{100}{1 + 99e^{-t}}$$

$$= \frac{100}{1 + 99(0)}$$

$$= 100 \text{ people}$$

Therefore,

$$\begin{aligned}\lim_{t \rightarrow \infty} (100 - y) &= 100 - \lim_{t \rightarrow \infty} y \\ &= 100 - 100 \\ &= 0 \text{ people.}\end{aligned}$$

In general,

$$\lim_{t \rightarrow \infty} y = N \text{ and } \lim_{t \rightarrow \infty} (N - y) = 0.$$

11. (a) The differential equation is

$$\frac{dy}{dt} = a(N - y)y.$$

$$y_0 = 100; y = 400 \text{ when } t = 10,$$

$$N = 20,000.$$

The solution is Equation 11 in Example 4, which is

$$y = \frac{N}{1 + be^{-kt}},$$

where $b = \frac{N - y_0}{y_0}$ and $k = aN$.

Since $y_0 = 100$ and $N = 20,000$,

$$b = \frac{20,000 - 100}{100} = 199; k = 20,000a.$$

Therefore,

$$y = \frac{20,000}{1 + 199e^{-20,000at}}.$$

$$y = 400 \text{ when } t = 10.$$

$$400 = \frac{20,000}{1 + 199e^{-20,000(10)a}}$$

$$400 + 400(199)e^{-200,000a} = 20,000$$

$$e^{-200,000a} = \frac{19,600}{400(199)}$$

$$= 0.2462312$$

$$a = \frac{\ln(0.2462312)}{-200,000}$$

$$= 7 \times 10^{-6}$$

$$k = 20,000a = 20,000(7 \times 10^{-6}) = 0.14$$

Therefore,

$$y = \frac{20,000}{1 + 199e^{-0.14t}} \text{ or } \frac{20,000^{0.14t}}{e^{0.14t} + 199}.$$

- (b) Half the community is $y = 10,000$. Find t for $y = 10,000$.

$$10,000 = \frac{20,000}{1 + 199e^{-0.14t}}$$

$$10,000 + 10,000(199)e^{-0.14t} = 20,000$$

$$e^{-0.14t} = \frac{10,000}{10,000(199)} = 0.005$$

$$\begin{aligned}t &= \frac{\ln(0.005)}{-0.14} \\ &= 37.77\end{aligned}$$

Half the community will be infected in about 38 days.

12. (a) $\frac{dy}{dt} = kye^{-at}$

$$a = 0.02, \text{ so}$$

$$\frac{dy}{dt} = kye^{-0.02t}.$$

$$y = 50 \text{ when } t = 0;$$

$$y = 300 \text{ when } t = 10.$$

$$\frac{1}{y} dy = ke^{-0.02t} dt$$

$$\ln y = \frac{k}{-0.02} e^{-0.02t} + C$$

To find k and C , solve the system

$$\ln 50 = \frac{k}{-0.02} + C \quad (1)$$

$$\ln 300 = \frac{k}{-0.02} e^{-0.2} + C. \quad (2)$$

Subtract equation (1) from equation (2).

$$\ln 300 - \ln 50 = \frac{k}{-0.02} (e^{-0.2} - 1)$$

$$\ln\left(\frac{300}{50}\right) = \ln 6 = \frac{k}{0.02} (1 - e^{-0.2})$$

$$k = \frac{0.02 \ln 6}{1 - e^{-0.2}} \approx 0.1977$$

Therefore,

$$\begin{aligned}C &= \ln 50 + \frac{k}{0.02} \\ &\approx \ln 50 + \frac{0.1977}{0.02} \\ &\approx 13.7970.\end{aligned}$$

So

$$\begin{aligned}\ln y &= \frac{0.1977}{-0.02} e^{-0.02t} + 13.7970 \\ &\approx -9.885e^{-0.02t} + 13.7970.\end{aligned}$$

Thus,

$$\begin{aligned}y &\approx e^{-9.885e^{-0.02t} + 13.7970} \\ &= e^{13.7970} e^{-9.885e^{-0.02t}} \\ &\approx 982,000e^{-9.89e^{-0.02t}}\end{aligned}$$

(b) Find t when

$$\begin{aligned}y &= \frac{1}{2}(10,000) = 5000. \\ \ln 5000 &= -9.89e^{-0.02t} + 13.7970 \\ e^{-0.02t} &= \frac{\ln 5000 - 13.7970}{-9.89} \\ t &= \frac{1}{-0.02} \ln \left[\frac{\ln 5000 - 13.7970}{-9.89} \right] \\ &\approx 31 \text{ days}\end{aligned}$$

13. (a) $\frac{dy}{dt} = -ay + b(f - y)Y$

$$\begin{aligned}a &= 1, b = 1, f = 0.5, Y = 0.01; \\ y &= 0.02 \text{ when } t = 0.\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= -y + 1(0.5 - y)(0.01) \\ &= -1.010y + 0.005\end{aligned}$$

$$\int \frac{dy}{-1.010y + 0.005} = \int dt$$

$$\begin{aligned}\frac{1}{-1.010} \ln |-1.010y + 0.005| &= t + C_2 \\ \ln |-1.010y + 0.005| &= -1.010t + C_1 \\ |-1.010y + 0.005| &= e^{-1.010t + C_1} \\ &= e^{C_1} e^{-1.010t}\end{aligned}$$

$$\begin{aligned}-1.010y + 0.005 &= Ce^{-1.010t} \\ y &= 0.005 - 0.990Ce^{-1.010t}\end{aligned}$$

Since $y = 0.02$ when $t = 0$,

$$\begin{aligned}0.02 &= 0.005 - 0.990Ce^0 \\ -0.990C &= 0.015.\end{aligned}$$

Therefore,

$$y = 0.005 + 0.015e^{-1.010t}.$$

(b) $\frac{dY}{dt} = -AY + B(F - Y)y$

$$\begin{aligned}A &= 1, B = 1, y = 0.1, F = 0.03; \\ Y &= 0.01 \text{ when } t = 0.\end{aligned}$$

$$\begin{aligned}\frac{dY}{dt} &= -Y + 1(0.03 - Y)(0.1) \\ &= -1.1Y + 0.003\end{aligned}$$

$$\begin{aligned}\frac{dY}{-1.1Y + 0.003} &= dt \\ -\frac{1}{1.1} \ln |-1.1Y + 0.003| &= t + C_2\end{aligned}$$

$$\begin{aligned}\ln |-1.1Y + 0.003| &= -1.1t + C_1 \\ |-1.1Y + 0.003| &= e^{-1.1t + C_1} \\ &= e^{C_1} e^{-1.1t} \\ -1.1Y + 0.003 &= Ce^{-1.1t}\end{aligned}$$

$$\begin{aligned}Y &= \frac{C}{-1.1} e^{-1.1t} - \frac{0.003}{-1.1} \\ &= 0.909Ce^{-1.1t} + 0.00273\end{aligned}$$

Since $Y = 0.01$ when $t = 0$,

$$\begin{aligned}0.01 &= -0.909Ce^0 + 0.00273 \\ -0.909C &= 0.00727.\end{aligned}$$

Therefore,

$$Y = 0.00727e^{-1.1t} + 0.00273.$$

14. From equations (4) and (5) in Section 1 of this chapter, the solution to

$$\frac{dy}{dt} = a(N - y)y$$

is

$$y = \frac{N}{1 + be^{-aNt}},$$

where y = number of people who have heard the information and

$$b = \frac{N - y_0}{y_0}.$$

If $y_0 = 3$, $N = 500$, and $y = 50$ when $t = 2$, find y .

$$b = \frac{500 - 3}{3} = \frac{497}{3}, \text{ so}$$

$$y = \frac{500}{1 + \frac{497}{3} e^{-500at}}$$

$$50 = \frac{500}{1 + \frac{497}{3} e^{-1000a}}$$

$$1 + \frac{497}{3}e^{-1000a} = 10$$

$$e^{-1000a} = \frac{27}{497}$$

$$a = -\frac{1}{1000} \ln\left(\frac{27}{497}\right) = \frac{\ln \frac{497}{27}}{1000}$$

$$500a \approx 1.456$$

The number of people who have heard the rumor in t days is given by

$$y = \frac{500}{1 + \frac{497}{3}e^{-1.456t}}$$

$$= \frac{1500}{3 + 497e^{-1.456t}}.$$

When $t = 5$,

$$y = \frac{1500}{3 + 497e^{-1.456(5)}}$$

$$\approx 449,$$

so in 5 days, about 449 people have heard the rumor.

15. (a) $\frac{dy}{dt} = a(N - y)y$

$$y_0 = 3; y = 12 \text{ when } t = 3; N = 45$$

From equations (4) and (5) in Section 1 of this chapter, the solution is

$$y = \frac{N}{1 + be^{-kt}} \text{ where } b = \frac{N - y_0}{y_0} \text{ and } k = aN.$$

Since $y_0 = 3$ and $N = 45$,

$$b = \frac{45 - 3}{3} = 14; k = 45a.$$

$$y = \frac{45}{1 + 14e^{-45at}}.$$

$y = 12$ when $t = 3$, so

$$12 = \frac{45}{1 + 14e^{-135a}}$$

$$12 + 168e^{-135a} = 45$$

$$e^{-135a} = \frac{33}{168} = \frac{11}{56}$$

$$-135a = \ln \frac{11}{56} = -1.627$$

$$-45a = -0.542.$$

Therefore, $y \approx \frac{45}{1 + 14e^{-0.54t}}.$

(b) When $y = 30$,

$$30 = \frac{45}{1 + 14e^{-0.54t}}$$

$$30 + 450e^{-0.54t} = 45$$

$$e^{-0.54t} = \frac{15}{450} = \frac{1}{30}$$

$$t = -\frac{1}{0.54} \ln \frac{1}{30} = 6.30.$$

In about 6 days, 30 employees have heard the rumor.

16. $y_0 = 5$, $N = 100$, and when $t = 1$, $y = 20$.

$$b = \frac{100 - 5}{5} = 19, \text{ so}$$

$$y = \frac{100}{1 + 19e^{-100at}}$$

$$20 = \frac{100}{1 + 19e^{-100a}}$$

$$1 + 19e^{-100a} = 5$$

$$e^{-100a} = \frac{4}{19}$$

$$a = -\frac{1}{100} \ln \frac{4}{19} = \frac{\ln\left(\frac{19}{4}\right)}{100}$$

$$100a \approx 1.558$$

The number of people who have heard the news in t days is given by

$$y = \frac{100}{1 + 19e^{-1.558t}}.$$

When $t = 3$,

$$y = \frac{100}{1 + 19e^{-1.558(3)}}$$

$$\approx 85,$$

so after 3 days, about 85 people have heard the news.

17. (a) $\frac{dy}{dt} = kye^{-at}; a = 0.1; y = 5$ when $t = 0$;
 $y = 15$ when $t = 3$.

$$\int \frac{dy}{y} = k \int e^{-0.1t} dt$$

$$\ln |y| = -10ke^{-0.1t} + C_1$$

$$|y| = e^{-10ke^{-0.1t} + C_1}$$

$$= e^{C_1} e^{-10ke^{-0.1t}}$$

$$y = Ce^{-10ke^{-0.1t}}$$

Since $y = 5$ when $t = 0$,

$$5 = Ce^{-10k}$$

$$C = 5e^{10k}.$$

Since $y = 15$ when $t = 3$,

$$15 = Ce^{-10ke^{-0.3}} = Ce^{-7.41k}$$

$$C = 15e^{7.41k}.$$

Solve the system

$$C = 5e^{10k}$$

$$C = 15e^{7.41k}.$$

$$5e^{10k} = 15e^{7.41k}$$

$$e^{10k} = 3e^{7.41k}$$

Take natural logarithms on both sides.

$$10k \ln e = \ln 3 + 7.41k$$

$$2.59k = \ln 3$$

$$k = \frac{1}{2.59} \ln 3 = 0.424$$

$$C = 5e^{10(0.424)} = 347$$

Therefore, $y = 347e^{-10(0.424)e^{-0.1t}}$

$$= 347e^{-4.24e^{-0.1t}}.$$

(b) If $y = 30$,

$$30 = 347e^{-4.24e^{-0.1t}}$$

$$e^{-4.24e^{-0.1t}} = \frac{30}{347} = 0.0865$$

$$-4.24e^{-0.1t} \ln e = \ln 0.0865$$

$$e^{-0.1t} = -\frac{1}{4.24} \ln 0.0865$$

$$= 0.5773$$

$$-0.1t \ln e = \ln 0.5773$$

$$t = -10 \ln 0.5773$$

$$= 5.493.$$

30 employees have heard the rumor in about 5.5 days.

18. Let y = amount (in pounds) of salt at time t .

At $t = 0$, $y = 20$ lb and $V = 100$ gallons.

(a) Rate of Salt In

$$= (2 \text{ gal/min}) \cdot (2 \text{ lb/gal})$$

$$= 4 \text{ lb/min.}$$

Rate of Salt Out

$$= (2 \text{ gal/min}) \cdot \left(\frac{y}{V} \text{ lb/gal} \right)$$

$$= \frac{2y}{V} (\text{lb/min})$$

rate of liquid in

$$= \text{rate of liquid out}$$

$$= 2 \text{ gal/min}$$

So $\frac{dV}{dt} = 0$ and $V = 100$ at all times t .

Therefore,

$$\frac{dy}{dt} = 4 - \frac{2y}{100} = \frac{200 - y}{50} \text{ lb/min}$$

$$\frac{dy}{200 - y} = \frac{dt}{50}.$$

$$-\ln(200 - y) = \frac{t}{50} + C$$

$$-\ln 180 = C$$

$$-\ln(200 - y) = \frac{t}{50} - \ln 180$$

$$\ln(200 - y) = \ln 180 - 0.02t$$

$$200 - y = e^{\ln 180 - 0.02t}$$

$$= 180e^{-0.02t}$$

$$y = 200 - 180e^{-0.02t}$$

(b) One hour later, $t = 60$ min,

$$y = 200 - 180e^{0.02(60)} \approx 146 \text{ lb.}$$

(c) $\lim_{x \rightarrow \infty} (200 - 180e^{-0.02t})$

$$= 200 - 180(0) = 200 \text{ lb}$$

As $t \rightarrow \infty$, y increases, approaching 200 lb.

19. Let y = the amount of salt present at time t .

(a) $\frac{dy}{dt} = (\text{rate of salt in})$

$$- (\text{rate of salt out})$$

$$\text{rate of salt in} = (3 \text{ gal/min})(2 \text{ lb/gal})$$

$$= 6 \text{ lb/min}$$

$$\text{rate of salt out} = \left(\frac{y}{V} \text{ lb/gal} \right) (2 \text{ gal/min})$$

$$= \left(\frac{2y}{V} \text{ lb/min} \right)$$

$$\frac{dy}{dt} = 6 - \frac{2y}{V}; y(0) = 20 \text{ lb}$$

$$\begin{aligned}\frac{dV}{dt} &= (\text{rate of liquid in}) \\ &\quad - (\text{rate of liquid out}) \\ &= 3 \text{ gal/min} - 2 \text{ gal/min} \\ &= 1 \text{ gal/min}\end{aligned}$$

$$\begin{aligned}\frac{dV}{dt} &= 1 \\ V &= t + C_1\end{aligned}$$

When $t = 0$, $V = 100$. Thus,

$$\begin{aligned}C_1 &= 100. \\ V &= t + 100.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dy}{dt} &= 6 - \frac{2y}{t + 100} \\ \frac{dy}{dt} + \frac{2}{t + 100}y &= 6. \\ I(t) &= e^{\int 2 dt/(t+100)} \\ &= e^{2 \ln |t + 100|} \\ &= (t + 100)^2 \\ \frac{dy}{dt}(t + 100)^2 + 2y(t + 100) &= 6(t + 100)^2 \\ D_t[y(t + 100)^2] &= 6(t + 100)^2 \\ y(t + 100)^2 &= 6 \int (t + 100)^2 dt \\ y(t + 100)^2 &= 2(t + 100)^3 + C \\ y &= 2(t + 100) + \frac{C}{(t + 100)^2}\end{aligned}$$

Since $t = 0$ when $y = 20$,

$$\begin{aligned}20 &= 2(100) + \frac{C}{100^2} \\ C &= -1,800,000. \\ y &= 2(t + 100) - \frac{1,800,000}{(t + 100)^2} \\ &= \frac{2(t + 100)^3 - 1,800,000}{(t + 100)^2}.\end{aligned}$$

(b) $t = 1 \text{ hr} = 60 \text{ min}$

$$y = \frac{2(160)^3 - 1,800,000}{(160)^2} = 249.69$$

After 1 hr, about 250 lb of salt are present.

(c) As time increases, salt concentration continues to increase.

20. (a) Change rate of salt in to (1 gal/min)
1 lb/gal = 2 lb/min and rate of liquid in
to 1 gal/min, so $\frac{dV}{dt} = 1 - 2 = -1$ gal/min.

$V = -t + C$ and $V = 100$ when $t = 0$,
so $V = 100 - t$.

Therefore,

$$\begin{aligned}\frac{dy}{dt} &= 2 - \frac{2y}{100 - t} \\ \frac{dy}{dt} + \frac{2y}{100 - t} &= 2. \\ I(x) &= e^{\int 2/(100-t) dt} \\ &= e^{-2 \ln (100-t)} \\ &= \frac{1}{(100 - t)^2} \\ \frac{1}{(100 - t)^2}y &= \int \frac{2}{(100 - t)^2} dt \\ &= \int 2(100 - t)^{-2} + dt \\ &= 2(100 - t)^{-1} + C\end{aligned}$$

$$y = 2(100 - t) + C(100 - t)^2$$

$$20 = 2(100) + C(100)^2$$

$$C = -0.018$$

$$y = 2(100 - t) - 0.018(100 - t)^2$$

(b) When $t = 60$ min,

$$\begin{aligned}y &= 2(100 - 60) - 0.018(100 - 60)^2 \\ &\approx 51 \text{ lb}.\end{aligned}$$

(c) The graph of

$$y = 2(100 - t) - 0.018(100 - t)^2$$

is a parabola opening downward with vertex
at $t = \frac{400}{9}$ and t -intercepts of $-\frac{100}{9}$ and
100 since $y = 20 + 1.6t - 0.018t^2$.

Thus, y increases from $t = 0$ to $t = \frac{400}{9}$
and decreases from $t = \frac{400}{9}$ to $t = 100$.

21. Let y = the amount of salt present at time t minutes.

$$(a) \quad \frac{dy}{dt} = (\text{rate of salt in}) - (\text{rate of salt out})$$

$$\text{rate of salt in} = 0$$

$$\begin{aligned} \text{rate of salt out} &= \left(\frac{y}{V} \text{ lb/gal}\right)(2 \text{ gal/min}) \\ &= \frac{2y}{V} \text{ lb/min} \end{aligned}$$

$$\frac{dV}{dt} = -\frac{2y}{V}; y(0) = 20$$

$$\begin{aligned} \frac{dV}{dt} &= (\text{rate of liquid in}) \\ &\quad - (\text{rate of liquid out}) \\ &= 2 \text{ gal/min} - 2 \text{ gal/min} = 0 \end{aligned}$$

$$\begin{aligned} \frac{dV}{dt} &= 0 \\ V &= C_1 \end{aligned}$$

$$\text{When } t = 0, V = 100, \text{ so } C_1 = 100.$$

Therefore,

$$\frac{dy}{dt} = -\frac{2y}{100} = -0.02y$$

$$\frac{dy}{y} = -0.02 dt$$

$$\ln |y| = -0.02t + C_1$$

$$\begin{aligned} |y| &= e^{-0.02t+C_1} = e^{C_1}e^{-0.02t} \\ &= Ce^{-0.02t} \end{aligned}$$

$$\text{Since } t = 0 \text{ when } y = 20,$$

$$20 = Ce^0$$

$$C = 20.$$

$$y = 20e^{-0.02t}$$

- (b) $t = 1 \text{ hr} = 60 \text{ min}$

$$y = 20e^{-0.02(60)} = 6.024$$

After 1 hr, about 6 lb of salt are present.

- (c) As time increases, salt concentration continues to decrease.

22. Let y = amount (in grams) of chemical at time t .
At $t = 0$, $y = 5 \text{ g}$ and $V = 100$ liters.

$$(a) \quad \begin{aligned} \text{rate of chemical in} &= 0 \\ \text{rate of chemical out} \end{aligned}$$

$$= (1 \text{ liter/min})\left(\frac{y}{V} \text{ g/liter}\right)$$

$$= \frac{y}{V} \text{ g/min}$$

$$\begin{aligned} \text{rate of liquid in} \\ &= 2 \text{ liters/min;} \end{aligned}$$

$$\begin{aligned} \text{rate of liquid out} \\ &= 1 \text{ liter/min; so} \end{aligned}$$

$$\frac{dV}{dt} = 1 \text{ liter/min.}$$

Therefore,

$$V = t + C \text{ and } 100 = C, \text{ so } V = t + 100.$$

$$\frac{dy}{dt} = \frac{-y}{t + 100}$$

$$\frac{dy}{y} = \frac{-dt}{t + 100}$$

$$\ln y = -\ln(t + 100) + C$$

$$\ln 5 = -\ln 100 + C$$

$$\ln 5 + \ln 100 = C$$

$$C = \ln(500)$$

$$\ln y = -\ln(t + 100) + \ln(500)$$

$$= \ln\left(\frac{500}{t + 100}\right)$$

$$y = \frac{500}{t + 100}$$

- (b) When $t = 30$,

$$y = \frac{500}{130} \approx 3.8 \text{ g.}$$

23. Let y = amount of the chemical at time t .

$$(a) \quad \begin{aligned} \frac{dy}{dt} &= (\text{rate of chemical in}) \\ &\quad - (\text{rate of chemical out}) \end{aligned}$$

$$\begin{aligned} \text{rate of chemical in} \\ &= (2 \text{ liters/min})(0.25 \text{ g/liter}) \\ &= 0.5 \text{ g/min} \end{aligned}$$

rate of chemical out

$$= \left(\frac{y}{V} \text{ g/liter} \right) (1 \text{ liter/min})$$

$$= \frac{y}{V} \text{ g/liter}$$

$$\frac{dy}{dt} = 0.5 - \frac{y}{V}; y(0) = 5$$

$$\frac{dV}{dt} = (\text{rate of liquid in})$$

$$- (\text{rate of liquid out})$$

$$= 2 \text{ liter/min} - 1 \text{ liter/min}$$

$$= 1 \text{ liter/min}$$

$$\frac{dV}{dt} = 1$$

$$V = t + C_1$$

When $t = 0, V = 100$, so $C_1 = 100$.

$$V = t + 100$$

Therefore,

$$\frac{dy}{dt} = 0.5 - \frac{y}{t + 100}$$

$$\frac{dy}{dt} + \frac{1}{t + 100} \cdot y = 0.5$$

$$I(t) = e^{\int dt/(t+100)} = e^{\ln|t+100|}$$

$$= t + 100$$

$$\frac{dy}{dt}(t + 100) + y = 0.5(t + 100)$$

$$D_x(t + 100)y = 0.5(t + 100)$$

$$(t + 100)y = \int 0.5(t + 100) dt$$

$$(t + 100)y = 0.25(t + 100)^2 + C$$

$$y = 0.25(t + 100) + \frac{C}{t + 100}$$

$$t = 0, y = 5$$

$$5 = 0.25(100) + \frac{C}{100}$$

$$500 = 2500 + C$$

$$C = -2000$$

Therefore,

$$y = 0.25(t + 100) + \frac{-2000}{t + 100}$$

$$= \frac{0.25(t + 100)^2 - 2000}{t + 100}.$$

(b) When $t = 30$ min,

$$y = \frac{0.25(130)^2 - 2000}{130}$$

$$= 17.115,$$

After 30 min, about 17.1 g of chemical are present.

24. Let y = amount (in pounds) of soap at time t .
At $t = 0, y = 4$ pounds and $V = 200$ gallons.

rate of soap in = 0;

rate of soap out

$$= (8 \text{ gal/min}) \left(\frac{y}{V} \text{ lb/gal} \right)$$

$$= \frac{8y}{V} \text{ lb/min}$$

rate of liquid in

= rate of liquid out

= 8 gal/min,

so $\frac{dV}{dt} = 0$, and $V = 200$ for all t .

$$\frac{dy}{dt} = -\frac{8y}{200}$$

$$\frac{1}{y} dy = -\frac{1}{25} dt$$

$$\ln y = -\frac{t}{25} + C$$

$$\ln 4 = C$$

$$\ln y = -\frac{t}{25} + \ln 4$$

$$y = 4e^{-0.04t}$$

If $y = 1$,

$$1 = 4e^{-0.04t}$$

$$e^{-0.04t} = \frac{1}{4}$$

$$t = -\frac{1}{0.04} \ln \left(\frac{1}{4} \right)$$

$$= 25 \ln 4 \approx 34.7 \text{ min.}$$

Chapter 10 Review Exercises

1. True
2. False: No; $y = e^{2x}$ satisfies the given differential equation.
3. True
4. False: No; the term $y(dy/dx)$ makes the equation nonlinear.
5. False: Many (most) differential equations are neither separable nor linear.
6. True
7. False: There is no way to separate the variables in this equation.
8. True
9. False: The integrating factor is x^5 .
10. False: Euler's method finds a numerical approximation to a particular solution over some interval.
11. True
12. True
17. $y \frac{dy}{dx} = 2x + y$

Since you cannot separate the variables, that is, rewrite the equation in the form $g(y)dy = f(x)dx$ where g is a function of y alone and f is a function of x alone, then the equation is not separable. Since you cannot rewrite the equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$, then the equation is not a linear first-order differential equation. Therefore, the equation is neither linear nor separable.

18. $\frac{dy}{dx} + y^2 = xy^2$

Since the equation can be rewritten in the form $\frac{dy}{y^2} = (x - 1)dx$, then the equation is separable, but since it cannot be rewritten in the form $\frac{dy}{dx} + P(x)y = Q(x)$, then the equation is not linear.

19. $\sqrt{x} \frac{dy}{dx} = \frac{1 + \ln x}{y}$

Since you can rewrite the equation in the form $y dy = \frac{1 + \ln x}{\sqrt{x}} dx$, then the equation is separable, but since it cannot be rewritten in the form $\frac{dy}{dx} + P(x)y = Q(x)$, then the equation is not linear.

20. $\frac{dy}{dx} = xy + e^x$

Since the equation can be rewritten in the form $\frac{dy}{dx} + (-x)y = e^x$, then the equation is linear. Since the equation cannot be rewritten in the form $g(y)dy = f(x)dx$, then it is not separable.

21. $\frac{dy}{dx} + x = xy$

Since you can rewrite the equation in the form $\frac{dy}{dx} + (-x)y = -x$, then the equation is linear. Since the equation can be rewritten in the form $\frac{dy}{y-1} = x dx$, then it is also separable. Therefore, it is both linear and separable.

22. $\frac{x dy}{y dx} = 4 + x^{3/2}$

Since the equation can be rewritten in the form $\frac{dy}{dx} + \left(-\frac{4+x^{3/2}}{x}\right)y = 0$, then the equation is linear. Since it can be rewritten in the form $\frac{dy}{y} = \frac{4+x^{3/2}}{x} dx$, then it is also separable. Therefore, it is both linear and separable.

23. $x \frac{dy}{dx} + y = e^x(1 + y)$

Since the equation can be rewritten in the form

$$\frac{dy}{dx} + y \left(\frac{1 - e^x}{x} \right) = \frac{e^x}{x},$$

then the equation is linear. Since the equation cannot be rewritten in the form $g(y)dy = f(x)dx$, then the equation is not separable.

24. $\frac{dy}{dx} = x^2 + y^2$

Since the equation cannot be rewritten in either form $\frac{dy}{dx} + P(x)y = Q(x)$ or the form $g(y)dy = f(x)dx$, then the equation is neither nor separable.

25. $\frac{dy}{dx} = 3x^2 + 6x$
 $dy = (3x^2 + 6x)dx$
 $y = x^3 + 3x^2 + C$

26. $\frac{dy}{dx} = 4x^3 + 6x^5$
 $y = x^4 + x^6 + C$

27. $\frac{dy}{dx} = 4e^{2x}$
 $dy = 4e^{2x}dx$
 $y = 2e^{2x} + C$

28. $\frac{dy}{dx} = \frac{1}{3x + 2}$
 $y = \frac{1}{3} \ln |3x + 2| + C$

29. $\frac{dy}{dx} = \frac{3x + 1}{y}$
 $y dy = (3x + 1) dx$
 $\frac{y^2}{2} = \frac{3x^2}{2} + x + C_1$
 $y^2 = 3x^2 + 2x + C$

30. $\frac{dy}{dx} = \frac{e^x + x}{y - 1}$
 $(y - 1) dy = (e^x + x) dx$
 $\frac{y^2}{2} - y = e^x + \frac{x^2}{2} + C$

31. $\frac{dy}{dx} = \frac{2y + 1}{x}$
 $\frac{dy}{2y + 1} = \frac{dx}{x}$
 $\frac{1}{2} \left(\frac{2 dy}{2y + 1} \right) = \frac{dx}{x}$
 $\frac{1}{2} \ln |2y + 1| = \ln |x| + C_1$
 $\ln |2y + 1|^{1/2} = \ln |x| + \ln k$
 Let $\ln k = C_1$
 $\ln |2y + 1|^{1/2} = \ln k |x|$

$$|2y + 1|^{1/2} = k|x|$$

$$2y + 1 = k^2 x^2$$

$$2y + 1 = Cx^2$$

$$2y = Cx^2 - 1$$

$$y = \frac{Cx^2 - 1}{2}$$

32. $\frac{dy}{dx} = \frac{3 - y}{e^x}$
 $\frac{1}{3 - y} dy = e^{-x} dx$
 $-\ln |3 - y| = -e^{-x} + C$
 $\ln |3 - y| = e^{-x} - C$
 $3 - y = ke^{e^{-x}}$
 $y = 3 - ke^{e^{-x}}$
 or $y = 3 + Me^{e^{-x}}$

33. $\frac{dy}{dx} + y = x$
 $I(x) = e^{\int dx} = e^x$
 $e^x \frac{dy}{dx} + e^x y = xe^x$
 $D_x(e^x y) = xe^x$

Integrate both sides, integrating the right side by parts.

$$e^x y = xe^x - e^x + C$$

$$y = x - 1 + Ce^{-x}$$

$$34. \quad x^4 \frac{dy}{dx} + 3x^3 y = 1$$

$$\frac{dy}{dx} + \frac{3}{x} y = \frac{1}{x^4}$$

$$I(x) = e^{\int \frac{3}{x} dx} = x^3$$

$$x^3 \frac{dy}{dx} + 3x^2 y = \frac{1}{x}$$

$$D_x(x^3 y) = \frac{1}{x}$$

Integrate both sides.

$$x^3 y = \ln x + C$$

$$y = \frac{\ln x + C}{x^3}$$

$$35. \quad x \ln x \frac{dy}{dx} + y = 2x^2$$

$$\frac{dy}{dx} + \frac{1}{x \ln x} y = \frac{2x}{\ln x}$$

$$\begin{aligned} I(x) &= e^{\int \frac{1}{x \ln x} dx} \\ &= e^{\int \frac{1}{\ln x} \left(\frac{1}{x} dx \right)} \\ &= e^{\ln(\ln x)} \\ &= \ln x \end{aligned}$$

Multiply each term by the integrating factor, express the left side as the derivative of a product and integrate on both sides.

$$\ln x \frac{dy}{dx} + \frac{1}{x} y = 2x$$

$$D_x[(\ln x)y] = 2x$$

$$(\ln x)y = x^2 + C$$

$$y = \frac{x^2 + C}{\ln x}$$

$$36. \quad x \frac{dy}{dx} + 2y - e^{2x} = 0$$

$$\frac{dy}{dx} + \frac{2}{x} y = \frac{1}{x} e^{2x}$$

$$I(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

$$x^2 \frac{dy}{dx} + 2xy = xe^{2x}$$

$$D_x(x^2 y) = xe^{2x}$$

$$x^2 y = \int xe^{2x} dx$$

$$= \int xe^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C$$

$$y = \frac{e^{2x}}{2x} - \frac{e^{2x}}{4x^2} + \frac{C}{x^2}$$

$$37. \quad \frac{dy}{dx} = x^2 - 6x; y = 3 \text{ when } x = 0.$$

$$dy = (x^2 - 6x) dx$$

$$y = \frac{x^3}{3} - 3x^2 + C$$

When $x = 0$, $y = 3$.

$$3 = 0 - 0 + C$$

$$C = 3$$

$$y = \frac{x^3}{3} - 3x^2 + 3$$

$$38. \quad \frac{dy}{dx} = 5(e^{-x} - 1); y = 17 \text{ when } x = 0.$$

$$dy = 5(e^{-x} - 1) dx$$

$$y = 5(-e^{-x} - x) + C$$

$$= -5e^{-x} - 5x + C$$

$$17 = -5e^0 - 0 + C$$

$$17 = -5 + C$$

$$22 = C$$

$$y = -5e^{-x} - 5x + 22$$

$$39. \quad \frac{dy}{dx} = (x + 2)^3 e^y; y = 0 \text{ when } x = 0.$$

$$e^{-y} dy = (x + 2)^3 dx$$

$$-e^{-y} = \frac{1}{4}(x + 2)^4 + C$$

$$-1 = \frac{1}{4}(2)^4 + C$$

$$C = -5$$

$$e^{-y} = 5 - \frac{1}{4}(x + 2)^4$$

$$y = -\ln \left[5 - \frac{1}{4}(x + 2)^4 \right]$$

Notice that $x < \sqrt[4]{20} - 2$.

40. $(3 - 2x)y = \frac{dy}{dx}; y = 5 \text{ when } x = 0.$

$$(3 - 2x)dx = \frac{dy}{y}$$

$$3x - x^2 + C = \ln|y|$$

$$e^{3x-x^2+C} = y$$

$$e^{3x-x^2} \cdot e^C = y$$

$$Me^{3x-x^2} = y$$

$$Me^0 = 5$$

$$M = 5$$

$$y = 5e^{3x-x^2}$$

41. $\frac{dy}{dx} = \frac{1-2x}{y+3}; y = 16 \text{ when } x = 0.$

$$(y+3)dy = (1-2x)dx$$

$$\frac{y^2}{2} + 3y = x - x^2 + C$$

$$\frac{16^2}{2} + 3(16) = 0 + C$$

$$176 = C$$

$$\frac{y^2}{2} + 3y = x - x^2 + 176$$

$$y^2 + 6y = 2x - 2x^2 + 352$$

42. $\sqrt{x} \frac{dy}{dx} = xy; y = 4 \text{ when } x = 1.$

$$\frac{1}{y} dy = \frac{x}{\sqrt{x}} dx$$

$$\int \frac{1}{y} dy = \int x^{1/2} dx$$

$$\ln|y| = \frac{x^{3/2}}{\frac{3}{2}} + C_1$$

$$\ln|y| = \frac{2}{3}x^{3/2} + C_1$$

$$|y| = e^{(2/3)x^{3/2}} + C_1$$

$$|y| = e^{C_1} e^{(2/3)x^{3/2}}$$

$$y = \pm e^{C_1} e^{(2/3)x^{3/2}}$$

$$y = Ce^{(2/3)x^{3/2}}$$

Since $y = 4$ when $x = 1$,

$$4 = Ce^{(2/3)(1)^{3/2}}$$

$$C = \frac{4}{e^{2/3}}$$

$$C \approx 2.054$$

$$y = 2.054e^{(2/3)x^{3/2}}$$

43. $e^x \frac{dy}{dx} - e^x y = x^2 - 1; y = 42 \text{ when } x = 0.$

$$\frac{dy}{dx} - y = (x^2 - 1)e^{-x}$$

$$I(x) = e^{\int -1 dx} = e^{-x}$$

$$e^{-x}y = \int (x^2 - 1)e^{-x} \cdot e^{-x} dx$$

$$= \int (x^2 - 1)e^{-2x} dx$$

Integration by parts:

Let $u = x^2 - 1$ $dv = e^{-2x} dx$

$$du = 2x dx \quad v = -\frac{1}{2}e^{-2x}$$

$$e^{-x}y = \frac{(x^2 - 1)}{2}e^{-2x} + \int xe^{-2x} dx$$

Let $u = x$ $dv = e^{-2x} dx$

$$du = dx \quad v = -\frac{1}{2}e^{-2x}$$

$$e^{-x}y = -\frac{(x^2 - 1)}{2}e^{-2x} - \frac{x}{2}e^{-2x} + \int \frac{1}{2}e^{-2x} dx$$

$$= -\frac{(x^2 - 1)}{2}e^{-2x} - \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$

$$y = -\frac{(x^2 - 1)}{2}e^{-x} - \frac{x}{2}e^{-x} - \frac{1}{4}e^{-x} + Ce^x$$

$$42 = \frac{1}{2} - 0 - \frac{1}{4} + C$$

$$C = 41.75$$

$$y = -\frac{(x^2 - 1)}{2}e^{-x} - \frac{x}{2}e^{-x} - \frac{1}{4}e^{-x} + 41.75e^x$$

$$= e^{-x} \left[-\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} \right] + 41.75e^x$$

$$= \frac{-x^2 e^{-x}}{2} - \frac{x e^{-x}}{2} + \frac{e^{-x}}{4} + 41.75e^x$$

44. $\frac{dy}{dx} + 3x^2y = x^2$; $y = 2$ when $x = 0$.

$$\begin{aligned} I(x) &= e^{\int 3x^2 dx} = e^{x^3} \\ e^{x^3} \frac{dy}{dx} + x^2 e^{x^3} y &= x^2 e^{x^3} \\ D_x(e^{x^3} y) &= x^2 e^{x^3} \\ e^{x^3} y &= \int x^2 e^{x^3} dx \\ &= \frac{1}{3} e^{x^3} + C \\ y &= \frac{1}{3} + C e^{-x^3} \end{aligned}$$

Since $x = 0$ when $y = 2$,

$$\begin{aligned} 2 &= \frac{1}{3} + C e^0 \\ C &= \frac{5}{3}. \end{aligned}$$

Therefore,

$$y = \frac{1}{3} + \frac{5}{3} e^{-x^3}.$$

45. $x \frac{dy}{dx} - 2x^2y + 3x^2 = 0$
 $y = 15$ when $x = 0$.

$$\begin{aligned} \frac{dy}{dx} - 2xy &= -3x \\ I(x) &= e^{\int -2x dx} = e^{-x^2} \\ e^{-x^2} y - 2xe^{-x^2} y &= -3xe^{-x^2} \\ D_x(e^{-x^2} y) &= -3xe^{-x^2} \\ e^{-x^2} y &= \int -3xe^{-x^2} dx \\ &= \frac{3}{2} e^{-x^2} + C \\ y &= \frac{3}{2} + C e^{x^2} \end{aligned}$$

Since $x = 0$ when $y = 15$,

$$\begin{aligned} 15 &= \frac{3}{2} + C e^0 \\ C &= \frac{27}{2}. \end{aligned}$$

Therefore,

$$y = \frac{3}{2} + \frac{27e^{x^2}}{2}.$$

46. $x^2 \frac{dy}{dx} + 4xy - e^{2x^3} = 0$; $y = e^2$
 when $x = 1$.

$$\begin{aligned} \frac{dy}{dx} + \frac{4}{x} y &= \frac{1}{x^2} e^{2x^3} \\ I(x) &= e^{\int 4/x dx} = e^{4 \ln x} = x^4 \\ x^4 y &= \int x^4 \cdot \frac{1}{x^2} e^{2x^3} dx \\ &= \int x^2 e^{2x^3} dx \\ &= \frac{1}{6} e^{2x^3} + C \\ 1 \cdot e^2 &= \frac{1}{6} e^2 + C \\ C &= \frac{5}{6} e^2 \\ y &= \frac{e^{2x^3} + 5e^2}{6x^4} \end{aligned}$$

48. $\frac{dy}{dx} = x + y^{-1}$; $y(0) = 1$; $h = 0.2$;

$$\begin{aligned} g(x, y) &= x + \frac{1}{y} \\ x_0 &= 0; y_0 = 1 \\ g(x_0, y_0) &= 0 + \frac{1}{1} = 1 \\ x_1 &= 0.2; y_1 = 1 + 1(0.2) = 1.2 \\ g(x_1, y_1) &= 0.2 + \frac{1}{1.2} = 1.0333 \\ x_2 &= 0.4; y_2 = 1.2 + 1.0333(0.2) \\ &= 1.4067 \\ g(x_2, y_2) &= 0.4 + \frac{1}{1.4067} = 1.1109 \\ x_3 &= 0.6; y_3 = 1.4067 + 1.1109(0.2) \\ &= 1.6289 \\ g(x_3, y_3) &= 0.6 + \frac{1}{1.6289} \\ &= 1.2139 \\ x^4 &= 0.8; y_4 = 1.6289 + 1.2139(0.2) \\ &= 1.8717 \\ g(x_4, y_4) &= 0.8 + \frac{1}{1.8717} = 1.3343 \\ x_5 &= 1.0; y_5 = 1.8719 + 1.3343(0.2) \\ y_5 &= 2.13855 \\ y(1) &\approx 2.138 \end{aligned}$$

49. $\frac{dy}{dx} = e^x + y$; $y(0) = 1$, $h = 0.2$; find $y(0.6)$.

$$g(x, y) = e^x + y$$

$$x_0 = 0; y_0 = 1$$

$$g(x_0, y_0) = e^0 + 1 = 2$$

$$x_1 = 0.2; y_1 = y_0 + g(x_0, y_0)h \\ = 1 + 2(0.2) = 1.4$$

$$g(x_1, y_1) = e^2 + 1.4 \approx 2.6214$$

$$x_2 = 0.4; y_2 = y_1 + g(x_1, y_1)h \\ = 1.4 + 2.6214(0.2) \\ \approx 1.9243$$

$$g(x_2, y_2) = e^{0.4} + 1.9243 \approx 3.4161$$

$$x_3 = 0.6; y_3 = y_2 + g(x_2, y_2)h \\ = 1.9243 + 3.4161(0.2) \\ \approx 2.6075$$

x_i	y_i
0	1
0.2	1.4
0.4	1.9243
0.6	2.608

So, $y(0.6) \approx 2.608$.

50. $\frac{dy}{dx} = \frac{x}{2} + 4$; $y(0) = 0$; $h = 0.1$,

$$g(x, y) = \frac{x}{2} + 4$$

$$x_0 = 0; y_0 = 0$$

$$g(x_0, y_0) = \frac{0}{2} + 4 = 4$$

$$x_1 = 0.1; y_1 = 0 + 4(0.1) = 0.4$$

$$g(x_1, y_1) = \frac{0.1}{2} + 4 = 4.05$$

$$x_2 = 0.2; y_2 = 0.4 + 4.05(0.1) \\ = 0.805$$

$$g(x_2, y_2) = \frac{0.2}{2} + 4 = 4.1$$

$$x_3 = 0.3; y_3 = 0.805 + 4.1(0.1) \\ = 1.215$$

Solving the differential equation gives

$$\frac{dy}{dx} = \frac{x}{2} + 4$$

$$\int dy = \int \left(\frac{x}{2} + 4 \right) dx$$

$$y = \frac{x^2}{4} + 4x + C.$$

Since $x = 0$ when $y = 0$, $C = 0$.

$$y = \frac{x^2}{4} + 4x$$

$$y(x_3) = y(0.3)$$

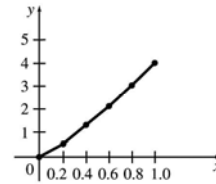
$$= \frac{0.3^2}{4} + 4(0.3) = 1.223$$

$$y_3 - y(x_3) = 1.215 - 1.223 = -0.008$$

51. $\frac{dy}{dx} = 3 + \sqrt{y}$, $y(0) = 0$, $h = 0.2$, find $y(1)$.

x_i	y_i
0	0
0.2	0.6
0.4	1.354919
0.6	2.187722
0.8	3.083541
1	4.034741

Therefore, $y(1) \approx 4.035$.



53. (a) $\frac{dy}{dx} = 6e^{0.3x}$

$$dy = 6e^{0.3x} dx$$

$$y = 20e^{0.3x} + C$$

When $x = 0$, $y = 0$

$$0 = 20e^0 + C$$

$$C = -20$$

$$y = 20e^{0.3x} - 20.$$

When $x = 6$,

$$y = 20e^{1.8} - 20 \\ \approx 100.99.$$

Sales are \$10,099.

(b) When $x = 12$,

$$y = 20e^{3.6} - 20 \\ \approx 711.96.$$

Sales are \$71,196.

54. $\frac{dy}{dx} = 0.1(150 - y); y = 15$

when $x = 0; y \leq 150$

$$\begin{aligned} \frac{1}{150 - y} dy &= 0.1 dx \\ -\ln(150 - y) &= 0.1x + C \\ -\ln 135 &= C \\ -\ln(150 - y) &= 0.1x - \ln 135 \\ \ln(150 - y) &= \ln 135 - 0.1x \\ 150 - y &= 135e^{-0.1x} \\ y &= 150 - 135e^{-0.1x} \end{aligned}$$

(a) If $x = 10$,

$$y = 150 - 135e^{-0.1(10)} \\ \approx 100 \text{ items.}$$

(b) $100 = 150 - 135e^{-0.1t}$

$$50 = 135e^{-0.1t} \\ e^{-0.1t} = \frac{50}{135} = \frac{10}{27}$$

$$-0.1t = \ln\left(\frac{10}{27}\right)$$

$$t = -10 \ln\left(\frac{10}{27}\right)$$

$$\approx 9.83$$

It will take 10 days.

55. $A = 300,000$ when $t = 0; r = 0.05$,
 $D = -20,000$

(a) $\frac{dA}{dt} = 0.05A - 20,000$

(b) $\frac{1}{0.05A - 20,000} dA = dt$

$$\frac{1}{0.05} \ln|0.05A - 20,000| = t + C$$

$$\ln|0.05A - 20,000| = 0.05t + k$$

$$\ln|0.05(300,000) - 20,000| = k$$

$$k = \ln|-5000| = \ln 5000$$

$$\ln|0.05A - 20,000| = 0.05t + \ln 5000$$

$$|0.05A - 20,000| = 5000e^{0.05t}$$

Since $0.05A < 20,000$,

$$|0.05A - 20,000| = 20,000 - 0.05A$$

$$20,000 - 0.05A = 5000e^{0.05t}$$

$$\begin{aligned} A &= \frac{1}{0.05} (20,000 - 5000e^{0.05t}) \\ &= 100,000(4 - e^{0.05t}). \end{aligned}$$

When $t = 10$,

$$\begin{aligned} A &= 100,000(4 - e^{0.05(10)}) \\ &\approx \$235,127.87. \end{aligned}$$

56. $\frac{dA}{dt} = rA - D; t = 0, A = \$300,000$;
 $r = 0.05; D = \$20,000$

$$\frac{dA}{dt} = 0.05A - 20,000$$

$$\begin{aligned} \int \frac{dA}{0.05A - 20,000} &= \int dt \\ \frac{1}{0.05} \ln|0.05A - 20,000| &= t + C_2 \end{aligned}$$

$$\ln|0.05A - 20,000| = 0.05t + C_1$$

$$0.05A - 20,000 = Ce^{0.05t}$$

$$A = 20Ce^{0.05t} + 400,000$$

Since $t = 0$ when $A = 300,000$,

$$300,000 = 20C(1) + 400,000$$

$$C = -5000.$$

$$A = 20(-5000)e^{0.05t} + 400,000$$

$$A = 400,000 - 100,000e^{0.05t}.$$

Find t for $A = 0$.

$$0 = 400,000 - 100,000e^{0.05t}$$

$$e^{0.05t} = \frac{400,000}{100,000} = 4$$

$$\ln e^{0.05t} = \ln 4$$

$$t = 20 \ln 4 = 27.73$$

It will take about 27.7 yr.

57. $\frac{dy}{dt} = \frac{-10}{1 + 5t}; y = 50$ when $t = 0$.

$$y = -2 \ln(1 + 5t) + C$$

$$50 = -2 \ln 1 + C$$

$$= C$$

$$y = 50 - 2 \ln(1 + 5t)$$

(a) If $t = 24$,

$$y = 50 - 2 \ln[1 + 5t(24)] \\ \approx 40 \text{ insects.}$$

(b) If $y = 0$,

$$50 = 2 \ln(1 + 5t) \\ 1 + 5t = e^{25} \\ t = \frac{e^{25} - 1}{5} \\ \approx 1.44 \times 10^{10} \text{ hours} \\ \approx 6 \times 10^8 \text{ days} \\ \approx 1.6 \text{ million years.}$$

58. $\frac{dy}{dt} = ky$; $k = 0.1$, $t = 0$, $y = 120$

$$\frac{dy}{y} = k dt \\ \ln|y| = kt + C_1 \\ |y| = e^{kt+C_1} \\ y = Me^{kt} \\ y = Me^{0.1t} \\ 120 = Me^0 \\ M = 120 \\ y = 120e^{0.1t}$$

Let $t = 6$ and find y .

$$y = 120e^{0.6} \\ \approx 219$$

After 6 weeks, about 219 are present.

59. $\frac{dx}{dt} = 0.2x - 0.5xy$

$$\frac{dy}{dt} = -0.3y + 0.4xy$$

$$\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-0.3y + 0.4xy}{0.2x - 0.5xy}$$

$$= \frac{y(-0.3 + 0.4x)}{x(0.2 - 0.5y)}$$

$$\frac{0.2 - 0.5y}{y} dy = \frac{-0.3 + 0.4x}{x} dx$$

$$\left(\frac{0.2}{y} - 0.5\right) dy = \left(\frac{-0.3}{x} + 0.4\right) dx$$

$$0.2 \ln y - 0.5y = -0.3 \ln x + 0.4x + C$$

$$0.3 \ln x + 0.2 \ln y - 0.4x - 0.5y = C$$

Both growth rates are 0 if

$$0.2x - 0.5xy = 0 \text{ and } -0.3y + 0.4xy = 0.$$

If $x \neq 0$ and $y \neq 0$, we have

$$0.2 - 0.5y = 0 \text{ and } -0.3 + 0.4x = 0, \text{ so}$$

$$x = \frac{3}{4} \text{ units and } y = \frac{2}{5} \text{ units.}$$

60. $\frac{dy}{dt} = \text{rate of smoke in} - \text{rate of smoke out}$

$$\text{rate of smoke in} = 0$$

$$\text{rate of smoke out} = \left(\frac{y}{V}\right)(1200) = \frac{1200y}{V}$$

$$\frac{dy}{dt} = \frac{-1200y}{V}$$

$$\frac{dV}{dt} = 1200 - 1200 = 0$$

$$V(t) = C_1; \text{ at } t = 0, V = 15,000.$$

$$C_1 = 15,000$$

$$V(t) = 15,000$$

$$\frac{dy}{dt} = \frac{-1200y}{15,000} = -0.08y$$

$$\frac{dy}{y} = -0.08 dt$$

$$\ln y = -0.08t + C$$

$$y = ke^{-0.08t}$$

$$\text{At } t = 0, y = 20.$$

$$20 = ke^0$$

$$k = 20$$

Therefore,

$$y = 20e^{-0.08t}.$$

At $y = 5$,

$$5 = 20e^{-0.08t}$$

$$.25 = e^{-0.08t}$$

$$t = \frac{-\ln(0.25)}{0.08}$$

$$= 17.3 \text{ min.}$$

61. Let y = the amount in parts per million (ppm) of smoke at time t .

When $t = 0$, $y = 20$ ppm, $V = 15,000 \text{ ft}^3$.

rate of smoke in = 5 ppm,

rate of smoke out

$$= (1200 \text{ ft}^3/\text{min}) \left(\frac{y}{V} \text{ ppm}/\text{ft}^3 \right)$$

Rate of air in = rate of air out,

so $V = 15,000 \text{ ft}^3$ for all t .

$$\begin{aligned} \frac{dy}{dt} &= 5 - \frac{1200y}{15,000} = 5 - \frac{2y}{25} \\ &= \frac{125 - 2y}{25} \end{aligned}$$

$$\begin{aligned} \frac{1}{125 - 2y} dy &= \frac{dt}{25} \\ -\frac{1}{2} \ln(125 - 2y) &= \frac{t}{25} + C \end{aligned}$$

$$-\frac{1}{2} \ln(125 - 2(20)) = C$$

$$C = -\frac{1}{2} \ln 85$$

If $y = 10$,

$$\begin{aligned} -\frac{1}{2} \ln[125 - 2(10)] &= \frac{t}{25} - \frac{1}{2} \ln 85 \\ \ln 105 &= \ln 85 - \frac{2t}{25} \\ t &= \frac{25}{2} [\ln 85 - \ln 105], \end{aligned}$$

Which is negative.

It is impossible to reduce y to 10 ppm.

62. (a) The differential equation for y , the number of individuals infected, is

$$\frac{dy}{dt} = a(N - y)y.$$

From Example 3 in Section 4 of this chapter, the solution is

$$y = \frac{N}{1 + (N - 1)e^{-aNy}}; t \text{ is in weeks.}$$

The number of individuals uninfected at time t is

$$\begin{aligned} y &= N - \frac{N}{1 + (N - 1)e^{-aNy}} \\ &= \frac{N(N - 1)}{N - 1 + e^{aNy}}. \end{aligned}$$

Substitute $N = 700$.

$$\begin{aligned} y &= \frac{700(699)}{699 + e^{700at}} \\ &= \frac{489,300}{699 + e^{700at}} \end{aligned}$$

Substitute $t = 6$, $y = 300$.

$$\begin{aligned} 300 &= \frac{489,300}{699 + e^{700(6)a}} \\ &= \frac{489,300}{699 + e^{4200a}} \\ 209,700 + 300e^{4200a} &= 489,300 \\ e^{4200a} &= 932 \\ a &= \frac{\ln 932}{4200} \approx 0.00163 \\ 700a &\approx 700(0.00163) \approx 1.140 \end{aligned}$$

Therefore,

$$y = \frac{489,300}{699 + e^{1.140t}}.$$

- (b) At $t = 7$ weeks,

$$\begin{aligned} y &= \frac{489,300}{699 + e^{1.140(7)}} \\ &\approx 135 \text{ people.} \end{aligned}$$

- (c) From Example 3,

$$\begin{aligned} t_m &= \frac{\ln(N - 1)}{aN} \\ &\approx \frac{\ln(700 - 1)}{(0.00163)(700)} \\ t_m &\approx 5.7 \text{ wk.} \end{aligned}$$

63. $y = \frac{N}{1 + be^{-kt}}$; $y = y_i$ when $x = x_i$,
 $i = 1, 2, 3$.

t_1, t_2, t_3 are equally spaced:

$$t_3 = 2t_2 - t_1, \text{ so } t_1 + t_3 = 2t_2,$$

$$\text{or } t_2 = \frac{t_1 + t_3}{2}.$$

$$\text{Show } N = \frac{\frac{1}{y_1} + \frac{1}{y_3} - \frac{2}{y_2}}{\frac{1}{y_1 y_3} - \frac{1}{y_2^2}}.$$

Let

$$\begin{aligned}
 A &= \frac{1}{y_1} + \frac{1}{y_3} - \frac{2}{y_2} \\
 &= \frac{1 + be^{-kt_1}}{N} + \frac{1 + be^{-kt_3}}{N} - \frac{2(1 + be^{-kt_2})}{N} \\
 &= \frac{1}{N} (1 + be^{-kt_1} + 1 + be^{-kt_3} - 2 - 2be^{-kt_2}) \\
 &= \frac{b}{N} [e^{-kt_1} + e^{-kt_3} - 2e^{-kt_2}]
 \end{aligned}$$

Let

$$\begin{aligned}
 B &= \frac{1}{y_1 y_3} - \frac{1}{y_2^2} \\
 &= \frac{(1 + be^{-kt_1})(1 + be^{-kt_3})}{N^2} \\
 &\quad - \frac{(1 + be^{-kt_2})^2}{N^2} \\
 &= \frac{1}{N^2} [1 + be^{-kt_1} + e^{-kt_3} + b^2 e^{-k(t_1+t_3)} \\
 &\quad - 1 - 2be^{-kt_2} - b^2 e^{-2kt_2}] \\
 &= \frac{b}{N^2} [e^{-kt_1} + be^{-kt_3} + be^{-k(2t_2)} \\
 &\quad - 2e^{-kt_2} - be^{-2kt_2}] \\
 &= \frac{b}{N^2} [e^{-kt_1} + e^{-kt_3} - 2e^{-kt_2}]
 \end{aligned}$$

Clearly, $\frac{A}{B} = N$.

Hence,

$$N = \frac{\frac{1}{y_1} + \frac{1}{y_3} - \frac{2}{y_2}}{\frac{1}{y_1 y_3} - \frac{1}{y_2^2}}.$$

$$64. \quad N = \frac{\frac{1}{y_1} + \frac{1}{y_3} - \frac{2}{y_2}}{\frac{1}{y_1 y_3} - \frac{1}{y_2^2}}$$

(a) Using the formula for N with $y_1 = 5.3$, $y_2 = 23.2$, and $y_3 = 76.0$, $N \approx 185$. So, the upper bound that the U.S. population was approaching during these years was approximately 185 million.

(b) Using the formula for N with $y_1 = 23.2$, $y_2 = 76.0$, and $y_3 = 150.7$, $N \approx 207$. So, the upper bound that the U.S. population was approaching during these years was approximately 207 million.

(c) Using the formula for N with $y_1 = 39.8$, $y_2 = 105.7$, and $y_3 = 203.3$, $N \approx 326$. So, the upper bound that the U.S. population was approaching during these years was approximately 326 million.

65. From Exercise 64,

$$N = \frac{\frac{1}{40} + \frac{1}{204} - \frac{2}{106}}{\frac{1}{40(204)} - \frac{1}{(106)^2}} \approx 329.$$

(a) $y_0 = 40$,

$$b = \frac{N - y_0}{y_0} = \frac{329 - 40}{40} \approx 7.23.$$

If 1920 corresponds to $t = 5$ decades, then

$$\begin{aligned}
 106 &= \frac{329}{1 + 7.23e^{-5k}} \\
 1 + 7.23e^{-5k} &= \frac{329}{106} \\
 e^{-5k} &= \frac{1}{7.23} \left(\frac{329}{106} - 1 \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{so } k &= -\frac{1}{5} \ln \left[\frac{1}{7.23} \left(\frac{329}{106} - 1 \right) \right] \\
 &\approx 0.247.
 \end{aligned}$$

$$(b) \quad y = \frac{329}{1 + 7.23e^{-0.247t}}$$

In 2010, $t = 14$. If $t = 14$,

$$y = \frac{329}{1 + 7.23e^{-3.458}} \approx 268 \text{ million}$$

The predicated population is 268 million which is less than the table value of 308.7 million.

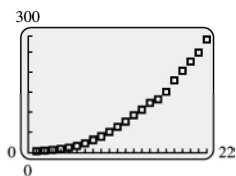
(c) In 2030, $t = 16$, so

$$y = \frac{329}{1 + 7.23e^{-0.247(16)}} \approx 289 \text{ million.}$$

In 2050, $t = 18$, so

$$y = \frac{329}{1 + 7.23e^{-0.247(18)}} \approx 303 \text{ million.}$$

66. (a)



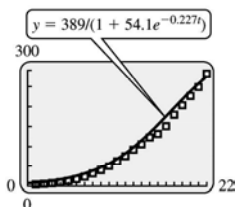
The points suggest the lower portion of a logistic growth curve, so yes, a logistic function seems appropriate.

- (b) A calculator with a logistic regression function determines that

$$y = \frac{389}{1 + 54.1e^{-0.227t}}$$

best fits the data.

(c)



The model does produce appropriate y -values for the given t -values.

- (d) As t gets larger and larger, $e^{-0.227t}$ approaches 0, so y approaches

$$\frac{389}{1 + 54.1 \cdot 0} = \frac{389}{1} = 389.$$

Therefore, according to this model, the U.S. population has a limiting size of 389 million.

67. (a) $\frac{dx}{dt} = 1 - kx$

Separate the variables and integrate.

$$\int \frac{dx}{1 - kx} = \int dt$$

$$-\frac{1}{k} \ln |1 - kx| = t + C_1$$

Solve for x .

$$\ln |1 - kx| = -kt - kC_1$$

$$|1 - kx| = e^{-kt} e^{-kC_1}$$

$$1 - kx = Me^{-kt}, \text{ where } M = \pm e^{-kC_1}.$$

$$x = -\frac{1}{k}(Me^{-kt} - 1) = \frac{1}{k} + Ce^{-kt},$$

$$\text{where } C = -\frac{M}{k}.$$

- (b) Write the linear first-order differential equation in the linear form

$$\frac{dx}{dt} + kx = 1.$$

The integrating factor is $I(t) = e^{\int k dt} = e^{kt}$.

Multiply both sides of the differential equation by $I(t)$.

$$\frac{dx}{dt} e^{kt} + kxe^{kt} = e^{kt}$$

Replace the left side of this equation by

$$D_t(xe^{kt}) = \frac{dx}{dt} e^{kt} + kxe^{kt}.$$

$$D_t(xe^{kt}) = e^{kt}$$

Integrate both sides with respect to t .

$$xe^{kt} = \int e^{kt} dt = \frac{1}{k} e^{kt} + C$$

Solve for x .

$$x = \frac{1}{k} + Ce^{-kt}.$$

- (c) Since $k > 0$, then as t gets larger and larger, Ce^{-kt} approaches 0, so $\lim_{x \rightarrow \infty} \left(\frac{1}{k} + Ce^{-kt} \right) = \frac{1}{k}$.

68. (a) $\frac{dy}{dx} = a(N - y)y$; $N = 200$;
 $y_0 = 10$; $y = 35$ when $x = 3$.

The solution to the differential equation is equation 6 in Example 3 of Section 4 in this chapter, which is

$$y = \frac{N}{1 + be^{-kx}},$$

where $b = \frac{N - y_0}{y_0}$ and $k = aN$.

$$b = \frac{200 - 10}{10} = 19; k = 200a$$

$$y = \frac{200}{1 + 19e^{-200ax}}$$

Since $y = 35$ when $x = 3$,

$$\begin{aligned}
 35 &= \frac{200}{1 + 19e^{-200a(3)}} \\
 35 + 665e^{-600a} &= 200 \\
 e^{-600a} &= \frac{165}{665} = \frac{33}{133} \\
 -600a &= \ln\left(\frac{33}{133}\right) \\
 -200a &= \frac{1}{3} \ln\left(\frac{33}{133}\right) \approx -0.465.
 \end{aligned}$$

Therefore,

$$y = \frac{200}{1 + 19e^{-0.465x}}.$$

(b) For $x = 5$ days,

$$\begin{aligned}
 y &= \frac{200}{1 + 19e^{-0.465(5)}} \\
 y &\approx 69.98
 \end{aligned}$$

In 5 days, about 70 people have heard the rumor.

69. $\frac{dT}{dt} = k(T - T_F)$

$$\frac{dT}{dt} = k(T - 300^\circ)$$

$$T(0) = 40^\circ$$

$$T(1) = 150^\circ$$

$$\frac{dT}{T - 300} = k dt$$

$$\ln|T - 300| = kt + C_1$$

$$|T - 300| = e^{kt+C_1}$$

$$T - 300 = Ce^{kt}$$

$$T = Ce^{kt} + 300$$

Since $T(0) = 40^\circ$,

$$40 = Ce^0 + 300$$

$$C = -260.$$

Therefore,

$$T = -260e^{kt} + 300.$$

At $T(1) = 150^\circ$,

$$150 = -260e^k + 300$$

$$-150 = -260e^k$$

$$\frac{15}{26} = e^k$$

$$\ln\left(\frac{15}{26}\right) = k \ln e$$

$$k = \ln\left(\frac{15}{26}\right)$$

$$k \approx -0.55.$$

Therefore,

$$T = -260e^{-0.55t} + 300.$$

At $t = 2$,

$$\begin{aligned}
 T &= -260e^{-0.55(2)} + 300 \\
 &= 213^\circ.
 \end{aligned}$$

70. $\frac{dT}{dt} = k(T - T_F); T_F = 300; T = 40$

when $t = 0$.

$T = 150$ when $t = 1$.

From Exercise 57 in Section 1 of this chapter, the solution to the differential equation is

$$T = Ce^{kt} + T_M$$

where C is a constant ($-k$ has been replaced by k in this exercise.)

Here, $T = Ce^{kt} + 300$.

$$40 = C + 300$$

$$C = -260$$

$$T = 300 - 260e^{kt}$$

$$150 = 300 - 260e^k$$

$$e^k = \frac{15}{26}$$

$$k = \ln\left(\frac{15}{26}\right) \approx -0.55$$

$$T = 300 - 260e^{-0.55t}$$

$$250 = 300 - 260e^{-0.55t}$$

$$e^{-0.55t} = \frac{5}{26}$$

$$t = -\frac{1}{0.55} \ln\left(\frac{5}{26}\right) \approx 3 \text{ hr}$$

71. (a) $\frac{dv}{dt} = G^2 - K^2v^2$
 $dv = (G^2 - K^2v^2)dt$
 $\frac{1}{G^2 - K^2v^2}dv = dt$
 $\int \frac{1}{G^2 - K^2v^2}dv = \int dt$
 $\frac{1}{K^2} \int \frac{1}{\left(\frac{G}{K}\right)^2 - v^2} = \int dt$

Use entry 7 from the table of integrals.

$$\frac{1}{K^2} \cdot \frac{1}{2\frac{G}{K}} \ln \left| \frac{\frac{G}{K} + v}{\frac{G}{K} - v} \right| = t + C_1$$

$$\frac{1}{2GK} \ln \left| \frac{G + Kv}{G - Kv} \right| = t + C_1$$

$$\ln \left| \frac{G + Kv}{G - Kv} \right| = 2GKt + C_2$$

Since $v < \frac{G}{K}$, $Kv < G$ and $\frac{G+Kv}{G-Kv}$ is positive.

$$\ln \left(\frac{G + Kv}{G - Kv} \right) = 2GK(t) + C_2$$

When $t = 0$, $v = 0$, so

$$\ln \left(\frac{G + 0}{G - 0} \right) = 2GK(0) + C_2$$

$$\ln 1 = C_2$$

$$C_2 = 0.$$

Thus,

$$\ln \left(\frac{G + Kv}{G - Kv} \right) = 2GKt$$

$$\frac{G + Kv}{G - Kv} = e^{2GKt}$$

$$G + Kv = Ge^{2GKt} - Kve^{2GKt}$$

$$Kv + Kve^{2GKt} = Ge^{2GKt} - G$$

$$vK(e^{2GKt} + 1) = G(e^{2GKt} - 1)$$

$$v = \frac{G(e^{2GKt} - 1)}{K(e^{2GKt} + 1)}$$

$$v = \frac{G}{K} \cdot \frac{e^{2GKt} - 1}{e^{2GKt} + 1}$$

(b) $\lim_{t \rightarrow \infty} v = \lim_{t \rightarrow \infty} \left(\frac{G}{K} \cdot \frac{e^{2GKt} - 1}{e^{2GKt} + 1} \right)$

$$\lim_{t \rightarrow \infty} v = \frac{G}{K} \lim_{t \rightarrow \infty} \frac{e^{2GKt} - 1}{e^{2GKt} + 1}$$

$$\lim_{t \rightarrow \infty} v = \frac{G}{K} \cdot 1 = \frac{G}{K}$$

A falling object in the presence of air resistance has a limiting velocity, $\frac{G}{K}$.

(c) $\lim_{t \rightarrow \infty} v = \frac{G}{k}$

$$88 = \frac{G}{k}$$

$$k = \frac{G}{88}$$

$$Gk = \frac{G^2}{88}$$

$$Gk = \frac{32}{88}$$

$$2Gk = \frac{64}{88}$$

$$2Gk \approx 0.727$$

$$v = \frac{G}{k} \frac{e^{2Gkt} - 1}{e^{2Gkt} + 1}$$

$$v = 88 \frac{e^{0.727t} - 1}{e^{0.727t} + 1}$$

Extended Application: Pollution of the Great Lakes

1. $t = \frac{1}{k} \ln \left(\frac{P_L(0)}{P_L(t)} \right)$

(a) $\frac{1}{k} = 2.6; \frac{P_L(t)}{P_L(0)} = 0.4$

$$t = 2.6 \ln \left(\frac{1}{0.4} \right) = 2.4 \text{ yr}$$

(b) $\frac{1}{k} = 2.6; \frac{P_L(t)}{P_L(0)} = 0.3$

$$t = 2.6 \ln \left(\frac{1}{0.3} \right) = 3.1 \text{ yr}$$

$$2. \quad t = \frac{1}{k} \ln \left(\frac{P_L(0)}{P_L(t)} \right)$$

$$(a) \quad \frac{1}{k} = 30.8; \frac{P_L(t)}{P_L(0)} = 0.4$$

$$t = 30.8 \ln \left(\frac{1}{0.4} \right) = 28.2 \text{ yr}$$

$$(b) \quad \frac{1}{k} = 30.8; \frac{P_L(t)}{P_L(0)} = 0.3$$

$$t = 30.8 \ln \left(\frac{1}{0.3} \right) = 37.1 \text{ yr}$$

$$3. \quad t = \frac{1}{k} \ln \left(\frac{P_L(0)}{P_L(t)} \right)$$

$$(a) \quad \frac{1}{k} = 189; \frac{P_L(t)}{P_L(0)} = 0.4$$

$$t = 189 \ln \left(\frac{1}{0.4} \right) = 173 \text{ yr}$$

$$(b) \quad \frac{1}{k} = 189; \frac{P_L(t)}{P_L(0)} = 0.3$$

$$t = 189 \ln \left(\frac{1}{0.3} \right) = 228 \text{ yr}$$

4. (a)

$$e^{kt} P_L(t) = P_L(0) + k \int_0^t P_i(x) e^{kx} dx$$

$$e^{kx} \frac{dP_L}{dt} + ke^{kt} P_L(t) = kP_i(t) e^{kt}$$

$$\frac{dP_L}{dt} + kP_L(t) = kP_i(t)$$

(b) Substituting $t = 0$ in both sides of

$$P_L(t) = e^{-kt} \left[P_L(0) + k \int_0^t P_i(x) e^{kx} dx \right]$$

produces $P_L(0)$ on both sides. On the right side, e^{-kt} becomes $e^0 = 1$, and the integral goes to zero (because lower limit = upper limit).

(c) Since $30.8 = \frac{1}{k}$, $k = \frac{1}{30.8} \approx 0.0325$.

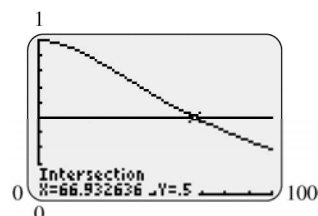
This represents $\frac{r}{V} = \frac{\text{flow rate}}{\text{lake volume}}$, which

means that every year, the total flow is 3.25% of the lake's volume. That is the percentage replaced. The higher k is, the lower $\frac{1}{k}$ is. So the lake with the biggest turnover is Lake Erie.

5. Substitute $P_i(x) = P_L(0)e^{-px}$ in $\int_0^t P_i(x)e^{kx} dx$.

$$\begin{aligned} \int_0^t P_L(0)e^{-px}e^{kx} dx &= P_L(0) \int_0^t e^{(k-p)x} dx \\ &= P_L(0) \left[\frac{1}{k-p} e^{(k-p)x} \right]_0^t \\ &= \frac{1}{k-p} P_L(0) [e^{(k-p)t} - 1] \end{aligned}$$

6. (a) For Lake Michigan, $k \approx 0.0325$. Graph $y_1 = \frac{1}{0.0125}(0.0325e^{-0.02x} - 0.02e^{-0.0325x})$ and $y_2 = 0.50$, and find the intersection point:



It will take about 67 years to reduce pollution to 50% of its current value. This compares with 21 years assuming pollution-free inflow.

(b) With $p = 0$, the inflow is constant at the same pollution level as the lake. The ratio

$$\frac{P_L(t)}{P_L(0)} = \frac{1}{k-p}(ke^{-pt} - pe^{-kt})$$

becomes

$$\frac{1}{k-0}(ke^0 - 0) = 1,$$

which means that the lake pollution level is constant, as would be expected.

Chapter 11

PROBABILITY AND CALCULUS

11.1 Continuous Probability Models

Your Turn 1

$$f(x) = \frac{2}{x^2} \text{ on } [1, 2]$$

Condition 1 holds because $f(x) \geq 0$ on $[1, 2]$.

$$\begin{aligned} \text{Condition 2 holds because } \int_1^2 f(x) dx &= \int_1^2 \frac{2}{x^2} dx \\ &= -\frac{2}{x} \Big|_1^2 \\ &= -1 - (-2) \\ &= 1. \end{aligned}$$

Thus $f(x)$ is a probability density function for the interval $[1, 2]$.

$$\begin{aligned} P(3/2 \leq X \leq 2) &= \int_{3/2}^2 \frac{2}{x^2} dx \\ &= -\frac{2}{x} \Big|_{3/2}^2 \\ &= -1 - \left(-\frac{2}{3/2}\right) \\ &= -1 + \frac{4}{3} \\ &= \frac{1}{3} \end{aligned}$$

Your Turn 2

The function $f(x) = kx^3$ will be nonnegative on the interval $[0, 4]$ for any positive k .

$$\begin{aligned} \int_0^4 kx^3 dx &= \frac{k}{4} x^4 \Big|_0^4 \\ &= \frac{k}{4} 4^4 \\ &= 64k \end{aligned}$$

Thus $f(x)$ will be a probability density function on $[0, 4]$ when $64k = 1$, or $k = 1/64$.

Your Turn 3

$$\begin{aligned} P(0 \leq X \leq 1) &= \int_0^1 2xe^{-x^2} dx \\ &= \left(-e^{-x^2}\right) \Big|_0^1 \\ &= -\frac{1}{e} - (-1) \\ &= 1 - \frac{1}{e} \approx 0.6321 \end{aligned}$$

Your Turn 4

The cumulative distribution function from Example 5 is $F(x) = -e^{-x^2} + 1$. The probability that there is a bird's nest within 1 km of the given point is

$$F(1) = -\frac{1}{e} + 1 \approx 0.6321.$$

11.1 Exercises

1. $f(x) = \frac{1}{9}x - \frac{1}{18}; [2, 5]$

Show that condition 1 holds.

Since $2 \leq x \leq 5$,

$$\begin{aligned} \frac{2}{9} &\leq \frac{1}{9}x \leq \frac{5}{9} \\ \frac{1}{6} &\leq \frac{1}{9}x - \frac{1}{18} \leq \frac{1}{2}. \end{aligned}$$

Hence, $f(x) \geq 0$ on $[2, 5]$.

Show that condition 2 holds.

$$\begin{aligned} \int_2^5 \left(\frac{1}{9}x - \frac{1}{18}\right) dx &= \frac{1}{9} \int_2^5 \left(x - \frac{1}{2}\right) dx \\ &= \frac{1}{9} \left(\frac{x^2}{2} - \frac{1}{2}x\right) \Big|_2^5 \\ &= \frac{1}{9} \left(\frac{25}{2} - \frac{5}{2} - \frac{4}{2} + 1\right) \\ &= \frac{1}{9}(8 + 1) \\ &= 1 \end{aligned}$$

Yes, $f(x)$ is a probability density function.

2. $f(x) = \frac{1}{3}x - \frac{1}{6}; [3, 4]$

Show that condition 1 holds.

Since $3 \leq x \leq 4$,

$$1 \leq \frac{1}{3}x \leq \frac{4}{3}$$

$$\frac{5}{6} \leq \frac{1}{3}x - \frac{1}{6} \leq \frac{7}{6}.$$

Hence, $f(x) \geq 0$ on $[3, 4]$.

Show that condition 2 holds.

$$\int_3^4 \left(\frac{1}{3}x - \frac{1}{6} \right) dx = \left(\frac{x^2}{6} - \frac{x}{6} \right) \Big|_3^4$$

$$= \frac{1}{6}(16 - 4 - 9 + 3)$$

$$= 1$$

Yes, $f(x)$ is a probability density function.

3. $f(x) = \frac{1}{21}x^2; [1, 4]$

Since $x^2 \geq 0$, $f(x) \geq 0$ on $[1, 4]$.

$$\frac{1}{21} \int_1^4 x^2 dx = \frac{1}{21} \left(\frac{x^3}{3} \right) \Big|_1^4$$

$$= \frac{1}{21} \left(\frac{64}{3} - \frac{1}{3} \right) = 1$$

Yes, $f(x)$ is a probability density function.

4. $f(x) = \frac{3}{98}x^2; [3, 5]$

Since $x^2 \geq 0$, $f(x) \geq 0$ on $[3, 5]$.

$$\int_3^5 \frac{3}{98}x^2 dx = \frac{x^3}{98} \Big|_3^5$$

$$= \frac{1}{98}(125 - 27)$$

$$= 1$$

Yes, $f(x)$ is a probability density function.

5. $f(x) = 4x^3; [0, 3]$

$$4 \int_0^3 x^3 dx = 4 \left(\frac{x^4}{4} \right) \Big|_0^3$$

$$= 4 \left(\frac{81}{4} - 0 \right)$$

$$= 81 \neq 1$$

No, $f(x)$ is not a probability density function.

6. $f(x) = \frac{x^3}{81}; [0, 3]$

$$\int_0^3 \frac{x^3}{81} dx = \frac{x^4}{324} \Big|_0^3$$

$$= \frac{1}{4}$$

$$\neq 1$$

No, $f(x)$ is not a probability density function.

7. $f(x) = \frac{x^2}{16}; [-2, 2]$

$$\frac{1}{16} \int_{-2}^2 x^2 dx = \frac{1}{16} \left(\frac{x^3}{3} \right) \Big|_{-2}^2$$

$$= \frac{1}{16} \left(\frac{8}{3} + \frac{8}{3} \right)$$

$$= \frac{1}{3} \neq 1$$

No, $f(x)$ is not a probability density function.

8. $f(x) = 2x^2; [-1, 1]$

$$\int_{-1}^1 2x^2 dx = \frac{2}{3}x^3 \Big|_{-1}^1$$

$$= \frac{2}{3}(1 + 1)$$

$$= \frac{4}{3} \neq 1$$

No, $f(x)$ is not a probability density function.

9. $f(x) = \frac{5}{3}x^2 - \frac{5}{90}; [-1, 1]$

Let $x = 0$. Then $f(x) = f(0) = -\frac{5}{90} < 0$.

So $f(x) < 0$ for at least one x -value in $[-1, 1]$.

No, $f(x)$ is not a probability density function.

10. $f(x) = \frac{3}{13}x^2 - \frac{12}{13}x + \frac{45}{52}; [0, 4]$

Let $x = 2$. Then $f(x) = f(2) = -\frac{3}{52} < 0$.

So $f(x) < 0$ for at least one x -value in $[0, 4]$.

No, $f(x)$ is not a probability density function.

11. $f(x) = kx^{1/2}; [1, 4]$

$$\begin{aligned}\int_1^4 kx^{1/2} dx &= \frac{2}{3}kx^{3/2} \Big|_1^4 \\ &= \frac{2}{3}k(8 - 1) \\ &= \frac{14}{3}k\end{aligned}$$

If $\frac{14}{3}k = 1$,

$$k = \frac{3}{14}.$$

Notice that $f(x) = \frac{3}{4}x^{1/2} \geq 0$ for all x in $[1, 4]$.

12. $f(x) = kx^{3/2}; [4, 9]$

$$\begin{aligned}\int_4^9 kx^{3/2} dx &= \frac{2k}{5}x^{5/2} \Big|_4^9 \\ &= \frac{2k}{5}(243 - 32) \\ &= \frac{422k}{5}\end{aligned}$$

If $\frac{422k}{5} = 1$, $k = \frac{5}{422}$.

Notice that $f(x) = \frac{5}{422}x^{3/2} \geq 0$ for all x in $[4, 9]$.

13. $f(x) = kx^2; [0, 5]$

$$\begin{aligned}\int_0^5 kx^2 dx &= k \frac{x^3}{3} \Big|_0^5 \\ &= k \left(\frac{125}{3} - 0 \right) \\ &= k \left(\frac{125}{3} \right)\end{aligned}$$

If $k \left(\frac{125}{3} \right) = 1$,

$$k = \frac{3}{125}.$$

Notice that $f(x) = \frac{3}{125}x^2 \geq 0$ for all x in $[0, 5]$.

14. $f(x) = kx^2; [-1, 2]$

$$\begin{aligned}\int_{-1}^2 kx^2 dx &= \frac{k}{3}x^3 \Big|_{-1}^2 \\ &= \frac{k}{3}(8 + 1) \\ &= 3k = 1 \\ k &= \frac{1}{3}\end{aligned}$$

Notice that $f(x) = \frac{1}{3}x^2 \geq 0$ for all x in $[-1, 2]$.

15. $f(x) = kx; [0, 3]$

$$\begin{aligned}\int_0^3 kx dx &= k \frac{x^2}{2} \Big|_0^3 \\ &= k \left(\frac{9}{2} - 0 \right) \\ &= \frac{9}{2}k\end{aligned}$$

If $\frac{9}{2}k = 1$,

$$k = \frac{2}{9}.$$

Notice that $f(x) = \frac{2}{9}x \geq 0$ for all x in $[0, 3]$.

16. $f(x) = kx$; $[2, 3]$

$$\int_2^3 kx \, dx = \left. \frac{k}{2} x^2 \right|_2^3 = \frac{k}{2}(9 - 4) = \frac{5k}{2} = 1$$

$$k = \frac{2}{5}$$

Notice that $f(x) = \frac{2}{5}x \geq 0$ for all x in $[2, 3]$.

17. $f(x) = kx$; $[1, 5]$

$$\begin{aligned} \int_1^5 kx \, dx &= \left. k \frac{x^2}{2} \right|_1^5 \\ &= k \left(\frac{25}{2} - \frac{1}{2} \right) \\ &= 12k \end{aligned}$$

If $12k = 1$,

$$k = \frac{1}{12}.$$

Notice that $f(x) = \frac{1}{12}x \geq 0$ for all x in $[1, 5]$.

18. $f(x) = kx^3$; $[2, 4]$

$$\begin{aligned} \int_2^4 kx^3 \, dx &= \left. \frac{k}{4} x^4 \right|_2^4 \\ &= \frac{k}{4}(256 - 16) \\ &= 60k = 1 \\ k &= \frac{1}{60} \end{aligned}$$

Notice that $f(x) = \frac{1}{60}x^3 \geq 0$ for all x in $[2, 4]$.

19. For the probability density function $f(x) = \frac{1}{9}x - \frac{1}{18}$ on $[2, 5]$, the cumulative distribution function is

$$\begin{aligned} F(x) &= \int_a^x f(t) \, dt \\ &= \int_2^x \left(\frac{1}{9}t - \frac{1}{18} \right) dt \\ &= \left(\frac{1}{18}t^2 - \frac{1}{18}t \right) \Big|_2^x \\ &= \frac{1}{18}[(x^2 - x) - (4 - 2)] \\ &= \frac{1}{18}(x^2 - x - 2), 2 \leq x \leq 5. \end{aligned}$$

20. For the probability density function

$f(x) = \frac{1}{3}x - \frac{1}{6}$ on $[3, 4]$, the cumulative distribution function is

$$\begin{aligned} F(x) &= \int_a^x f(t) \, dt \\ &= \int_3^x \left(\frac{1}{3}t - \frac{1}{6} \right) dt \\ &= \left(\frac{1}{6}t^2 - \frac{1}{6}t \right) \Big|_3^x \\ &= \frac{1}{6}[(x^2 - x) - (9 - 3)] \\ &= \frac{1}{6}(x^2 - x - 6), 3 \leq x \leq 4. \end{aligned}$$

21. For the probability density function $f(x) = \frac{x^2}{21}$ on $[1, 4]$, the cumulative distribution function is

$$\begin{aligned} F(x) &= \int_1^x \frac{t^2}{21} dt \\ &= \left. \frac{t^3}{63} \right|_1^x \\ &= \frac{1}{63}(x^3 - 1), 1 \leq x \leq 4. \end{aligned}$$

22. For the probability density function $f(x) = \frac{3}{98}x^2$ on $[3, 5]$, the cumulative distribution function is

$$\begin{aligned} F(x) &= \int_3^x \frac{3}{98}t^2 dt \\ &= \left. \frac{t^3}{98} \right|_3^x \\ &= \frac{1}{98}(x^3 - 27), 3 \leq x \leq 5. \end{aligned}$$

23. The value of k was found to be $\frac{3}{14}$. For the probability density function $f(x) = \frac{3}{14}x^{1/2}$ on $[1, 4]$, the cumulative distribution function is

$$\begin{aligned} F(x) &= \int_1^x \frac{3}{14}t^{1/2} dt \\ &= \frac{3}{14} \cdot \frac{2}{3} t^{3/2} \Big|_1^x \\ &= \frac{1}{7}(x^{3/2} - 1), 1 \leq x \leq 4. \end{aligned}$$

24. The value of k was found to be $\frac{5}{422}$. For the probability density function $f(x) = \frac{5}{422}x^{3/2}$ on $[4, 9]$, the cumulative distribution is

$$\begin{aligned} F(x) &= \int_4^x \frac{5}{422} t^{3/2} dt \\ &= \frac{5}{422} \cdot \frac{2}{5} t^{5/2} \Big|_4^x \\ &= \frac{1}{211} (x^{5/2} - 32), 4 \leq x \leq 9. \end{aligned}$$

25. The total area under the graph of a probability density function always equals 1.

29. $f(x) = \frac{1}{2}(1+x)^{-3/2}; [0, \infty)$

$$\begin{aligned} \frac{1}{2} \int_0^\infty (1+x)^{-3/2} dx &= \lim_{a \rightarrow \infty} \frac{1}{2} \int_0^a (1+x)^{-3/2} dx \\ &= \lim_{a \rightarrow \infty} \frac{1}{2} (1+x)^{-1/2} \left(\frac{-2}{1} \right) \Big|_0^a \\ &= \lim_{a \rightarrow \infty} [-(1+a)^{-1/2} + 1] \\ &= \lim_{a \rightarrow \infty} \left(\frac{-1}{\sqrt{1+a}} + 1 \right) \\ &= 0 + 1 = 1 \end{aligned}$$

Since $x \geq 0$, $f(x) \geq 0$.

$f(x)$ is a probability density function.

- (a) $P(0 \leq X \leq 2)$

$$\begin{aligned} &= \frac{1}{2} \int_0^2 (1+x)^{-3/2} dx \\ &= -(1+x)^{-1/2} \Big|_0^2 \\ &= -3^{-1/2} + 1 \\ &\approx 0.4226 \end{aligned}$$

- (b) $P(1 \leq X \leq 3)$

$$\begin{aligned} &= \frac{1}{2} \int_1^3 (1+x)^{-3/2} dx \\ &= -(1+x)^{-1/2} \Big|_1^3 \\ &= -4^{-1/2} + 2^{-1/2} \\ &\approx 0.2071 \end{aligned}$$

- (c) $P(X \geq 5)$

$$\begin{aligned} &= \frac{1}{2} \int_5^\infty (1+x)^{-3/2} dx \\ &= \lim_{a \rightarrow \infty} \frac{1}{2} \int_5^a (1+x)^{-3/2} dx \\ &= \lim_{a \rightarrow \infty} [-(1+x)^{-1/2}] \Big|_5^a \\ &= \lim_{a \rightarrow \infty} [-(1+a)^{-1/2} + 6^{-1/2}] \\ &= \lim_{a \rightarrow \infty} \left(\frac{-1}{\sqrt{1+a}} + 6^{-1/2} \right) \\ &\approx 0 + 0.4082 \\ &= 0.4082 \end{aligned}$$

30. $f(x) = e^{-x}; [0, \infty)$

$$\begin{aligned} \int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left(-e^{-x} \Big|_0^b \right) \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{e^b} \right) = 1 \end{aligned}$$

$f(x) \geq 0$ for all x .

$f(x)$ is a probability density function.

- (a) $P(0 \leq X \leq 1) = \int_0^1 e^{-x} dx$

$$\begin{aligned} &= -e^{-x} \Big|_0^1 \\ &= 1 - \frac{1}{e} \approx 0.6321 \end{aligned}$$

- (b) $P(1 \leq X \leq 2) = \int_1^2 e^{-x} dx$

$$\begin{aligned} &= -e^{-x} \Big|_1^2 \\ &= \frac{1}{e} - \frac{1}{e^2} \approx 0.2325 \end{aligned}$$

- (c) $P(X \leq 2) = \int_0^2 e^{-x} dx$

$$\begin{aligned} &= -e^{-x} \Big|_0^2 \\ &= 1 - \frac{1}{e^2} \approx 0.8647 \end{aligned}$$

Notice that

$$\begin{aligned} P(X \leq 2) &= P(0 \leq X \leq 1) \\ &\quad + P(1 \leq X \leq 2). \end{aligned}$$

$$31. \quad f(x) = \frac{1}{2}e^{-x/2}; [0, \infty)$$

$$\begin{aligned} & \frac{1}{2} \int_0^{\infty} e^{-x/2} dx \\ &= \lim_{a \rightarrow \infty} \frac{1}{2} \int_0^a e^{-x/2} dx \\ &= \lim_{a \rightarrow \infty} \frac{1}{2} \left(\frac{-2}{1} e^{-x/2} \right) \Big|_0^a \\ &= \lim_{a \rightarrow \infty} -e^{-x/2} \Big|_0^a \\ &= \lim_{a \rightarrow \infty} \left(\frac{-1}{e^{a/2}} + 1 \right) \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

$f(x) > 0$ for all x .

$f(x)$ is a probability density function.

$$\begin{aligned} \text{(a)} \quad P(0 \leq X \leq 1) &= \frac{1}{2} \int_0^1 e^{-x/2} dx \\ &= -e^{-x/2} \Big|_0^1 \\ &= \frac{-1}{e^{1/2}} + 1 \\ &\approx 0.3935 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(1 \leq X \leq 3) &= \frac{1}{2} \int_1^3 e^{-x/2} dx \\ &= -e^{-x/2} \Big|_1^3 \\ &= \frac{-1}{e^{3/2}} + \frac{1}{e^{1/2}} \\ &\approx 0.3834 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(X \geq 2) &= \frac{1}{2} \int_2^{\infty} e^{-x/2} dx \\ &= \lim_{a \rightarrow \infty} \frac{1}{2} \int_2^a e^{-x/2} dx \\ &= \lim_{a \rightarrow \infty} (-e^{-x/2}) \Big|_2^a \\ &= \lim_{a \rightarrow \infty} \left(\frac{-1}{e^{a/2}} + \frac{1}{e} \right) \\ &\approx 0.3679 \end{aligned}$$

$$32. \quad f(x) = \frac{20}{(x+20)^2}; [0, \infty)$$

$$\begin{aligned} & \int_0^{\infty} \frac{20}{(x+20)^2} dx \\ &= \lim_{b \rightarrow \infty} \int_0^b 20(x+20)^{-2} dx \\ &= \lim_{b \rightarrow \infty} -20(x+20)^{-1} \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{20}{b+20} \right) = 1 \end{aligned}$$

$f(x) \geq 0$ for all x .

Therefore $f(x)$ is a probability density function.

$$\begin{aligned} \text{(a)} \quad P(0 \leq X \leq 1) &= \int_0^1 20(x+20)^{-2} dx \\ &= -20(x+20)^{-1} \Big|_0^1 \\ &= -20 \left(\frac{1}{21} - \frac{1}{20} \right) \\ &= \frac{1}{21} \approx 0.0476 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(1 \leq X \leq 5) &= \int_1^5 20(x+20)^{-2} dx \\ &= -20(x+20)^{-1} \Big|_1^5 \\ &= -20 \left(\frac{1}{25} - \frac{1}{21} \right) \\ &= \frac{16}{105} \approx 0.1524 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(X \geq 5) &= \int_5^{\infty} 20(x+20)^{-2} dx \\ &= \lim_{b \rightarrow \infty} \int_5^b 20(x+20)^{-2} dx \\ &= \lim_{b \rightarrow \infty} [-20(x+20)^{-1}] \Big|_5^b \\ &= \lim_{b \rightarrow \infty} \left[-20 \left(\frac{1}{b+20} - \frac{1}{25} \right) \right] \\ &= \frac{4}{5} = 0.8 \end{aligned}$$

Alternatively,

$$\begin{aligned} P(X \geq 5) &= 1 - P(0 \leq X \leq 5) \\ &= 1 - (0.0476 + 0.1524) \\ &= 1 - 0.2 = 0.8. \end{aligned}$$

$$33. \quad f(x) = \begin{cases} \frac{x^3}{12} & \text{if } 0 \leq x \leq 2 \\ \frac{16}{3x^3} & \text{if } x > 2 \end{cases}$$

First, note that $f(x) > 0$ for $x > 0$. Next,

$$\begin{aligned} \int_0^\infty f(x) dx &= \int_0^2 \frac{x^3}{12} dx + \lim_{a \rightarrow \infty} \int_2^a \frac{16}{3x^3} dx \\ &= \left(\frac{x^4}{48} \right) \Big|_0^2 + \lim_{a \rightarrow \infty} \left(-\frac{8}{3x^2} \right) \Big|_2^a \\ &= \left(\frac{1}{3} - 0 \right) + \left[\lim_{a \rightarrow \infty} \left(-\frac{8}{3a^2} \right) - \left(-\frac{8}{12} \right) \right] \\ &= \frac{1}{3} + \frac{2}{3} \\ &= 1. \end{aligned}$$

Therefore, $f(x)$ is a probability density function.

$$\begin{aligned} \text{(a)} \quad P(0 \leq X \leq 2) &= \int_0^2 f(x) dx \\ &= \left(\frac{x^4}{48} \right) \Big|_0^2 \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(X \geq 2) &= P(X > 2) \\ &= \int_2^\infty \frac{16}{3x^3} dx \\ &= \lim_{a \rightarrow \infty} \int_2^a \frac{16}{3x^3} dx \\ &= \lim_{a \rightarrow \infty} \left(-\frac{8}{3x^2} \right) \Big|_2^a \\ &= \lim_{a \rightarrow \infty} \left(-\frac{8}{3a^2} \right) - \left(-\frac{8}{3 \cdot 2^2} \right) \\ &= 0 - \left(-\frac{2}{3} \right) \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(1 \leq X \leq 3) &= \int_1^2 \frac{x^3}{12} dx + \int_2^3 \frac{16}{3x^3} dx \end{aligned}$$

$$\begin{aligned} &= \left(\frac{x^4}{48} \right) \Big|_1^2 + \left(-\frac{8}{3x^2} \right) \Big|_2^3 \\ &= \left(\frac{1}{3} - \frac{1}{48} \right) + \left(-\frac{8}{27} + \frac{2}{3} \right) \\ &= \frac{295}{432} \end{aligned}$$

$$34. \quad f(x) = \begin{cases} \frac{20x^4}{9} & \text{if } 0 \leq x \leq 1 \\ \frac{20}{9x^5} & \text{if } x > 1 \end{cases}$$

First, note that $f(x) > 0$ for all $x > 0$. Next,

$$\begin{aligned} \int_0^\infty f(x) dx &= \int_0^1 \frac{20x^4}{9} dx + \lim_{a \rightarrow \infty} \int_1^a \frac{20}{9x^5} dx \\ &= \left(\frac{4x^5}{9} \right) \Big|_0^1 + \lim_{a \rightarrow \infty} \left(-\frac{5}{9x^4} \right) \Big|_1^a \\ &= \left(\frac{4}{9} - 0 \right) + \left[\lim_{a \rightarrow \infty} \left(-\frac{5}{9a^4} \right) - \left(-\frac{5}{9} \right) \right] \\ &= \frac{4}{9} + \frac{5}{9} \\ &= 1. \end{aligned}$$

Therefore, $f(x)$ is a probability density function.

$$\begin{aligned} \text{(a)} \quad P(0 \leq X \leq 1) &= \int_0^1 \frac{20x^4}{9} dx \\ &= \left(\frac{4x^5}{9} \right) \Big|_0^1 \\ &= \frac{4}{9} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(X \geq 1) &= P(X > 1) \\ &= \lim_{a \rightarrow \infty} \int_1^a \frac{20}{9x^5} dx \\ &= \lim_{a \rightarrow \infty} \left(-\frac{5}{9x^4} \right) \Big|_1^a \\ &= \lim_{a \rightarrow \infty} \left(-\frac{5}{9a^4} \right) - \left(-\frac{5}{9} \right) \\ &= \frac{5}{9} \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad P(0 \leq X \leq 2) &= \int_0^1 \frac{20x^4}{9} dx + \int_1^2 \frac{20}{9x^5} dx \\
 &= \left(\frac{4x^5}{9} \right) \Big|_0^1 + \left(-\frac{5}{9x^4} \right) \Big|_1^2 \\
 &= \left(\frac{4}{9} - 0 \right) + \left(-\frac{5}{144} + \frac{5}{9} \right) \\
 &= \frac{139}{144}
 \end{aligned}$$

$$35. \quad f(t) = \frac{1}{2}e^{-t/2}; [0, \infty)$$

$$\begin{aligned}
 \text{(a)} \quad P(0 \leq T \leq 12) &= \frac{1}{2} \int_0^{12} e^{-t/2} dt \\
 &= -e^{-t/2} \Big|_0^{12} \\
 &= \frac{-1}{e^6} + 1 \\
 &\approx 0.9975
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P(12 \leq T \leq 20) &= \frac{1}{2} \int_{12}^{20} e^{-t/2} dt \\
 &= -e^{-t/2} \Big|_{12}^{20} \\
 &= \frac{-1}{e^{10}} + \frac{1}{e^6} \\
 &\approx 0.0024
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad F(t) &= \int_0^t \frac{1}{2} e^{-s/2} ds \\
 &= \frac{1}{2} (-2) e^{-s/2} \Big|_0^t \\
 &= -(e^{-t/2} - 1) \\
 &= 1 - e^{-t/2}, \quad t \geq 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad F(6) &= 1 - e^{-6/2} \\
 &= 1 - e^{-3} \\
 &\approx 0.9502
 \end{aligned}$$

The probability is 0.9502.

$$36. \quad f(t) = \frac{1}{11} \left(1 + \frac{3}{\sqrt{t}} \right); [4, 9]$$

$$\begin{aligned}
 \text{(a)} \quad P(6 \leq T \leq 9) &= \int_6^9 \frac{1}{11} (1 + 3t^{-1/2}) dt \\
 &= \frac{1}{11} (t + 6t^{1/2}) \Big|_6^9 \\
 &= \frac{1}{11} (9 + 18 - 6 - 6\sqrt{6}) \\
 &= \frac{1}{11} (21 - 6\sqrt{6}) \\
 &\approx 0.5730
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P(4 \leq T \leq 5) &= \int_4^5 \frac{1}{11} (1 + 3t^{-1/2}) dt \\
 &= \frac{1}{11} (t + 6t^{1/2}) \Big|_4^5 \\
 &= \frac{1}{11} (5 + 6\sqrt{5} - 4 - 12) \\
 &= \frac{1}{11} (6\sqrt{5} - 11) \\
 &\approx 0.2197
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad P(4 \leq T \leq 7) &= \int_4^7 \frac{1}{11} (1 + 3t^{-1/2}) dt \\
 &= \frac{1}{11} (t + 6t^{1/2}) \Big|_4^7 \\
 &= \frac{1}{11} (7 + 6\sqrt{7} - 4 - 12) \\
 &= \frac{1}{11} (6\sqrt{7} - 9) \approx 0.6250
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad F(t) &= \int_4^t \frac{1}{11} \left(1 + \frac{3}{\sqrt{s}} \right) ds \\
 &= \frac{1}{11} (s + 6\sqrt{s}) \Big|_4^t \\
 &= \frac{1}{11} [(t + 6\sqrt{t}) - (4 + 6\sqrt{4})] \\
 &= \frac{1}{11} (t + 6\sqrt{t} - 16), \quad 4 \leq t \leq 9.
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad F(8) &= \frac{1}{11} (8 + 6\sqrt{8} - 16) \\
 &\approx 0.8155
 \end{aligned}$$

The probability is 0.8155.

37. If $f(x)$ is proportional to $(10 + x)^{-2}$, then, for some value of k , $f(x) = k(10 + x)^{-2}$ on $[0, 40]$. Find k . We know the total probability must equal 1.

$$\begin{aligned}\int_0^{40} k(10 + x)^{-2} dx &= -k(10 + x)^{-1} \Big|_0^{40} \\ &= -k(50^{-1} - 10^{-1})^{-1} \\ &= -k \left(\frac{1}{50} - \frac{1}{10} \right) \\ &= \frac{2}{25}k\end{aligned}$$

If $\frac{2}{25}k = 1$, then $k = \frac{25}{2}$. Therefore

$$f(x) = \frac{25}{2}(10 + x)^{-2}, 0 \leq x \leq 40$$

So the probability distribution function is

$$\begin{aligned}F(x) &= \int_0^x \frac{25}{2}(10 + t)^{-2} dt \\ &= -\frac{25}{2}(10 + t)^{-1} \Big|_0^x \\ &= -\frac{25}{2}[(10 + x)^{-1} - 10^{-1}] \\ &= -\frac{25}{2} \left(\frac{1}{10 + x} - \frac{1}{10} \right) \\ &= \frac{25}{2} \left(\frac{1}{10} - \frac{1}{10 + x} \right) \\ F(6) &= \frac{25}{2} \left(\frac{1}{10} - \frac{1}{106} \right) \\ &\approx 0.47\end{aligned}$$

The correct answer choice is **c**.

38. We are told that the payment is the loss minus the deductible. Therefore,

$$\begin{aligned}P(\text{payment} < 0.5) &= P(X - C < 0.5) \\ &= P(X < C + 0.5) \\ &= \int_0^{C+0.5} 2x dx \\ &= x^2 \Big|_0^{C+0.5} \\ &= (C + 0.5)^2 - 0 \\ &= C^2 + C + 0.25.\end{aligned}$$

We are given that the probability is 0.64.

$$C^2 + C + 0.25 = 0.64$$

$$C^2 + C - 0.39 = 0$$

$$(C + 1.3)(C - 0.3) = 0$$

$$C = -1.3 \text{ or } C = 0.3$$

The solution $C = -1.3$ is extraneous since $0 < C$. Thus, $C = 0.3$. The correct answer choice is **b**.

$$39. f(x) = \frac{1}{2\sqrt{x}}; [1, 4]$$

$$\begin{aligned}\text{(a)} \quad P(3 \leq X \leq 4) &= \int_3^4 \left(\frac{1}{2\sqrt{x}} \right) dx \\ &= \frac{1}{2} \int_3^4 x^{-1/2} dx \\ &= \frac{1}{2} (2)x^{1/2} \Big|_3^4 \\ &= 2 - 3^{1/2} \approx 0.2679\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad P(1 \leq X \leq 2) &= \int_1^2 \left(\frac{1}{2\sqrt{x}} \right) dx \\ &= \frac{1}{2} (2)x^{1/2} \Big|_1^2 \\ &= 2^{1/2} - 1 = 0.4142\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad P(2 \leq X \leq 3) &= \int_2^3 \left(\frac{1}{2\sqrt{x}} \right) dx \\ &= \frac{1}{2} (2)x^{1/2} \Big|_2^3 \\ &= 3^{1/2} - 2^{1/2} = 0.3178\end{aligned}$$

$$40. f(t) = \frac{1}{(\ln 20)t}; [1, 20]$$

$$\begin{aligned}\text{(a)} \quad P(1 \leq T \leq 5) &= \int_1^5 \frac{1}{(\ln 20)t} dt \\ &= \frac{1}{\ln 20} \ln t \Big|_1^5 \\ &= \frac{\ln 5}{\ln 20} \approx 0.5372\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad P(10 \leq T \leq 20) &= \int_{10}^{20} \frac{1}{\ln 20 t} dt \\ &= \frac{1}{\ln 20} \ln t \Big|_{10}^{20} \\ &= 1 - \frac{\ln 10}{\ln 20} \approx 0.2314\end{aligned}$$

41. $f(x) = 1.185 \cdot 10^{-9} x^{4.5222} - 0.049846x$

(a) $P(0 \leq X \leq 150)$

$$= \int_0^{150} 1.185 \cdot 10^{-9} x^{4.5222} e^{-0.049846x} dx$$

$$\approx 0.8131$$

(b) $P(100 \leq x \leq 200)$

$$= \int_{100}^{200} 1.185 \cdot 10^{-9} x^{4.5222} e^{-0.049846x} dx$$

$$\approx 0.4901$$

42. $p(x, t) = \frac{e^{-x^2/(4Dt)}}{\int_0^L e^{-u^2/(4Dt)} du}$

(a) Let $t = 12$, $L = 6$, and $D = 38.3$.

$$p(x, 12) = \frac{e^{-x^2/1838.4}}{\int_0^6 e^{-u^2/1838.4} du}$$

Evaluate this (and other) integrals using the integration feature on a graphing calculator.

$$p(x, 12) \approx \frac{1}{5.9611} e^{-x^2/1838.4}$$

$$P(0 \leq X \leq 3)$$

$$= \int_0^3 p(x, 12) dx$$

$$\approx \int_0^3 \frac{1}{5.9611} e^{-x^2/1838.4} dx$$

$$\approx 0.5024$$

The probability that a flea beetle will be recaptured within 3 meters of the release point is about 0.50.

(b) $P(1 \leq X \leq 5)$

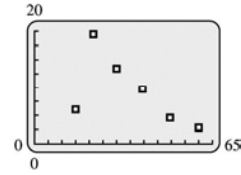
$$= \int_1^5 p(x, 12) dx$$

$$\approx \int_1^5 \frac{1}{5.9611} e^{-x^2/1838.4} dx$$

$$\approx 0.6673$$

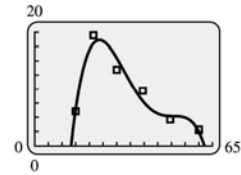
The probability that a flea beetle will be recaptured between 1 and 5 meters of the release point is about 0.6673.

43. (a)



A polynomial function could fit the data.

(b) $N(t) = -0.00007445t^4 + 0.01243t^3$
 $- 0.7419t^2 + 18.18t - 137.5$



(c) $\int_{13.4}^{62.0} N(t) dt \approx 466.26$, as found using a calculator. Thus the density function corresponding to the quartic fit is

$$S(t) = \frac{1}{466.26} N(t)$$

$$= \frac{1}{466.26} (-0.00007445t^4$$

$$+ 0.01243t^3 - 0.7419t^2$$

$$+ 18.18t - 137.5).$$

(d) Use a calculator to compute the integrals needed in (d).

$$P(35 \leq \text{age} < 45) = \int_{35}^{45} S(t) dt$$

$$\approx 0.1688;$$

actual relative frequency

$$= \frac{9.701}{56.065} \approx 0.1730$$

$$P(18 \leq \text{age} < 35) = \int_{18}^{35} S(t) dt$$

$$\approx 0.5896;$$

actual relative frequency

$$= \frac{19.461 + 13.423}{56.065} \approx 0.5865$$

$$P(45 \leq \text{age}) = \int_{45}^{62} S(t) dt$$

$$\approx 0.1610;$$

actual relative frequency

$$= \frac{4.582 + 2.849}{56.065} \approx 0.1325$$

44. $f(t) = \frac{8}{7(t-2)^2}; [3, 10]$

$$\begin{aligned} \text{(a)} \quad P(3 \leq T \leq 4) &= \frac{8}{7} \int_3^4 (t-2)^{-2} dt \\ &= -\frac{8}{7} (t-2)^{-1} \Big|_3^4 \\ &= -\frac{8}{7} \left(\frac{1}{2} - 1 \right) \\ &= -\frac{8}{7} \left(-\frac{1}{2} \right) \\ &= \frac{4}{7} \approx 0.5714 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(5 \leq T \leq 10) &= \frac{8}{7} \int_5^{10} (t-2)^{-2} dt \\ &= -\frac{8}{7} (t-2)^{-1} \Big|_5^{10} \\ &= \frac{8}{7} \left(\frac{1}{8} - \frac{1}{3} \right) \\ &= -\frac{8}{7} \left(-\frac{5}{24} \right) \\ &= \frac{5}{21} \approx 0.2381 \end{aligned}$$

45. $f(x) = \frac{5.5-x}{15}; [0, 5]$

$$\begin{aligned} \text{(a)} \quad P(3 \leq X \leq 5) &= \int_3^5 \frac{5.5-x}{15} dx \\ &= \left(\frac{5.5}{15}x - \frac{1}{15} \cdot \frac{x^2}{2} \right) \Big|_3^5 \\ &= \left(\frac{5.5}{15} \cdot 5 - \frac{1}{15} \cdot \frac{5^2}{2} \right) \\ &\quad - \left(\frac{5.5}{15} \cdot 3 - \frac{1}{15} \cdot \frac{3^2}{2} \right) \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(0 \leq X \leq 2) &= \int_0^2 \frac{5.5-x}{15} dx \\ &= \left(\frac{5.5}{15}x - \frac{1}{15} \cdot \frac{x^2}{2} \right) \Big|_0^2 \\ &= -\left(\frac{5.5}{15} \cdot 2 - \frac{1}{15} \cdot \frac{2^2}{2} \right) \\ &\quad - \left(\frac{5.5}{15} \cdot 0 - \frac{1}{15} \cdot \frac{0^2}{2} \right) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(1 \leq X \leq 4) &= \int_1^4 \frac{5.5-x}{15} dx \\ &= \left(\frac{5.5}{15}x - \frac{1}{15} \cdot \frac{x^2}{2} \right) \Big|_1^4 \\ &= \left(\frac{5.5}{15} \cdot 4 - \frac{1}{15} \cdot \frac{4^2}{2} \right) \\ &\quad - \left(\frac{5.5}{15} \cdot 1 - \frac{1}{15} \cdot \frac{1^2}{2} \right) \\ &= 0.6 \end{aligned}$$

46. $f(t) = \frac{1}{960}e^{-t/960}$

$$\begin{aligned} \text{(a)} \quad P(T < 365) &= \int_0^{365} \frac{1}{960} e^{-t/960} dt \\ &= -e^{-t/960} \Big|_0^{365} \\ &= -e^{-365/960} + 1 \\ &\approx 0.3163 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(T > 960) &= \int_{960}^{\infty} \frac{1}{960} e^{-t/960} dt \\ &= \lim_{b \rightarrow \infty} \int_{960}^b e^{-t/960} dt \\ &= \lim_{b \rightarrow \infty} (-e^{-t/960}) \Big|_{960}^b \\ &= \lim_{b \rightarrow \infty} (-e^{-b/960} + e^{-1}) \\ &= 0 + e^{-1} \\ &\approx 0.3679 \end{aligned}$$

$$47. \quad f(t) = \frac{1}{3650.1} e^{-t/3650.1}$$

$$\begin{aligned} \text{(a)} \quad P(365 < T < 1095) &= \int_{365}^{1095} \frac{1}{3650.1} e^{-t/3650.1} dt \\ &= \left(-e^{-t/3650.1} \right) \Big|_{365}^{1095} \\ &= -e^{-1095/3650.1} + e^{-365/3650.1} \\ &\approx 0.1640 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(T > 7300) &= \int_{7300}^{\infty} \frac{1}{3650.1} e^{-t/3650.1} dt \\ &= \lim_{b \rightarrow \infty} \int_{7300}^b \frac{1}{3650.1} e^{-t/3650.1} dt \\ &= \lim_{b \rightarrow \infty} \left(-e^{-t/3650.1} \right) \Big|_{7300}^b \\ &= \lim_{b \rightarrow \infty} \left(-e^{-b/3650.1} + e^{-7300/3650.1} \right) \\ &= 0 + e^{-7300/3650.1} \approx 0.1353 \end{aligned}$$

$$48. \quad f(t) = \frac{4.045}{t^{1.532}}$$

$$\begin{aligned} \text{(a)} \quad P(16 \leq T \leq 25) &= \int_{16}^{25} f(t) dt \\ &= -\frac{4.045}{0.532} t^{-0.532} \Big|_{16}^{25} \\ &\approx -7.6034(25^{-0.532} - 16^{-0.532}) \\ &\approx 0.3676 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(35 \leq T \leq 80) &= \int_{35}^{80} f(t) dt \\ &= -\frac{4.045}{0.532} t^{-0.532} \Big|_{35}^{80} \\ &\approx -7.6034(80^{-0.532} - 35^{-0.532}) \\ &\approx 0.4081 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(21 \leq T \leq 30) &= \int_{21}^{30} f(t) dt \\ &= -\frac{4.045}{0.532} t^{-0.532} \Big|_{21}^{30} \\ &\approx -7.6034(30^{-0.532} - 21^{-0.532}) \\ &\approx 0.2601 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad F(t) &= \int_{16}^t f(s) ds \\ &= -\frac{4.045}{0.532} s^{-0.532} \Big|_{16}^t \\ &= -\frac{4.045}{0.532} (t^{-0.532} - 16^{-0.532}) \\ &= 1.7395 - 7.6034t^{-0.532}, \\ &\quad 16 \leq t \leq 80 \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad F(21) &= 1.7395 - (7.6034)(21^{-0.532}) \\ &\approx 0.2343 \end{aligned}$$

$$49. \quad f(t) = 0.06049e^{-0.03211t}; [16, 84]$$

$$\begin{aligned} \text{(a)} \quad P(16 \leq T \leq 25) &= \int_{16}^{25} f(t) dt \\ &= \int_{16}^{25} 0.06049e^{-0.03211t} dt \\ &= \frac{0.06049}{-0.03211} (e^{-0.03211t}) \Big|_{16}^{25} \\ &\approx -1.88384(e^{-0.03211 \cdot 25} - e^{-0.03211 \cdot 16}) \\ &\approx 0.2829 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(35 \leq T \leq 84) &= \int_{35}^{84} 0.06049e^{-0.03211t} dt \\ &\approx -1.88384(e^{-0.03211 \cdot 84} - e^{-0.03211 \cdot 35}) \\ &\approx 0.4853 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(21 \leq T \leq 30) &= \int_{21}^{30} 0.06049e^{-0.03211t} dt \\ &\approx -1.88384(e^{-0.03211 \cdot 30} - e^{-0.03211 \cdot 21}) \\ &\approx 0.2409 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad F(t) &= \int_{16}^t 0.06049e^{-0.03211s} ds \\ &= 0.06049 \cdot \frac{1}{-0.03211} e^{-0.03211s} \Big|_{16}^t \\ &= -1.8838(e^{-0.03211t} - e^{-0.03211 \cdot 16}) \\ &= 1.8838(0.5982 - e^{-0.03211t}), \\ &\quad 16 \leq t \leq 84 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad F(21) &= 1.8838(0.5982 - e^{-0.03211 \cdot 21}) \\
 &= 1.8838(0.0887) \\
 &= 0.1671
 \end{aligned}$$

The probability is 0.1671.

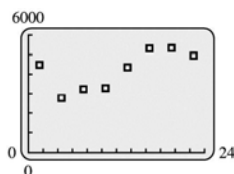
50. $f(t) = 3t^{-4}; [1, \infty)$

$$\begin{aligned}
 \text{(a)} \quad P(1 \leq T \leq 2) &= \int_1^2 3t^{-4} dt \\
 &= -t^{-3} \Big|_1^2 \\
 &= 1 - \frac{1}{8} \\
 &= \frac{7}{8} = 0.875
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P(3 \leq T \leq 5) &= \int_3^5 3t^{-4} dt \\
 &= -t^{-3} \Big|_3^5 \\
 &= \frac{1}{27} - \frac{1}{125} \approx 0.0290
 \end{aligned}$$

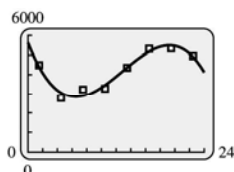
$$\begin{aligned}
 \text{(c)} \quad P(T \geq 3) &= \int_3^\infty 3t^{-4} dt \\
 &= \lim_{b \rightarrow \infty} \int_3^b 3t^{-4} dt \\
 &= \lim_{b \rightarrow \infty} (-t^{-3}) \Big|_3^b \\
 &= \lim_{b \rightarrow \infty} \left(\frac{1}{27} - \frac{1}{b^3} \right) \\
 &= \frac{1}{27} \approx 0.0370
 \end{aligned}$$

51. (a)



A polynomial function could fit the data.

$$\begin{aligned}
 \text{(b)} \quad T(t) &= -2.564t^3 + 99.11t^2 \\
 &\quad - 964.6t + 5631
 \end{aligned}$$



$$\begin{aligned}
 \text{(c)} \quad \int_0^{24} T(t) dt &\approx 101,370, \text{ as found using a} \\
 &\text{calculator. Thus the density function} \\
 &\text{corresponding to the cubic fit is}
 \end{aligned}$$

$$\begin{aligned}
 S(t) &= \frac{1}{101,370} T(t) \\
 &= \frac{1}{101,370} (-2.564t^3 + 99.11t^2 \\
 &\quad - 964.6t + 5631).
 \end{aligned}$$

(d) Use a calculator to compute the integrals needed in (d).

$$\begin{aligned}
 P(12 \text{ am} \leq \text{time} \leq 2 \text{ am}) \\
 &= \int_0^2 S(t) dt \\
 &\approx 0.09457
 \end{aligned}$$

$$\begin{aligned}
 P(4 \text{ pm} \leq \text{time} \leq 5:30 \text{ pm}) \\
 &= \int_{16}^{17.5} S(t) dt \\
 &\approx 0.07732
 \end{aligned}$$

11.2 Expected Value and Variance of Continuous Random Variables

Your Turn 1

Find the expected value and the variance of the random variable X with probability density function

$$f(x) = \frac{8}{3x^3} \text{ on } [1, 2].$$

$$\begin{aligned}
 \mu &= \int_1^2 x f(x) dx \\
 &= \int_1^2 x \frac{8}{3x^3} dx \\
 &= \int_1^2 \frac{8}{3} x^{-2} dx \\
 &= -\frac{8}{3} x^{-1} \Big|_1^2 \\
 &= -\frac{8}{6} - \left(-\frac{8}{3} \right) \\
 &= \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
\text{Var}(x) &= \int_1^2 \left(x - \frac{4}{3}\right)^2 \frac{8}{3x^3} dx \\
&= \int_1^2 \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) \frac{8}{3x^3} dx \\
&= \int_1^2 \left(\frac{8}{3}x^{-1} - \frac{64}{9}x^{-2} + \frac{128}{27}x^{-3}\right) dx \\
&= \left(\frac{8}{3}\ln(x) + \frac{64}{9x} - \frac{64}{27x^2}\right) \Big|_1^2 \\
&= \left(\frac{8}{3}\ln 2 + \frac{32}{9} - \frac{16}{27}\right) - \left(\frac{64}{9} - \frac{64}{27}\right) \\
&= \frac{8}{3}\ln 2 - \frac{16}{9} \approx 0.0706
\end{aligned}$$

Your Turn 2

Use the alternative formula to find the variance of the random variable X with probability density function

$f(x) = \frac{4}{x^5}$ for $x \geq 1$. First find the mean μ .

$$\begin{aligned}
\mu &= \int_1^\infty x \left(\frac{4}{x^5}\right) dx \\
&= \int_1^\infty 4x^{-4} dx \\
&= \lim_{b \rightarrow \infty} \int_1^b 4x^{-4} dx \\
&= \lim_{b \rightarrow \infty} \left(-\frac{4}{3}x^{-3} \Big|_1^b\right) \\
&= \lim_{b \rightarrow \infty} \left(-\frac{4}{3b^3} - \left(-\frac{4}{3}\right)\right) \\
&= \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= \int_1^\infty x^2 f(x) dx - \mu^2 \\
&= \int_1^\infty x^2 \frac{4}{x^5} dx - \left(\frac{4}{3}\right)^2 \\
&= 2 \int_1^\infty 2x^{-3} dx - \frac{16}{9} \\
&= 2 \lim_{b \rightarrow \infty} \left(-x^{-2} \Big|_1^b\right) - \frac{16}{9} \\
&= 2 \left(1 - \lim_{b \rightarrow \infty} b^{-2}\right) - \frac{16}{9} \\
&= 2 - \frac{16}{9} = \frac{2}{9}
\end{aligned}$$

Your Turn 3

Find the median m for the probability density function

$f(x) = \frac{4}{x^5}$ for $x \geq 1$.

$$\begin{aligned}
\int_1^m \frac{4}{x^5} dx &= \frac{1}{2} \\
\left(-\frac{1}{x^4}\right) \Big|_1^m &= \frac{1}{2} \\
-\frac{1}{m^4} - (-1) &= \frac{1}{2} \\
m^4 &= 2 \\
m &= \sqrt[4]{2} \approx 1.1892
\end{aligned}$$

11.2 Exercises

1. $f(x) = \frac{1}{4}; [3, 7]$

$$\begin{aligned}
E(X) = \mu &= \int_3^7 \frac{1}{4}x dx = \frac{1}{4} \left(\frac{x^2}{2}\right) \Big|_3^7 \\
&= \frac{49}{8} - \frac{9}{8} \\
&= 5
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= \int_3^7 (x-5)^2 \left(\frac{1}{4}\right) dx \\
&= \frac{1}{4} \cdot \frac{(x-5)^3}{3} \Big|_3^7 \\
&= \frac{8}{12} + \frac{8}{12} \\
&= \frac{4}{3} \approx 1.33 \\
\sigma &\approx \sqrt{\text{Var}(X)} \\
&= \sqrt{\frac{4}{3}} \\
&\approx 1.15
\end{aligned}$$

2. $f(x) = \frac{1}{10}; [0, 10]$

$$\begin{aligned}
E(X) = \mu &= \int_0^{10} x \left(\frac{1}{10}\right) dx \\
&= \frac{x^2}{20} \Big|_0^{10} = 5
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= \int_0^{10} (x-5)^2 \left(\frac{1}{10}\right) dx \\
&= \frac{1}{10} \cdot \frac{(x-5)^3}{3} \Big|_0^{10} \\
&= \frac{25}{6} + \frac{25}{6} \\
&= \frac{25}{3} \approx 8.33 \\
\sigma &\approx \sqrt{\text{Var}(X)} \approx 2.89
\end{aligned}$$

3. $f(x) = \frac{x}{8} - \frac{1}{4}; [2, 6]$

$$\begin{aligned}
\mu &= \int_2^6 x \left(\frac{x}{8} - \frac{1}{4}\right) dx \\
&= \int_2^6 \left(\frac{x^2}{8} - \frac{x}{4}\right) dx \\
&= \left(\frac{x^3}{24} - \frac{x^2}{8}\right) \Big|_2^6 \\
&= \left(\frac{216}{24} - \frac{36}{8}\right) - \left(\frac{8}{24} - \frac{4}{8}\right) \\
&= \frac{208}{24} - 4 \\
&= \frac{26}{3} - 4 \\
&= \frac{14}{3} \approx 4.67
\end{aligned}$$

Use the alternative formula to find

$$\begin{aligned}
\text{Var}(X) &= \int_2^6 x^2 \left(\frac{x}{8} - \frac{1}{4}\right) dx - \left(\frac{14}{3}\right)^2 \\
&= \int_2^6 \left(\frac{x^3}{8} - \frac{x^2}{4}\right) dx - \frac{196}{9} \\
&= \left(\frac{x^4}{32} - \frac{x^3}{12}\right) \Big|_2^6 - \frac{196}{9} \\
&= \left(\frac{1296}{32} - \frac{216}{12}\right) \\
&\quad - \left(\frac{16}{32} - \frac{8}{12}\right) - \frac{196}{9} \\
&\approx 0.89. \\
\sigma &= \sqrt{\text{Var}(X)} \approx \sqrt{0.89} \approx 0.94
\end{aligned}$$

4. $f(x) = 2(1-x); [0, 1]$

$$\begin{aligned}
\mu &= \int_0^1 2x(1-x) dx = \int_0^1 (2x - 2x^2) dx \\
&= \left(x^2 - \frac{2x^3}{3}\right) \Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3} \approx 0.33
\end{aligned}$$

Use the alternative formula to find

$$\begin{aligned}
\text{Var}(X) &= \int_0^1 2x^2(1-x) dx - \left(\frac{1}{3}\right)^2 \\
&= \int_0^1 (2x^2 - 2x^3) dx - \frac{1}{9} \\
&= \left(\frac{2x^3}{3} - \frac{x^4}{2}\right) \Big|_0^1 - \frac{1}{9} \\
&= \frac{2}{3} - \frac{1}{2} - \frac{1}{9} = \frac{1}{18} \approx 0.06. \\
\sigma &= \sqrt{\text{Var}(X)} \approx 0.24
\end{aligned}$$

5. $f(x) = 1 - \frac{1}{\sqrt{x}}; [1, 4]$

$$\begin{aligned}
\mu &= \int_1^4 x(1 - x^{-1/2}) dx \\
&= \int_1^4 (x - x^{1/2}) dx \\
&= \left(\frac{x^2}{2} - \frac{2x^{3/2}}{3}\right) \Big|_1^4 \\
&= \frac{16}{2} - \frac{16}{3} - \frac{1}{2} + \frac{2}{3} \\
&= \frac{17}{6} \approx 2.83
\end{aligned}$$

$$\begin{aligned}
\text{Var}(X) &= \int_1^4 x^2(1 - x^{-1/2}) dx - \left(\frac{17}{6}\right)^2 \\
&= \int_1^4 (x^2 - x^{3/2}) dx - \frac{289}{36} \\
&= \left(\frac{x^3}{3} - \frac{2x^{5/2}}{5}\right) \Big|_1^4 - \frac{289}{36} \\
&= \frac{64}{3} - \frac{64}{5} - \frac{1}{3} + \frac{2}{5} - \frac{289}{36} \\
&\approx 0.57 \\
\sigma &\approx \sqrt{\text{Var}(X)} \approx 0.76
\end{aligned}$$

$$6. \quad f(x) = \frac{1}{11} \left(1 + \frac{3}{\sqrt{x}} \right); [4, 9]$$

$$\begin{aligned} \mu &= \int_4^9 \frac{x}{11} \left(1 + \frac{3}{\sqrt{x}} \right) dx \\ &= \int_4^9 \frac{1}{11} (x + 3x^{1/2}) dx \\ &= \frac{1}{11} \left(\frac{x^2}{2} + 2x^{3/2} \right) \Big|_4^9 \\ &= \frac{1}{11} \left(\frac{81}{2} + 54 - 8 - 16 \right) \\ &= \frac{141}{22} \approx 6.41 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= \int_4^9 \frac{x^2}{11} \left(1 + \frac{3}{\sqrt{x}} \right) dx - \mu^2 \\ &= \int_4^9 \frac{1}{11} (x^2 + 3x^{3/2}) dx - \mu^2 \\ &= \frac{1}{11} \left(\frac{x^3}{3} + \frac{6}{5} x^{5/2} \right) \Big|_4^9 - \mu^2 \\ &= \frac{1}{11} \left(243 + \frac{1458}{5} - \frac{64}{3} - \frac{192}{5} \right) \\ &\quad - \left(\frac{141}{22} \right)^2 \\ &\approx 2.09 \\ \sigma &\approx \sqrt{\text{Var}(X)} \approx 1.45 \end{aligned}$$

$$7. \quad f(x) = 4x^{-5}; [1, \infty)$$

$$\begin{aligned} \mu &= \int_1^\infty x(4x^{-5}) dx \\ &= \lim_{a \rightarrow \infty} \int_1^a 4x^{-4} dx \\ &= \lim_{a \rightarrow \infty} \left(\frac{4x^{-3}}{-3} \right) \Big|_1^a \\ &= \lim_{a \rightarrow \infty} \left(\frac{-4}{3a^3} + \frac{4}{3} \right) \\ &= \frac{4}{3} \approx 1.33 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= \int_1^\infty x^2(4x^{-5}) dx - \left(\frac{4}{3} \right)^2 \\ &= \lim_{a \rightarrow \infty} \int_1^a 4x^{-3} dx - \frac{16}{9} \\ &= \lim_{a \rightarrow \infty} \left(\frac{4x^{-2}}{-2} \right) \Big|_1^a - \frac{16}{9} \\ &= \lim_{a \rightarrow \infty} \left(\frac{-2}{a^2} + 2 \right) - \frac{16}{9} \\ &= 2 - \frac{16}{9} = \frac{2}{9} \approx 0.22 \\ \sigma &= \sqrt{\text{Var}(X)} = \sqrt{\frac{2}{9}} \approx 0.47 \end{aligned}$$

$$8. \quad f(x) = 3x^{-4}; [1, \infty)$$

$$\begin{aligned} \mu &= \int_1^\infty x(3x^{-4}) dx \\ &= \lim_{b \rightarrow \infty} \int_1^b 3x^{-3} dx \\ &= \lim_{b \rightarrow \infty} \left(-\frac{3x^{-2}}{2} \right) \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \left(\frac{3}{2} - \frac{3}{2b^2} \right) = \frac{3}{2} = 1.5 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= \int_1^\infty x^2(3x^{-4}) dx - \left(\frac{3}{2} \right)^2 \\ &= \lim_{b \rightarrow \infty} \int_1^b 3x^{-2} dx - \frac{9}{4} \\ &= \lim_{b \rightarrow \infty} (-3x^{-1}) \Big|_1^b - \frac{9}{4} \\ &= \lim_{b \rightarrow \infty} \left(3 - \frac{3}{b} \right) - \frac{9}{4} \\ &= 3 - \frac{9}{4} = \frac{3}{4} = 0.75 \\ \sigma &= \sqrt{\text{Var}(X)} \approx 0.87 \end{aligned}$$

11. $f(x) = \frac{\sqrt{x}}{18}; [0, 9]$

$$\begin{aligned} \text{(a)} \quad E(X) = \mu &= \int_0^9 \frac{x\sqrt{x}}{18} dx \\ &= \int_0^9 \frac{x^{3/2}}{18} dx \\ &= \frac{2x^{5/2}}{90} \Big|_0^9 = \frac{x^{5/2}}{45} \Big|_0^9 \\ &= \frac{243}{45} = \frac{27}{5} = 5.40 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{Var}(X) &= \int_0^9 \frac{x^2\sqrt{x}}{18} dx - \left(\frac{27}{5}\right)^2 \\ &= \int_0^9 \frac{x^{5/2}}{18} dx - \left(\frac{27}{5}\right)^2 \\ &= \frac{x^{7/2}}{63} \Big|_0^9 - \left(\frac{27}{5}\right)^2 \\ &= \frac{2187}{63} - \left(\frac{27}{5}\right)^2 \approx 5.55 \end{aligned}$$

(c) $\sigma = \sqrt{\text{Var}(X)} \approx 2.36$

(d) $P(5.40 < X \leq 9)$

$$\begin{aligned} &= \int_{5.4}^9 \frac{x^{1/2}}{18} dx \\ &= \frac{x^{3/2}}{27} \Big|_{5.4}^9 \\ &= \frac{27}{27} - \frac{(5.4)^{1.5}}{27} \\ &\approx 0.5352 \end{aligned}$$

(e) $P(5.40 - 2.36 \leq X \leq 5.40 + 2.36)$

$$\begin{aligned} &= \int_{3.04}^{7.76} \frac{x^{1/2}}{18} dx \\ &= \frac{x^{3/2}}{27} \Big|_{3.04}^{7.76} \\ &= \frac{7.76^{3/2}}{27} - \frac{3.04^{3/2}}{27} \\ &\approx 0.6043 \end{aligned}$$

12. $f(x) = \frac{x^{-1/3}}{6}; [0, 8]$

$$\begin{aligned} \text{(a)} \quad \mu &= \int_0^8 x \left(\frac{x^{-1/3}}{6} \right) dx \\ &= \int_0^8 \frac{x^{2/3}}{6} dx \\ &= \frac{x^{5/3}}{10} \Big|_0^8 \\ &= \frac{16}{5} = 3.2 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{Var}(X) &= \int_0^8 x^2 \left(\frac{x^{-1/3}}{6} \right) dx - \left(\frac{16}{5}\right)^2 \\ &= \int_0^8 \frac{x^{5/3}}{6} dx - \frac{256}{25} \\ &= \frac{x^{8/3}}{16} \Big|_0^8 - \frac{256}{25} \\ &= 16 - \frac{256}{25} = \frac{144}{25} = 5.76 \end{aligned}$$

(c) $\sigma = \sqrt{\text{Var}(X)} = 2.4$

(d) $P(X > \mu) = \int_{\mu}^8 \frac{x^{-1/3}}{6} dx$

$$\begin{aligned} &= \frac{x^{2/3}}{4} \Big|_{3.2}^8 \\ &= 1 - \frac{(3.2)^{2/3}}{4} \approx 0.4571 \end{aligned}$$

(e) $P(3.2 - 2.4 < X < 3.2 + 2.4)$
 $= P(0.8 < X < 5.6)$

$$\begin{aligned} &= \int_{0.8}^{5.6} \frac{x^{-1/3}}{6} dx \\ &= \frac{x^{2/3}}{4} \Big|_{0.8}^{5.6} \\ &= \frac{1}{4} [(5.6)^{2/3} - (0.8)^{2/3}] \approx 0.5729 \end{aligned}$$

13. $f(x) = \frac{1}{4}x^3; [0, 2]$

(a) $E(X) = \mu = \int_0^2 \frac{1}{4}x^4 dx = \frac{x^5}{20} \Big|_0^2$

$$= \frac{32}{20} = \frac{8}{5} = 1.6$$

(b) $\text{Var}(X) = \int_0^2 \frac{1}{4}x^5 dx - \frac{64}{25}$

$$= \frac{x^6}{24} \Big|_0^2 - \frac{64}{25}$$

$$= \frac{8}{3} - \frac{64}{25}$$

$$= \frac{8}{75} \approx 0.11$$

(c) $\sigma = \sqrt{\text{Var}(X)}$

$$= \sqrt{\frac{8}{75}}$$

$$\approx 0.3266$$

$$\approx 0.33$$

(d) $P(8/5 < X \leq 2) = \int_{8/5}^2 \frac{x^3}{4} dx$

$$= \frac{x^4}{16} \Big|_{8/5}^2$$

$$= 1 - \frac{256}{625}$$

$$= \frac{369}{625} \approx 0.5904$$

(e) Use a four-place value for the standard deviation.

$$P(1.6 - 0.3266 \leq X \leq 1.6 + 0.3266)$$

$$= \int_{1.2734}^{1.9266} \frac{x^3}{4} dx = \frac{x^4}{16} \Big|_{1.2734}^{1.9266}$$

$$= \frac{1.9266^4}{16} - \frac{0.1.2734^4}{16}$$

$$\approx 0.6967$$

14. $f(x) = \frac{3}{16}(4 - x^2); [0, 2]$

(a) $\mu = \int_0^2 \frac{3}{16}x(4 - x^2)dx$

$$= \int_0^2 \frac{3}{16}(4x - x^3)dx$$

$$= \left(\frac{3}{8}x^2 - \frac{3}{64}x^4 \right) \Big|_0^2$$

$$= \frac{3}{4} = 0.75$$

(b) $\text{Var}(X) = \int_0^2 \frac{3}{16}x^2(4 - x^2)dx - \left(\frac{3}{4} \right)^2$

$$= \int_0^2 \frac{3}{16}(4x^2 - x^4)dx - \frac{9}{16}$$

$$= \frac{3}{16} \left(\frac{4x^3}{3} - \frac{x^5}{5} \right) \Big|_0^2 - \frac{9}{16}$$

$$= \frac{3}{16} \left(\frac{32}{3} - \frac{32}{5} \right) - \frac{9}{16}$$

$$= \frac{4}{5} - \frac{9}{16} = \frac{19}{80}$$

$$\approx 0.24$$

(c) $\sigma = \sqrt{\text{Var}(X)} \approx 0.4873$

$$\approx 0.49$$

(d) $P(X > \mu) = \int_{\mu}^2 \frac{3}{16}(4 - x^2)dx$

$$= \frac{3}{16} \left(4x - \frac{x^3}{3} \right) \Big|_{3/4}^2$$

$$= \frac{3}{16} \left(8 - \frac{8}{3} - 3 + \frac{9}{64} \right)$$

$$= \frac{475}{1024}$$

$$\approx 0.4639$$

(e) Use a four-place value for the standard deviation.

$$P(0.75 - 0.4873 < X < 0.75 + 0.4873)$$

$$= P(0.2627 < X < 1.2373)$$

$$= \int_{0.2627}^{1.2373} \frac{3}{16}(4 - x^2)dx$$

$$= \frac{3}{16} \left(4x - \frac{x^3}{3} \right) \Big|_{0.2627}^{1.2373}$$

$$\begin{aligned}
&= \frac{3}{16} \left[(4)(1.2373) - \frac{(1.2373)^3}{3} \right. \\
&\quad \left. - (4)(0.2627) + \frac{(0.2627)^3}{3} \right] \\
&\approx 0.6137
\end{aligned}$$

15. $f(x) = \frac{1}{4}; [3, 7]$

(a) $m = \text{median: } \int_3^m \frac{1}{4} dx = \frac{1}{2}$

$$\begin{aligned}
\frac{1}{4}x \Big|_3^m &= \frac{1}{2} \\
\frac{m}{4} - \frac{3}{4} &= \frac{1}{2} \\
m - 3 &= 2 \\
m &= 5
\end{aligned}$$

(b) $E(X) = \mu = 5$ (from Exercise 1)

$$P(X = 5) = \int_5^5 \frac{1}{4} dx = 0$$

16. $f(x) = \frac{1}{10}; [0, 10]$

(a) $\int_0^m \frac{1}{10} dx = \frac{x}{10} \Big|_0^m = \frac{m}{10}$

$$= \frac{1}{2} \text{ when } m = 5.$$

(b) $E(X) = \mu = 5$ (from Exercise 2)

$$P(5 \leq X \leq 5) = P(X = 5) = 0$$

17. $f(x) = \frac{x}{8} - \frac{1}{4}; [2, 6]$

(a) $m = \text{median:}$

$$\begin{aligned}
\int_2^m \left(\frac{x}{8} - \frac{1}{4} \right) dx &= \frac{1}{2} \\
\left(\frac{x^2}{16} - \frac{x}{4} \right) \Big|_2^m &= \frac{1}{2} \\
\frac{m^2}{16} - \frac{m}{4} - \frac{1}{4} + \frac{1}{2} &= \frac{1}{2} \\
m^2 - 4m - 4 + 8 &= 8 \\
m^2 - 4m - 4 &= 0 \\
m &= \frac{4 \pm \sqrt{16 + 16(1)}}{2}
\end{aligned}$$

Reject $\frac{4 - \sqrt{32}}{2}$ since it is not in $[2, 6]$.

$$\begin{aligned}
m &= \frac{4 + \sqrt{32}}{2} \\
&= 2 + 2\sqrt{2} \approx 4.8284
\end{aligned}$$

(b) $E(X) = \mu = \frac{14}{3}$ (from Exercise 3)

$$\begin{aligned}
P\left(\frac{14}{3} \leq X \leq 2 + 2\sqrt{2}\right) &= \int_{14/3}^{2+2\sqrt{2}} \left(\frac{x}{8} - \frac{1}{4} \right) dx \\
&= \left(\frac{x^2}{16} - \frac{x}{4} \right) \Big|_{14/3}^{2+2\sqrt{2}} \\
&= \frac{(2 + 2\sqrt{2})^2}{16} - \frac{2 + 2\sqrt{2}}{4} \\
&\quad - \frac{(14/3)^2}{16} + \frac{14/3}{4} \\
&= \frac{1}{18} \approx 0.0556
\end{aligned}$$

If you do the integration on a calculator using rounded values for the limits you may get a slightly different answer, such as 0.0553.

18. $f(x) = 2(1 - x); [0, 1]$

(a) $\int_0^m 2(1 - x) dx = (2x - x^2) \Big|_0^m$

$$\begin{aligned}
&= 2m - m^2 \\
&= \frac{1}{2} \\
2m^2 - 4m + 1 &= 0 \\
m &= \frac{2 - \sqrt{2}}{2} \\
&\approx 0.2929
\end{aligned}$$

(Reject $m = \frac{2 + \sqrt{2}}{2}$, which is not in the interval $[0, 1]$.)

(b) $E(X) = \mu = \frac{1}{3}$ (from Exercise 4)

$$\begin{aligned} P\left(\frac{2-\sqrt{2}}{2} < X < \frac{1}{3}\right) &= \int_{1/3}^{(2-\sqrt{2})/2} 2(1-x)dx \\ &= (2x - x^2)\Big|_{1/3}^{(2-\sqrt{2})/2} \\ &= 2\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)^2 - 2\left(\frac{2-\sqrt{2}}{2}\right) + \left(\frac{2-\sqrt{2}}{2}\right)^2 \\ &= \frac{1}{18} \approx 0.0556 \end{aligned}$$

If you do the integration on a calculator using rounded values for the limits you may get a slightly different answer, such as 0.0553.

19. $f(x) = 4x^{-5}; [1, \infty)$

(a) $m = \text{median:}$

$$\begin{aligned} \int_1^m 4x^{-5} dx &= \frac{1}{2} \\ \frac{4x^{-4}}{-4} \Big|_1^m &= \frac{1}{2} \\ -m^{-4} + 1 &= \frac{1}{2} \\ 1 - \frac{1}{m^4} &= \frac{1}{2} \\ 2m^4 - 2 &= m^4 \\ m^4 &= 2 \\ m &= \sqrt[4]{2} \approx 1.189 \end{aligned}$$

(b) $E(X) = \mu = \frac{4}{3}$ (from Exercise 7)

$$\begin{aligned} P\left(1.19 \leq X \leq \frac{4}{3}\right) &\approx \int_{1.189}^{1.333} 4x^{-5} dx \\ &\approx -x^{-4} \Big|_{1.189}^{1.333} \\ &\approx -\frac{1}{1.333^4} + \frac{1}{1.189^4} \\ &\approx 0.1836 \end{aligned}$$

20. $f(x) = 3x^{-4}; [1, \infty)$

(a) $\int_1^m 3x^{-4} dx = -x^{-3} \Big|_1^m$

$$\begin{aligned} &= 1 - \frac{1}{m^3} = \frac{1}{2} \\ m^3 &= 2 \\ m &= \sqrt[3]{2} \\ &\approx 1.260 \end{aligned}$$

(b) $E(X) = \mu = 1.5$ (from Exercise 8)

$$\begin{aligned} P(1.26 < X < 1.5) &= \int_{1.26}^{1.5} 3x^{-4} dx \\ &= -x^{-3} \Big|_{1.26}^{1.5} \\ &= \frac{1}{(1.26)^3} - \frac{1}{(1.5)^3} \\ &\approx 0.2036 \end{aligned}$$

21. $f(x) = \begin{cases} \frac{x^3}{12} & \text{if } 0 \leq x \leq 2 \\ \frac{16}{3x^3} & \text{if } x > 2 \end{cases}$

Expected value:

$$\begin{aligned} E(X) = \mu &= \int_0^\infty x f(x) dx \\ &= \int_0^2 x \left(\frac{x^3}{12}\right) dx + \lim_{a \rightarrow \infty} \int_2^a x \left(\frac{16}{3x^3}\right) dx \\ &= \int_0^2 \frac{x^4}{12} dx + \lim_{a \rightarrow \infty} \int_2^a \frac{16}{3x^2} dx \\ &= \left(\frac{x^5}{60}\right) \Big|_0^2 + \lim_{a \rightarrow \infty} \left(-\frac{16}{3x}\right) \Big|_2^a \\ &= \left(\frac{8}{15} - 0\right) + \left[\lim_{a \rightarrow \infty} \left(-\frac{16}{3a}\right) - \left(-\frac{16}{6}\right)\right] \\ &= \frac{16}{5} \end{aligned}$$

Variance:

$$\begin{aligned}\text{Var}(X) &= \int_0^{\infty} x^2 f(x) dx - \mu^2 \\ &= \int_0^2 x^2 \left(\frac{x^3}{12} \right) dx + \int_2^{\infty} \left(\frac{16}{3x^3} \right) dx - \left(\frac{16}{5} \right)^2\end{aligned}$$

Examine the second integral.

$$\begin{aligned}\int_2^{\infty} x^2 \left(\frac{16}{3x^3} \right) dx &= \lim_{a \rightarrow \infty} \int_2^a x^2 \left(\frac{16}{3x^3} \right) dx \\ &= \lim_{a \rightarrow \infty} \int_2^a \frac{16}{3x} dx \\ &= \lim_{a \rightarrow \infty} \frac{16}{3} \ln |a| - \frac{16}{3} \ln |2|\end{aligned}$$

Since the limit diverges, neither the variance nor the standard deviation exists.

$$22. \quad f(x) = \begin{cases} \frac{20x^4}{9} & \text{if } 0 \leq x \leq 1 \\ \frac{20}{9x^5} & \text{if } x > 1 \end{cases}$$

Expected value:

$$\begin{aligned}E(X) = \mu &= \int_0^{\infty} x f(x) dx \\ &= \int_0^1 x \frac{20x^4}{9} dx + \lim_{a \rightarrow \infty} \int_1^a x \frac{20}{9x^5} dx \\ &= \int_0^1 \frac{20x^5}{9} dx + \lim_{a \rightarrow \infty} \int_1^a \frac{20}{9x^4} dx \\ &= \left(\frac{20x^6}{54} \right) \Big|_0^1 + \lim_{a \rightarrow \infty} \left(-\frac{20}{27x^3} \right) \Big|_1^a \\ &= \left(\frac{10}{27} - 0 \right) + \left[\lim_{a \rightarrow \infty} \left(-\frac{20}{27a^3} \right) - \left(-\frac{20}{27} \right) \right] \\ &= \frac{10}{27} + \frac{20}{27} \\ &= \frac{10}{9}\end{aligned}$$

Variance:

$$\begin{aligned}\text{Var}(X) &= \int_0^{\infty} x^2 f(x) dx - \mu^2 \\ &= \int_0^1 x^2 \frac{20x^4}{9} dx \\ &\quad + \lim_{a \rightarrow \infty} \int_1^a x^2 \frac{20}{9x^5} dx - \left(\frac{10}{9} \right)^2\end{aligned}$$

$$\begin{aligned}&= \int_0^1 \frac{20x^6}{9} dx + \lim_{a \rightarrow \infty} \int_1^a \frac{20}{9x^3} dx \\ &\quad - \frac{100}{81} \\ &= \left(\frac{20x^7}{63} \right) \Big|_0^1 + \lim_{a \rightarrow \infty} \left(-\frac{10}{9x^2} \right) \Big|_1^a - \frac{100}{81} \\ &= \left(\frac{20}{63} - 0 \right) + \left[\lim_{a \rightarrow \infty} \left(-\frac{10}{9a^2} \right) - \left(-\frac{10}{9} \right) \right] - \frac{100}{81} \\ &= \frac{20}{63} + \frac{10}{9} - \frac{100}{81} \\ &= \frac{110}{567}\end{aligned}$$

Standard deviation:

$$\sigma = \sqrt{\text{Var}(X)} \approx 0.4405$$

$$23. \quad f(x) = \begin{cases} \frac{|x|}{10} & \text{for } -2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

First, note that

$$|x| = \begin{cases} -x & \text{for } -2 \leq x \leq 0 \\ x & \text{for } 0 \leq x \leq 4 \end{cases}$$

The expected value is

$$\begin{aligned}E(X) = \mu &= \int_{-2}^4 x \cdot \frac{|x|}{10} dx \\ &= \int_{-2}^0 x \cdot \frac{-x}{10} dx + \int_0^4 x \cdot \frac{x}{10} dx \\ &= \int_{-2}^0 -\frac{x^2}{10} dx + \int_0^4 \frac{x^2}{10} dx \\ &= -\frac{x^3}{30} \Big|_{-2}^0 + \frac{x^3}{30} \Big|_0^4 \\ &= -\left(0 - \frac{-8}{30} \right) + \left(\frac{64}{30} - 0 \right) \\ &= \frac{56}{30} = \frac{28}{15}\end{aligned}$$

The correct answer choice is **d**.

24. $f(t) = \frac{1}{58\sqrt{t}}; [1, 900]$

(a)
$$\begin{aligned} E(T) &= \int_1^{900} t \cdot \frac{1}{58\sqrt{t}} dt \\ &= \frac{1}{58} \int_1^{900} t^{1/2} dt \\ &= \frac{1}{87} t^{3/2} \Big|_1^{900} \\ &= \frac{1}{87} (900^{3/2} - 1^{3/2}) \\ &\approx 310.3 \text{ hr} \end{aligned}$$

(b)
$$\begin{aligned} \text{Var}(T) &= \int_1^{900} t^2 \frac{1}{58\sqrt{t}} dt - (310.3)^2 \\ &= \frac{1}{58} \int_1^{900} t^{3/2} dt - (310.3)^2 \\ &= \frac{1}{145} t^{5/2} \Big|_1^{900} - (310.3)^2 \\ &= \frac{1}{145} (900^{5/2} - 1^{5/2}) - (310.3)^2 \\ &= 71,300.11 \\ \sigma &= \sqrt{71,300.11} \approx 267.0 \text{ hr} \end{aligned}$$

(c)
$$\begin{aligned} P(310.3 + 267 < T \leq 900) &= \int_{577.3}^{900} \frac{1}{58\sqrt{t}} dt \\ &= \frac{2}{58} t^{1/2} \Big|_{577.3}^{900} \\ &= \frac{2}{58} (900^{1/2} - 577.3^{1/2}) \approx 0.2059 \end{aligned}$$

(d) The median life of the bulbs is the value of m such that $\int_a^m f(t) dt = \frac{1}{2}$.

$$\begin{aligned} \int_1^m \frac{1}{58\sqrt{t}} dt &= \frac{1}{2} \\ \frac{1}{29} \sqrt{t} \Big|_1^m &= \frac{1}{2} \\ \frac{1}{29} (\sqrt{m} - 1) &= \frac{1}{2} \\ \sqrt{m} - 1 &= \frac{29}{2} \\ \sqrt{m} &= \frac{31}{2} \\ m &= \frac{961}{4} = 240.3 \end{aligned}$$

The median life of the bulbs is 240.3 hours.

25. $f(t) = \frac{1}{11} \left(1 + \frac{3}{\sqrt{t}} \right); [4, 9]$

(a) From Exercise 6, $\mu = \frac{141}{22} \text{ yr}$
 $\approx 6.409 \text{ yr.}$

(b) $\sigma = 1.447 \text{ yr.}$

(c)
$$\begin{aligned} P\left(T > \frac{141}{22}\right) &= \int_{141/22}^9 \frac{1}{11} (1 + 3t^{-1/2}) dt \\ &= \frac{1}{11} (t + 6t^{1/2}) \Big|_{141/22}^9 \\ &= \frac{1}{11} \left[9 + 18 - \frac{141}{22} - 6 \left(\frac{141}{22} \right)^{1/2} \right] \\ &\approx 0.4910 \end{aligned}$$

26. $f(t) = \frac{1}{2} e^{-t/2}; [0, \infty)$

(a)
$$\begin{aligned} E(T) = \mu &= \int_0^{\infty} t \cdot \frac{1}{2} e^{-t/2} dt \\ &= \frac{1}{2} \lim_{a \rightarrow \infty} \int_0^a t e^{-t/2} dt \end{aligned}$$

Use integration by parts.

$$\begin{aligned} &= \frac{1}{2} \lim_{a \rightarrow \infty} (-2te^{-t/2}) \Big|_0^a \\ &\quad - \frac{1}{2} \lim_{a \rightarrow \infty} \int_0^a -2e^{-t/2} dt \\ &= \frac{1}{2} \lim_{a \rightarrow \infty} (-2te^{-t/2}) \Big|_0^a \\ &\quad + \lim_{a \rightarrow \infty} \int_0^a e^{-t/2} dt \\ &= - \lim_{a \rightarrow \infty} (te^{-t/2}) \Big|_0^a \\ &\quad - 2 \lim_{a \rightarrow \infty} (e^{-t/2}) \Big|_0^a \\ &= - \lim_{a \rightarrow \infty} \left(\frac{a}{e^{a/2}} - \frac{0}{e^0} \right) \\ &\quad - 2 \lim_{a \rightarrow \infty} \left(\frac{1}{e^{a/2}} - \frac{1}{e^0} \right) \\ &= (0 - 0) - 2(0 - 1) \\ &= -(0) + 2 \\ E(T) &= 2 \text{ mo} \end{aligned}$$

$$(b) \quad \text{Var}(T) = \frac{1}{2} \int_0^\infty t^2 e^{-t/2} dt - 2^2$$

Use integration by parts twice.

$$\begin{aligned} & \frac{1}{2} \int t^2 e^{-t/2} dt \\ &= \frac{1}{2} (-2t^2 e^{-t/2} + 4 \int t e^{-t/2} dt) \\ &= \frac{1}{2} [-2t^2 e^{-t/2} + 4(-2t e^{-t/2} \\ & \quad + 2 \int e^{-t/2} dt)] \\ &= \frac{1}{2} (-2t^2 e^{-t/2} - 8te^{-t/2} - 16e^{-t/2}) \\ &= -t^2 e^{-t/2} - 4te^{-t/2} - 8e^{-t/2} \end{aligned}$$

$\text{Var}(T)$

$$\begin{aligned} &= \lim_{a \rightarrow \infty} (-t^2 e^{-t/2} - 4te^{-t/2} - 8e^{-t/2}) \Big|_0^a - 4 \\ &= \lim_{a \rightarrow \infty} (-a^2 e^{-a/2} - 4ae^{-a/2} - 8e^{-a/2} \\ & \quad + 0 + 0 + 8e^0) - 4 \\ &= -0 - 0 - 0 + 0 + 0 + 8 - 4 \end{aligned}$$

$$\text{Var}(T) = 4 \text{ mo}$$

$$\sigma = \sqrt{4} = 2 \text{ mo}$$

$$(c) \quad P(0 < T < 2)$$

$$\begin{aligned} &= \frac{1}{2} \int_0^2 e^{-t/2} dt = -e^{-t/2} \Big|_0^2 \\ &= -e^{-1} + 1 \approx 0.6321 \end{aligned}$$

$$(d) \quad \text{The median life is the value of } m \text{ such that}$$

$$\int_a^m f(t) dt = \frac{1}{2}.$$

$$\begin{aligned} & \int_0^m \frac{1}{2} e^{-t/2} dt = \frac{1}{2} \\ & e^{-t/2} \Big|_0^m = \frac{1}{2} \\ & -e^{-m/2} - (-e^0) = \frac{1}{2} \\ & -e^{-m/2} + 1 = \frac{1}{2} \\ & -e^{-m/2} = \frac{1}{2} \\ & m = -2 \ln \left(\frac{1}{2} \right) \approx 1.386 \end{aligned}$$

The median life of the parts is 1.386 months.

27. Using the hint, we have

$$\text{loss not paid} = \begin{cases} x & \text{for } 0.6 < x < 2 \\ 2 & \text{for } x > 2 \end{cases}$$

Therefore, the mean of the manufacturer's annual losses not paid will be

$$\begin{aligned} \mu &= \int_{0.6}^0 x \cdot f(x) dx + \int_2^\infty 2 \cdot f(x) dx \\ &= \int_{0.6}^2 x \frac{2.5(0.6)^{2.5}}{x^{3.5}} dx \\ & \quad + \int_2^\infty 2 \frac{2.5(0.6)^{2.5}}{x^{3.5}} dx \\ &= 2.5(0.6)^{2.5} \int_{0.6}^2 \frac{1}{x^{2.5}} dx \\ & \quad + 5(0.6)^{2.5} \int_2^\infty \frac{1}{x^{3.5}} dx \\ &= 2.5(0.6)^{2.5} \left(\frac{1}{-1.5} \right) \frac{1}{x^{1.5}} \Big|_{0.6}^2 \\ & \quad + 5(0.6)^{2.5} \left(\frac{1}{-2.5} \right) \frac{1}{x^{2.5}} \Big|_2^\infty \\ &= -\frac{5}{3} (0.6)^{2.5} \left(\frac{1}{2^{1.5}} - \frac{1}{0.6^{1.5}} \right) \\ & \quad - 2(0.6)^{2.5} \left(0 - \frac{1}{2^{2.5}} \right) \\ &\approx 0.8357 + 0.0986 \approx 0.93 \end{aligned}$$

The correct answer choice is c.

28. Using the hint, we have

$$\text{benefit paid} = \begin{cases} y & \text{for } 1 < y < 10 \\ 10 & \text{for } y > 10 \end{cases}$$

Therefore, the expected value of the benefit paid will be

$$\begin{aligned} E(Y) &= \int_1^{10} y \cdot f(y) dy + \int_{10}^\infty 10 \cdot f(y) dy \\ &= \int_1^{10} y \frac{2}{y^3} + dy + \int_{10}^\infty 10 \frac{2}{y^3} dy \\ &= 2 \int_1^{10} \frac{1}{y^2} dy + 20 \int_{10}^\infty \frac{1}{y^3} dy \\ &= 2 \left(-\frac{1}{y} \right) \Big|_1^{10} + 20 \left(-\frac{1}{2y^2} \right) \Big|_{10}^\infty \end{aligned}$$

$$\begin{aligned}
 &= 2\left(-\frac{1}{10} + 1\right) + 20\left(0 + \frac{1}{200}\right) \\
 &= \frac{9}{5} + \frac{1}{10} = 1.9.
 \end{aligned}$$

The correct choice is **d**.

29. Since the probability density function is proportional to $(1+x)^{-4}$, we have $f(x) = k(1+x)^{-4}$, $0 < x < \infty$. To determine k , solve the equation $\int_0^\infty k f(x) dx = 1$.

$$\begin{aligned}
 \int_0^\infty k(1+x)^{-4} dx &= 1 \\
 k\left(-\frac{1}{3}\right)(1+x)^{-3} \Big|_0^\infty &= 1 \\
 -\frac{k}{3}(0-1) &= 1 \\
 \frac{k}{3} &= 1 \\
 k &= 3
 \end{aligned}$$

Thus, $f(x) = 3(1+x)^{-4}$, $0 < x < \infty$.

The expected monthly claims are

$$\int_0^\infty x \cdot 3(1+x)^{-4} dx = 3 \int_0^\infty \frac{x}{(1+x)^4} dx$$

The antiderivative can be found using the substitution $u = 1+x$.

$$\begin{aligned}
 \int \frac{x}{(1+x)^4} dx &= \int \frac{u-1}{u^4} du \\
 &= \int \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du \\
 &= -\frac{1}{2u^2} + \frac{1}{3u^3}
 \end{aligned}$$

Resubstitute $u = 1+x$.

$$\begin{aligned}
 3 \int_0^\infty \frac{x}{(1+x)^4} dx &= 3 \left[-\frac{1}{2(1+x)^2} + \frac{1}{3(1+x)^3} \right] \Big|_0^\infty \\
 &= 3 \left[0 - \left(-\frac{1}{2} + \frac{1}{3} \right) \right] = 3 \left(\frac{1}{6} \right) = \frac{1}{2}
 \end{aligned}$$

The correct answer choice is **c**.

30. The median benefit is the value of m such that $\int_a^m f(x) dx = \frac{1}{2}$.

First, we need to determine the value of c . Since f is a probability density function,

$$\begin{aligned}
 \int_0^\infty ce^{-0.004x} dx &= 1 \\
 c \frac{1}{-0.004} e^{-0.004x} \Big|_0^\infty &= 1 \\
 -\frac{c}{0.004} (e^{-\infty} - e^0) &= 1 \\
 -\frac{c}{0.004} (0-1) &= 1 \\
 \frac{c}{0.004} &= 1 \\
 c &= 0.004
 \end{aligned}$$

Thus, $f(x) = 0.004e^{-0.004x}$, $x \geq 0$.

Now solve for m .

$$\begin{aligned}
 \int_0^m 0.004e^{-0.004x} dx &= \frac{1}{2} \\
 0.004 \left(\frac{1}{-0.004} \right) e^{-0.004x} \Big|_0^m &= \frac{1}{2} \\
 -(e^{-0.004m} - 1) &= \frac{1}{2} \\
 e^{-0.004m} &= \frac{1}{2} \\
 -0.004m &= \ln\left(\frac{1}{2}\right) \\
 m &= \frac{1}{-0.004} \ln\left(\frac{1}{2}\right) \\
 m &\approx 173
 \end{aligned}$$

The correct answer choice is **c**.

31. $f(t) = \frac{1}{(\ln 20)t}$; $[1, 20]$

$$\begin{aligned}
 \text{(a)} \quad \mu &= \int_1^{20} t \cdot \frac{1}{(\ln 20)t} dt \\
 &= \int_1^{20} \frac{1}{\ln 20} dt \\
 &= \frac{t}{\ln 20} \Big|_1^{20} \\
 &= \frac{19}{\ln 20} \approx 6.3424 \approx 6.342 \text{ seconds}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \text{Var}(T) &= \int_1^{20} t^2 \cdot \frac{1}{(\ln 20)t} dt - \mu^2 \\
 &= \int_1^{20} \frac{t}{\ln 20} dt - \mu^2 \\
 &= \frac{t^2}{2 \ln 20} \Big|_1^{20} - (6.3424)^2 \\
 &= \frac{399}{2 \ln 20} - (6.3424)^2 \\
 &\approx 26.3687 \\
 \sigma &\approx \sqrt{26.3687} \\
 &\approx 5.1350 \text{ sec}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad &P(6.3424 - 5.1350 < T < 6.3424 + 5.1350) \\
 &= P(1.2074 < T < 11.4774) \\
 &= \int_{1.2074}^{11.4774} \frac{1}{(\ln 20)t} dt \\
 &= \frac{\ln t}{\ln 20} \Big|_{1.2074}^{11.4774} \\
 &= \frac{1}{\ln 20} (\ln 11.4774 - \ln 1.2074) \\
 &\approx 0.7517
 \end{aligned}$$

If you do the integration on a calculator using differently rounded values for the limits you may get a slightly different answer, such as 0.7518.

- (d) The median clotting time is the value of m such that $\int_a^m f(t) dt = \frac{1}{2}$.

$$\begin{aligned}
 \int_1^m \frac{1}{(\ln 20)t} dt &= \frac{1}{2} \\
 \frac{1}{\ln 20} \ln t \Big|_1^m &= \frac{1}{2} \\
 \frac{1}{\ln 20} (\ln m - 0) &= \frac{1}{2} \\
 \ln m &= \frac{\ln 20}{2} \\
 m &= e^{\ln 20/2} \\
 &\approx 4.472
 \end{aligned}$$

$$32. \quad f(x) = \frac{3}{32}(4x - x^2); [0, 4]$$

$$\begin{aligned}
 \text{(a)} \quad E(X) &= \int_0^4 \frac{3x}{32}(4x - x^2) dx \\
 &= \int_0^4 \left(\frac{3}{8}x^2 - \frac{3x^3}{32} \right) dx \\
 &= \left(\frac{x^3}{8} - \frac{3x^4}{128} \right) \Big|_0^4 \\
 &= 8 - \frac{768}{128} = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \text{Var}(X) &= \int_0^4 \frac{3x^2}{32}(4x - x^2) dx - 4 \\
 &= \int_0^4 \left(\frac{3x^3}{8} - \frac{3x^4}{32} \right) dx - 4 \\
 &= \left(\frac{3x^4}{32} - \frac{3x^5}{160} \right) \Big|_0^4 \\
 &= \frac{3(4^4)}{32} - \frac{3(4^5)}{160} - 4 \\
 &= 24 - 19.2 - 4 = 0.8 \\
 \sigma &= \sqrt{\text{Var}(X)} \approx 0.89443 \\
 \sigma &\approx 0.8944
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad &P(2 - 0.89443 \leq X \leq 2 + 0.89443) \\
 &\approx \int_{1.10557}^{2.89443} \frac{3}{32}(4x - x^2) dx \\
 &= \left(\frac{3}{16}x^2 - \frac{x^3}{32} \right) \Big|_{1.10557}^{2.89443} \\
 &= \frac{3(2.89443)^2}{16} - \frac{2.89443^3}{32} \\
 &\quad - \frac{3(1.10557)^2}{16} + \frac{1.10557^3}{32} \\
 &\approx 0.6261
 \end{aligned}$$

$$33. \quad f(x) = \frac{1}{2\sqrt{x}}; [1, 4]$$

$$\begin{aligned} \text{(a)} \quad \mu &= \int_1^4 x \cdot \frac{1}{2\sqrt{x}} dx \\ &= \int_1^4 \frac{x^{1/2}}{2} dx = \frac{x^{3/2}}{3} \Big|_1^4 \\ &= \frac{1}{3}(8 - 1) \\ &= \frac{7}{3} \approx 2.333 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{Var}(X) &= \int_1^4 x^2 \cdot \frac{1}{2\sqrt{x}} dx - \left(\frac{7}{3}\right)^2 \\ &= \int_1^4 \frac{x^{3/2}}{2} dx - \frac{49}{9} \\ &= \frac{x^{5/2}}{5} \Big|_1^4 - \frac{49}{9} \\ &= \frac{1}{5}(32 - 1) - \frac{49}{9} \\ &\approx 0.7556 \\ \sigma &= \sqrt{\text{Var}(X)} \\ &\approx 0.8692 \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(X > 2.33 + 2(0.87)) \\ &= P(X > 4.07) \\ &= 0 \end{aligned}$$

The probability is 0 since two standard deviations falls out of the given interval $[1, 4]$.

(d) The median petal length is the value of m such that $\int_a^m f(x)dx = \frac{1}{2}$.

$$\begin{aligned} \int_1^m \frac{1}{2\sqrt{x}} dx &= \frac{1}{2} \\ \sqrt{x} \Big|_1^m &= \frac{1}{2} \\ \sqrt{m} - 1 &= \frac{1}{2} \\ \sqrt{m} &= \frac{3}{2} \\ m &= \frac{9}{4} = 2.25 \end{aligned}$$

The median petal length is 2.25 cm.

$$34. \quad p(x, t) = \frac{e^{-x^2/(4Dt)}}{\int_0^L e^{-u^2/(4Dt)} du}$$

Letting $t = 12$, $L = 6$, and $D = 38.3$,

$$\begin{aligned} p(x, 12) &= \frac{e^{-x^2/(4 \cdot 38.3 \cdot 12)}}{\int_0^6 e^{-u^2/(4 \cdot 38.3 \cdot 12)} du} \\ &= \frac{e^{-x^2/1838.4}}{\int_0^6 e^{-u^2/(1838.4)} du} \\ &\approx \frac{1}{5.9611} e^{-x^2/1838.4} \end{aligned}$$

The integral in the denominator was evaluated using the integration feature on the calculator.

$$\begin{aligned} E(X) &= \int_0^L x \cdot p(x, 12) dx \\ &= \frac{1}{5.9611} \int_0^6 x e^{-x^2/1838.4} dx \\ &= \frac{1}{5.9611} \cdot \left(-\frac{1}{2}\right) \cdot 1838.4 e^{-x^2} \Big|_0^6 \\ &\approx 2.990 \end{aligned}$$

The expected recapture distance is 2.990 m.

$$35. \quad f(x) = 1.185 \cdot 10^{-9} x^{4.5222} e^{-0.049846x}$$

$$E(X) = \int_1^{1000} x f(x) dx$$

Using the integration function on our calculator.

$$E(X) \approx 110.80$$

The expected size is about 111.

$$36. \quad \text{The median } m \text{ satisfies } \int_3^m \frac{8}{7(t-2)^2} dt = \frac{1}{2}.$$

$$\begin{aligned} \int_3^m \frac{8}{7(t-2)^2} dt &= \frac{1}{2} \\ \frac{8}{7} \left(-\frac{1}{t-2} \right) \Big|_3^m &= \frac{1}{2} \\ \frac{8}{7} \left[-\frac{1}{m-2} - \left(-\frac{1}{1} \right) \right] &= \frac{1}{2} \\ -\frac{1}{m-2} &= \frac{7}{16} - 1 = -\frac{9}{16} \\ m-2 &= \frac{16}{9} \\ m &= \frac{34}{9} \approx 3.778 \end{aligned}$$

The median time is about 3.778 min.

$$37. \quad S(t) = \frac{1}{466.26}(-0.00007445t^4 + 0.01243t^3 - 0.7419t^2 + 18.18t - 137.5)$$

for t in the interval $[13.4, 62.0]$.

Evaluate the required integrals with a calculator.

$$\mu = \int_{13.4}^{62.0} tS(t)dt$$

$$\approx 31.75 \text{ years}$$

$$\text{Var}(X) = \int_{13.4}^{62.0} t^2 S(t)dt - (31.75)^2$$

$$\approx 133.44$$

$$\sigma \approx \sqrt{133.44}$$

$$\approx 11.55 \text{ years}$$

$$38. \quad f(t) = \frac{1}{960}e^{-t/960}; [0, \infty)$$

$$E(T) = \int_0^{\infty} \frac{t}{960}e^{-t/960}dt$$

$$= \lim_{b \rightarrow \infty} \int_0^b \frac{t}{960}e^{-t/960}dt$$

Integrating by parts, choose

$$dv = e^{-t/960}dt \text{ and } u = \frac{t}{960}.$$

Then

$$v = -960e^{-t/960} \text{ and } du = \frac{1}{960}dt.$$

So,

$$\int_0^b \frac{t}{960}e^{-t/960}dt = uv - \int v du$$

$$= -te^{-t/960} \Big|_0^b - \int_0^b -e^{-t/960}dt$$

$$= -te^{-t/960} \Big|_0^b - 960e^{-t/960} \Big|_0^b$$

$$= -(t + 960)e^{-t/960} \Big|_0^b$$

$$= -(b + 960)e^{-b/960} + 960$$

Thus,

$$E(T) = \lim_{b \rightarrow \infty} \int_0^b \frac{t}{960}e^{-t/960}dt$$

$$= \lim_{b \rightarrow \infty} [-(b + 960)e^{-b/960} + 960]$$

$$= 960 \text{ days}$$

$$\text{Var}(T) = \int_0^{\infty} \frac{t^2}{960}e^{-t/960}dt - 960^2$$

$$= \frac{1}{960} \left[\lim_{b \rightarrow \infty} \int_0^b t^2 e^{-t/960}dt \right] - 960^2$$

Use the column method for integration by parts.

Let $u = t^2$ and $dv = e^{-t/960}dt$.

D	I
t^2	$+$ $e^{-t/960}$
$2t$	$-$ $-960e^{-t/960}$
2	$+$ $960^2e^{-t/960}$
0	$-$ $-960^3e^{-t/960}$

$$\int_0^b t^2 e^{-t/960}dt$$

$$= -960t^2e^{-t/960} - 2t(960^2e^{-t/960})$$

$$+ 2(-960^3e^{-t/960}) \Big|_0^b$$

$$= -960e^{-t/960}(t^2 + 2(960)t + 2(960^2)) \Big|_0^b$$

$$= -960e^{-b/960}(b^2 + 2(960)b + 2(960^2)) + 2(960^3)$$

$$\text{Var}(T) = \frac{1}{960} \left[\lim_{b \rightarrow \infty} \int_0^b t^2 e^{-t/960}dt \right] - 960^2$$

$$= \frac{1}{960} [\lim_{b \rightarrow \infty} [-960e^{-b/960}(b^2 + 2(960)b + 2(960^2)) + 2(960^3)]] - 960^2$$

$$= \frac{1}{960} \cdot 2(960^3) - 960^2$$

$$= 960^2$$

$$\sigma = \sqrt{\text{Var}(T)} = \sqrt{960^2} = 960 \text{ days}$$

$$39. \quad f(x) = \frac{5.5 - x}{15}; [0, 5]$$

$$(a) \quad \mu = \int_0^5 x \left(\frac{5.5 - x}{15} \right) dx$$

$$= \int_0^5 \left(\frac{5.5}{15}x - \frac{1}{15}x^2 \right) dx$$

$$= \frac{5.5}{30}x^2 - \frac{1}{45}x^3 \Big|_0^5$$

$$= \left(\frac{5.5}{30} \cdot 25 - \frac{1}{45} \cdot 125 \right) - 0$$

$$\approx 1.806$$

$$\begin{aligned}
 \text{(b)} \quad \text{Var}(X) &= \int_0^5 x^2 \left(\frac{5.5-x}{15} \right) dx - \mu^2 \\
 &= \int_0^5 \left(\frac{5.5}{15} x^2 - \frac{1}{15} x^3 \right) dx - \mu^2 \\
 &= \left(\frac{5.5}{45} x^3 - \frac{1}{60} x^4 \right) \Big|_0^5 - \mu^2 \\
 &= \frac{5.5}{45} \cdot 125 - \frac{1}{60} \cdot 625 - 0 - \mu^2 \\
 &\approx 1.60108 \\
 \sigma &= \sqrt{\text{Var}(X)} \approx 1.265
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad P(X \leq \mu - \sigma) &= P(X \leq 1.806 - 1.265) \\
 &= P(X \leq 0.541) \\
 &= \int_0^{0.541} \frac{5.5-x}{15} dx \\
 &= \left(\frac{5.5}{15} x - \frac{1}{30} x^2 \right) \Big|_0^{0.541} \\
 &= \left(\frac{5.5}{15} (0.541) - \frac{1}{30} (0.541)^2 - 0 \right) \\
 &\approx 0.1886
 \end{aligned}$$

40. $f(t) = \frac{4.045}{t^{1.532}}$ for t in $[16, 80]$.

$$\begin{aligned}
 \text{(a)} \quad \mu &= \int_{16}^{80} t f(t) dt \\
 &= \int_{16}^{80} t \left(\frac{4.045}{t^{1.532}} \right) dt \\
 &= \int_{16}^{80} 4.045 t^{-0.532} dt \\
 &= \frac{4.045}{0.468} t^{0.468} \Big|_{16}^{80} \\
 &= \frac{4.045}{0.468} (80^{0.468} - 16^{0.468}) \\
 &\approx 35.55 \text{ years}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \text{Var}(T) &= \int_{16}^{80} t^2 f(t) dt - (35.55)^2 \\
 &= \int_{16}^{80} t^2 \left(\frac{4.045}{t^{1.532}} \right) dt - (35.55)^2 \\
 &= \int_{16}^{80} 4.045 t^{0.468} dt - (35.55)^2 \\
 &= \frac{4.045}{1.468} (t^{1.468}) \Big|_{16}^{80} - (35.55)^2 \\
 &= \frac{4.045}{1.468} (80^{1.468} - 16^{1.468}) - (35.55)^2 \\
 &\approx 288.496 \\
 \sigma &\approx \sqrt{288.496} \\
 &\approx 16.98 \text{ years}
 \end{aligned}$$

(c) $\mu - \sigma = 35.55 - 16.98 = 18.57$

$$\begin{aligned}
 P(T < 18.57) &= \int_{16}^{18.57} f(t) dt \\
 &= \int_{16}^{18.57} \frac{4.045}{t^{1.532}} dt \\
 &= -\frac{4.045}{0.532} (t^{-0.532}) \Big|_{16}^{18.57} \\
 &= -\frac{4.045}{0.532} (18.57^{-0.532} - 16^{-0.532}) \\
 &\approx 0.1325
 \end{aligned}$$

(d) The median age m satisfies

$$\int_{16}^m \frac{4.045}{t^{1.532}} dt = \frac{1}{2}.$$

Use the integral found in (c).

$$\begin{aligned}
 \int_{16}^m \frac{4.045}{t^{1.532}} dt &= \frac{1}{2} \\
 -\frac{4.045}{0.532} (t^{-0.532}) \Big|_{16}^m &= \frac{1}{2} \\
 -\frac{4.045}{0.532} (m^{-0.532} - 16^{-0.532}) &= \frac{1}{2}
 \end{aligned}$$

$$m^{-0.532} = -\frac{0.532}{4.045} \left(\frac{1}{2} \right) + 16^{-0.532}$$

$$m^{-0.532} = 0.16301$$

$$e^{-0.532 \ln m} = 0.16301$$

$$-0.532 \ln m = \ln 0.16301$$

$$\ln m = \frac{-1.8139}{-0.532}$$

$$\ln m = 3.4096$$

$$m = e^{3.4096}$$

$$m \approx 30.35 \text{ years}$$

41. $f(t) = 0.06049e^{-0.03211t}$; $[16, 84]$

(a) Expected value:

$$\begin{aligned} E(T) &= \mu \\ &= \int_{16}^{84} t(0.06049e^{-0.03211t})dt \end{aligned}$$

Use integration by parts.

$$\begin{aligned} &\int_{16}^{84} t(0.06049e^{-0.03211t})dt \\ &\approx -1.88384(t + 31.14295)e^{-0.03211t} \Big|_{16}^{84} \\ &= -1.88384[(84 + 31.14295)e^{-0.03211 \cdot 84} \\ &\quad - (16 + 31.14295)e^{-0.03211 \cdot 16}] \\ &\approx 38.512 \\ &\approx 38.51 \end{aligned}$$

The expected value is 38.51 years.

(b) $\text{Var}(T)$

$$= \int_{16}^{84} t^2(0.06049e^{-0.03211t})dt - \mu^2$$

Use integration by parts twice.

$$\begin{aligned} &\int_{16}^{84} t^2(0.06049e^{-0.03211t})dt - \mu^2 \\ &\approx -1.88384(t^2 + 62.28589t \\ &\quad + 1939.76619)e^{-0.03211t} \Big|_{16}^{84} - 38.512^2 \\ &\approx 308.305 \end{aligned}$$

$$\sigma = \sqrt{\text{Var}(T)} = \sqrt{308.290} \approx 17.558$$

$$s \approx 17.56 \text{ years}$$

$$P(16 \leq T \leq 38.512 - 17.558)$$

$$\begin{aligned} &= \int_{16}^{20.954} 0.06049e^{-0.03211t}dt \\ &= -1.88384e^{-0.03211t} \Big|_{16}^{20.954} \\ &\approx 0.1657 \end{aligned}$$

If you do the integration on a calculator using differently rounded values for the limits you may get a slightly different answer, such as 0.1.

(d) To find the median age, find the value of m

such that $\int_a^m f(t)dt = \frac{1}{2}$.

$$\begin{aligned} \int_{16}^m 0.06049e^{-0.03211t}dt &= \frac{1}{2} \\ \frac{0.06049}{-0.03211} e^{-0.03211t} \Big|_{16}^m &= \frac{1}{2} \\ \frac{0.06049}{-0.03211} (e^{-0.03211m} - e^{-0.03211(16)}) &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} e^{-0.03211m} - e^{-0.51376} &= -\frac{0.03211}{2 \cdot 0.06049} \\ e^{-0.03211m} &= e^{-0.51376} - \frac{0.03211}{2 \cdot 0.06049} \end{aligned}$$

$$-0.03211m = \ln \left(e^{-0.51376} - \frac{0.03211}{2 \cdot 0.06049} \right)$$

$$\begin{aligned} m &= \frac{1}{-0.03211} \ln \left(e^{-0.51376} - \frac{0.03211}{2 \cdot 0.06049} \right) \\ m &\approx 34.26 \end{aligned}$$

The median age is 34.26 years.

42. $f(t) = 3t^{-4}$ for t in $[1, \infty)$.

$$\begin{aligned} \mu &= \int_1^{\infty} t(3t^{-4})dt \\ &= \lim_{b \rightarrow \infty} \int_1^b 3t^{-3}dt \\ &= \lim_{b \rightarrow \infty} \left(-\frac{3}{2}t^{-2} \Big|_1^b \right) \\ &= \lim_{b \rightarrow \infty} \left(-\frac{3}{2}b^{-3} - \left(-\frac{3}{2} \right) \right) \\ &= \frac{3}{2} \end{aligned}$$

The expected length of a phone call is 1.5 minutes.

$$43. \quad S(t) = \frac{1}{101,370}(-2.564t^3 + 99.11t^2 - 964.6t + 5631) \text{ for } t \text{ in } [0, 24]$$

$$\begin{aligned} \mu &= \int_0^{24} tS(t) dt \\ &= \frac{1}{101,370} \int_0^{24} (-2.564t^4 + 99.11t^3 - 964.6t^2 + 5631t) dt \end{aligned}$$

Use a calculator to evaluate the integral.

$$\mu \approx 12.964$$

The expected time of day at which a fatal accident will occur is about 1 pm.

11.3 Special Probability Density Functions

Your Turn 1

$42 - 27 = 15$, so the uniform distribution for the maximum daily temperature T is $f(t) = \frac{1}{15}$ for t in $[27, 42]$.

$$\begin{aligned} \mu &= \frac{27 + 42}{2} \\ &= \frac{69}{2} \\ &= 34.5 \end{aligned}$$

The expected maximum daily temperature is 34.5°C .

$$\begin{aligned} \sigma &= \frac{1}{\sqrt{12}}(42 - 27) \\ &= \frac{15}{\sqrt{12}} \\ &= \frac{5\sqrt{3}}{2} \approx 4.33 \end{aligned}$$

A temperature one standard deviation below the mean is

$$34.5 - \frac{5\sqrt{3}}{2} \approx 30.16987. \quad \text{A temperature one standard}$$

deviation above the mean is $34.5 + \frac{5\sqrt{3}}{2} \approx 38.83013$.

$$P(\mu - \sigma \leq T \leq \mu + \sigma) = \int_{30.16987}^{38.83013} \frac{1}{15} dx$$

This integral is $1/15$ times the difference of the limits, but this difference is just twice the standard deviation, so the probability is

$$\frac{5\sqrt{3}}{2} \cdot \frac{2}{15} = \frac{\sqrt{3}}{3} \approx 0.5774$$

Your Turn 2

$$f(t) = \frac{1}{25}e^{-t/25} \text{ for } t \geq 0$$

- (a) The probability that a randomly selected battery has a useful life less than 100 hours is

$$\begin{aligned} P(T \leq 100) &= \int_0^{100} \frac{1}{25}e^{-t/25} dt \\ &= \frac{1}{25}(-25e^{-t/25}) \Big|_0^{100} \\ &= -(e^{-100/25} - e^0) \\ &= 1 - e^{-4} \\ &\approx 0.9817. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \mu &= \frac{1}{1/25} = 25 \\ \sigma &= \frac{1}{1/25} = 25 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P(T > 40) &= \int_{40}^{\infty} \frac{1}{25}e^{-t/25} dt \\ &= \lim_{b \rightarrow \infty} \frac{1}{25}(-25e^{-t/25}) \Big|_{40}^b \\ &= \lim_{b \rightarrow \infty} (-e^{-b/25} + e^{-40/25}) \\ &= e^{-8/5} \\ &\approx 0.2019 \end{aligned}$$

Your Turn 3

- (a) The z -score for an age of 79 is

$$\begin{aligned} z &= \frac{79 - \mu}{\sigma} \\ &= \frac{79 - 75}{16} \\ &= 0.25. \end{aligned}$$

$$\begin{aligned} P(X > 79) &= P(z > 0.25) \\ &= 1 - P(z \leq 0.25) \\ &= 1 - 0.5987 \\ &= 0.4013 \end{aligned}$$

(We find the value 0.5987 in the row for 0.2 and the column for 0.05 in the table giving the area under the normal curve.)

- (b) Compute the z -scores for 67 and 83.

$$\begin{aligned} z &= \frac{67 - 75}{16} = -0.5 \\ z &= \frac{83 - 75}{16} = 0.5 \end{aligned}$$

$$\begin{aligned}
 P(67 \leq X \leq 83) &= P(-0.5 \leq z \leq 0.5) \\
 &= 0.6915 - 0.3085 \\
 &= 0.3830
 \end{aligned}$$

(We find the value 0.6915 in the row for 0.5 and the column for 0.00 in the normal table, and the value 0.3085 in the row for -0.5 and the column for 0.00.)

11.3 Exercises

1. $f(x) = \frac{5}{7}$ for x in $[3, 4.4]$

This is a uniform distribution: $a = 3, b = 4.4$.

(a) $\mu = \frac{1}{2}(4.4 + 3) = \frac{1}{2}(7.4)$
 $= 3.7$ cm

(b) $\sigma = \frac{1}{\sqrt{12}}(4.4 - 3)$
 $= \frac{1}{\sqrt{12}}(1.4)$
 ≈ 0.4041 cm

(c) $P(3.7 < X < 3.7 + 0.4041)$
 $= P(3.7 < X < 4.1041)$
 $= \int_{3.7}^{4.1041} \frac{5}{7} dx$
 $= \frac{5}{7}x \Big|_{3.7}^{4.1041}$
 ≈ 0.2886

2. $f(x) = 4$ for x in $[2.75, 3]$

This is a uniform distribution: $a = 2.75, b = 3$.

(a) $\mu = \frac{1}{2}(3 + 2.75)$
 $= \frac{1}{2}(5.75)$
 $= 2.875$ hundred dollars, or \$287.50

(b) $\sigma = \frac{1}{\sqrt{12}}(3 - 2.75)$
 $= \frac{1}{\sqrt{12}}(0.25)$
 ≈ 0.0722 hundred dollars, or \$7.22

(c) $P(2.875 < X < 2.875 + 0.0722)$
 $= P(2.875 < X < 2.9472)$
 $= \int_{2.875}^{2.9472} 4 dx$
 $= 4x \Big|_{2.875}^{2.9472}$
 ≈ 0.2888

3. $f(t) = 4e^{-4t}$ for t in $[0, \infty)$

This is an exponential distribution: $a = 4$.

(a) $\mu = \frac{1}{4} = 0.25$ year

(b) $\sigma = \frac{1}{4} = 0.25$ year

(c) $P(0.25 < T < 0.25 + 0.25)$
 $= P(0.25 < T < 0.5)$
 $= \int_{0.25}^{0.5} 4e^{-4t} dt$
 $= -e^{-4t} \Big|_{0.25}^{0.5}$
 $= -\frac{1}{e^{-2}} + \frac{1}{e^{-1}}$
 ≈ 0.2325

4. $f(t) = 0.05e^{-0.05t}$ for $[0, \infty)$

This is a exponential distribution.

(a) $\mu = \frac{1}{0.05} = 20$ yr

(b) $\sigma = \frac{1}{0.05} = 20$ yr

(c) $P(20 < X < 20 + 20)$
 $= P(20 < x < 40)$
 $= \int_{20}^{40} 0.05e^{-0.05t} dt$
 $= -e^{-0.05t} \Big|_{20}^{40}$
 $= e^{-1} - e^{-2}$
 ≈ 0.2325

5. $f(t) = \frac{e^{-t/3}}{3}$ for t in $[0, \infty)$

This is an exponential distribution: $a = \frac{1}{3}$.

(a) $\mu = \frac{1}{\frac{1}{3}} = 3$ days

(b) $\sigma = \frac{1}{\frac{1}{3}} = 3$ days

(c) $P(3 < T < 3 + 3) = P(3 < T < 6)$

$$= \int_3^6 \frac{e^{-t/3}}{3} dt$$

$$= e^{-t/3} \Big|_3^6$$

$$= -\frac{1}{e^{-2}} + \frac{1}{e^{-1}}$$

$$\approx 0.2325$$

6. $f(x) = 0.1e^{-0.1x}$ for $[0, \infty)$

This is an exponential distribution.

(a) $\mu = \frac{1}{0.1} = 10$ m

(b) $\sigma = \frac{1}{0.1} = 10$ m

(c) $P(10 < X < 10 + 10)$
 $= P(10 < X < 20)$

$$= \int_{10}^{20} 0.1e^{-0.1x} dx$$

$$= -e^{-0.1x} \Big|_{10}^{20}$$

$$= e^{-1} - e^{-2} \approx 0.2325$$

In Exercise 7–14, use the table in the Appendix for areas under the normal curve.

7. $z = 3.50$

Area to the left of $z = 3.50$ is 0.9998. Given mean $\mu = z - 0$, so area to left of μ is 0.5.

Area between μ and z is

$$0.9998 - 0.5 = 0.4998.$$

Therefore, this area represents 49.98% of total area under normal curve.

8. $z = 1.68$

Area between mean $z = 0$ and $z = 1.68$ is

$$0.9535 - 0.5000 = 0.4535.$$

Percent of area = 45.35%

9. Between $z = 1.28$ and $z = 2.05$

Area to left of $z = 2.05$ is 0.9798 and area to left of $z = 1.28$ is 0.8997.

$$0.9798 - 0.8997 = 0.0801$$

Percent of total area = 8.01%

10. Area between $z = -2.13$ and $z = -0.04$ is

$$0.4840 - 0.0166 = 0.4674.$$

Percent of area = 46.74%

11. Since 10% = 0.10, the z -score that corresponds to the area of 0.10 to the left of z is -1.28 .

12. Since 2% = 0.02, the z -score that corresponds to the area of 0.02 to the left of z is -2.05 .

13. 18% of the total area to the right of z means $1 - 0.18$ of the total area is to the left of z .

$$1 - 0.18 = 0.82$$

The closest z -score that corresponds to the area of 0.82 is 0.92

14. 22% of the total area to the right of z means $1 - 0.22$ of the total area to the left of z .

$$1 - 0.22 = 0.78$$

The closest z -score that corresponds to the area of 0.78 is 0.77.

18. For the uniform distribution, $f(x) = \frac{1}{b-a}$ for x in $[a, b]$.

If m is the median,

$$P(X \leq m) = \frac{1}{2}$$

$$\int_a^m \frac{1}{b-a} dx = \frac{1}{2}$$

$$\frac{1}{b-a} \int_a^m dx = \frac{1}{2}$$

Multiply both sides by $b - a$.

$$\begin{aligned}
\int_a^m dx &= \frac{1}{2}b - \frac{1}{2}a \\
x \Big|_a^m &= \frac{1}{2}b - \frac{1}{2}a \\
m - a &= \frac{1}{2}b - \frac{1}{2}a \\
m &= \frac{1}{2}b + \frac{1}{2}a \\
m &= \frac{b+a}{2}
\end{aligned}$$

19. Let m be the median of the exponential distribution $f(x) = ae^{-ax}$ for $[0, \infty)$.

$$\begin{aligned}
\int_0^m ae^{-ax} dx &= 0.5 \\
-e^{-ax} \Big|_0^m &= 0.5 \\
-e^{-am} + 1 &= 0.5 \\
0.5 &= e^{-am} \\
-am &= \ln 0.5 \\
m &= -\frac{\ln 0.5}{a} \\
\text{or } -am &= \ln \frac{1}{2} \\
-am &= -\ln 2 \\
m &= \frac{\ln 2}{a}
\end{aligned}$$

20. $f(x) = ae^{-ax}$ for $[0, \infty)$ with $a > 0$.

$$\mu = \int_0^\infty xae^{-ax} dx$$

We first consider $\int xae^{-ax} dx$ using integration by parts.

$$\begin{aligned}
\text{Let } u &= x \quad \text{and} \quad dv = ae^{-ax} dx; \\
du &= dx \quad \text{and} \quad v = -e^{-ax}.
\end{aligned}$$

$$\begin{aligned}
\int xae^{-ax} dx &= -xe^{-ax} - \int (-e^{-ax}) dx \\
&= -xe^{-ax} - \frac{1}{a}e^{-ax} + C \\
&= -\frac{ax+1}{ae^{ax}} + C
\end{aligned}$$

$$\begin{aligned}
\mu &= \int_0^\infty xae^{-ax} dx = \lim_{b \rightarrow \infty} \int_0^b xae^{-ax} dx \\
&= \lim_{b \rightarrow \infty} \left(-\frac{ax+1}{ae^{ax}} \right) \Big|_0^b = \lim_{b \rightarrow \infty} \left(-\frac{ab+1}{ae^{ab}} + \frac{1}{a} \right) \\
\mu &= \frac{1}{a}
\end{aligned}$$

since e^{ab} grows more rapidly than ab .

$$\text{Var}(X) = \int_0^\infty x^2 ae^{-ax} dx - \left(\frac{1}{a} \right)^2$$

We first consider $\int x^2 ae^{-ax} dx$ using integration by parts.

$$\begin{aligned}
\text{Let } u &= x \quad \text{and} \quad dv = xae^{-ax} dx \\
du &= dx \quad \text{and} \quad v = -xe^{-ax} - \frac{1}{a}e^{-ax}.
\end{aligned}$$

from the previous integration by parts.

$$\begin{aligned}
&\int x^2 ae^{-ax} dx \\
&= x \left(-xe^{-ax} - \frac{1}{a}e^{-ax} \right) \\
&\quad - \int \left(-xe^{-ax} - \frac{1}{a}e^{-ax} \right) dx \\
&= -x^2 e^{-ax} - \frac{1}{a}xe^{-ax} \\
&\quad + \int xe^{-ax} dx + \frac{1}{a} \int e^{-ax} dx \\
&= -x^2 e^{-ax} - \frac{1}{a}xe^{-ax} \\
&\quad + \frac{1}{a} \left(-xe^{-ax} - \frac{1}{a}e^{-ax} \right) - \frac{1}{a^2}e^{-ax} \\
&= -x^2 e^{-ax} - \frac{1}{a}xe^{-ax} \\
&\quad - \frac{1}{a}xe^{-ax} - \frac{1}{a^2}e^{-ax} - \frac{1}{a^2}e^{-ax} \\
&= -\frac{a^2 x^2 + 2ax + 2}{a^2 e^{ax}}
\end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= \int_0^\infty x^2 a e^{-ax} dx - \frac{1}{a^2} \\
 &= \lim_{b \rightarrow \infty} \int_0^b x^2 a e^{-ax} dx - \frac{1}{a^2} \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{a^2 x^2 + 2ax + 2}{a^2 e^{ax}} \right]_0^b - \frac{1}{a^2} \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{a^2 x^2 + 2ax + 2}{a^2 e^{ab}} + \frac{2}{a^2} \right] - \frac{1}{a^2} \\
 &= \frac{2}{a^2} - \frac{1}{a^2},
 \end{aligned}$$

since e^{ab} grows more rapidly than $a^2 b^2 + 2ab + 2$.

$$\text{Var}(X) = \frac{1}{a^2}$$

Thus,

$$\sigma = \sqrt{\frac{1}{a^2}} = \frac{1}{a}.$$

21. The area that is to the left of x is

$$A = \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt.$$

Let $u = \frac{(t-\mu)}{\sigma}$. Then $du = \frac{1}{\sigma} dt$ and $dt = \sigma du$.

If $t = x$,

$$u = \frac{x - \mu}{\sigma} = z.$$

As $t \rightarrow -\infty$, $u \rightarrow -\infty$.

Therefore,

$$\begin{aligned}
 A &= \int_{-\infty}^z \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)u^2} \sigma du \\
 &= \frac{\sigma}{\sigma} \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \\
 &= \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.
 \end{aligned}$$

This is the area to the left of z for the standard normal curve.

$$\begin{aligned}
 22. \quad f(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \\
 f'(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \left[-2 \frac{(x-\mu)}{2\sigma^2} \right] \\
 f'(x) &= \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi} e^{(x-\mu)^2/2\sigma^2}} \\
 f''(x) &= \frac{1}{\sigma^3\sqrt{2\pi}} \left[\frac{e^{(x-\mu)^2/2\sigma^2}(-1) + (x-\mu)e^{(x-\mu)^2/2\sigma^2} \frac{(x-\mu)}{\sigma^2}}{e^{(x-\mu)^2/\sigma^2}} \right] \\
 f''(x) &= \frac{-e^{(x-\mu)^2/2\sigma^2} \left(1 - \frac{(x-\mu)^2}{\sigma^2} \right)}{\sigma^3\sqrt{2\pi} e^{(x-\mu)^2/\sigma^2}} \\
 f''(x) &= -\frac{1 - \frac{(x-\mu)^2}{\sigma^2}}{\sigma^3\sqrt{2\pi} e^{(x-\mu)^2/2\sigma^2}}
 \end{aligned}$$

If $f''(x) = 0$, then $1 - \frac{(x-\mu)^2}{\sigma^2} = 0$

$$\frac{(x-\mu)^2}{\sigma^2} = 1$$

$$(x-\mu)^2 = \sigma^2$$

$$(x-\mu) = \pm\sigma$$

$$x = \mu \pm \sigma.$$

If $x < \mu - \sigma$, $f''(x) > 0$.

If $\mu - \sigma < x < \mu + \sigma$, $f''(x) < 0$.

If $x > \mu + \sigma$, $f''(x) > 0$.

Therefore, there are points of inflection when $x = \mu - \sigma$ and when $x = \mu + \sigma$.

In Exercise 23, use Simpson's rule with $n = 140$ or the integration feature on a graphing calculator to approximate the integrals. Answers may vary slightly from those given here depending on the method that is used.

$$23. \quad (a) \quad \int_0^{35} 0.5e^{-0.5x} dx \approx 1.00000$$

$$(b) \quad \int_0^{35} 0.5xe^{-0.5x} dx \approx 1.99999$$

$$(c) \quad \int_0^{35} 0.5x^2e^{-0.5x} dx \approx 8.00000$$

24. The exponential distribution with $a = 0.5$ has $\mu = \frac{1}{0.5} = 2$ and $\sigma = \frac{1}{0.5} = 2$. The answer in part (b) of Exercise 23 approximates the mean very closely. If we approximated σ using the integrals in parts (b) and (c) of Exercise 23 we would get $\sqrt{8.00000 - 1.99999^2} \approx 2.00001$, again very close to the exact value.

25. Use Simpson's rule with $n = 40$ and limits of -6 and 6 to approximate the mean and standard deviation of a normal probability distribution.

$$\begin{aligned} \text{(a)} \quad & \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx \\ & \approx \int_{-6}^6 \frac{x}{\sqrt{2\pi}} e^{-x^2/2} dx \end{aligned}$$

$$\mu = 0$$

(For the integral of an odd function over an interval symmetric to 0, Simpson's rule will give 0; there is no need to do any calculation.)

$$\begin{aligned} \text{(b)} \quad & \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx \\ & \approx \int_{-6}^6 \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx \\ \sigma & \approx 0.9999999224 \quad (\text{Simpson's rule}) \\ \sigma & \approx 0.9999999251 \quad (\text{calculator}) \end{aligned}$$

26. Use $f(x) = abx^{b-1}e^{-ax^b}$ with $a = 4$ and $b = 1.5$.

$$\begin{aligned} f(x) &= (4)(1.5)x^{1.5-1}e^{-4x^{1.5}} \\ &= 6x^{0.5}e^{-4x^{1.5}} \end{aligned}$$

$$\begin{aligned} \text{(a)} \quad \mu &= \int_0^{\infty} x(6x^{0.5}e^{-4x^{1.5}})dx \\ &= \int_0^{\infty} 6x^{1.5}e^{-4x^{1.5}}dx \\ \mu &\approx \int_0^3 6x^{1.5}e^{-4x^{1.5}}dx \end{aligned}$$

Using the integration feature on a graphing calculator,

$$\mu \approx 0.3583.$$

$$\text{(b)} \quad \sigma^2 \approx \int_0^3 (x - 0.3583)^2 6x^{0.5}e^{-4x^{1.5}} dx$$

Using the integration feature on a graphing calculator,

$$\sigma^2 \approx 0.05916$$

$$\sigma \approx 0.2432.$$

27. The probability density function for the uniform distribution is $f(x) = \frac{1}{b-a}$ for x in $[a, b]$.

The cumulative distribution function for f is

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_a^x f(t) dt \\ &= \int_a^x \frac{1}{b-a} dt \\ &= \frac{1}{b-a} t \Big|_a^x \\ &= \frac{1}{b-a} (x-a) \\ &= \frac{x-a}{b-a}, a \leq x \leq b. \end{aligned}$$

28. The probability density function for the uniform distribution is $f(x) = ae^{-ax}$ for x in $[0, \infty)$.

The cumulative distribution function for f is

$$\begin{aligned} F(x) &= P(T \leq x) \\ &= \int_0^x f(t) dt \\ &= \int_0^x ae^{-at} dt \\ &= -e^{-at} \Big|_0^x \\ &= -e^{-ax} + -e^{-a(0)} \\ &= 1 - e^{-ax}, x \geq 0. \end{aligned}$$

29. For a uniform distribution,

$$f(x) = \frac{1}{b-a} \text{ for } [a, b].$$

Thus, we have

$$f(x) = \frac{1}{85-10} = \frac{1}{75}$$

for $[10, 85]$.

$$\begin{aligned} \text{(a)} \quad \mu &= \frac{1}{2}(10 + 85) = \frac{1}{2}(95) \\ &= 47.5 \text{ thousands} \end{aligned}$$

Therefore, the agent sells \$47,500 in insurance.

$$\begin{aligned} \text{(b)} \quad P(50 < X < 85) &= \int_{50}^{85} \frac{1}{75} dx \\ &= \left. \frac{x}{75} \right|_{50}^{85} \\ &= \frac{85}{75} - \frac{50}{75} \\ &= \frac{35}{75} = 0.4667 \end{aligned}$$

30. We have an exponential distribution with mean $\mu = 5$.

$$\begin{aligned} \mu &= \frac{1}{a} = 5 \\ a &= 0.2 \end{aligned}$$

$$\text{(a)} \quad f(x) = 0.2e^{-0.2x} \text{ for } [0, \infty)$$

$$\begin{aligned} \text{(b)} \quad P(2 < X < 6) &= \int_2^6 0.2e^{-0.2x} dx \\ &= -e^{-0.2x} \Big|_2^6 \\ &= e^{-0.4} - e^{-1.2} \\ &\approx 0.3691 \end{aligned}$$

31. (a) Since we have exponential distribution with $\mu = 4.25$,

$$\begin{aligned} \mu &= \frac{1}{a} = 4.25 \\ a &= 0.235. \end{aligned}$$

Therefore, $f(x) = 0.235e^{-0.235x}$ on $[0, \infty)$.

$$\begin{aligned} \text{(b)} \quad P(X > 10) &= \int_{10}^{\infty} 0.235e^{-0.235x} dx \\ &= \lim_{a \rightarrow \infty} \int_{10}^a 0.235e^{-0.235x} dx \\ &= \lim_{a \rightarrow \infty} (-e^{-0.235x}) \Big|_{10}^a \\ &= \lim_{a \rightarrow \infty} \left(-\frac{1}{e^{0.235a}} + \frac{1}{e^{2.35}} \right) \\ &= \frac{1}{e^{2.35}} = 0.0954 \end{aligned}$$

32. We have a normal distribution with $\mu = 32.8$, $\sigma = 1.1$.

Let x = number of ounce of juice.

$$\begin{aligned} \text{(a)} \quad P(X < 32) &= P\left(\frac{x - 32.8}{1.1} < \frac{32 - 32.8}{1.1}\right) \\ &= P(z < -0.73) \\ &= 0.2327 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(X > 33) &= P\left(\frac{x - 32.8}{1.1} < \frac{33 - 32.8}{1.1}\right) \\ &= P(z > 0.18) \\ &= 1 - P(z \leq 0.18) \\ &= 1 - 0.5714 \\ &= 0.4286 \end{aligned}$$

33. (a) $\mu = 2.5$, $\sigma = 0.2$, $x = 2.7$

$$z = \frac{2.7 - 2.5}{0.2} = 1$$

Area to the right of $z = 1$ is

$$1 - 0.8413 = 0.1587.$$

Probability = 0.1587

- (b) Within 1.2 standard deviations of the mean is the area between $z = -1.2$ and $z = 1.2$.

Area to left of $z = 1.2 = 0.8849$

Area to the left of the $z = -1.2 = 0.1151$

$$0.8849 - 0.1151 = 0.7698$$

Probability = 0.7698

34. We have a normal distribution, with $\mu = 54.40$, $\sigma = 13.50$.

$$\begin{aligned} P(-a < z < a) &= \frac{1}{2} \\ P(z < -a) &= 0.25 \end{aligned}$$

Since the closest value to 0.25 is 0.2514, we use $z = -0.67$.

$$-0.67 < z < 0.67$$

$$-0.67 < \frac{x - 54.40}{13.50} < 0.67$$

$$45.36 < x < 63.45$$

Therefore, $P(45.36 < X < 63.45) = \frac{1}{2}$, and the middle 50% of the customers spend between \$45.36 and \$63.45.

35. If X has a uniform distribution on $[0, 1000]$, then its density function is $f(x) = \frac{1}{1000}$ for x in $[0, 1000]$. The expected payment with no deductible is

$$\begin{aligned} E(X) &= \int_0^{1000} x \cdot \frac{1}{1000} dx \\ &= \frac{1}{1000} \cdot \frac{1}{2} x^2 \Big|_0^{1000} \\ &= \frac{1}{2000} \cdot (1000^2 - 0) \\ &= 500. \end{aligned}$$

Now, let the deductible be D . According to the hint,

$$\text{payment} = \begin{cases} 0 & \text{for } x \leq D \\ x - D & \text{for } x > D. \end{cases}$$

The expected payment with the deductible is therefore

$$\begin{aligned} E(X) &= \int_0^D 0 \cdot \frac{1}{1000} dx + \int_D^{1000} (x - D) \cdot \frac{1}{1000} dx \\ &= 0 + \frac{1}{1000} \cdot \left(\frac{1}{2} x^2 - Dx \right) \Big|_D^{1000} \\ &= \frac{1}{1000} \cdot \left[\left(\frac{1}{2} 1000^2 - 1000D \right) - \left(\frac{1}{2} D^2 - D^2 \right) \right] \\ &= 500 - D + \frac{1}{2000} D^2. \end{aligned}$$

For this amount to be 25% of the amount with no deductible, we must have

$$\begin{aligned} 500 - D + \frac{1}{2000} D^2 &= 0.25 \cdot 500 \\ \frac{1}{2000} D^2 - D + 500 &= 125 \\ \frac{1}{2000} D^2 - D + 375 &= 0 \\ D^2 - 2000D + 750,000 &= 0 \\ (D - 1500)(D - 500) &= 0 \\ D = 1500 \text{ or } D = 500 \end{aligned}$$

We reject $D = 1500$ since it is not in $[0, 1000]$.

Therefore, $D = 500$. The correct answer choice is **c**.

36. Let the random variable X be the number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is

involved in an accident. Then it has exponential distribution $f(x) = ae^{-ax}$ for $x \geq 0$.

We are given $P(0 \leq X \leq 50) = 0.3$ so that

$$\begin{aligned} \int_0^{60} ae^{-ax} dx &= 0.3 \\ -e^{-ax} \Big|_0^{50} &= 0.3 \\ -e^{-50a} + 1 &= 0.3 \\ e^{-50a} &= 0.7 \\ -50a &= \ln 0.7 \\ a &= -\frac{1}{50} \ln 0.7 \approx 0.007133 \end{aligned}$$

Thus, $f(x) = 0.007133e^{-0.007133x}$ for $x \geq 0$.

The portion expected to be involved in an accident during the first 80 days is

$$\begin{aligned} \int_0^{80} 0.007133e^{-0.007133x} dx \\ &= -e^{-0.007133x} \Big|_0^{80} \\ &= -e^{-0.007133(80)} + 1 \\ &\approx 0.43 \end{aligned}$$

The correct answer choice is **c**.

37. Let the random variable X be the lifetime of the printer in a years. Then it has exponential distribution $f(x) = ae^{-ax}$ for $x \geq 0$.

If the mean is 2 years, then $\frac{1}{a} = 2$, or $a = \frac{1}{2}$ and the function is $f(x) = \frac{1}{2}e^{-x/2}$ for $x \geq 0$.

We wish to find $P(0 \leq X \leq 1)$ and $P(1 \leq X \leq 2)$.

$$\begin{aligned} P(0 \leq X \leq 1) &= \int_0^1 \frac{1}{2} e^{-x/2} dx \\ &= -e^{-x/2} \Big|_0^1 \\ &= -e^{-1/2} + 1 \end{aligned}$$

$$\begin{aligned} P(1 \leq X \leq 2) &= \int_1^2 \frac{1}{2} e^{-x/2} dx \\ &= -e^{-x/2} \Big|_1^2 \\ &= -e^{-1} + e^{-1/2} \end{aligned}$$

If 100 printers are sold, then $(1 - e^{-1/2} + 1)(100)$ will fail in the first year, and $(e^{-1/2} - e^{-1})(100)$ will fail in the second year.

The manufacturer pays a full refund on those failing first year and one-half refund on those failing during the second year.

$$\begin{aligned}\text{Refunds} &= (1 - e^{-1/2} + 1)(100)(\$200) \\ &\quad + (e^{-1/2} - e^{-1})(100)(\$100) \\ &\approx \$10,255.90\end{aligned}$$

The correct answer choice is **d**.

- 38.** Let the random variable X be the life time to failure. Then it has exponential distribution $f(x) = ae^{-ax}$ for $x \geq 0$.

If the mean is 4 hours, then

$$\begin{aligned}\int_0^4 ae^{-ax} dx &= \frac{1}{2} \\ -e^{-ax} \Big|_0^4 &= \frac{1}{2} \\ -e^{-4a} + 1 &= \frac{1}{2} \\ -e^{-4a} &= -\frac{1}{2} \\ -4a &= \ln \frac{1}{2} \\ a &= -\frac{1}{4} \ln \frac{1}{2} \approx 0.1733\end{aligned}$$

So the exponential function for X is

$f(x) = 0.1733e^{-0.1733x}$ for $x \geq 0$. The probability that the component will work without failing for at least 5 hours is,

$$\begin{aligned}P(5 \leq X \leq \infty) &= \int_5^\infty 0.1733e^{-0.1733x} dx \\ &= -e^{-0.1733x} \Big|_5^\infty \\ &= 0 + e^{-0.1733(5)} \\ &\approx 0.4204\end{aligned}$$

The correct answer choice is **d**.

- 39.** For a uniform distribution,

$$f(x) = \frac{1}{b-a} \text{ for } x \text{ in } [a, b].$$

$$f(x) = \frac{1}{36-20} = \frac{1}{16} \text{ for } x \text{ in } [20, 36]$$

$$\begin{aligned}\text{(a)} \quad \mu &= \frac{1}{2}(20 + 36) = \frac{1}{2}(56) \\ &= 28 \text{ days}\end{aligned}$$

$$\text{(b)} \quad P(30 < X \leq 36)$$

$$\begin{aligned}&= \int_{30}^{36} \frac{1}{16} dx = \frac{1}{16} x \Big|_{30}^{36} \\ &= \frac{1}{16}(36 - 30) \\ &= 0.375\end{aligned}$$

- 40.** For an exponential distribution, $f(x) = ae^{-ax}$ for $[0, \infty)$. Since $a = 2$, $f(x) = 2e^{-2x}$ for $[0, 1]$.

- (a)** The expected proportion is

$$\mu = \frac{1}{2} = 0.5.$$

$$\begin{aligned}\text{(b)} \quad P(0 < X < \frac{1}{3}) &= \int_0^{1/3} 2e^{-2x} dx \\ &= -e^{-2x} \Big|_0^{1/3} \\ &= -\frac{1}{e^{2/3}} + 1 \\ &\approx 0.4866\end{aligned}$$

- 41.** We have an exponential distribution, with $a = 1$. $f(t) = e^{-t}$, $[0, \infty)$

$$\text{(a)} \quad \mu = \frac{1}{1} = 1 \text{ hr}$$

$$\text{(b)} \quad P(T < 30 \text{ min})$$

$$\begin{aligned}&= \int_0^{0.5} e^{-t} dt \\ &= -e^{-t} \Big|_0^{0.5} \\ &= 1 - e^{-0.5} \approx 0.3935\end{aligned}$$

$$\text{42.} \quad \mu = 3.2 \text{ ft}, \sigma = 0.2 \text{ ft}$$

If we wish to find the middle 50%, or 0.50, we want to find the value of z corresponding to a probability of 0.25 in the standard normal distribution table. This is about 0.675. Since the desired area is symmetric about the mean, we use ± 0.675 .

$$\begin{aligned}
 z &= \frac{x - \mu}{\sigma} \\
 \pm 0.675 &= \frac{x - 3.2}{0.2} \\
 \pm 0.135 &= x - 3.2 \\
 x &= 3.2 \pm 0.135 \\
 x &= 3.065, 3.335
 \end{aligned}$$

The largest height is 3.335 ft, and the smallest height is 3.065 ft.

43. $f(x) = ae^{-ax}$ for $[0, \infty]$

Since $\mu = 25$ and $\mu = \frac{1}{a}$,

$$a = \frac{1}{25} = 0.04.$$

This, $f(x) = 0.04e^{-0.04x}$.

(a) We must find t such that $P(X \leq t) = 0.90$.

$$\begin{aligned}
 \int_0^t 0.04e^{-0.04x} dx &= 0.90 \\
 -e^{-0.04x} \Big|_0^t &= 0.90 \\
 -e^{-0.04t} + 1 &= 0.90 \\
 0.10 &= -e^{-0.04t} \\
 -0.04t &= \ln 0.10 \\
 t &= \frac{\ln 0.10}{-0.04} \\
 t &\approx 57.56
 \end{aligned}$$

The longest time within which the predator will be 90% certain of finding a prey is approximately 58 min.

(b) $P(X \geq 60)$

$$\begin{aligned}
 &= \int_{60}^{\infty} 0.04e^{-0.04x} dx \\
 &= \lim_{b \rightarrow \infty} \int_{60}^b 0.04e^{-0.04x} dx \\
 &= \lim_{b \rightarrow \infty} (e^{-0.04x}) \Big|_{60}^b \\
 &= \lim_{b \rightarrow \infty} [-e^{-0.04b} + e^{-0.04(60)}] \\
 &= 0 + e^{-2.4} \\
 &\approx 0.0907
 \end{aligned}$$

The probability that the predator will have to spend more than one hour looking for a prey is approximately 0.0907.

44. For an exponential distribution,

$f(x) = ae^{-ax}$ for x in $[0, \infty)$.

Since $\mu = \frac{1}{a} = 26.1$, $a = \frac{1}{26.1}$.

(a) $P(X \geq 25)$

$$\begin{aligned}
 &= \int_{25}^{\infty} \frac{1}{26.1} e^{-x/26.1} dx \\
 &= \lim_{b \rightarrow \infty} \int_{25}^b \frac{1}{26.1} e^{-x/26.1} dx \\
 &= \lim_{b \rightarrow \infty} \left(\frac{1}{26.1} e^{-x/26.1} \Big|_{25}^b \right) \\
 &= \lim_{b \rightarrow \infty} (-e^{-b/26.1} + e^{-25/26.1}) \\
 &= e^{-25/26.1} \\
 &\approx 0.3837
 \end{aligned}$$

(b) $P(X < 20) = \int_0^{20} \frac{1}{26.1} e^{-x/26.1} dx$

$$\begin{aligned}
 &= (-e^{-x/26.1}) \Big|_0^{20} \\
 &= -e^{-20/26.1} + 1 \\
 &\approx 0.5353
 \end{aligned}$$

45. For an exponential distribution,

$f(x) = ae^{-ax}$ for x in $[0, \infty)$.

Since $\mu = \frac{1}{a} = 12.3$, $a = \frac{1}{12.3}$.

(a) $P(X \geq 20) = \int_{20}^{\infty} \frac{1}{12.3} e^{-x/12.3} dx$

$$\begin{aligned}
 &= \lim_{b \rightarrow \infty} \int_{20}^b \frac{1}{12.3} e^{-x/12.3} dx \\
 &= \lim_{b \rightarrow \infty} \left(\frac{1}{12.3} e^{-x/12.3} \Big|_{20}^b \right) \\
 &= \lim_{b \rightarrow \infty} (-e^{-b/12.3} + e^{-20/12.3}) \\
 &= e^{-20/12.3} \\
 &\approx 0.1967
 \end{aligned}$$

(b) $P(10 \leq X \leq 20) = \int_{10}^{20} \frac{1}{12.3} e^{-x/12.3} dx$

$$\begin{aligned}
 &= (-e^{-x/12.3}) \Big|_{10}^{20} \\
 &= -e^{-20/12.3} + e^{-10/12.3} \\
 &\approx 0.2468
 \end{aligned}$$

46. (a) We have a normal distribution with $\mu = 6.9$ and $\sigma = 4.6$.

$$\begin{aligned} P(X > 6) &= P\left(z \geq \frac{6 - 6.9}{4.6}\right) \\ &= P(z \geq -0.19565) \\ &\approx 0.5793 \end{aligned}$$

No, this is insufficient evidence.

- (b) We have a normal distribution with $\mu = 0.6$ and $\sigma = 0.3$.

$$\begin{aligned} P(X \geq 6) &= P\left(z \geq \frac{6 - 0.6}{0.3}\right) \\ &= P(z \geq 18) \\ &\approx 0 \end{aligned}$$

Yes, by today's standards there is sufficient evidence to conclude that Andrew Jackson suffered from mercury poisoning.

47. We have an exponential distribution, with $a = 0.229$.

So $f(t) = 0.229e^{-0.229t}$, for $[0, \infty)$.

- (a) The life expectancy is

$$\mu = \frac{1}{a} = \frac{1}{0.229} \approx 4.37 \text{ millennia.}$$

The standard deviation is

$$\sigma = \frac{1}{a} = \frac{1}{0.229} \approx 4.37 \text{ millennia.}$$

$$\begin{aligned} \text{(b)} \quad P(T \geq 2) &= \int_2^\infty 0.229e^{-0.229t} dt \\ &= 1 - \int_0^2 0.229e^{-0.229t} dt \\ &= 1 + \left(e^{-0.229t} \Big|_0^2 \right) \\ &= 1 + \left[e^{-0.229(2)} - 1 \right] \\ &= e^{-0.458} \approx 0.6325 \end{aligned}$$

48. Uniform distribution on $[32, 44]$

$$f(x) = \frac{1}{44 - 32} = \frac{1}{12} \text{ for } [32, 44]$$

$$\text{(a)} \quad \mu = \frac{1}{2}(32 + 44) = 38 \text{ inches}$$

- (b) $P(38 < X < 40)$

$$\begin{aligned} &= \int_{38}^{40} \frac{1}{12} dx \\ &= \frac{x}{12} \Big|_{38}^{40} \\ &= \frac{1}{6} \approx 0.1667 \end{aligned}$$

49. For an exponential distribution, $f(x) = ae^{-ax}$ for $[0, \infty)$.

Since $\mu = \frac{1}{a} = 8$, $a = \frac{1}{8}$.

$$\begin{aligned} \text{(a)} \quad P(X \geq 10) &= \int_{10}^\infty \frac{1}{8} e^{-x/8} dx \\ &= 1 - \int_0^{10} \frac{1}{8} e^{-x/8} dx \\ &= 1 + \left(e^{-x/8} \Big|_0^{10} \right) \\ &= 1 + [e^{-10/8} - 1] \\ &= e^{-10/8} \approx 0.2865 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad P(X < 2) &= \int_0^2 \frac{1}{8} e^{-x/8} dx \\ &= -e^{-x/8} \Big|_0^2 \\ &= -e^{-2/8} + 1 \\ &\approx 0.2212 \end{aligned}$$

50. We have an exponential distribution, $f(x) = ae^{-ax}$ for $[0, \infty)$. Since

$$a = \frac{1}{609.5}, f(x) = \frac{1}{609.5} e^{-x/609.5}.$$

- (a) The expected number of days is $\mu = \frac{1}{a} = 609.5$. The standard deviation is $\sigma = \frac{1}{a} = 609.5$.

$$\begin{aligned} \text{(b)} \quad P(x > 365) &= \int_{365}^\infty \frac{1}{609.5} e^{-x/609.5} dx \\ &= 1 - \int_0^{365} \frac{1}{609.5} e^{-x/609.5} dx \\ &= 1 + \left(e^{-x/609.5} \Big|_0^{365} \right) \\ &= 1 + (e^{-365/609.5} - 1) \\ &= e^{-365/609.5} \approx 0.5494 \end{aligned}$$

51. We have an exponential distribution $f(x)$
 $= ae^{-ax}$ for $x \geq 0$. Since $a = \frac{1}{90}$, $f(x) =$
 $\frac{1}{90}e^{-x/90}$ for $x \geq 0$.

- (a) The probability that the time for a goal is no more than 71 minutes is

$$\begin{aligned} P(0 < X < 71) &= \int_0^{71} \frac{1}{90} e^{-x/90} dx \\ &= -e^{-x/90} \Big|_0^{71} \\ &= -e^{-71/90} + 1 \\ &\approx 0.5457. \end{aligned}$$

- (b) The probability that the time for a goal is 499 minutes or more is

$$\begin{aligned} P(X \geq 499) &= \int_{499}^{\infty} \frac{1}{90} e^{-x/90} dx \\ &= e^{-x/90} \Big|_{499}^{\infty} \\ &= 0 + e^{-499/90} \\ &\approx 0.0039. \end{aligned}$$

52. We have a normal distribution with $\mu = 0$ and $\sigma = 13.861$. The probability that the margin of victory over the point spread is greater than -3 is

$$\begin{aligned} P(X > -3) &= P\left(z \geq \frac{-3 - 0}{13.861}\right) \\ &= P(z \geq -0.2164) \\ &= P(z \leq 0.2164). \end{aligned}$$

Using the normal table in the book with $z = 0.22$, we have the probability is about 0.5871.

6. True
7. True
8. True
9. False: The normal distribution is symmetrical; the exponential distribution has a long tail to the right.
10. False: The expected value is 0 and the standard deviation is 1.
11. In a probability function, the y-values (or function values) represent probabilities.
13. A probability density function f for $[a, b]$ must satisfy the following two conditions:
- (1) $f(x) \geq 0$ for all x in the interval $[a, b]$;
- (2) $\int_a^b f(x) dx = 1$.
14. In a probability density function, the probability that X equals a specific value, $P(X = c)$, is zero.

15. $f(x) = \sqrt{x}$; $[4, 9]$

$$\begin{aligned} \int_4^9 x^{1/2} dx &= \frac{2}{3} x^{3/2} \Big|_4^9 \\ &= \frac{2}{3} (27 - 8) \\ &= \frac{38}{3} \neq 1 \end{aligned}$$

$f(x)$ is not a probability density function.

16. $f(x) = \frac{1}{27}(2x + 4)$; $[1, 4]$

$$\begin{aligned} \int_1^4 \frac{1}{27}(2x + 4) dx &= \frac{1}{27} (x^2 + 4x) \Big|_1^4 \\ &= \frac{1}{27} (32 - 5) \\ &= 1 \end{aligned}$$

Since $1 \leq x \leq 4$, $f(x) \geq 0$.

Therefore, $f(x)$ is a probability density function.

Chapter 11 Review Exercises

- True
- True
- True
- False: A density function is always nonnegative.
- False: If the random variable takes on negative values the expectation may also be negative.

17. $f(x) = 0.7e^{-0.7x}; [0, \infty)$

$$\begin{aligned}\int_0^{\infty} 0.7e^{-0.7x} dx &= -e^{-0.7x} \Big|_0^{\infty} \\ &= \lim_{b \rightarrow \infty} (-e^{-0.7b}) + e^0 \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{e^{0.7b}} \right) + 1 \\ &= 0 + 1 = 1\end{aligned}$$

$f(x) \geq 0$ for all x in $[0, \infty)$.

Therefore, $f(x)$ is a probability density function.

18. $f(x) = 0.4; [4, 6.5]$

$$\begin{aligned}\int_4^{6.5} 0.4 dx &= 0.4x \Big|_4^{6.5} \\ &= 0.4(6.5 - 4) = 1\end{aligned}$$

$f(x) \geq 0$ for all x in $[4, 6.5]$.

Therefore, $f(x)$ is a probability density function.

19. $f(x) = kx^2; [1, 4]$

$$\begin{aligned}\int_1^4 kx^2 dx &= \frac{kx^3}{3} \Big|_1^4 \\ &= 21k\end{aligned}$$

Since $f(x)$ is a probability density function,

$$\begin{aligned}21k &= 1 \\ k &= \frac{1}{21}.\end{aligned}$$

20. $f(x) = k\sqrt{x}; [4, 9]$

$$\begin{aligned}\int_4^9 k\sqrt{x} dx &= \int_4^9 kx^{1/2} dx \\ &= \frac{2}{3} kx^{3/2} \Big|_4^9 \\ &= \frac{2}{3} k(27 - 8) \\ &= \frac{38}{3} k\end{aligned}$$

Since $f(x)$ is a probability density function,

$$\begin{aligned}\frac{38}{3} k &= 1 \\ k &= \frac{3}{38}.\end{aligned}$$

21. $f(x) = \frac{1}{10}$ for $[10, 20]$

(a) $P(10 \leq X \leq 12)$

$$\begin{aligned}&= \int_{10}^{12} \frac{1}{10} dx \\ &= \frac{x}{10} \Big|_{10}^{12} \\ &= \frac{1}{5} = 0.2\end{aligned}$$

(b) $P\left(\frac{31}{2} \leq X \leq 20\right)$

$$\begin{aligned}&= \int_{31/2}^{20} \frac{1}{10} dx \\ &= \frac{x}{10} \Big|_{31/2}^{20} \\ &= 2 - \frac{31}{20} \\ &= \frac{9}{20} = 0.45\end{aligned}$$

(c) $P(10.8 \leq X \leq 16.2)$

$$\begin{aligned}&= \int_{10.8}^{16.2} \frac{1}{10} dx \\ &= \frac{x}{10} \Big|_{10.8}^{16.2} = 0.54\end{aligned}$$

22. $f(x) = 1 - \frac{1}{\sqrt{x-1}}; [2, 5]$

(a) $P(3 \leq X \leq 5)$

$$\begin{aligned}&= \int_3^5 [1 - (x-1)^{-1/2}] dx \\ &= [x - 2(x-1)^{1/2}] \Big|_3^5 \\ &= 5 - 2(2) - 3 + 2\sqrt{2} \\ &\approx 0.8284\end{aligned}$$

(b) $P(2 \leq X \leq 4)$

$$\begin{aligned}&= \int_2^4 [1 - (x-1)^{-1/2}] dx \\ &= [x - 2(x-1)^{1/2}] \Big|_2^4 \\ &= 4 - 2\sqrt{3} - 2 + 2 \\ &\approx 0.5359\end{aligned}$$

$$(c) \quad P(3 \leq X \leq 4)$$

$$\begin{aligned} &= \int_3^4 [1 - (x-1)^{-1/2}] dx \\ &= [x - 2(x-1)^{1/2}]_3^4 \\ &= 4 - 2\sqrt{3} - 3 + 2\sqrt{2} \\ &\approx 0.3643 \end{aligned}$$

23. If we consider the probabilities as weights, the expected value or mean of a probability distribution represents the point at which the distribution balances.

24. The distribution that is tallest or most peaked has the smallest standard deviation. This is the distribution pictured in graph (b).

$$25. \quad f(x) = \frac{2}{9}(x-2); [2, 5]$$

$$\begin{aligned} (a) \quad \mu &= \int_2^5 \frac{2x}{9}(x-2) dx \\ &= \int_2^5 \frac{2}{9}(x^2 - 2x) dx \\ &= \frac{2}{9} \left(\frac{x^3}{3} - x^2 \right) \Big|_2^5 \\ &= \frac{2}{9} \left(\frac{125}{3} - 25 - \frac{8}{3} + 4 \right) = 4 \end{aligned}$$

$$\begin{aligned} (b) \quad \text{Var}(X) &= \int_2^5 \frac{2x^2}{9}(x-2) dx - (4)^2 \\ &= \int_2^5 \frac{2}{9}(x^3 - 2x^2) dx - 16 \\ &= \frac{2}{9} \left(\frac{x^4}{4} - \frac{2x^3}{3} \right) \Big|_2^5 - 16 \\ &= \frac{2}{9} \left(\frac{625}{4} - \frac{250}{3} - 4 + \frac{16}{3} \right) - 16 \\ &= 0.5 \end{aligned}$$

$$(c) \quad \sigma = \sqrt{0.5} \approx 0.7071$$

$$(d) \quad \int_2^m \frac{2}{9}(x-2) dx = \frac{1}{2}$$

$$\frac{1}{9}(m-2)^2 \Big|_2^m = \frac{1}{2}$$

$$\frac{1}{9}[(m-2)^2 - 0] = \frac{1}{2}$$

$$m^2 - 4m + 4 = \frac{9}{2}$$

$$m^2 - 4m - \frac{1}{2} = 0$$

$$m = \frac{4 \pm 3\sqrt{2}}{2}$$

$$\approx -0.121, 4.121$$

We reject -0.121 since it is not in $[2, 5]$. So,
 $m = 4.121$

$$\begin{aligned} (e) \quad \int_2^x \frac{2}{9}(t-2) dt &= \frac{1}{9}(t-2)^2 \Big|_2^x \\ &= \frac{1}{9}[(x-2)^2 - 0] \\ &= \frac{(x-2)^2}{9}, 2 \leq x \leq 5 \end{aligned}$$

$$26. \quad f(x) = \frac{1}{5}; [4, 9]$$

$$\begin{aligned} (a) \quad E(X) = \mu &= \int_4^9 x \left(\frac{1}{5} \right) dx = \frac{x^2}{10} \Big|_4^9 \\ &= \frac{81}{10} - \frac{16}{10} = \frac{65}{10} = 6.5 \end{aligned}$$

$$\begin{aligned} (b) \quad \text{Var}(X) &= \int_4^9 \frac{1}{5}(x-6.5)^2 dx \\ &= \frac{1}{15}(x-6.5)^3 \Big|_4^9 \\ &= \frac{1}{15}(2.5^3 + 2.5^3) \\ &\approx 2.083 \end{aligned}$$

$$(c) \quad \sigma = \sqrt{\text{Var}(X)} \approx 1.443$$

$$(d) \int_4^m \frac{1}{5} dx = \frac{1}{2}$$

$$\frac{1}{5}x \Big|_4^m = \frac{1}{2}$$

$$\frac{1}{5}(m-4) = \frac{1}{2}$$

$$m-4 = \frac{5}{2}$$

$$m = \frac{13}{2} = 6.5$$

$$(e) F(x) = \int_4^x \frac{1}{5} dt$$

$$= \frac{1}{5}t \Big|_4^x$$

$$= \frac{x-4}{5}, 4 \leq x \leq 9.$$

$$27. f(x) = 5x^{-6}; [1, \infty)$$

$$(a) \mu = \int_1^\infty x \cdot 5x^{-6} dx = \int_1^\infty 5x^{-5} dx$$

$$= \lim_{b \rightarrow \infty} \int_1^b 5x^{-5} dx = \lim_{b \rightarrow \infty} \frac{5x^{-4}}{-4} \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} \frac{5}{4} \left(1 - \frac{1}{b^4} \right) = \frac{5}{4}$$

$$(b) \text{Var}(X) = \int_1^\infty x^2 \cdot 5x^{-6} dx - \left(\frac{5}{4} \right)^2$$

$$= \lim_{b \rightarrow \infty} \int_1^b 5x^{-4} dx - \frac{25}{16}$$

$$= \lim_{b \rightarrow \infty} \frac{5x^{-3}}{-3} \Big|_1^b - \frac{25}{16}$$

$$= \lim_{b \rightarrow \infty} \frac{5}{3} \left(1 - \frac{1}{b^3} \right) - \frac{25}{16}$$

$$= \frac{5}{3} - \frac{25}{16} = \frac{5}{48} \approx 0.1042$$

$$(c) \sigma \approx \sqrt{\text{Var}(X)} \approx 0.3227$$

$$(d) \int_1^m 5x^{-6} dx = \frac{1}{2}$$

$$-x^{-5} \Big|_1^m = \frac{1}{2}$$

$$-m^{-5} + 1 = \frac{1}{2}$$

$$m^{-5} = \frac{1}{2}$$

$$m^5 = 2$$

$$m = \sqrt[5]{2} \approx 1.149$$

$$(e) \int_1^x 5t^{-6} dt = -t^{-5} \Big|_1^x$$

$$= -x^{-5} + 1$$

$$= 1 - \frac{1}{x^5}, x \geq 1$$

$$28. f(x) = \frac{1}{20} \left(1 + \frac{3}{\sqrt{x}} \right); [1, 9]$$

$$(a) E(X) = \mu = \int_1^9 x \left[\frac{1}{20} (1 + 4x^{-1/2}) \right] dx$$

$$= \frac{1}{20} \int_1^9 (x + 3x^{1/2}) dx$$

$$= \frac{1}{20} \left(\frac{x^2}{2} + 2x^{3/2} \right) \Big|_1^9$$

$$= \frac{1}{20} \left(\frac{81}{2} + 54 - \frac{1}{2} - 2 \right)$$

$$= \frac{92}{20} = 4.6$$

$$(b) \text{Var}(X) = \int_1^9 \frac{x^2}{20} (1 + 3x^{-1/2}) dx - (4.6)^2$$

$$= \int_1^9 \left(\frac{x^2}{20} + \frac{3}{20} x^{3/2} \right) dx - (4.6)^2$$

$$= \left(\frac{x^3}{60} + \frac{3}{50} x^{5/2} \right) \Big|_1^9 - (4.6)^2$$

$$= \left(\frac{729}{60} + \frac{729}{50} - \frac{1}{60} - \frac{3}{50} \right)$$

$$- (4.6)^2$$

$$\approx 5.4933$$

$$\approx 5.493$$

$$\begin{aligned} \text{(c)} \quad \sigma &= \sqrt{\text{Var}(X)} \approx 2.3438 \\ \sigma &\approx 2.344 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int_1^m \frac{1}{20} \left(1 + \frac{3}{\sqrt{x}} \right) dx &= \frac{1}{2} \\ \frac{1}{20} (x + 6x^{1/2}) \Big|_1^m &= \frac{1}{2} \\ \frac{1}{20} m + \frac{3}{10} m^{1/2} - \frac{7}{20} &= \frac{1}{2} \\ m + 6\sqrt{m} - 17 &= 0 \end{aligned}$$

To solve this equation, let $u = \sqrt{m}$ so $u^2 = m$.

$$\begin{aligned} u^2 + 6u - 17 &= 0 \\ u &= \frac{-6 \pm \sqrt{36 + 68}}{2} \\ &= -3 \pm \sqrt{26} \\ m = u^2 &\approx 4.4059, 65.5941 \end{aligned}$$

We reject 65.5941 since it is not in $[1, 9]$. So, $m \approx 4.406$.

$$\begin{aligned} \text{(e)} \quad \int_1^x \frac{1}{20} \left(1 + \frac{3}{\sqrt{t}} \right) dt &= \frac{1}{20} (t + 6t^{1/2}) \Big|_1^x \\ &= \frac{1}{20} [(x + 6\sqrt{x}) - (1 + 6)] \\ &= \frac{1}{20} (x + 6\sqrt{x} - 7), \quad 1 \leq x \leq 9 \end{aligned}$$

$$29. \quad f(x) = 4x - 3x^2; [0, 1]$$

$$\begin{aligned} \text{(a)} \quad \mu &= \int_0^1 x(4x - 3x^2) dx \\ &= \int_0^1 (4x^2 - 3x^3) dx \\ &= \left(\frac{4x^3}{3} - \frac{3x^4}{4} \right) \Big|_0^1 \\ &= \frac{4}{3} - \frac{3}{4} = \frac{7}{12} \\ &\approx 0.5833 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{Var}(X) &= \int_0^1 x^2(4x - 3x^2) dx - \left(\frac{7}{12} \right)^2 \\ &= \int_0^1 (4x^3 - 3x^4) dx - \left(\frac{7}{12} \right)^2 \\ &= \left(x^4 - \frac{3x^5}{5} \right) \Big|_0^1 - \left(\frac{7}{12} \right)^2 \\ &= 1 - \frac{3}{5} - \left(\frac{7}{12} \right)^2 \\ &\approx 0.0597 \\ \sigma &\approx \sqrt{\text{Var}(X)} \\ &\approx 0.2444 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad P\left(0 \leq X \leq \frac{7}{12}\right) &= \int_0^{7/12} (4x - 3x^3) dx \\ &= (2x^2 - x^3) \Big|_0^{7/12} \\ &= 2 \left(\frac{7}{12} \right)^2 - \left(\frac{7}{12} \right)^3 \\ &\approx 0.4821 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad P(\mu - \sigma \leq X \leq \mu + \sigma) &\approx P(0.339 \leq x \leq 0.827) \\ &= \int_{0.339}^{0.827} (4x - 3x^3) dx \\ &= (2x^2 - x^3) \Big|_{0.339}^{0.827} \\ &= 2(0.827)^2 - (0.827)^3 \\ &\quad - 2(0.339)^2 + (0.339)^3 \\ &\approx 0.6114 \end{aligned}$$

$$30. \quad f(x) = 4x - 3x^2; [0, 1]$$

$$\begin{aligned} \mu &= \int_0^1 x(4x - 3x^2) dx \\ &= \int_0^1 (4x^2 - 3x^3) dx \\ &= \left(\frac{4x^3}{3} - \frac{3x^4}{4} \right) \Big|_0^1 \\ &= \frac{4}{3} - \frac{3}{4} = \frac{7}{12} \approx 0.5833 \end{aligned}$$

Find m such that

$$\int_0^m (4x - 3x^2) dx = \frac{1}{2}.$$

$$\begin{aligned}\int_0^m (4x - 3x^2) dx &= (2x^2 - x^3) \Big|_0^m \\ &= 2m^2 - m^3 = \frac{1}{2}\end{aligned}$$

Therefore,

$$2m^3 - 4m^2 + 1 = 0.$$

This equation has no rational roots, but trial and error used with synthetic division reveals that $m \approx -0.4516, 0.5970$, and 1.855 . The only one of these in $[0, 1]$ is 0.5970 .

$$\begin{aligned}P\left(\frac{7}{12} < X < 0.5970\right) &= \int_{7/12}^{0.5970} (4x - 3x^2) dx \\ &= 2x^2 - x^3 \Big|_{7/12}^{0.5970} \\ &= 2(0.5970)^2 - (0.5970)^3 - 2\left(\frac{7}{12}\right)^2 \\ &\quad + \left(\frac{7}{12}\right)^3 \approx 0.0180\end{aligned}$$

- 31.** $f(x) = 0.01e^{-0.01x}$ for $[0, \infty)$ is an exponential distribution.

(a) $\mu = \frac{1}{0.01} = 100$

(b) $\sigma = \frac{1}{0.01} = 100$

(c)
$$\begin{aligned}P(100 - 100 < X < 100 + 100) &= P(0 < X < 200) \\ &= \int_0^{200} 0.01e^{-0.01x} dx \\ &= -e^{-0.01x} \Big|_0^{200} \\ &= 1 - e^{-2} \approx 0.8647\end{aligned}$$

32. $f(x) = \frac{5}{112}(1 - x^{-3/2}); [1, 25]$

(a)
$$\begin{aligned}\mu &= \int_1^{25} \frac{5x}{112}(1 - x^{-3/2}) dx \\ &= \frac{5}{112} \int_1^{25} (x - x^{-1/2}) dx \\ &= \frac{5}{112} \left(\frac{x^2}{2} - 2x^{1/2} \right) \Big|_1^{25} \\ &= \frac{5}{112} \left(\frac{625}{2} - 10 - \frac{1}{2} + 2 \right) \\ &= \frac{95}{7} \approx 13.5714 \\ &\approx 13.6\end{aligned}$$

(b)
$$\begin{aligned}\text{Var}(X) &= \int_1^{25} \frac{5x^2}{112}(1 - x^{-3/2}) dx - \left(\frac{95}{7}\right)^2 \\ &= \frac{5}{112} \int_1^{25} (x^2 - x^{1/2}) dx - \left(\frac{95}{7}\right)^2 \\ &= \frac{5}{112} \left(\frac{x^3}{3} - \frac{2}{3}x^{3/2} \right) \Big|_1^{25} - \left(\frac{95}{7}\right)^2 \\ &= \frac{5}{112} \left(\frac{25^3}{3} - \frac{250}{3} - \frac{1}{3} + \frac{2}{3} \right) \\ &\quad - \left(\frac{95}{7}\right)^2 \\ &= \frac{6560}{147} \\ \sigma &= \sqrt{\text{Var}(X)} \approx 6.6803 \\ \sigma &\approx 6.7\end{aligned}$$

(c)
$$\begin{aligned}P(\mu - \sigma \leq X \leq \mu + \sigma) &= P(6.8911 \leq X \leq 20.2517) \\ &= \int_{6.8911}^{20.2517} \frac{5}{112}(1 - x^{-3/2}) dx \\ &= \frac{5}{112} (x + 2x^{-1/2}) \Big|_{6.8911}^{20.2517} \\ &= \frac{5}{112} \left(20.2517 + \frac{2}{\sqrt{20.2517}} - 6.8911 - \frac{2}{\sqrt{6.8911}} \right) \\ &\approx 0.5823\end{aligned}$$

If you do the integration on a calculator using differently rounded values for μ and you may get a slightly different answer, such as 0.5840 .

For Exercises 33–40, use the table in the Appendix for the areas under the normal curve.

33. Area to the left of $z = -0.43$ is 0.3336.

Percent of area is 33.36%.

34. Area to right of $z = 1.62$ is

$$1 - 0.9474 = 0.0526 \text{ or } 5.26\%.$$

35. Area between $z = -1.17$ and $z = -0.09$ is

$$0.4641 - 0.1210 = 0.3431.$$

Percent of area is 34.31%.

36. Area to left of $z = 1.28$ is 0.8997.

Area to left of $z = -1.39$ is equivalent to area to right of $z = 1.39$:

$$1 - 0.9177 = 0.0823.$$

Area between is

$$0.8997 - 0.0823 = 0.8174 \text{ or } 81.74\%.$$

37. The region up to 1.2 standard deviations below the mean is the region to the left of $z = -1.2$. The area is 0.1151, so the percent of area is 11.51%.

38. $\sigma = 2.5$, $\mu = 0$, $z = 0 + 2.5$

Area to left of $z = 2.5$ is 0.9938 or 99.38%.

39. 52% of area is to the right implies that 48% is to the left.

$$P(z < a) = 0.48 \text{ for } a = -0.05$$

Thus, 52% of the area lies to the right of $z = -0.05$.

40. We want to find the z -score for 21% of the area under the normal curve to the left of z . We note that $21\% < 50\%$, so z must be negative. The z -score for the value in the table nearest 0.21 is $z \approx -0.81$.

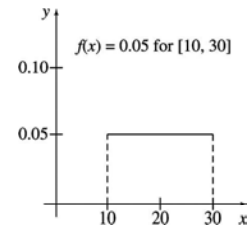
41. $f(x) = 0.05$ for $[10, 30]$

(a) This is a uniform distribution.

(b) The domain of f is $[10, 30]$.

The range of f is $[0, 0.05]$.

(c)



(d) For a uniform distribution,

$$\mu = \frac{1}{2}(b + a) \text{ and}$$

$$\text{Var}(X) = \frac{b^2 - 2ab + a^2}{12}.$$

Thus,

$$\mu = \frac{1}{2}(30 + 10) = \frac{1}{2}(40) = 20$$

$$\begin{aligned} \text{Var}(X) &= \frac{30^2 - 2(10)(30) + 10^2}{12} \\ &= \frac{400}{12}. \end{aligned}$$

$$\sigma = \sqrt{\frac{400}{12}} \approx 5.77$$

$$\begin{aligned} \text{(e)} \quad P(\mu - \sigma \leq X \leq \mu + \sigma) &= P(20 - 5.77 \leq X \leq 20 + 5.77) \\ &= P(14.23 \leq X \leq 25.77) \\ &= \int_{14.23}^{25.77} 0.05 \, dx \\ &= 0.05x \Big|_{14.23}^{25.77} \\ &= 0.05(25.77 - 14.23) \\ &\approx 0.577 \end{aligned}$$

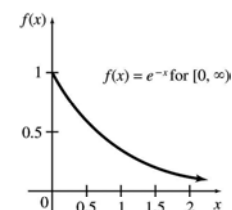
42. $f(x) = e^{-x}$ for $[0, \infty)$

$$f(x) = 1e^{-1x}$$

(a) This is an exponential distribution with $a = 1$.

(b) The domain of f is $[0, \infty)$.
The range of f is $(0, 1]$.

(c)



- (d) For an exponential distribution, $\mu = \frac{1}{a}$ and $\sigma = \frac{1}{a}$.

Thus

$$\mu = \frac{1}{1} = 1 \text{ and } \sigma = \frac{1}{1} = 1.$$

- (e) $P(\mu - \sigma \leq X \leq \mu + \sigma)$
 $= P(1 - 1 \leq X \leq 1 + 1)$
 $= P(0 \leq X \leq 2)$
 $= \int_0^2 e^{-x} dx$
 $= -e^{-x} \Big|_0^2$
 $= -e^{-2} + 1$
 ≈ 0.8647

43. $f(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$ for $(-\infty, \infty)$

- (a) Since the exponent of e in $f(x)$ may be written

$$-x^2 = \frac{-(x-0)^2}{2\left(\frac{1}{\sqrt{2}}\right)^2},$$

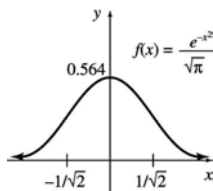
and

$$\frac{1}{\sqrt{\pi}} = \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2\pi}},$$

$f(x)$ is a normal distribution with $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2}}$.

- (b) The domain of f is $(-\infty, \infty)$.
 The range of f is $\left(0, \frac{1}{\sqrt{\pi}}\right]$.

- (c)



- (d) For this normal distribution, $\mu = 0$ and $\sigma = \frac{1}{\sqrt{2}}$.
- (e) $P(\mu - \sigma \leq X \leq \mu + \sigma)$
 $= 2P(0 \leq X \leq \mu + \sigma)$
 $= 2P\left(0 \leq X \leq \frac{1}{\sqrt{2}}\right)$

$$\text{If } x = \frac{1}{\sqrt{2}}, z = \frac{\frac{1}{\sqrt{2}} - 0}{\frac{1}{\sqrt{2}}} = 1.00.$$

Thus,

$$\begin{aligned} P(\mu - \sigma \leq X \leq \mu + \sigma) &= 2P(0 \leq z \leq 1.00) \\ &= 2(0.3413) \\ &\approx 0.6826 \end{aligned}$$

44. $f(x) = \frac{xe^{-x/2}}{4}$ for x in $[0, \infty)$

(a) $P(0 \leq X \leq \infty) = \int_0^\infty \frac{xe^{-x/2}}{4} dx$

For all $x \geq 0$, $e^{-x/2} > 0$ so that $f(x) \geq 0$ for x in $[0, \infty)$.

Evaluate $\int xe^{-x/2} dx$ using integration by parts.

Let $u = x$ and $dv = e^{-x/2} dx$
 Then $du = dx$ and $v = -2e^{-x/2}$

$$\begin{aligned} \frac{1}{4} \int xe^{-x/2} dx &= \frac{1}{4} \left(-2xe^{-x/2} + \int 2e^{-x/2} dx \right) \\ &= \frac{1}{4} \left(-2xe^{-x/2} - 4e^{-x/2} \right) \\ &= -\frac{1}{2} xe^{-x/2} - e^{-x/2} \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty \frac{xe^{-x/2}}{4} dx &= \left(-\frac{1}{2} xe^{-x/2} - e^{-x/2} \right) \Big|_0^\infty \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{2} be^{-b/2} - e^{-b/2} \right) - \left(-\frac{1}{2} (0)e^{-0/2} - e^{-0/2} \right) \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{2} be^{-b/2} - e^{-b/2} \right) - (0 - 1) \end{aligned}$$

$\lim_{b \rightarrow \infty} be^{-b/2} = 0$. Therefore,

$$\int_0^\infty \frac{xe^{-x/2}}{4} dx = 0 - (-1) = 1.$$

$$\begin{aligned}
 \text{(b)} \quad P(0 \leq X \leq 3) &= \int_0^3 \frac{xe^{-x/2}}{4} dx \\
 &= \left(-\frac{1}{2}xe^{-x/2} - e^{-x/2} \right) \Big|_0^3 \\
 &= \left(-\frac{3}{2}e^{-3/2} - e^{-3/2} \right) \\
 &\quad - (0 - 1) \\
 &= 1 - \frac{5}{2}e^{-3/2} \\
 &\approx 0.4422
 \end{aligned}$$

$$45. \quad f(x) = \frac{x^{-1/2}e^{-x/2}}{\sqrt{2\pi}} \text{ for } x \text{ in } (0, \infty)$$

(a) Using integration by parts:

$$\text{Let } u = e^{-x/2} \quad \text{and } dv = x^{-1/2}dx$$

$$\text{Then } du = -\frac{1}{2}e^{-x/2} \quad \text{and } v = 2x^{1/2}.$$

$$\begin{aligned}
 &\frac{1}{\sqrt{2\pi}} \int x^{-1/2}e^{-x/2} dx \\
 &= \frac{1}{\sqrt{2\pi}} \left[2x^{1/2}e^{-x/2} - \int 2x^{1/2} \left(-\frac{1}{2} \right) e^{-x/2} dx \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[2x^{1/2}e^{-x/2} + \int x^{1/2}e^{-x/2} dx \right]
 \end{aligned}$$

Thus,

$$\begin{aligned}
 P(0 < X \leq b) &= \frac{1}{\sqrt{2\pi}} \int_0^b x^{-1/2}e^{-x/2} dx \\
 &= \frac{1}{\sqrt{2\pi}} \left[2x^{1/2}e^{-x/2} \Big|_0^b + \int_0^b x^{1/2}e^{-x/2} dx \right].
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P(0 < X \leq 1) &= \frac{1}{\sqrt{2\pi}} \left[2x^{1/2}e^{-x/2} \Big|_0^1 \right. \\
 &\quad \left. + \int_0^1 x^{1/2}e^{-x/2} dx \right]
 \end{aligned}$$

Notice that

$$\begin{aligned}
 &2x^{1/2}e^{-x/2} \Big|_0^1 \\
 &= 2e^{-1/2} - 0 \\
 &\approx 1.2131.
 \end{aligned}$$

Using Simpson's rule with $n = 12$ to evaluate the improper integral, we have

$$\int_0^1 x^{1/2}e^{-x/2} dx \approx 0.4962$$

Therefore,

$$\begin{aligned}
 P(0 \leq X \leq 1) &= \frac{1}{\sqrt{2\pi}} \int_0^1 x^{-1/2}e^{-x/2} dx \\
 &\approx \frac{1}{\sqrt{2\pi}} (1.2131 + 0.4962) \\
 &\approx 0.6819.
 \end{aligned}$$

$$\text{(c)} \quad P(0 < X \leq 10)$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi}} \left[2x^{1/2}e^{-x/2} \Big|_0^{10} \right. \\
 &\quad \left. + \int_0^{10} x^{1/2}e^{-x/2} dx \right]
 \end{aligned}$$

First,

$$\begin{aligned}
 &2x^{1/2}e^{-x/2} \Big|_0^{10} \\
 &= 2\sqrt{10}e^{-5} - 0 \\
 &\approx 0.0426.
 \end{aligned}$$

Using Simpson's rule with $n = 12$ to evaluate the improper integral, we have

$$\int_0^{10} x^{1/2}e^{-x/2} dx \approx 2.3928$$

Therefore,

$$\begin{aligned}
 P(0 \leq X \leq 1) &= \frac{1}{\sqrt{2\pi}} \int_0^{10} x^{-1/2}e^{-x/2} dx \\
 &\approx \frac{1}{\sqrt{2\pi}} (0.0426 + 2.3928) \\
 &\approx 0.9716.
 \end{aligned}$$

(d) Since $f(x)$ is a probability density function, the limit as $b \rightarrow \infty$ should be 1. The previous results do support this conclusion.

46. (a) $f(x) = \frac{3}{4}(x^2 - 16x + 65)$ for $[8, 9]$

$$\begin{aligned} P(8 \leq X \leq 8.50) &= \int_8^{8.5} \frac{3}{4}(x^2 - 16x + 65)dx \\ &= \frac{3}{4} \left(\frac{x^3}{3} - 8x^2 + 65x \right) \Big|_8^{8.5} \\ &= \frac{3}{4} \left[\frac{8.5^3}{3} - 8(8.5)^2 + 65(8.5) \right. \\ &\quad \left. - \frac{8^3}{3} + 8(8)^2 - 65(8) \right] \\ &\approx 0.4063 \end{aligned}$$

(b) $\mu = \frac{3}{4} \int_8^9 x(x^2 - 16x + 65)dx$

$$\begin{aligned} &= \frac{3}{4} \int_8^9 (x^3 - 16x^2 + 65x)dx \\ &= \frac{3}{4} \left(\frac{x^4}{4} - \frac{16}{3}x^3 + \frac{65}{2}x^2 \right) \Big|_8^9 \\ &= \frac{3}{4} \left(\frac{6561}{4} - 3888 + \frac{5265}{2} \right. \\ &\quad \left. - 1024 + \frac{8192}{3} - 2080 \right) \\ &= \frac{137}{16} \\ &\approx 8.56 \end{aligned}$$

The expected value of the price is \$8.56.

(c)

$$\begin{aligned} \text{Var} &= \frac{3}{4} \int_8^9 x^2(x^2 - 16x + 65)dx - \left(\frac{137}{16} \right)^2 \\ &= \frac{3}{4} \int_8^9 (x^4 - 16x^3 + 65x^2)dx - \left(\frac{137}{16} \right)^2 \\ &= \frac{3}{4} \left(\frac{x^5}{5} - 4x^4 + \frac{65}{3}x^3 \right) \Big|_8^9 - \left(\frac{137}{16} \right)^2 \end{aligned}$$

$$\begin{aligned} &= \frac{3}{4} \left(\frac{59,045}{5} - 26,244 + 15,795 \right. \\ &\quad \left. - \frac{32,768}{5} + 16,384 - \frac{33,280}{3} \right) \\ &\quad - \frac{18,769}{256} \\ &= \frac{107}{1280} \\ \sigma &= \sqrt{\frac{107}{1280}} \approx 0.29 \end{aligned}$$

The standard deviation of the price is \$0.29.

47. (a) $f(t) = \frac{5}{112}(1 - t^{-3/2})$; $[1, 25]$

$$\begin{aligned} P(\text{No repairs in years 1-3}) &= P(\text{First repair needed in years 4-25}) \\ &= \int_4^{25} \frac{5}{112}(1 - t^{-3/2})dt \\ &= \frac{5}{112} \left(t + 2t^{-1/2} \right) \Big|_4^{25} \\ &= \frac{5}{112} \left[25 + \frac{2}{5} - 4 - 1 \right] \\ &= \frac{51}{56} \approx 0.9107 \end{aligned}$$

(b) $\mu = \frac{5}{112} \int_1^{25} t(1 - t^{-3/2})dt$

$$\begin{aligned} &= \frac{5}{112} \int_1^{25} (t - t^{-1/2})dt \\ &= \frac{5}{112} \left(\frac{t^2}{2} - 2t^{1/2} \right) \Big|_1^{25} \\ &= \frac{5}{112} \left(\frac{625}{2} - 10 - \frac{1}{2} + 2 \right) \\ &= \frac{95}{7} \\ &\approx 13.57 \end{aligned}$$

The expected value for the number of years before the machine requires repairs is 13.57 years.

(c)

$$\begin{aligned}
 \text{Var} &= \frac{5}{112} \int_1^{25} t^2(1 - t^{-3/2}) dt - \left(\frac{95}{7}\right)^2 \\
 &= \frac{5}{112} \int_1^{25} (t^2 - t^{1/2}) dt - \left(\frac{95}{7}\right)^2 \\
 &= \frac{5}{112} \left(\frac{t^3}{3} - \frac{2}{3} t^{3/2} \right) \bigg|_1^{25} - \left(\frac{95}{7}\right)^2 \\
 &= \frac{5}{112} \left(\frac{15,625}{3} - \frac{250}{3} - \frac{1}{3} + \frac{2}{3} \right) - \frac{9025}{49} \\
 &= \frac{6560}{147} \\
 \sigma &= \sqrt{\frac{6560}{147}} \\
 &\approx 6.68
 \end{aligned}$$

The standard deviation of the number of years before repairs are required is 6.68 years.

48. $f(x) = \frac{1}{6}e^{-x/6}$ for $[0, \infty)$ is an exponential distribution.

(a) $\mu = \frac{1}{\frac{1}{6}} = 6$

(b) $\sigma = \frac{1}{\frac{1}{6}} = 6$

(c)
$$\begin{aligned}
 P(X > 6) &= \int_6^\infty \frac{1}{6}e^{-x/6} dx \\
 &= \lim_{b \rightarrow \infty} \int_6^b \frac{1}{6}e^{-x/6} dx \\
 &= \lim_{b \rightarrow \infty} -e^{-x/6} \bigg|_6^b \\
 &= \lim_{b \rightarrow \infty} \left(e^{-1} - \frac{1}{e^{b/6}} \right) \\
 &= \frac{1}{e} \approx 0.3679
 \end{aligned}$$

49. (a) $\mu = 8$

$$\frac{1}{a} = 8$$

$$a = \frac{1}{8}$$

$$f(x) = \frac{1}{8}e^{-x/8} \text{ for } [0, \infty)$$

- (b) Expected number = $\mu = 8$

(c) $\sigma = \mu = 8$

(d)
$$\begin{aligned}
 P(5 \leq X \leq 10) &= \int_5^{10} \frac{1}{8}e^{-x/8} dx \\
 &= -e^{-x/8} \bigg|_5^{10} \\
 &= -e^{-10/8} + e^{-5/8} \\
 &\approx 0.2488
 \end{aligned}$$

50. $\mu = 46.2, \sigma = 15.8, x = 60$

$$z = \frac{x - \mu}{\sigma} = \frac{60 - 46.2}{15.8} \approx 0.8734$$

0.8734 is the z-score for the area of about 0.8078 (from the table).

$$\begin{aligned}
 P(X \geq 60) &\approx P(z \geq 0.8734) \\
 &\approx 1 - 0.8078 \\
 &= 0.1922
 \end{aligned}$$

51. Let the random variable X be the number of printers. We have an exponential distribution

$f(t) = ae^{-at}$ for t in $[0, \infty)$. Since

$\mu = 10, a = \frac{1}{\mu} = 0.1$ so that

$$f(t) = 0.1e^{-0.1t}, t \geq 0.$$

We need to determine the portion of the printers sold in the first year and in the second and third year. In other words, we need to calculate $P(0 \leq X \leq 1)$ and $P(1 \leq X \leq 3)$.

$$\begin{aligned}
 P(0 \leq X \leq 1) &= \int_0^1 0.1e^{-0.1t} dt \\
 &= -e^{-0.1t} \bigg|_0^1 \\
 &= -e^{-0.1} + e^0 \\
 &= 1 - e^{-0.1}
 \end{aligned}$$

$$\begin{aligned}
 P(1 \leq X \leq 3) &= \int_1^3 0.1e^{-0.1t} dt \\
 &= -e^{-0.1t} \bigg|_1^3 \\
 &= -e^{-0.3} + e^{-0.1} \\
 &= e^{-0.1} - e^{-0.3}
 \end{aligned}$$

Since

$$\text{payment} = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 0.5x & \text{for } 1 \leq x \leq 3 \\ 0 & \text{for } x > 3 \end{cases}$$

the expected payment will be

$$\begin{aligned}
 E(X) &= (1 - e^{-0.1})(x) \\
 &\quad + (e^{-0.1} - e^{-0.3})(0.5x) + 0 \\
 &= x - 1.5xe^{-0.1} - 0.5xe^{-0.3}.
 \end{aligned}$$

To determine the level x must be set for this to be 1000, solve

$$\begin{aligned}
 E(X) &= x - 1.5xe^{-0.1} - 0.5xe^{-0.3} \\
 &= 1000.
 \end{aligned}$$

Using our calculators, we find $x \approx 5644$. The correct answer choice is **d**.

$$52. \quad f(x) = \frac{8}{7}x^{-2} \text{ for } [1, 8]$$

$$\begin{aligned}
 \text{(a)} \quad \mu &= \int_1^8 \frac{8}{7}x^{-2}(x) dx \\
 &= \int_1^8 \frac{8}{7}x^{-1} dx \\
 &= \frac{8}{7} \ln x \Big|_1^8 \\
 &= \frac{8}{7}(\ln 8 - \ln 1) \approx 2.377 \text{ g}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \text{Var}(X) &= \int_1^8 \frac{8}{7}x^{-2}(x^2) dx - 2.377^2 \\
 &= \frac{8}{7}x \Big|_1^8 - 2.377^2 \\
 &= \frac{64}{7} - \frac{8}{7} - 2.377^2 \\
 &\approx 2.350 \\
 \sigma &= \sqrt{\text{Var}(X)} \approx 1.533 \text{ g}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad P(\mu - \sigma \leq X \leq \mu + \sigma) &= P(0.844 \leq X \leq 3.91) \\
 &= P(1 \leq X \leq 3.91) \text{ since } [1, 8] \\
 &= \int_1^{3.91} \frac{8}{7}x^{-2} dx \\
 &= \frac{-8}{7}x^{-1} \Big|_1^{3.91} dx \\
 &= \frac{-8}{7} \left(\frac{1}{3.91} \right) + \frac{8}{7} \\
 &\approx 0.8506
 \end{aligned}$$

$$53. \quad f(x) = 0.01e^{-0.01x} \text{ for } [0, \infty) \text{ is an exponential distribution.}$$

$$\begin{aligned}
 P(0 \leq X \leq 100) &= \int_0^{100} 0.01e^{-0.01x} dx \\
 &= -e^{-0.01x} \Big|_0^{100} \\
 &= 1 - \frac{1}{e} \\
 &\approx 0.6321
 \end{aligned}$$

$$54. \quad f(x) = \frac{1}{b-a}; \text{ for } [a, b]$$

$$f(x) = \frac{1}{30-2} = \frac{1}{28}$$

This is uniform distribution for $a = 2, b = 30$.

$$\begin{aligned}
 \text{(a)} \quad E(X) &= \mu = \frac{1}{2}(b + a) \\
 &= \frac{1}{2}(30 + 2) = 16 \text{ in.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P(20 < X \leq 30) &= \int_{20}^{30} \frac{1}{28} dx \\
 &= \frac{1}{28}x \Big|_{20}^{30} \\
 &= \frac{30}{28} - \frac{20}{28} \\
 &= \frac{10}{28} \approx 0.3571
 \end{aligned}$$

$$55. \quad f(x) = \frac{3}{19,696}(x^2 + x) \text{ for } x \text{ in } [38, 42]$$

$$\begin{aligned}
 \text{(a)} \quad \mu &= \frac{3}{19,696} \int_{38}^{42} x(x^2 + x) dx \\
 &= \frac{3}{19,696} \int_{38}^{42} (x^3 + x^2) dx \\
 &= \frac{3}{19,696} \left(\frac{x^4}{4} + \frac{x^3}{3} \right) \Big|_{38}^{42} \\
 &= \frac{3}{19,696} \left(\frac{(42)^4}{4} + \frac{(42)^3}{3} \right. \\
 &\quad \left. - \frac{(38)^4}{4} - \frac{(38)^3}{3} \right) \\
 &\approx 40.07
 \end{aligned}$$

The expected body temperature of the species is 40.07°C .

$$\begin{aligned}
 \text{(b)} \quad P(X \leq \mu) &= \frac{3}{19,696} \int_{38}^{40.07} (x^2 + x) dx \\
 &= \frac{3}{19,696} \left(\frac{x^3}{3} + \frac{x^2}{2} \right)_{38}^{40.07} \\
 &= \frac{3}{19,696} \left(\frac{(40.07)^3}{3} + \frac{(40.07)^2}{2} \right. \\
 &\quad \left. - \frac{(38)^3}{3} - \frac{(38)^2}{2} \right) \\
 &\approx 0.4928
 \end{aligned}$$

The probability of a body temperature below the mean is 0.4928.

56. $\mu = 7.8 \text{ lb}, \sigma = 1.1 \text{ lb}, x = 9 \text{ lb}$

$$z = \frac{x - \mu}{\sigma} = \frac{9 - 7.8}{1.1} \approx 1.09$$

$$\begin{aligned}
 P(X > 9) &\approx P(z > 1.09) \\
 &= 1 - 0.8621 \\
 &\approx 0.1379
 \end{aligned}$$

57. Normal distribution,
 $\mu = 2.2 \text{ g}, \sigma = 0.4 \text{ g}, X = \text{tension}$

$$\begin{aligned}
 P(X < 1.9) &= P\left(\frac{x - 2.2}{0.4} < \frac{1.9 - 2.2}{0.4}\right) \\
 &= P(z < -0.75) \\
 &\approx 0.2266.
 \end{aligned}$$

58. For an exponential distribution,
 $f(x) = ae^{-ax}$ for x in $[0, \infty)$.

Since $\mu = \frac{1}{a} = 17.0$, $a = \frac{1}{17.0}$.

$$\begin{aligned}
 \text{(a)} \quad P(X \geq 15) &= \int_{15}^{\infty} \frac{1}{17.0} e^{-x/17.0} dx \\
 &= \lim_{b \rightarrow \infty} \int_{15}^b \frac{1}{17.0} e^{-x/17.0} dx \\
 &= \lim_{b \rightarrow \infty} \left(\frac{1}{17.0} e^{-x/17.0} \Big|_{15}^b \right) \\
 &= \lim_{b \rightarrow \infty} \left(-e^{-b/17.0} + e^{-15/17.0} \right) \\
 &= e^{-15/17.0} \\
 &\approx 0.4138
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P(X < 10) &= \int_0^{10} \frac{1}{17.0} e^{-x/17.0} dx \\
 &= (-e^{-x/17.0}) \Big|_0^{10} \\
 &= -e^{-10/17.0} + e^0 \\
 &\approx 0.4447
 \end{aligned}$$

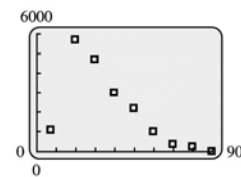
59. For an exponential distribution,
 $f(x) = ae^{-ax}$ for x in $[0, \infty)$.

Since $\mu = \frac{1}{a} = 32.5$, $a = \frac{1}{32.5}$.

$$\begin{aligned}
 \text{(a)} \quad P(X \geq 40) &= \int_{40}^{\infty} \frac{1}{32.5} e^{-x/32.5} dx \\
 &= \lim_{b \rightarrow \infty} \int_{40}^b \frac{1}{32.5} e^{-x/32.5} dx \\
 &= \lim_{b \rightarrow \infty} \left(\frac{1}{32.5} e^{-x/32.5} \Big|_{40}^b \right) \\
 &= \lim_{b \rightarrow \infty} \left(-e^{-b/32.5} + e^{-40/32.5} \right) \\
 &= e^{-40/32.5} \\
 &\approx 0.2921
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad P(30 \leq X < 50) &= \int_{30}^{50} \frac{1}{32.5} e^{-x/32.5} dx \\
 &= (-e^{-x/32.5}) \Big|_{30}^{50} \\
 &= -e^{-50/32.5} + e^{-30/32.5} \\
 &\approx 0.1826
 \end{aligned}$$

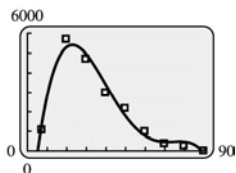
60. (a)



A polynomial function could fit the data.

Your answers to (b) through (f) may differ slightly from the answers given here, depending on how many places you keep in each of the coefficients of N and S .

$$\begin{aligned}
 \text{(b)} \quad N(t) &= -0.002154802t^4 + 0.493396t^3 \\
 &\quad - 38.50574t^2 + 1090.5874t \\
 &\quad - 4695.1476
 \end{aligned}$$



- (c) $\int_{5.2}^{88.9} N(t) dt \approx 197,023$, as found using a calculator. Thus the density function corresponding to the quartic fit is

$$S(t) = \frac{1}{197,023}(-0.002154802t^4 + 0.493396t^3 - 38.50574t^2 + 1090.5874t - 4695.1476).$$

Use a calculator to compute the integrals needed in (d), (e), and (f).

- (d) $P(\text{age} < 25) = \int_{5.2}^{25} S(t) dt \approx 0.3971$;

$$\text{actual relative frequency} = \frac{1096 + 5729}{18,488} \approx 0.3692$$

$$P(45 \leq \text{age} < 65) = \int_{45}^{65} S(t) dt \approx 0.1320;$$

$$\text{actual relative frequency} = \frac{2207 + 1011}{18,488} \approx 0.1741$$

$$P(75 \leq \text{age}) = \int_{75}^{88.9} S(t) dt \approx 0.0265;$$

$$\text{actual relative frequency} = \frac{274 + 32}{18,488} \approx 0.0166$$

- (e) $\mu = \int_{5.2}^{88.9} tS(t) dt \approx 31.80$

The expected age at which a person will die by assault is 31.80 years.

- (f) $\text{Var} = \int_{5.2}^{88.9} t^2 S(t) dt - (31.80)^2$
 ≈ 267.59
 $\sigma = \sqrt{267.59}$
 ≈ 16.36

The standard deviation of the distribution of age from death by assault is 16.36 years.

61. $f(t) = \frac{1}{3650.1} e^{-t/3650.1}$

This is an exponential distribution with

$$a = \frac{1}{3650.1}.$$

So the expected value is $\mu = \frac{1}{a} = 3650.1$ days.

The standard deviation is $\sigma = \frac{1}{a} = 3650.1$ days.

62. Normal distribution, $\mu = 40$, $\sigma = 13$,
 $x = \text{"take"}$

$$\begin{aligned} P(X > 50) &= P\left(\frac{X - 40}{13} > \frac{50 - 40}{13}\right) \\ &= P(Z > 0.77) \\ &= 1 - P(Z \leq 0.77) \\ &= 1 - 0.7794 = 0.2206 \end{aligned}$$

Extended Application: Exponential Waiting Times

1. If the density function is continuous, the probability of any event of the form $(X = c)$ is 0, so it does not matter whether we use strict or inclusive inequalities.
2. We already know that the probability of a gap of 30 minutes or longer is about 0.05. Over 40 trips, the expected number of waits that are 30 minutes or longer is $40(0.05) = 2$.

3.
$$\begin{aligned} P(t > 20) &= \int_{20}^{\infty} \frac{1}{10} e^{-t/10} dt \\ &= 1 - \int_0^{20} \frac{1}{10} e^{-t/10} dt \\ &= 1 - \left(-e^{-t/10} \Big|_0^{20}\right) \\ &= 1 - (-e^{-2} + 1) \\ &= e^{-2} \approx 0.135 \end{aligned}$$

4. For the exponential schedule,

$$\begin{aligned} P(9 < t < 10) &= \int_9^{10} \frac{1}{10} e^{-t/10} dt \\ &= -e^{-t/10} \Big|_9^{10} \\ &= -e^{-1} + e^{-9/10} \\ &\approx 0.039 \end{aligned}$$

For the uniform schedule,

$$\begin{aligned}
 P(9 < t < 10) &= \int_9^{10} \frac{1}{10} dt \\
 &= \left. \frac{1}{10} t \right|_9^{10} \\
 &= 1 - \frac{9}{10} = 0.1
 \end{aligned}$$

5. For the exponential model,

$$\begin{aligned}
 P(8 < t < 12) &= \int_8^{12} \frac{1}{10} e^{-t/10} dt \\
 &= -e^{-t/10} \Big|_8^{12} \\
 &= -e^{-12/10} + e^{-8/10} \\
 &\approx 0.148
 \end{aligned}$$

For the uniform model, *all* interarrival times are 10 minutes, so they all meet the ± 2 criterion.

6. The trick used at the HotBits site is the following: they measure times between successive *pairs* of decay events; if the first wait is shorter than the second, the generator omits a 0, and if the first wait is longer than the second, it omits a 1. For more details, see <http://www.fourmilab.ch/hotbits/how/html>.

Chapter 12

SEQUENCES AND SERIES

12.1 Geometric Sequences

Your Turn 1

$$\text{If } = 1, a_n = 3(1) - 6 = 3 - 6 = -3.$$

$$\text{If } = 2, a_n = 3(2) - 6 = 6 - 6 = 0.$$

$$\text{If } = 3, a_n = 3(3) - 6 = 9 - 6 = 3.$$

$$\text{If } = 4, a_n = 3(4) - 6 = 12 - 6 = 6.$$

Your Turn 2

Verify that $r = -3$.

$$-\frac{6}{2} = \frac{18}{-6} = \frac{-54}{18} = -3$$

$$\begin{aligned} a_7 &= 2(-3)^{7-1} \\ &= 2(-3)^6 \\ &= 2(729) \\ &= 1458 \end{aligned}$$

Your Turn 3

Since the machine loses 10% of its value each year, at the end of each year its value is 0.9 times its value at the end of the year before. Thus its values form a geometric sequence with $r = 0.9$ and $a_1 = 10,000$. Its value at the end of 10 years is its value at $n = 11$.

$$\begin{aligned} a_{11} &= a_0 r^{11-1} \\ &= 10,000(0.9)^{10} \\ &= 3486.78 \end{aligned}$$

The value of the machine at the end of its tenth year is \$3486.78.

Your Turn 4

Since $-8/2 = -4$, this is a geometric sequence with $a_1 = 2$ and $r = -4$.

$$\begin{aligned} S_7 &= \frac{2[(-4)^7 - 1]}{-4 - 1} \\ &= \frac{2(-16,384 - 1)}{-5} \\ &= 6554 \end{aligned}$$

Your Turn 5

Here $a = 81$ and $r = 1/3$. The index of summation i runs from 0 to 4, so $n = 5$.

$$\begin{aligned} \sum_{i=0}^4 81 \left(\frac{1}{3}\right)^i &= S_5 \\ &= \frac{81 \left[\left(\frac{1}{3}\right)^5 - 1 \right]}{\frac{1}{3} - 1} \\ &= \frac{81 \left(-\frac{242}{243} \right)}{-\frac{2}{3}} \\ &= \frac{-\frac{242}{3}}{-\frac{2}{3}} = 121 \end{aligned}$$

12.1 Exercises

1. $a_1 = 2, r = 3, n = 4$

Since $n = 4$, we must find a_1, a_2, a_3 , and a_4 with $a = a_1 = 2$.

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 2(3)^{2-1} = 2(3)^1 = 2(3) = 6 \\ a_3 &= 2(3)^{3-1} = 2(3)^2 = 2(9) = 18 \\ a_4 &= 2(3)^{4-1} = 2(3)^3 = 2(27) = 54 \end{aligned}$$

The first four terms of this geometric sequence are 2, 6, 18, and 54.

2. $a_1 = 4, r = 2, n = 5$

Since $n = 5$, we must find a_1, a_2, a_3, a_4 , and a_5 with $a = a_1 = 4$.

$$\begin{aligned} a_1 &= 4 \\ a_2 &= 4(2)^{2-1} = 4(2)^1 = 4(2) = 8 \\ a_3 &= 4(2)^{3-1} = 4(2)^2 = 4(4) = 16 \\ a_4 &= 4(2)^{4-1} = 4(2)^3 = 4(8) = 32 \\ a_5 &= 4(2)^{5-1} = 4(2)^4 = 4(16) = 64 \end{aligned}$$

The first five terms of this geometric sequence are 4, 8, 16, 32, and 64.

3. $a_1 = \frac{1}{2}, r = 4, n = 4$

Since $n = 4$, we must find a_1, a_2, a_3 , and a_4 , with $a = a_1 = \frac{1}{2}$.

$$\begin{aligned} a_1 &= \frac{1}{2} \\ a_2 &= \frac{1}{2}(4)^{2-1} = \frac{1}{2}(4)^1 = \frac{1}{2}(4) = 2 \\ a_3 &= \frac{1}{2}(4)^{3-1} = \frac{1}{2}(4)^2 = \frac{1}{2}(16) = 8 \\ a_4 &= \frac{1}{2}(4)^{4-1} = \frac{1}{2}(4)^3 = \frac{1}{2}(64) = 32 \end{aligned}$$

The first four terms of this geometric sequence are $\frac{1}{2}, 2, 8$, and 32 .

4. $a_1 = \frac{2}{3}, r = 6, n = 3$

Since $n = 3$, we must find a_1, a_2 , and a_3 with $a = a_1 = \frac{2}{3}$.

$$\begin{aligned} a_1 &= \frac{2}{3} \\ a_2 &= \frac{2}{3}(6)^{2-1} = \frac{2}{3}(6)^1 = \frac{2}{3}(6) = 4 \\ a_3 &= \frac{2}{3}(6)^{3-1} = \frac{2}{3}(6)^2 = \frac{2}{3}(36) = 24 \end{aligned}$$

The first three terms of this geometric sequence are $\frac{2}{3}, 4$, and 24 .

5. $a_3 = 6, a_4 = 12, n = 5$

Since $n = 5$, we must find a_1, a_2, a_3, a_4 and a_5 with $r = \frac{a_4}{a_3} = \frac{12}{6} = 2$. To find a , use $a_3 = 6$, $r = 2$, and $n = 3$ in the formula.

$$\begin{aligned} 6 &= a(2)^{3-1} \\ 6 &= a(2)^2 \\ 6 &= 4a \\ \frac{3}{2} &= \frac{3}{2} = a \end{aligned}$$

$$a_1 = \frac{3}{2}$$

$$a_2 = \frac{3}{2}(2)^{2-1} = \frac{3}{2}(2)^1 = \frac{3}{2}(2) = 3$$

$$a_3 = \frac{3}{2}(2)^{3-1} = \frac{3}{2}(2)^2 = \frac{3}{2}(4) = 6$$

$$a_4 = \frac{3}{2}(2)^{4-1} = \frac{3}{2}(2)^3 = \frac{3}{2}(8) = 12$$

$$a_5 = \frac{3}{2}(2)^{5-1} = \frac{3}{2}(2)^4 = \frac{3}{2}(16) = 24$$

The first five terms of this geometric sequence are $\frac{3}{2}, 3, 6, 12$, and 24 .

6. $a_2 = 9, a_3 = 3, n = 4$

Since $n = 4$, we must find a_1, a_2, a_3 , and a_4 with $r = \frac{a_3}{a_2} = \frac{3}{9} = \frac{1}{3}$. To find a , use $a_2 = 9$, $r = \frac{1}{3}$, and $n = 2$ in the formula.

$$\begin{aligned} 9 &= a\left(\frac{1}{3}\right)^{2-1} \\ 9 &= a\left(\frac{1}{3}\right)^1 \\ 9 &= a\left(\frac{1}{3}\right) \\ 27 &= a \end{aligned}$$

$$a_1 = 27$$

$$a_2 = 27\left(\frac{1}{3}\right)^{2-1} = 27\left(\frac{1}{3}\right)^1 = 27\left(\frac{1}{3}\right) = 9$$

$$a_3 = 27\left(\frac{1}{3}\right)^{3-1} = 27\left(\frac{1}{3}\right)^2 = 27\left(\frac{1}{9}\right) = 3$$

$$a_4 = 27\left(\frac{1}{3}\right)^{4-1} = 27\left(\frac{1}{3}\right)^3 = 27\left(\frac{1}{27}\right) = 1$$

The first four terms of this geometric sequence are $27, 9, 3$, and 1 .

7. $a_1 = 4, r = 3$

Since we want a_5 , use $n = 5$ in the formula with $a = a_1 = 4$ and $r = 3$.

$$a_5 = 4(3)^{5-1} = 4(3)^4 = 4(81) = 324$$

$$a_n = 4(3)^{n-1}$$

8. $a_1 = 8, r = 4$

Since we want a_5 , use $n = 5$ in the formula with $a = a_1 = 8$ and $r = 4$.

$$\begin{aligned} a_5 &= 8(4)^{5-1} = 8(4)^4 \\ &= 8(256) = 2048 \\ a_n &= 8(4)^{n-1} \end{aligned}$$

9. $a_1 = -3, r = -5$

Since we want a_5 , use $n = 5$ in the formula with $a = a_1 = -3$ and $r = -5$.

$$\begin{aligned} a_5 &= -3(-5)^{5-1} = -3(-5)^4 \\ &= -3(625) = -1875 \\ a_n &= -3(-5)^{n-1} \end{aligned}$$

10. $a_1 = -4, r = -2$

Since we want a_5 , use $n = 5$ in the formula with $a = a_1 = -4$ and $r = -2$.

$$\begin{aligned} a_5 &= -4(-2)^{5-1} = -4(-2)^4 \\ &= -4(16) = -64 \\ a_n &= -4(-2)^{n-1} \end{aligned}$$

11. $a_2 = 12, r = \frac{1}{2}$

To find a , use $a_2 = 12, r = \frac{1}{2}$, and $n = 2$ in the formula.

$$\begin{aligned} 12 &= a\left(\frac{1}{2}\right)^{2-1} = \frac{a}{2} \\ a &= 24 \end{aligned}$$

Since we want a_5 , use $n = 5$ in the formula with $a = 24$ and $r = \frac{1}{2}$.

$$\begin{aligned} a_5 &= 24\left(\frac{1}{2}\right)^{5-1} = 24\left(\frac{1}{2}\right)^4 = 24\left(\frac{1}{16}\right) = \frac{3}{2} \\ a_n &= 24\left(\frac{1}{2}\right)^{n-1} \text{ or } \frac{24}{2^{n-1}} \end{aligned}$$

12. $a_2 = 2, r = \frac{1}{3}$

To find a , use $a_3 = 2, r = \frac{1}{3}$, and $n = 3$ in the formula.

$$\begin{aligned} 2 &= a\left(\frac{1}{3}\right)^{3-1} = \frac{a}{9} \\ a &= 18 \end{aligned}$$

Since we want a_5 , use $n = 5$ in the formula with $a = 18$ and $r = \frac{1}{3}$.

$$\begin{aligned} a_5 &= 18\left(\frac{1}{3}\right)^{5-1} = 18\left(\frac{1}{3}\right)^4 = 18\left(\frac{1}{81}\right) = \frac{2}{9} \\ a_n &= 18\left(\frac{1}{3}\right)^{n-1} \text{ or } \frac{18}{3^{n-1}} \end{aligned}$$

13. $a_4 = 64, r = -4$

To find a , use $a_4 = 64, r = -4$, and $n = 4$ in the formula.

$$\begin{aligned} 64 &= a(-4)^{4-1} \\ 64 &= a(-4)^3 \\ 64 &= a(-64) \\ -1 &= a \end{aligned}$$

Since we want a_5 , use $n = 5$ in the formula with $a = -1$ and $r = -4$.

$$\begin{aligned} a_5 &= -1(-4)^{5-1} = -1(-4)^4 \\ &= -1(256) = -256 \\ a_n &= -(-4)^{n-1} \end{aligned}$$

14. $a_4 = 81, r = -3$

To find a , use $a_4 = 81, r = -3$, and $n = 4$ in the formula.

$$\begin{aligned} 81 &= a(-3)^{4-1} \\ 81 &= a(-3)^3 \\ 81 &= a(-27) \\ -3 &= a \end{aligned}$$

Since we want a_5 , use $n = 5$ in the formula with $a = -3$ and $r = -3$.

$$\begin{aligned} a_5 &= -3(-3)^{5-1} = -3(-3)^4 = -3(81) = -243 \\ a_n &= -3(-3)^{n-1} = (-3)^{1+(n-1)} = (-3)^n \end{aligned}$$

15. 6, 12, 24, 48, ...

$$r = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = 2$$

Since $r = 2$ and $a = a_1 = 6$, $a_n = 6(2)^{n-1}$.

16. 4, 16, 64, 256, ...

$$r = \frac{16}{4} = \frac{64}{16} = \frac{256}{64} = 4$$

Since $r = 4$ and $a = a_1 = 4$,

$$a_n = 4(4)^{n-1} = 4^n.$$

- 17.
- $\frac{3}{4}, \frac{3}{2}, 3, 6, 12, \dots$

$$r = \frac{\frac{3}{2}}{\frac{3}{4}} = \frac{3}{\frac{3}{2}} = \frac{6}{3} = \frac{12}{6} = 2$$

Since $r = 2$ and $a = a_1 = \frac{3}{4}$, $a_n = \frac{3}{4}(2)^{n-1}$.

18. -7, -5, -3, -1, 1, 3, ...

Since $\frac{-5}{-7} = \frac{5}{7}$ and $\frac{-3}{-5} = \frac{3}{5}$, the ratio is not constant, so the sequence is not geometric.

19. 4, 8, -16, 32, 64, -128, ...

Since $\frac{8}{4} = 2$ and $\frac{-16}{8} = -2$, the ratio is not constant, so the sequence is not geometric.

20. 6, 8, 10, 12, 14, ...

Since $\frac{8}{6} = \frac{4}{3}$ and $\frac{10}{8} = \frac{5}{4}$, the ratio is not constant, so the sequence is not geometric.

- 21.
- $-\frac{5}{8}, \frac{5}{12}, -\frac{5}{18}, \frac{5}{27}, \dots$

$$r = \frac{\frac{5}{12}}{-\frac{5}{8}} = \frac{-\frac{5}{18}}{\frac{5}{12}} = \frac{\frac{5}{27}}{-\frac{5}{18}} = -\frac{2}{3}$$

Since $r = -\frac{2}{3}$ and $a = a_1 = -\frac{5}{8}$,

$$a_n = -\frac{5}{8}\left(-\frac{2}{3}\right)^{n-1}.$$

- 22.
- $\frac{7}{4}, -\frac{7}{12}, \frac{7}{36}, -\frac{7}{108}, \dots$

$$r = \frac{-\frac{7}{12}}{\frac{7}{4}} = \frac{\frac{7}{36}}{-\frac{7}{12}} = \frac{-\frac{7}{108}}{\frac{7}{36}} = -\frac{1}{3}$$

Since $r = -\frac{1}{3}$ and $a = a_1 = \frac{7}{4}$,

$$a_n = \frac{7}{4}\left(-\frac{1}{3}\right)^{n-1}.$$

23. 3, 6, 12, 24, ...

Since $a = a_1 = 3$ and $r = \frac{6}{3} = 2$,

$$\begin{aligned} S_5 &= \frac{3(2^5 - 1)}{2 - 1} \\ &= \frac{3(32 - 1)}{1} \\ &= 93. \end{aligned}$$

The sum of the first five terms of this geometric sequence is 93.

24. 5, 20, 80, 320, ...

Since $a = a_1 = 5$ and $r = \frac{20}{5} = 4$,

$$\begin{aligned} S_5 &= \frac{5(4^5 - 1)}{4 - 1} \\ &= \frac{5(1024 - 1)}{3} \\ &= 1705. \end{aligned}$$

The sum of the first five terms of this geometric sequence is 1705.

25. 12, -6, 3,
- $-\frac{3}{2}, \dots$

Since $a = a_1 = 12$ and $r = \frac{-6}{12} = -\frac{1}{2}$,

$$\begin{aligned} S_5 &= \frac{12\left[\left(-\frac{1}{2}\right)^5 - 1\right]}{\left(-\frac{1}{2}\right) - 1} \\ &= \frac{12\left[-\frac{1}{32} - 1\right]}{-\frac{3}{2}} \\ &= \frac{33}{4}. \end{aligned}$$

The sum of the first five terms of this geometric sequence is $\frac{33}{4}$.

26. $18, -3, \frac{1}{2}, -\frac{1}{12}, \dots$

Since $a = a_1 = 18$ and $r = \frac{-3}{18} = -\frac{1}{6}$,

$$\begin{aligned} S_5 &= \frac{18 \left[\left(-\frac{1}{6} \right)^5 - 1 \right]}{-\frac{1}{6} - 1} \\ &= \frac{18 \left[-\frac{1}{7776} - 1 \right]}{-\frac{7}{6}} \\ &= \frac{1111}{72}. \end{aligned}$$

The sum of the first five terms of this geometric sequence is $\frac{1111}{72}$.

27. $a_1 = 3, r = -2$

Since $a = a_1 = 3$,

$$S_5 = \frac{3[(-2)^5 - 1]}{-2 - 1} = \frac{3(-32 - 1)}{-3}.$$

The sum of the first five terms of this geometric sequence is 33.

28. $a_1 = -5, r = 4$

Since $a = a_1 = -5$,

$$\begin{aligned} S_5 &= \frac{-5(4^5 - 1)}{4 - 1} \\ &= \frac{-5(1024 - 1)}{3} \\ &= -1705. \end{aligned}$$

The sum of the first five terms of this sequence is -1705 .

29. $a_1 = 6.324, r = 2.598$

Since $a = a_1 = 6.324$,

$$\begin{aligned} S_5 &= \frac{6.324(2.598^5 - 1)}{2.598 - 1} \\ &\approx \frac{6.324(118.3575 - 1)}{1.598} \\ &\approx 464.4. \end{aligned}$$

The sum of the first five terms of this sequence is about 464.4.

30. $a_1 = -2.772, r = -1.335$

Since $a = a_1 = -2.772$,

$$\begin{aligned} S_5 &= \frac{-2.772[(-1.335)^5 - 1]}{-1.335 - 1} \\ &\approx \frac{-2.772(-4.2404 - 1)}{-2.335} \\ &\approx -6.221. \end{aligned}$$

The sum of the first five terms of this sequence is about -6.221 .

31. For $\sum_{i=0}^7 8(2)^i$, use the formula with $a = 8$, $r = 2$, and $n = 8$.

$$\begin{aligned} S_8 &= \frac{8(2^8 - 1)}{2 - 1} \\ &= \frac{8(256 - 1)}{1} \\ &= 2040 \end{aligned}$$

32. For $\sum_{i=0}^6 4(3)^i$, use the formula with $a = 4$, $r = 3$, and $n = 7$.

$$\begin{aligned} S_7 &= \frac{4(3^7 - 1)}{3 - 1} \\ &= \frac{4(2187 - 1)}{2} \\ &= 4372 \end{aligned}$$

33. For $\sum_{i=0}^8 \frac{3}{2}(4)^i$, use the formula with $a = \frac{3}{2}$, $r = 4$, and $n = 9$.

$$\begin{aligned} S_9 &= \frac{\frac{3}{2}(4^9 - 1)}{4 - 1} \\ &= \frac{\frac{3}{2}(262,144 - 1)}{3} \\ &= \frac{262,143}{2} \end{aligned}$$

34. For $\sum_{i=0}^9 \frac{3}{4}(2)^i$, use the formula with $a = \frac{3}{4}$, $r = 2$, and $n = 10$.

$$\begin{aligned} S_{10} &= \frac{\frac{3}{4}(2^{10} - 1)}{2 - 1} \\ &= \frac{\frac{3}{4}(1024 - 1)}{1} \\ &= \frac{3069}{4} \end{aligned}$$

35. For $\sum_{i=0}^4 \frac{3}{4}(-3)^i$, use the formula with $a = \frac{3}{4}$, $r = -3$, and $n = 5$.

$$\begin{aligned} S_5 &= \frac{\frac{3}{4}[(-3)^5 - 1]}{-3 - 1} \\ &= \frac{\frac{3}{4}(-243 - 1)}{-4} \\ &= \frac{183}{4} \end{aligned}$$

36. For $\sum_{i=0}^5 \frac{3}{2}(-2)^i$, use the formula with $a = \frac{3}{2}$, $r = -2$, and $n = 6$.

$$\begin{aligned} S_6 &= \frac{\frac{3}{2}[(-2)^6 - 1]}{-2 - 1} \\ &= \frac{\frac{3}{2}(64 - 1)}{-3} \\ &= -\frac{63}{2} \end{aligned}$$

37. For $\sum_{i=0}^8 64\left(\frac{1}{2}\right)^i$, use the formula with $n = 9$, $r = \frac{1}{2}$, and $a = 64$.

$$\begin{aligned} S_9 &= \frac{64\left[\left(\frac{1}{2}\right)^9 - 1\right]}{\frac{1}{2} - 1} \\ &= \frac{64\left(\frac{1}{512} - 1\right)}{-\frac{1}{2}} \\ &= \frac{511}{4} \end{aligned}$$

$$\text{Therefore, } \sum_{i=0}^8 64\left(\frac{1}{2}\right)^i = \frac{511}{4}.$$

38. For $\sum_{i=0}^6 81\left(\frac{2}{3}\right)^i$, use the formula with $n = 7$, $r = \frac{2}{3}$, and $a = 81$.

$$S_7 = \frac{81\left[\left(\frac{2}{3}\right)^7 - 1\right]}{\frac{2}{3} - 1} = \frac{81\left(\frac{128}{2187} - 1\right)}{-\frac{1}{3}} = \frac{2059}{9}$$

$$\text{Therefore, } \sum_{i=0}^6 81\left(\frac{2}{3}\right)^i = \frac{2059}{9}.$$

39. (a) A machine that loses 20% of its value maintains 80% of its value. If the initial value of the machine is \$12,000, then its value at the end of the first year will be 80% of \$12,000, and its value at the end of each subsequent year will be 80% of its value at the end of the previous year. This means that the end-of-year values form a geometric sequence with $r = 0.80$. If we let $a_1 = 12,000$, then a_2 will represent the value at the end of the first year, a_3 will represent the value at the end of the second year, and so on. Thus, to find the value of the machine at the end of the fifth year, we are looking for a_6 in the geometric sequence with $n = 6$, $r = 0.80$ and $a = a_1 = 12,000$.

$$\begin{aligned} a_6 &= 12,000(0.80)^{6-1} = 12,000(0.80)^5 \\ &= 12,000(0.32768) = 3932.16 \end{aligned}$$

Therefore, the value of the machine at the end of the fifth year is about \$3932.

- (b) To find the value of the machine at the end of the eighth year, we are looking for a_9 in the geometric sequence.

$$\begin{aligned} a_9 &= 12,000(0.80)^{9-1} = 12,000(0.80)^8 \\ &= 12,000(0.16777216) \approx 2013.27 \end{aligned}$$

Therefore, the value of the machine at the end of the eighth year is about \$2013.

40. The yearly incomes produced by the well form a geometric sequence with $r = \frac{3}{4}$ and $a_1 = 4,000,000$. To determine the total amount produced in 8 years, use the formula to find S_n with $n = 8$, $r = \frac{3}{4}$, and $a = a_1 = 4,000,000$.

$$\begin{aligned} S_8 &= \frac{4,000,000\left[\left(\frac{3}{4}\right)^8 - 1\right]}{\frac{3}{4} - 1} \\ &= \frac{4,000,000\left(\frac{6561}{65,536} - 1\right)}{-\frac{1}{4}} \\ &\approx 14,398,193 \end{aligned}$$

The total amount of income produced by the well in eight years is \$14,398,193.

41. The amounts saved form a geometric sequence with $a_1 = 1$ and $r = 2$. To determine the amount saved on January 31, we want to use the

formula to find a_n with $n = 31$, $r = 2$, and $a = a_1 = 1$.

$$\begin{aligned} a_{31} &= 1(2)^{31-1} = 2^{30} \\ &= 1,073,741,824 \end{aligned}$$

To determine the total amount saved during January, we want to use the formula to find S_n with $n = 31$, $r = 2$, and $a = a_1 = 1$.

$$\begin{aligned} S_{31} &= \frac{1(2^{31} - 1)}{2 - 1} = \frac{2,147,483,648 - 1}{1} \\ &= 2,147,483,647 \end{aligned}$$

The savings on January 31 would be $\$2^{30}$ or $\$1,073,741,824$ and for the month of January would be $\$2^{31} - \1 or $\$2,147,483,647$.

42. If the machine loses 30% of its value, it maintains 70% of its value. With an initial value of $\$200,000$, its value at the end of the first year will be 70% of $\$200,000$, and its value at the end of each subsequent year will be 70% of its value at the end of the previous year. Thus, the end-of-year values form a geometric sequence with $r = 0.70$. If we let $a_1 = 200,000$, then a_2 represents the value of the machine at the end of the first year, a_3 is the value at the end of the second year, and so on. To find the value at the end of sixth year we are looking for a_7 in the geometric sequence with $n = 7$, $r = 0.70$, and $a = a_1 = 200,000$.

$$\begin{aligned} a_7 &= 200,000(0.70)^{7-1} \\ &= 200,000(0.70)^6 \\ &= 200,000(0.117649) \\ &= 23,529.8 \end{aligned}$$

The value of the machine at the end of the sixth year will be $\$23,530$.

43. If we let a_1 represent the initial bacteria population, then the population at the end of the first hour will be $a_1 + 5\%(a_1)$, or $105\%(a_1)$. The populations form a geometric sequence with $r = 105\% = 1.05$. To determine the percent increase in the population after 7 hours, we want to use the formula to find a_n with $n = 8$, $r = 1.05$, and $a = a_1$.

$$a_6 = a(1.05)^{8-1} \approx 1.40710a$$

Since $a = a_1$, $a_8 \approx 1.40710a_1$. This means that the population at the end of 7 hours is about 141% of the initial population.

After five hours, the population has increased by about 41%.

44. If 10^{15} molecules are present initially, then $\frac{1}{2}(10^{15})$ molecules will be present after 3 years. The numbers of molecules present at the end of each 3-year period form a geometric sequence with $r = \frac{1}{2}$ and $a_1 = 10^{15}$. To determine the number of molecules unchanged after 15 years, we want the number of molecules after five 3-year periods. If we let $a_1 = 10^{15}$, then a_2 represents the number of molecules after the first 3-year period, a_3 is the number present after the second 3-year period, and so on. To find the number of molecules after five 3-year periods, we are looking for a_6 . Use the formula to find a_n with $n = 6$, $r = \frac{1}{2}$, and $a = a_1 = 10^{15}$.

$$\begin{aligned} a_6 &= 10^{15} \left(\frac{1}{2} \right)^{6-1} \\ &= 10^{15} \left(\frac{1}{2} \right)^5 \\ &= 10^{15} \left(\frac{1}{32} \right) \\ &= 10^{15}(0.03125) \\ &= 3.125 \times 10^{13} \end{aligned}$$

After fifteen years, 3.125×10^{13} molecules will be unchanged.

45. The number of rotations form a geometric sequence with $r = \frac{3}{4}$ and $a_1 = 400$. If we let $a_1 = 400$, then a_2 represents the number of rotations at the end of the first minute, a_3 is the number of rotations at the end of the second minute, and so on. To determine the number of rotations in the fifth minute, we want a_6 . Use the formula to find a_n with $n = 6$, $r = \frac{3}{4}$, and $a = a_1 = 400$.

$$\begin{aligned} a_6 &= 400 \left(\frac{3}{4} \right)^{6-1} = 400 \left(\frac{3}{4} \right)^5 \\ &= 400 \left(\frac{243}{1024} \right) = 94.921875 \end{aligned}$$

In the fifth minute after the rider's feet are removed from the pedals, the wheel will rotate about 95 times.

46. (a) The thicknesses of the paper form a geometric sequence with $r = 2$ and $a_1 = 0.008$. If $a_1 = 0.008$, then a_2 represents the thickness after the first fold, a_3 is the thickness after the second fold, and so on. To determine the thickness after 12 folds, use the formula to find a_n with $n = 13$, $r = 2$, and $a = a_1 = 0.008$.

$$\begin{aligned} a_{13} &= 0.008(2)^{13-1} \\ &= 0.008(2)^{12} \\ &= 0.008(4096) \\ &= 32.768 \end{aligned}$$

After twelve folds, the final stack of paper is 32.77 inches thick.

- (b) $a_{51} = 0.008(2)^{50} \approx 9.007 \times 10^{12}$

The stack would be about 9.007×10^{12} inches thick, or about 142 million miles thick.

47. (a) Initially, there are 64 teams that play $a_1 = 32$ games in round one. After each round, the number of teams (and the number of games) decreases by $\frac{1}{2}$. At the end of the first round, there are 32 teams left to play $a_2 = 16$ games in round two. At the end of the second round, there will be 16 teams left to play $a_3 = 8$ games in round 3, and so on.

Note that $a_1 = 2^5$, $a_2 = 2^4$, $a_3 = 2^3$, so that by the sixth round there will be 2 teams left playing $a_6 = 2^0 = 1$ game to decide the champion. Therefore, to determine the total number of games, we need to sum the terms of the geometric sequence

$$\begin{aligned} a_1 &= 2^5, a_2 = 2^4, a_3 = 2^3, a_4 = 2^2, \\ a_5 &= 2^1, a_6 = 1 \text{ or } 1 + 2 + 2^2 + 2^3 \\ &\quad + 2^4 + 2^5. \end{aligned}$$

- (b) To find the sum from part (a), use the formula to find S_n with $n = 6$, $r = \frac{1}{2}$, and $a = a_1 = 2^5$.

$$\begin{aligned} S_6 &= \frac{2^5 \left[\left(\frac{1}{2} \right)^6 - 1 \right]}{\frac{1}{2} - 1} \\ &= \frac{\left(\frac{1}{2} - 32 \right)}{-\frac{1}{2}} = 63, \end{aligned}$$

so 63 games must be played to produce the champion.

- (c) To generalize the methods of (a) and (b) for 2^n teams rather than $64 = 2^6$ teams, first note that 2^n teams play 2^{n-1} games in the initial round. In part (a), note that it required 6 rounds to determine a champion when 2^6 teams are initially involved. Similarly, for 2^n teams it will require n rounds. So, the total number of games is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

with $a = a_1 = 2^{n-1}$, and $r = \frac{1}{2}$, which is the desired sum

$$2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 2^0.$$

$$\begin{aligned} S_n &= \frac{2^{n-1} \left[\left(\frac{1}{2} \right)^n - 1 \right]}{\frac{1}{2} - 1} = \frac{\frac{1}{2} - 2^{n-1}}{-\frac{1}{2}} \\ &= -2 \left(\frac{1}{2} - 2^{n-1} \right) \\ &= -1 + 2^n \\ &= 2^n - 1. \end{aligned}$$

48. (a) In round one, there are 81 players playing in $\frac{81}{3} = 27$ games that produce 27 winners to play in round two. In round two, there will be $\frac{27}{3} = 9$ games that produce 9 winners to play $\frac{9}{3} = 3$ games in round three.

These games will determine the 3 winners that will play in the championship in round four. Therefore, the total number of games is the sum $27 + 9 + 3 + 1$, or $3^3 + 3^2 + 3^1 + 3^0$, which is the sum of the geometric sequence $a_1 = 3^3$, $a_2 = 3^2$, $a_3 = 3^1$, $a_4 = 3^0$.

- (b) Notice that the ratio for the geometric sequence in part (a) is $r = \frac{3^2}{3^3} = \frac{1}{3}$. The number of players and the number of games decreases by $\frac{1}{3}$ from one round to the next. The total number of games therefore is the sum S_n with $n = 4$, $r = \frac{1}{3}$, and $a = a_1 = 27$, or

$$\begin{aligned} S_5 &= \frac{27 \left[\left(\frac{1}{3} \right)^4 - 1 \right]}{\frac{1}{3} - 1} \\ &= \frac{\frac{1}{3} - 27}{-\frac{2}{3}} \\ &= -\frac{3}{2} \left(\frac{1 - 81}{3} \right) \\ &= \frac{-80}{-2} \\ &= 40, \end{aligned}$$

so 40 games are required in all to determine a champion.

- (c) When there are $3^4 = 81$ players initially, we know from part (a) that 4 rounds are required to determine a champion. For 3^n players, n rounds are required. To determine the total number of games in such a tournament, we know that 3^n players will play 3^{n-1} games in the first round, and for each succeeding round, $\frac{1}{3}$ fewer games will be played. Therefore, the number of games played in each round forms a geometric sequence with $a_1 = a = 3^{n-1}$ and $r = \frac{1}{3}$.

The total number of games played is then

$$\begin{aligned} S_n &= \frac{3^{n-1} \left[\left(\frac{1}{3} \right)^n - 1 \right]}{\frac{1}{3} - 1} \\ &= \frac{\left(\frac{1}{3} - 3^{n-1} \right)}{-\frac{2}{3}} \\ &= -\frac{3}{2} \left(\frac{1}{3} - 3^{n-1} \right) \\ &= \frac{1}{2} (-1 + 3^n), \end{aligned}$$

or

$$S_n = \frac{3^n - 1}{2} = 3^{n-1} + \cdots + 3^2 + 3^1 + 1.$$

- (d) For a tournament where t^n players are initially present, if the total number of games played is a sum of a geometric sequence as in parts (a) and (c), then it must be that t^n players will play t^{n-1} games in the first round. This means the number of winners and the number of games played in subsequent rounds will decrease by a factor of $\frac{1}{t}$. We also know from parts (a) and (c) that a tournament that begins with t^n players will require n rounds to produce a champion. Therefore, the total number of games played is the sum

$$t^{n-1} + t^{n-2} + \cdots + t^2 + t + 1,$$

which is equal to

$$\begin{aligned} S_n &= \frac{t^{n-1} \left[\left(\frac{1}{t} \right)^n - 1 \right]}{\frac{1}{t} - 1} \\ &= \frac{\left(\frac{1}{t} - t^{n-1} \right)}{\frac{1-t}{t}} \\ &= \frac{t}{1-t} \left(\frac{1}{t} - t^{n-1} \right) \\ &= \frac{1 - t^n}{1 - t} \\ &= \frac{t^n - 1}{t - 1}. \end{aligned}$$

12.2 Annuities: An Application of Sequences

Your Turn 1

This is an ordinary annuity with $R = 22,000$, $n = 7$, and $i = 0.05$.

$$\begin{aligned} S &= 22,000 \left[\frac{(1.05)^7 - 1}{0.05} \right] \\ &= (22,000)(8.142008453125) \\ &\approx 179,124.19 \end{aligned}$$

Erin will have \$179,124.19 on deposit.

Your Turn 2

Since the interest is compounded monthly for 5 years, the number of periods n is $(5)(12) = 60$. The annual interest is 2%, so the interest per monthly compounding period is $0.02/12$. The initial deposit is \$125.

$$\begin{aligned}
 S &= 125 \left[\frac{\left(1 + \frac{0.02}{12}\right)^{60} - 1}{\frac{0.02}{12}} \right] \\
 &\approx (125)(63.047356) \\
 &\approx 7880.92
 \end{aligned}$$

The amount of the annuity is \$7880.92.

Your Turn 3

The only difference from Example 3 is that the interest rate per period is now $0.025/4$. There are still $(4)(3) = 12$ compounding periods.

$$\begin{aligned}
 2400 &= R \left[\frac{\left(1 + \frac{0.025}{4}\right)^{12} - 1}{\frac{0.025}{4}} \right] \\
 2400 &\approx R(12.421246) \\
 R &\approx \frac{2400}{12.421246} \approx 193.22
 \end{aligned}$$

Melissa must make twelve deposits of \$193.22.

Your Turn 4

We need to find the present value of an annuity of payments of \$2500 made at the end of each year for 12 years, with interest at 4.5% compounded annually.

$$\begin{aligned}
 P &= R \left[\frac{1 - (1 + i)^{-n}}{i} \right] \\
 &= 2500 \left[\frac{1 - (1.045)^{-12}}{0.045} \right] \\
 &\approx (2500)(9.118581) \\
 &\approx 22,796.45
 \end{aligned}$$

The amount accumulated at the end of 12 years will be \$22,796.45.

Your Turn 5

The present value is \$11,000, the interest rate is $0.06/12$ per period, or 0.005, and there will be 48 payments.

$$\begin{aligned}
 P &= R \left[\frac{1 - (1 + i)^{-n}}{i} \right] \\
 11,000 &= R \left[\frac{1 - (1 + 0.005)^{-48}}{0.005} \right] \\
 11,000 &\approx R(42.580318) \\
 R &\approx \frac{11,000}{42.580318} \approx 258.34
 \end{aligned}$$

Each payment will be \$258.34.

Your Turn 6

The present value is $\$256,000 - \$32,000 = \$224,000$.

The monthly interest rate is $0.049/12$ and there are $(12)(30) = 360$ periods.

$$\begin{aligned}
 P &= R \left[\frac{1 - (1 + i)^{-n}}{i} \right] \\
 224,000 &= R \left[\frac{1 - \left(1 + \frac{0.049}{12}\right)^{-360}}{\frac{0.049}{12}} \right] \\
 224,000 &\approx R(188.420888) \\
 R &\approx \frac{224,000}{188.420888} \\
 &\approx 1188.83
 \end{aligned}$$

Each monthly payment will be \$1188.83.

12.2 Exercises

1. $R = 120, i = 0.05, n = 10$

$$\begin{aligned}
 S &= R \cdot s_{\overline{n}|i} \\
 &= 120 \cdot s_{\overline{10}|0.05} \\
 &= 120 \left[\frac{(1.05)^{10} - 1}{0.05} \right] \\
 &\approx 120(12.57789254) \\
 &\approx 1509.35
 \end{aligned}$$

The amount is \$1509.35.

2. $R = 1500, i = 0.04, n = 12$

$$\begin{aligned}
 S &= R \cdot s_{\overline{n}|i} \\
 &= 1500 \cdot s_{\overline{12}|0.04} \\
 &= 1500 \left[\frac{(1.04)^{12} - 1}{0.04} \right] \\
 &\approx 1500(15.02580546) \\
 &\approx 22,538.71
 \end{aligned}$$

The amount is \$22,538.71.

3. $R = 9000, i = 0.06, n = 18$

$$\begin{aligned}
 S &= 9000 \cdot s_{\overline{18}|0.06} \\
 &= 9000 \left[\frac{(1.06)^{18} - 1}{0.06} \right] \\
 &\approx 9000(30.90565255) \\
 &\approx 278,150.87
 \end{aligned}$$

The amount is \$278,150.87.

4. $R = 80,000, i = 0.07, n = 24$

$$\begin{aligned} S &= 80,000 \cdot s_{\overline{24}|0.07} \\ &= 80,000 \left[\frac{(1.07)^{24} - 1}{0.07} \right] \\ &\approx 80,000(58.17667076) \\ &\approx 4,654,133.66 \end{aligned}$$

The amount is \$4,654,133.66.

5. $R = 11,500, i = 0.055, n = 30$

$$\begin{aligned} S &= 11,500 \cdot s_{\overline{30}|0.055} \\ &= 11,500 \left[\frac{(1.055)^{30} - 1}{0.055} \right] \\ &\approx 11,500(72.43547797) \\ &\approx 833,008.00 \end{aligned}$$

The amount is \$833,008.00

6. $R = 13,400, i = 0.045, n = 25$

$$\begin{aligned} S &= 13,400 \cdot s_{\overline{25}|0.045} \\ &= 13,400 \left[\frac{(1.045)^{25} - 1}{0.045} \right] \\ &\approx 13,400(44.56521015) \\ &\approx 597,173.82 \end{aligned}$$

The amount is \$597,173.82.

7. $R = 10,500$

Interest is earned semiannually for 7 years, so
 $i = \frac{0.10}{2} = 0.05$ and $n = 2 \cdot 7 = 14$ periods.

$$\begin{aligned} S &= 10,500 \left[\frac{(1.05)^{14} - 1}{0.05} \right] \\ &\approx 10,500(19.59863199) \\ &\approx 205,785.6359 \end{aligned}$$

The amount is \$205,785.64.

8. $R = 4200$

Interest is earned semiannually for 11 years, so
 $i = \frac{0.06}{2} = 0.03$ and $n = 2 \cdot 11 = 22$ periods.

$$\begin{aligned} S &= 4200 \left[\frac{(1.03)^{22} - 1}{0.03} \right] \\ &\approx 4200(30.5367803) \\ &\approx 128,254.48 \end{aligned}$$

The amount is \$128,254.48.

9. $R = 1800$

Interest is earned quarterly for 12 years, so
 $i = \frac{0.08}{4} = 0.02$ and $n = 4 \cdot 12 = 48$ periods.

$$\begin{aligned} S &= 1800 \left[\frac{(1.02)^{48} - 1}{0.02} \right] \\ &\approx 1800(79.35351927) \\ &\approx 142,836.33 \end{aligned}$$

The amount is \$142,836.33.

10. $R = 5300$

Interest is earned quarterly for 9 years, so
 $i = \frac{0.04}{4} = 0.01$ and $n = 4 \cdot 9 = 36$ periods.

$$\begin{aligned} S &= 5300 \left[\frac{(1.01)^{36} - 1}{0.01} \right] \\ &\approx 5300(43.07687836) \\ &\approx 228,307.46 \end{aligned}$$

The amount is \$228,307.46.

11. This describes an ordinary annuity with
 $S = 10,000$, $i = 0.08$, and $n = 12$ periods.

$$\begin{aligned} 10,000 &= R \cdot s_{\overline{12}|0.08} \\ 10,000 &= R(18.97712646) \\ R &\approx 526.95 \end{aligned}$$

The periodic payment should be \$526.95.

12. This describes an ordinary annuity with
 $S = 80,000$, $i = \frac{0.06}{2} = 0.03$, and
 $n = 2 \cdot 9 = 18$ periods.

$$\begin{aligned} S &= R \cdot s_{\overline{n}|i} \\ R &= \frac{80,000}{s_{\overline{18}|0.03}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{80,000}{\left[\frac{(1.03)^{18} - 1}{0.03} \right]} \\
 &\approx \frac{80,000}{23.41443537} \\
 &\approx 3416.70
 \end{aligned}$$

The periodic payment should be \$3416.70.

13. This describes an ordinary annuity with $S = 50,000$, $i = 0.03$ ($= \frac{12\%}{4}$), and $n = 8 \cdot 4 = 32$ periods.

$$\begin{aligned}
 50,000 &= R \cdot s_{\overline{32}|0.03} \\
 50,000 &\approx R(52.50275852) \\
 R &\approx 952.33
 \end{aligned}$$

The periodic payment should be \$952.33.

14. This describes an ordinary annuity with $S = 8000$, $i = 0.04/12$, and $n = (12)(5) = 60$ periods.

$$\begin{aligned}
 8,000 &= R \cdot s_{\overline{60}|(0.04/12)} \\
 8000 &= R \left[\frac{\left(1 + \frac{0.04}{12}\right)^{60} - 1}{\frac{0.04}{12}} \right] \\
 8000 &\approx R(66.298978) \\
 R &\approx 120.67
 \end{aligned}$$

The periodic payment should be \$120.67.

15. $R = 5000$, $i = 0.06$, and $n = 11$ payments.

$$a_{\overline{11}|0.06} = \left[\frac{1 - (1.06)^{-11}}{0.06} \right] = 7.886874577,$$

so

$$\begin{aligned}
 P &\approx 5000(7.886874577) \approx 39,434.37, \\
 &\text{or } \$39,434.37.
 \end{aligned}$$

16. $R = 1280$, $i = 0.07$, and $n = 9$ periods.

$$\begin{aligned}
 P &= 1280 \cdot a_{\overline{9}|0.07} \\
 &= 1280 \left[\frac{1 - (1.07)^{-9}}{0.07} \right] \\
 &\approx 1280(6.515232249) \\
 &\approx 8339.50
 \end{aligned}$$

The present value is \$8339.50.

17. $R = 1400$, $i = \frac{0.06}{2} = 0.03$, and $n = 2 \cdot 8 = 16$ periods.

$$\begin{aligned}
 P &= 1400 \left[\frac{1 - (1.03)^{-16}}{0.03} \right] \\
 &\approx 1400(12.56110203) \\
 &\approx 17,585.54
 \end{aligned}$$

The present value is \$17,585.54.

18. $R = 960$, $i = \frac{0.05}{2} = 0.025$, and $n = 2 \cdot 16 = 32$ periods.

$$\begin{aligned}
 P &= 960 \left[\frac{1 - (1.025)^{-32}}{0.025} \right] \\
 &\approx 960(21.84917796) \\
 &\approx 20,975.21
 \end{aligned}$$

The present value is \$20,975.21.

19. $R = 50,000$, $i = \frac{0.08}{4} = 0.02$, and $n = 10 \cdot 4 = 40$ payments.

$$\begin{aligned}
 a_{\overline{40}|0.02} &= \left[\frac{1 - (1.02)^{-40}}{0.02} \right] \\
 &\approx 27.35547924,
 \end{aligned}$$

so

$$\begin{aligned}
 P &\approx 50,000(27.35547924) \\
 &\approx 1,367,773.96,
 \end{aligned}$$

or \$1,367,773.96.

20. $R = 9800$, $i = \frac{0.04}{4} = 0.01$, and $n = 4 \cdot 15 = 60$ periods.

$$\begin{aligned}
 P &= 9800 \left[\frac{1 - (1.01)^{-60}}{0.01} \right] \\
 &\approx 9800(44.95503841) \\
 &\approx 440,559.38
 \end{aligned}$$

The present value is \$440,559.38.

21. $a_{\overline{15}|0.04} = \left[\frac{1 - (1.04)^{-15}}{0.04} \right] \approx 11.11838743$, so

$P \approx 10,000(11.11838743) \approx 111,183.87$, or \$111,183.87. A lump sum deposit of \$111,183.87 today at 4% compounded annually will yield the same total after 15 years as deposits of \$10,000 at the end of each year for 15 years at 4% compounded annually.

$$22. \quad a_{\overline{15}|0.005} = \left[\frac{1 - (1.05)^{-15}}{0.05} \right] \approx 10.37965804, \text{ so}$$

$P \approx 10,000(10.37965804) \approx 103,796.58$, or \$103,796.58.

A lump sum deposit of \$103,796.58 today at 5% compounded annually will yield the same total after 15 years as deposits of \$10,000 at the end of each year for 15 years at 5% compounded annually.

$$23. \quad a_{\overline{15}|0.06} = \left[\frac{1 - (1.06)^{-15}}{0.06} \right] \approx 9.712248988, \text{ so}$$

$P \approx 10,000(9.712248988) \approx 97,122.49$, or \$97,122.49. A lump sum deposit of \$97,122.49 today at 6% compounded annually will yield the same total after 15 years as deposits of \$10,000 at the end of each year for 15 years at 6% compounded annually.

$$24. \quad a_{\overline{15}|0.08} = \left[\frac{1 - (1.08)^{-15}}{0.08} \right] = 8.559478688, \text{ so}$$

$P \approx 10,000(8.559478688) \approx 85,594.79$, or \$85,594.79. A lump sum deposit of \$85,594.79 today at 8% compounded annually will yield the same total after 15 years as deposits of \$10,000 at the end of each year for 15 years at 8% compounded annually.

25. \$2500 is the present value of this annuity of R dollars, with 6 periods, and $i = \frac{16\%}{4} = 4\% = 0.04$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 2500 &= R \cdot a_{\overline{6}|0.04} \\ R &= \frac{2500}{a_{\overline{6}|0.04}} \\ &\approx \frac{2500}{5.242136857} \\ &\approx 476.90 \end{aligned}$$

Each payment is \$476.90.

26. \$1000 is the present value of this annuity of R dollars, with 9 periods, and $i = 8\% = 0.08$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 1000 &= R \cdot a_{\overline{9}|0.08} \end{aligned}$$

$$\begin{aligned} R &= \frac{1000}{a_{\overline{9}|0.08}} \\ &\approx \frac{1000}{6.246887911} \\ &\approx 160.08 \end{aligned}$$

Each payment is \$160.08.

27. \$90,000 is the present value of this annuity of R dollars, with 12 periods, and $i = 8\% = 0.08$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 90,000 &= R \cdot a_{\overline{12}|0.08} \\ R &= \frac{90,000}{a_{\overline{12}|0.08}} \\ &\approx \frac{90,000}{7.536078017} \\ &\approx 11,942.55 \end{aligned}$$

Each payment is \$11,942.55.

28. \$41,000 is the present value of this annuity of R dollars, with $n = 10$ periods, and $i = \frac{12\%}{2} = 6\% = 0.06$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 41,000 &= R \cdot a_{\overline{10}|0.06} \\ R &= \frac{41,000}{a_{\overline{10}|0.06}} \\ &\approx \frac{41,000}{7.36008705} \\ &\approx 5570.59 \end{aligned}$$

Each payment is \$5570.59.

29. \$55,000 is the present value of this annuity of R dollars, with $i = \frac{0.06}{12} = 0.005$, and $n = 36$ periods.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ R &= \frac{55,000}{a_{\overline{36}|0.005}} \\ &\approx \frac{55,000}{32.87101624} \\ &\approx 1673.21 \end{aligned}$$

Each payment is \$1673.21.

- 30.** \$6800 is the present value of this annuity of R dollars, $i = \frac{0.12}{12} = 0.01$, and $n = 24$ periods.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ R &= \frac{6800}{a_{\overline{24}|0.01}} \\ &\approx \frac{6800}{21.24338726} \\ &\approx 320.10 \end{aligned}$$

Each payment is \$320.10.

- 31. (a)** Sara's payments form an ordinary annuity with $R = 12,000$, $n = 9$, and $i = 0.05$. The amount of this annuity is

$$\begin{aligned} S &= 12,000s_{\overline{9}|0.05} \\ &= 12,000 \left[\frac{(1.05)^9 - 1}{0.05} \right] \\ &\approx (12,000)(11.026564) \\ &\approx 132,318.77, \end{aligned}$$

or \$132,318.77.

- (b)** Using her brother-in-law's bank gives her an annuity with $R = 12,000$, $n = 9$, and $i = 0.03$. The amount of this annuity is

$$\begin{aligned} S &= 12,000s_{\overline{9}|0.03} \\ S &= 12,000 \left[\frac{(1.03)^9 - 1}{0.03} \right] \\ &\approx (12,000)(10.159106) \\ &\approx 121,909.27, \end{aligned}$$

or \$121,909.27.

- (c)** The amount Sara would lose using the second option is $132,318.77 - 121,909.27 = 10,409.50$, or \$10,409.50.

- 32.** Tobi's payments form an ordinary annuity with $R = 100$, $n = (12)(8) = 96$, and $i = 0.06/12 = 0.005$.

- (a)** Tobi's payments total $(96)(100)$ or \$9600.

- (b)** The final amount in the annuity will be

$$\begin{aligned} S &= 100s_{\overline{96}|0.005} \\ &= 100 \left[\frac{(1.005)^{96} - 1}{0.005} \right] \\ &\approx (100)(122.828542) \\ &\approx 12,282.85, \end{aligned}$$

or \$12,282.85.

- (c)** The interest Tobi earned is $12,282.85 - 9600.00 = 2682.85$, or \$2682.25.

- 33. (a)** Steve's payments will form an ordinary annuity with final amount 20,000, R unknown, $n = (4)(8) = 32$, and $i = 0.06/4 = 0.015$.

$$\begin{aligned} 20,000 &= Rs_{\overline{32}|0.015} \\ &= R \left[\frac{(1.015)^{32} - 1}{0.015} \right] \\ &\approx R(40.688288) \\ R &\approx \frac{20,000}{40.688288} \\ &\approx 491.54 \end{aligned}$$

The required payments at the end of each quarter are \$491.54.

- (b)** In this scenario, $i = 0.04/4 = 0.01$.

$$\begin{aligned} 20,000 &= Rs_{\overline{32}|0.01} \\ &= R \left[\frac{(1.01)^{32} - 1}{0.01} \right] \\ &\approx R(37.494068) \\ R &\approx \frac{20,000}{37.494068} \approx 533.42 \end{aligned}$$

The required payments at the end of each quarter are \$533.42.

- 34.** Megan's payments will form an ordinary annuity with final amount 24,000, R unknown, $n = (4)(5) = 20$, and $i = 0.05/4 = 0.0125$.

$$\begin{aligned} 24,000 &= Rs_{\overline{20}|0.0125} \\ &= R \left[\frac{(1.0125)^{20} - 1}{0.0125} \right] \\ &\approx R(22.562979) \end{aligned}$$

$$R \approx \frac{24,000}{22.562979} \approx 1063.69$$

The required payments at the end of each quarter are \$1063.69.

- 35.** Harv's payments will form an ordinary annuity with final amount 12,000, R unknown, $n = (2)(4) = 8$, and $i = 0.04/2 = 0.02$.

$$\begin{aligned} 12,000 &= Rs_{\overline{8}|0.02} \\ &= R \left[\frac{(1.02)^8 - 1}{0.02} \right] \end{aligned}$$

$$\approx R(8.582969)$$

$$R \approx \frac{12,000}{8.582969} \approx 1398.12$$

The required payments at the end of each quarter are \$1398.12.

- 36. (a)** The firm's payments will form an ordinary annuity with final amount 40,000, R unknown, $n = 7$, and $i = 0.06$.

$$40,000 = Rs \overline{7}|_{0.06}$$

$$= R \left[\frac{(1.06)^7 - 1}{0.06} \right]$$

$$\approx R(8.393838)$$

$$R \approx \frac{40,000}{8.393838} \approx 4765.40$$

The required payments at the end of each quarter are \$4765.40.

- (b)** The firm's payments will form an ordinary annuity with final amount 40,000, R unknown, $n = 7$, and $i = 0.08$.

$$40,000 = Rs \overline{7}|_{0.08}$$

$$= R \left[\frac{(1.08)^7 - 1}{0.08} \right]$$

$$\approx R(8.922803)$$

$$R \approx \frac{40,000}{8.922803} \approx 4482.90$$

The required payments at the end of each quarter are \$4482.90.

- 37.** Interest of $\frac{6\%}{2} = 3\%$ is earned semiannually. In $65 - 40 = 25$ years, there are $25 \cdot 2 = 50$ semiannual periods. Since

$$s \overline{50}|_{0.03} = \left[\frac{(1.03)^{50} - 1}{0.03} \right]$$

$$\approx 112.7968673,$$

the \$1000 semiannual deposits will produce a total of

$$S = 1000(112.7968673)$$

$$\approx 112,796.87,$$

or \$112,796.87.

- 38.** Interest of $\frac{8\%}{2} = 4\%$ is earned semiannually. In $65 - 40 = 25$ years, there are $25 \cdot 2 = 50$ semiannual periods. Since

$$s \overline{50}|_{0.04} = \left[\frac{(1.04)^{50} - 1}{0.04} \right] \approx 152.6670837,$$

the \$1000 semiannual deposits will produce a total of

$$S = 1000(152.6670837) \approx 152,667.08,$$

or \$152,667.08.

- 39.** Interest of $\frac{10\%}{2} = 5\%$ is earned semiannually. In $65 - 40 = 25$ years, there are $25 \cdot 2 = 50$ semiannual periods. Since

$$s \overline{50}|_{0.05} = \left[\frac{(1.05)^{50} - 1}{0.05} \right] \approx 209.3479957,$$

the \$1000 semiannual deposits will produce a total of $S \approx 1000(209.3479957) \approx 209,348.00$, or \$209,348.00.

- 40.** Interest of $\frac{12\%}{2} = 6\%$ is earned semiannually. In $65 - 40 = 25$ years, there are $25 \cdot 2 = 50$ semiannual periods. Since

$$s \overline{50}|_{0.06} = \left[\frac{(1.06)^{50} - 1}{0.06} \right] \approx 290.3359046,$$

the \$1000 semiannual deposits will produce a total of

$$S \approx 1000(290.3359046) \approx 290,335.90,$$

or \$290,335.90.

- 41. (a)** The total amount of interest paid is $(60,000)(0.08)(7) = 33,600$, or \$33,600. This total is divided into $7 \cdot 4 = 28$ equal quarterly interest payments. Since $\frac{33,600}{28} = 1200$, each quarterly interest payment will be \$1200.

- (b)** This ordinary annuity will amount to \$60,000 in 7 years at 6% compounded semiannually. Thus, $S = 60,000$, $n = 7 \cdot 2 = 14$, and $i = \frac{6\%}{2} = 3\% = 0.03$, so

$$60,000 = R \cdot s \overline{14}|_{0.03}$$

$$R = \frac{60,000}{s \overline{14}|_{0.03}}$$

$$\approx \frac{60,000}{17.08632416} \approx 3511.58,$$

or \$3511.58.

(c)

Payment Number	Amount of Deposit	Interest Earned	Total in Account
1	\$3511.58	\$0	\$3511.58
2	\$3511.58	$(3511.58)(0.03) = \$105.35$	$3511.58 + 3511.58 + 105.35 = \7128.51
3	\$3511.58	$(7128.51)(0.03) = \$213.86$	$7128.51 + 3511.58 + 213.86 = \$10,853.95$
4	\$3511.58	$(10,853.95)(0.03) = \$325.62$	$10,853.95 + 3511.58 + 325.62 = \$14,691.15$
5	\$3511.58	$(14,691.15)(0.03) = \$440.73$	$14,691.15 + 3511.58 + 440.73 = \$18,643.46$
6	\$3511.58	$(18,643.46)(0.03) = \$559.30$	$18,643.46 + 3511.58 + 559.30 = \$22,714.34$
7	\$3511.58	$(22,714.34)(0.03) = \$681.43$	$22,714.34 + 3511.58 + 681.43 = \$26,907.35$
8	\$3511.58	$(26,907.35)(0.03) = \$807.22$	$26,907.35 + 3511.58 + 807.22 = \$31,226.15$
9	\$3511.58	$(31,226.15)(0.03) = \$936.78$	$31,226.15 + 3511.58 + 936.78 = \$35,674.51$
10	\$3511.58	$(35,674.51)(0.030) = \$1070.24$	$35,674.51 + 3511.58 + 1070.24 = \$40,256.33$
11	\$3511.58	$(40,256.33)(0.03) = \$1207.69$	$40,256.33 + 3511.58 + 1207.69 = \$44,975.60$
12	\$3511.58	$(44,975.60)(0.03) = \$1349.27$	$44,975.60 + 3511.58 + 1349.27 = \$49,836.45$
13	\$3511.58	$(49,836.45)(0.03) = \$1495.09$	$49,836.45 + 3511.58 + 1495.09 = \$54,843.12$
14	\$3511.58	$(54,843.12)(0.03) = \$1645.29$	$54,843.12 + 3511.58 + 1645.29 = \$59,999.99$

So that the final total in the account is \$60,000, add \$0.01 to the last amount of deposit.

Thus, line 14 of the table will be:

$$14 \quad \$3511.59 \quad (54,843.12)(0.03) = \$1645.29 \quad 54,843.12 + 3511.59 + 1645.29 = \$60,000.00$$

42. (a) The total amount of interest paid is $(4000)(0.06)(5) = \$1200$. This total is divided into $5 \cdot 2 = 10$ equal semiannual interest payments. Since $\frac{1200}{10} = 120$, each semiannual interest payment will be \$120.
- (b) This ordinary annuity will amount to \$4000 in 5 years at 8% compounded annually. Thus, $S = 4000$, $n = 5$, and $i = 0.08$, so

$$\begin{aligned}
 4000 &= R \cdot s_{\overline{5}|0.08} \\
 R &= \frac{4000}{s_{\overline{5}|0.08}} \approx \frac{4000}{5.86660096} \\
 &\approx 681.83.
 \end{aligned}$$

or \$681.83.

(c)

Payment Number	Amount of Deposit	Interest Earned	Total in Account
1	\$681.83	\$0	\$681.83
2	\$681.83	$(681.83)(0.08) = \$54.55$	$681.83 + 681.83 + 54.55 = \1418.21
3	\$681.83	$(1418.21)(0.08) = \$113.46$	$1418.21 + 681.83 + 113.46 = \2213.50
4	\$681.83	$(2213.50)(0.08) = \$177.08$	$2213.50 + 681.83 + 177.08 = \3072.41
5	\$681.83	$(3072.41)(0.08) = \$245.79$	$3072.41 + 681.83 + 245.79 = \4000.03

So that the final total in the account is \$4000, subtract \$0.03 from the last amount of deposit. Thus, line 5 of the table will be:

$$5 \quad \$681.80 \quad (3072.41)(0.08) = \$245.79 \quad 3072.41 + 681.80 + 245.79 = \$4000.00$$

43. \$150,000 is the future value of an annuity over 79 yr compounded quarterly. So, there are $79(4) = 316$ payment periods.

(a) The interest per quarter is $\frac{5.25\%}{4} = 1.3125\%$.

Thus, $S = 150,000$, $n = 316$, $i = 0.013125$, and we must find the quarterly payment R in the formula

$$S = R \left[\frac{(1+i)^n - 1}{i} \right]$$

$$150,000 = R \left[\frac{(1.013125)^{316} - 1}{0.013125} \right]$$

$$R \approx 32.4923796$$

She would have to put \$32.49 into her savings at the end of every three months.

(b) For a 2% interest rate, the interest per quarter is $\frac{2\%}{4} = 0.5\%$. Thus, $S = 150,000$, $n = 316$, $i = 0.005$, and we must find the quarterly payment R in the formula

$$S = R \left[\frac{(1+i)^n - 1}{i} \right]$$

$$150,000 = R \left[\frac{(1.005)^{316} - 1}{0.005} \right]$$

$$R \approx 195.5222794$$

She would have to put \$195.52 into her savings at the end of every three months. For a 7% interest rate, the interest per quarter is $\frac{7\%}{4} = 1.75\%$. Thus, $S = 150,000$, $n = 316$, $i = 0.0175$, and we must find the quarterly payment R in the formula

$$S = R \left[\frac{(1+i)^n - 1}{i} \right]$$

$$150,000 = R \left[\frac{(1.0175)^{316} - 1}{0.0175} \right]$$

$$R \approx 10.9663932$$

She would have to put \$10.97 into her savings at the end of every three months.

44. First, we want to find the amount of the annuity with $R = 1000$, $n = 8$, and $i = 0.06$.

$$S = 1000 \left[\frac{(1.06)^8 - 1}{0.06} \right]$$

The number in brackets, $s_{\overline{8}|0.06}$, is 9.897467909, so that

$$S \approx 1000(9.897467909) \approx 9897.47,$$

or \$9897.47.

Next, we want to find the lump sum that must be deposited today at 5% compounded annually for 8 years to provide \$9897.47.

$$A = 9897.47, i = 0.05, \text{ and } n = 8, \text{ so}$$

$$9897.47 = P(1.05)^8$$

$$P = \frac{9897.47}{(1.05)^8} \approx 6699.00,$$

or \$6699.00.

45. We want to find the present value of an annuity of \$2000 per year for 9 years at 8% compounded annually.

$$a_{\overline{9}|0.08} = \left[\frac{1 - (1.08)^{-9}}{0.08} \right] \approx 6.246887911, \text{ so}$$

$$P \approx 2000(6.246887911) \approx 12,493.78,$$

or \$12,493.78. A lump sum deposit of \$12,493.78 today at 8% compounded annually will yield the same total after 9 years deposits of \$2000 at the end of each year for 9 years at 8% compounded annually.

46. We want to find the present value of an annuity of \$50,000 per year for 20 years at 6% compounded annually.

$$a_{\overline{20}|0.06} = \left[\frac{1 - (1.06)^{-20}}{0.06} \right]$$

$$\approx 11.46992122, \text{ so}$$

$$P \approx 50,000(11.46992122)$$

$$\approx 573,496.06,$$

or \$573,496.06. A lump sum deposit of \$573,496.06 today at 6% compounded annually will yield the same total after 20 years as deposits of \$50,000 at the end of each year for 20 years at 6% compounded annually.

47. For parts (a) and (b), if \$1 million is divided into 20 equal payments, each payment is \$50,000.

(a) $i = 0.05, n = 20$

$$\begin{aligned} P &= R \left[\frac{1 - (1 + i)^{-n}}{i} \right] \\ &= 50,000 \left[\frac{1 - (1 + 0.05)^{-20}}{0.05} \right] \\ &\approx 623,110.52 \end{aligned}$$

The present value is \$623,110.52.

(b) $i = 0.09, n = 20$

$$\begin{aligned} P &= R \left[\frac{1 - (1 + i)^{-n}}{i} \right] \\ &= 50,000 \left[\frac{1 - (1 + 0.09)^{-20}}{0.09} \right] \\ &\approx 456,427.28 \end{aligned}$$

The present value is \$456,427.28.

For parts (c) and (d), if \$1 million is divided into 25 equal payments, each payment is \$40,000.

(c) $i = 0.05, n = 25$

$$\begin{aligned} P &= R \left[\frac{1 - (1 + i)^{-n}}{i} \right] \\ &= 40,000 \left[\frac{1 - (1 + 0.05)^{-25}}{0.05} \right] \\ &\approx 563,757.78 \end{aligned}$$

The present value is \$563,757.78.

(d) $i = 0.09, n = 25$

$$\begin{aligned} P &= R \left[\frac{1 - (1 + i)^{-n}}{i} \right] \\ &= 40,000 \left[\frac{1 - (1 + 0.09)^{-25}}{0.09} \right] \\ &\approx 392,903.18 \end{aligned}$$

The present value is \$392,903.18.

48. \$22,000 is the present value of this annuity with interest $i = \frac{0.09}{12} = 0.0075$ per month and of $n = 4 \cdot 12 = 48$ monthly payments.

$$\begin{aligned} S &= R \cdot a_{\overline{n}|i} \\ R &= \frac{22,000}{a_{\overline{48}|0.0075}} \approx \frac{22,000}{40.18478189} \approx 547.47 \end{aligned}$$

The amount of each payment is \$547.47.

Since 48 payments of \$547.47 each are made, $48(\$547.47) = \$26,278.56$ will be paid in total. This means that $\$26,278.56 - \$22,000 = \$4,278.56$ is the total amount of interest Kristina will pay.

49. The present value, P , is 249,560, $i = \frac{0.0775}{12} \approx 0.0064583333$, and $n = 12 \cdot 25 = 300$.

$$249,560 = R \cdot a_{\overline{300}|0.0064583333} = R \left[\frac{1 - (1 + 0.0064583333)^{-300}}{0.0064583333} \right] \approx R \left[\frac{1 - 0.1449639356}{0.0064583333} \right] \approx R \left[\frac{0.8550360644}{0.0064583333} \right]$$

or $R \approx 1885.00$.

Monthly payments of \$1885.00 will be required to amortize the loan.

Use the formula for the unpaid balance of a loan with $R = 1885$, $i \approx 0.0064583333$, $n = 12 \cdot 15 = 300$, and $x = 12 \cdot 5 = 60$.

$$y = R \left[\frac{1 - (1 + i)^{-(n-x)}}{i} \right] \approx 1885 \left[\frac{1 - (1 + 0.0064583333)^{-240}}{0.0064583333} \right] \approx 229,612.44$$

The unpaid balance after 5 years is \$229,612.44.

50. The present value, P , is 170,892, $i = \frac{0.0811}{12} \approx 0.0067583333$, and $n = 12 \cdot 30 = 360$.

$$170,892 = R \cdot a_{\overline{360}|0.0067583333} = R \left[\frac{1 - (1 + 0.0067583333)^{-360}}{0.0067583333} \right] \approx R \left[\frac{1 - 0.0884944586}{0.0067583333} \right] \approx R \left[\frac{0.9115055414}{0.0067583333} \right]$$

$R \approx 1267.07$.

Monthly payments of \$1267.07 will be required to amortize the loan.

Use the formula for the unpaid balance of a loan with $R = 1267.07$, $i \approx 0.0067583333$, $n = 12 \cdot 30 = 360$, and $x = 12 \cdot 5 = 60$.

$$y = R \left[\frac{1 - (1 + i)^{-(n-x)}}{i} \right] \approx 1267.07 \left[\frac{1 - (1 + 0.0067583333)^{-300}}{0.0067583333} \right] \approx 162,628.91$$

The unpaid balance after 5 years is \$162,628.91.

51. The present value, P , is 353,700, $i = \frac{0.0795}{12} = 0.006625$, and $n = 12 \cdot 30 = 360$ periods.

$$P = R \cdot a_{\overline{n}|i}$$

$$R = \frac{353,700}{a_{\overline{360}|0.006625}} = 353,700 \left[\frac{1 - (1.006625)^{-360}}{0.006625} \right] \approx \frac{353,700}{136.9334083} \approx 2583.01$$

Monthly payments of \$2583.01 will be required to amortize the loan.

To find the unpaid balance, use the formula for the unpaid balance of a loan with $R = 2583.01$, $i = 0.006625$, $n = 360$, and $x = 12 \cdot 5 = 60$.

$$y = R \cdot \left[\frac{1 - (1 + i)^{-(n-x)}}{i} \right] = 2583.01 \left[\frac{1 - (1 + 0.006625)^{-300}}{0.006625} \right] \approx 2583.01(130.1224488) \approx 336,107.59$$

The unpaid balance after 5 years is \$336,107.59.

52. The present value, P is 196,511, $i = \frac{0.0757}{12} \approx 0.0063083333$, and $n = 12 \cdot 25 = 300$.

$$196,511 = R \cdot a_{\overline{300}|0.0063083333} = R \left[\frac{1 - (1 + 0.0063083333)^{-300}}{0.0063083333} \right]$$

$$\approx R \left[\frac{1 - 0.1515930387}{0.0063083333} \right] = R \left[\frac{0.8484069613}{0.0063083333} \right]$$

$$R \approx 1461.16.$$

Monthly payments of \$1461.16 will be required to amortize the loan.

Use the formula for the unpaid balance of a loan with $R = 1461.16$, $i \approx 0.0063083333$, $n = 12 \cdot 25 = 300$, and $x = 12 \cdot 5 = 60$.

$$y = R \left[\frac{1 - (1 + i)^{-(n-x)}}{i} \right] \approx 1461.16 \left[\frac{1 - (1 + 0.0063083333)^{-240}}{0.0063083333} \right] \approx 180,417.11$$

The unpaid balance after 5 years is \$180,417.11.

53. (a) \$25,000 is the present value of this annuity of 8 years with interest of 6% per year.

$$a_{\overline{8}|0.06} = 6.209793811, \text{ so}$$

$$25,000 = R(6.209793811)$$

$$R \approx 4025.90$$

Annual withdrawals of \$4025.90 each will be needed.

- (b) \$25,000 is the present value of this annuity of 12 years with interest of 6% per year.

$$a_{\overline{12}|0.06} = 8.38384394, \text{ so}$$

$$25,000 = R(8.38384394)$$

$$R \approx 2981.93$$

Annual withdrawals of \$2981.93 each will be needed.

54. (a) \$150,000 is the present value of this annuity of $5 \cdot 2 = 10$ periods with interest of $\frac{6\%}{2} = 3\% = 0.03$ per year.

$$a_{\overline{10}|0.03} \approx 8.530202837, \text{ so}$$

$$150,000 = R(8.530202837)$$

$$R \approx 17,584.58$$

Withdrawals of \$17,584.58 at the end of each six-month period are needed.

- (b) \$150,000 is the present value of this annuity of $7 \cdot 2 = 14$ periods with interest $\frac{6\%}{2} = 3\% = 0.03$ per year.

$$a_{\overline{14}|0.03} \approx 11.29607314, \text{ so}$$

$$150,000 = R(11.29607314)$$

$$R \approx 13,278.95$$

Withdrawals of \$13,278.95 at the end of each six-month period are needed.

55. First, find the amount of each payment. \$4000 is the present value of an annuity of R dollars, with 4 periods, and $i = 0.08$ per period.

$$P = R \cdot a_{\overline{n}|i}$$

$$4000 = R \cdot a_{\overline{4}|0.08}$$

$$R = \frac{4000}{a_{\overline{4}|0.08}} \approx \frac{4000}{3.31212684} \approx 1207.68$$

Each payment is \$1207.68.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	\$4000.00
1	\$1207.68	$(4000)(0.08)(1) = \$320.00$	$1207.68 - 320.00 = \$887.68$	$4000.00 - 887.68 = \$3112.32$
2	\$1207.68	$(3112.32)(0.08)(1) = \$248.99$	$1207.68 - 248.99 = \$958.69$	$3112.32 - 958.69 = \$2153.63$
3	\$1207.68	$(2153.63)(0.08)(1) = \$172.29$	$1207.68 - 172.29 = \$1035.39$	$2153.63 - 1035.39 = \$1118.24$
4	\$1207.68	$(1118.24)(0.08)(1) = \$89.46$	$1207.68 - 89.46 = \$1118.22$	$1118.24 - 1118.22 = \$0.02$

So that the final principal is zero, add \$0.02 to the last payment. Thus, line 4 of the table will be:

$$4 \quad \$1207.70 \quad (1118.24)(0.08)(1) = \$89.46 \quad 1207.70 - 89.46 = \$1118.24 \quad 1118.24 - 1118.24 = \$0$$

56. First, find the amount of each payment. \$72,000 is the present value of an annuity of R dollars, with 9 periods, and $i = \frac{9.5\%}{2} = 4.75\% = 0.0475$ per period.

$$P = R \cdot a_{\overline{n}|i}$$

$$72,000 = R \cdot a_{\overline{9}|0.0475}$$

$$R = \frac{72,000}{a_{\overline{9}|0.0475}} \approx \frac{72,000}{7.187624181} \approx 10,017.22$$

Each payment is \$10,017.22.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	\$72,000.00
1	\$10,017.22	$72,000(0.0475) = \$3420.00$	$10,017.22 - 3420.00 = \$6597.22$	$72,000.00 - 6597.22 = \$65,402.78$
2	\$10,017.22	$65,402.78(0.0475) = \$3106.63$	$10,017.22 - 3106.63 = \$6910.59$	$65,402.78 - 6910.59 = \$58,492.19$
3	\$10,017.22	$58,492.19(0.0475) = \$2778.38$	$10,017.22 - 2778.38 = \$7238.84$	$58,492.19 - 7238.84 = \$51,253.35$
4	\$10,017.22	$51,253.35(0.0475) = \$2434.53$	$10,017.22 - 2434.53 = \$7582.69$	$51,253.35 - 7582.69 = \$43,670.66$
5	\$10,017.22	$43,670.66(0.0475) = \$2074.36$	$10,017.22 - 2074.36 = \$7942.86$	$43,670.66 - 7942.86 = \$35,727.80$
6	\$10,017.22	$35,727.80(0.0475) = \$1697.07$	$10,017.22 - 1697.07 = \$8320.15$	$35,727.80 - 8320.15 = \$27,407.65$
7	\$10,017.22	$27,407.65(0.0475) = \$1301.86$	$10,017.22 - 1301.86 = \$8715.36$	$27,407.65 - 8715.36 = \$18,692.29$
8	\$10,017.22	$18,692.29(0.0475) = \$887.88$	$10,017.22 - 887.88 = \$9129.34$	$18,692.29 - 9129.34 = \$9562.95$
9	\$10,017.22	$9562.95(0.0475) = \$454.24$	$10,017.22 - 454.24 = \$9562.98$	$9562.95 - 9562.98 = -\$0.03$

So that the final principal is zero, subtract \$0.03 from the last payment. Thus, line 9 of the table will be:

$$9 \quad \$10,017.19 \quad 9562.95(0.0475) = \$454.24 \quad 10,017.19 - 454.24 = \$9562.95 \quad 9562.95 - 9562.95 = \$0$$

57. The firm's total cost is $8(1048) = \$8384.00$. After making a down payment of \$1200, a balance of $8384 - 1200 = \$7184$ is owed. To amortize this balance, first find the amount of each payment. \$7184 is the present value of an annuity of R dollars, with $4 \cdot 12 = 48$ periods, and $i = \frac{10.5\%}{12} = .875\% = 0.00875$ per period.

$$P = R \cdot a_{\overline{n}|i}$$

$$7184 = R \cdot a_{\overline{48}|0.00875}$$

$$R = \frac{7184}{a_{\overline{48}|0.00875}} \approx \frac{7184}{39.05734359} \approx 183.93$$

Each payment is \$183.93.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	\$7184.00
1	\$183.93	$(7184.00)(0.105)\left(\frac{1}{12}\right) = \62.86	$183.93 - 62.86 = \$121.07$	$7184.00 - 121.07 = \$7062.93$
2	\$183.93	$(7062.93)(0.105)\left(\frac{1}{12}\right) = \61.80	$183.93 - 61.80 = \$122.13$	$7062.93 - 122.13 = \$6940.80$
3	\$183.93	$(6940.80)(0.105)\left(\frac{1}{12}\right) = \60.73	$183.93 - 60.73 = \$123.20$	$6940.80 - 123.20 = \$6817.60$
4	\$183.93	$(6817.60)(0.105)\left(\frac{1}{12}\right) = \59.65	$183.93 - 59.65 = \$124.28$	$6817.60 - 124.28 = \$6693.32$
5	\$183.93	$(6693.32)(0.105)\left(\frac{1}{12}\right) = \58.57	$183.93 - 58.57 = \$125.36$	$6693.32 - 125.36 = \$6567.96$
6	\$183.93	$(6567.96)(0.105)\left(\frac{1}{12}\right) = \57.47	$183.93 - 57.47 = \$126.46$	$6567.96 - 126.46 = \$6441.50$

58. Barbara's total cost is $14,000 + 7200 = \$21,200.00$. After making a down payment of \$1200, she owes a balance of $21,200 - 1200 = \$20,000$. To amortize this balance, first find the amount of each payment. \$20,000 is the present value of an annuity of R dollars, with $5 \cdot 2 = 10$ periods, and $i = \frac{8\%}{2} = 4\% = 0.04$ per period.

$$P = R \cdot a_{\overline{n}|i}$$

$$20,000 = R \cdot a_{\overline{10}|0.04}$$

$$R = \frac{20,000}{a_{\overline{10}|0.04}}$$

$$\approx \frac{20,000}{8.110895779}$$

$$\approx 2465.82$$

Each payment is \$2465.82.

Payment Number	Amount of Payment	Interest for Period	Portion to Principal	Principal at End of Period
0	—	—	—	\$20,000.00
1	\$2465.82	$(20,000)(0.08)\left(\frac{1}{2}\right) = \800.00	$2465.82 - 800.00 = \$1665.82$	$20,000.00 - 1665.82 = \$18,334.18$
2	\$2465.82	$(18,334.18)(0.08)\left(\frac{1}{2}\right) = \733.37	$2465.82 - 733.37 = \$1732.45$	$18,334.18 - 1732.45 = \$16,601.73$
3	\$2465.82	$(16,601.73)(0.08)\left(\frac{1}{2}\right) = \664.07	$2465.82 - 664.07 = \$1801.75$	$16,601.73 - 1801.75 = \$14,799.98$
4	\$2465.82	$(14,799.98)(0.08)\left(\frac{1}{2}\right) = \592.00	$2465.82 - 592.00 = \$1873.82$	$14,799.98 - 1873.82 = \$12,926.16$
5	\$2465.82	$(12,926.16)(0.08)\left(\frac{1}{2}\right) = \517.05	$2465.82 - 517.05 = \$1948.77$	$12,926.16 - 1948.77 = \$10,977.39$
6	\$2465.82	$(10,977.39)(0.08)\left(\frac{1}{2}\right) = \439.10	$2465.82 - 439.10 = \$2026.72$	$10,977.39 - 2026.72 = \$8950.67$
7	\$2465.82	$(8950.67)(0.08)\left(\frac{1}{2}\right) = \358.03	$2465.82 - 358.03 = \$2107.79$	$8950.67 - 2107.79 = \$6842.88$
8	\$2465.82	$(6842.88)(0.08)\left(\frac{1}{2}\right) = \273.72	$2465.82 - 273.72 = \$2192.10$	$6842.88 - 2192.10 = \$4650.78$
9	\$2465.82	$(4650.78)(0.08)\left(\frac{1}{2}\right) = \186.03	$2465.82 - 186.03 = \$2279.79$	$4650.78 - 2279.79 = \$2370.99$
10	\$2465.82	$(2370.99)(0.08)\left(\frac{1}{2}\right) = \94.84	$2465.82 - 94.84 = \$2370.98$	$2370.99 - 2370.98 = \$0.01$

So that the final principal is zero, add \$0.01 to the last payment. Thus, line 10 of the table will be:

$$10 \quad \$2465.83 \quad (2370.99)(0.08)\left(\frac{1}{2}\right) = \$94.84 \quad 2465.83 - 94.84 = \$2370.99 \quad 2370.99 - 2370.99 = \$0$$

12.3 Taylor Polynomials at 0

Your Turn 1

$$P_3(-0.15) = 1 + \frac{1}{1!}(-0.15) + \frac{1}{2!}(-0.15)^2 + \frac{1}{3!}(-0.15)^3 + \frac{1}{4!}(-0.15)^4 + \frac{1}{5!}(-0.15)^5$$

$$\approx 0.860708$$

Your Turn 2

The derivatives of $f(x) = \sqrt{x+4}$ are exactly as shown in Example 2 except that $x+1$ is replaced everywhere by $x+4$.

Derivative	Value at 0
$f(x) = (x+4)^{1/2}$	$f(0) = 2$
$f^{(1)}(x) = \frac{1}{2(x+4)^{1/2}}$	$f^{(1)}(0) = \frac{1}{4}$
$f^{(2)}(x) = \frac{-1}{4(x+4)^{3/2}}$	$f^{(2)}(0) = -\frac{1}{32}$
$f^{(3)}(x) = \frac{3}{8(x+4)^{5/2}}$	$f^{(3)}(0) = \frac{3}{256}$
$f^{(4)}(x) = \frac{-15}{16(x+4)^{7/2}}$	$f^{(4)}(0) = \frac{-15}{2048}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!} + \frac{f^{(2)}(0)}{2!} + \frac{f^{(3)}(0)}{3!} + \frac{f^{(4)}(0)}{4!} \\
 &= 2 + \frac{1/4}{1!} + \frac{-1/32}{2!} + \frac{3/256}{3!} + \frac{-15/2048}{4!} \\
 &= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 - \frac{5}{16,384}x^4
 \end{aligned}$$

Your Turn 3

To approximate $\sqrt{4.05}$ we evaluate $f(0.05)$ with $f(x) = \sqrt{x+4}$. Using the approximation from Your Turn 2,

$$\begin{aligned}
 \sqrt{4.05} &= f(0.05) \\
 &\approx 2 + \frac{1}{4}(0.05) - \frac{1}{64}(0.05)^2 + \frac{1}{512}(0.05)^3 - \frac{5}{16,384}(0.05)^4 \\
 &\approx 2.0124612.
 \end{aligned}$$

12.3 Exercises

1.	Derivative	Value at 0
	$f(x) = e^{-2x}$	$f(0) = 1$
	$f^{(1)}(x) = -2e^{-2x}$	$f^{(1)}(0) = -2$
	$f^{(2)}(x) = 4e^{-2x}$	$f^{(2)}(0) = 4$
	$f^{(3)}(x) = -8e^{-2x}$	$f^{(3)}(0) = -8$
	$f^{(4)}(x) = 16e^{-2x}$	$f^{(4)}(0) = 16$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{-2}{1!}x + \frac{4}{2!}x^2 + \frac{-8}{3!}x^3 + \frac{16}{4!}x^4 \\
 &= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4
 \end{aligned}$$

2.	Derivative	Value at 0
	$f(x) = e^{3x}$	$f(0) = 1$
	$f^{(1)}(x) = 3e^{3x}$	$f^{(1)}(0) = 3$
	$f^{(2)}(x) = 9e^{3x}$	$f^{(2)}(0) = 9$
	$f^{(3)}(x) = 27e^{3x}$	$f^{(3)}(0) = 27$
	$f^{(4)}(x) = 81e^{3x}$	$f^{(4)}(0) = 81$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{3}{1!}x + \frac{9}{2!}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4 \\
 &= 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4
 \end{aligned}$$

3.

Derivative	Value at 0
$f(x) = e^{x+1}$	$f(0) = e$
$f^{(1)}(x) = e^{x+1}$	$f^{(1)}(0) = e$
$f^{(2)}(x) = e^{x+1}$	$f^{(2)}(0) = e$
$f^{(3)}(x) = e^{x+1}$	$f^{(3)}(0) = e$
$f^{(4)}(x) = e^{x+1}$	$f^{(4)}(0) = e$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= e + \frac{e}{1!}x + \frac{e}{2!}x^2 + \frac{e}{3!}x^3 + \frac{e}{4!}x^4 \\
 &= e + ex + \frac{e}{2}x^2 + \frac{e}{6}x^3 + \frac{e}{24}x^4
 \end{aligned}$$

4.

Derivative	Value at 0
$f(x) = e^{-x}$	$f(0) = 1$
$f^{(1)}(x) = -e^{-x}$	$f^{(1)}(0) = -1$
$f^{(2)}(x) = e^{-x}$	$f^{(2)}(0) = 1$
$f^{(3)}(x) = -e^{-x}$	$f^{(3)}(0) = -1$
$f^{(4)}(x) = e^{-x}$	$f^{(4)}(0) = 1$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{-1}{1!}x + \frac{1}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{1}{4!}x^4 \\
 &= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4
 \end{aligned}$$

5.

Derivative	Value at 0
$f(x) = \sqrt{x+9} = (x+9)^{1/2}$	$f(0) = 3$
$f^{(1)}(x) = \frac{1}{2}(x+9)^{-1/2} = \frac{1}{2(x+9)^{1/2}}$	$f^{(1)}(0) = \frac{1}{6}$
$f^{(2)}(x) = -\frac{1}{4}(x+9)^{-3/2} = -\frac{1}{4(x+9)^{3/2}}$	$f^{(2)}(0) = -\frac{1}{108}$
$f^{(3)}(x) = \frac{3}{8}(x+9)^{-5/2} = \frac{3}{8(x+9)^{5/2}}$	$f^{(3)}(0) = \frac{1}{648}$
$f^{(4)}(x) = -\frac{15}{16}(x+9)^{-7/2} = -\frac{15}{16(x+9)^{7/2}}$	$f^{(4)}(0) = -\frac{5}{11,664}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 3 + \frac{1}{6}x + \frac{-\frac{1}{108}}{2!}x^2 + \frac{\frac{1}{648}}{3!}x^3 + \frac{-\frac{5}{11,664}}{4!}x^4 \\
 &= 3 + \frac{1}{6}x - \frac{1}{216}x^2 + \frac{1}{3888}x^3 - \frac{5}{279,936}x^4
 \end{aligned}$$

6.	Derivative	Value at 0
	$f(x) = \sqrt{x+16} = (x+16)^{1/2}$	$f(0) = 4$
	$f^{(1)}(x) = \frac{1}{2}(x+16)^{-1/2} = \frac{1}{2(x+16)^{1/2}}$	$f^{(1)}(0) = \frac{1}{8}$
	$f^{(2)}(x) = -\frac{1}{4}(x+16)^{-3/2} = -\frac{1}{4(x+16)^{3/2}}$	$f^{(2)}(0) = -\frac{1}{256}$
	$f^{(3)}(x) = \frac{3}{8}(x+16)^{-5/2} = \frac{3}{8(x+16)^{5/2}}$	$f^{(3)}(0) = \frac{3}{8192}$
	$f^{(4)}(x) = -\frac{15}{16}(x+16)^{-7/2} = -\frac{15}{16(x+16)^{7/2}}$	$f^{(4)}(0) = -\frac{15}{262,144}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 4 + \frac{\frac{1}{8}}{1!}x + \frac{-\frac{1}{256}}{2!}x^2 + \frac{\frac{3}{8192}}{3!}x^3 + \frac{-\frac{15}{262,144}}{4!}x^4 \\
 &= 4 + \frac{1}{8}x - \frac{1}{512}x^2 + \frac{1}{16,384}x^3 - \frac{5}{2,097,152}x^4
 \end{aligned}$$

7.	Derivative	Value at 0
	$f(x) = \sqrt[3]{x-1} = (x-1)^{1/3}$	$f(0) = -1$
	$f^{(1)}(x) = \frac{1}{3}(x-1)^{-2/3} = \frac{1}{3(x-1)^{2/3}}$	$f^{(1)}(0) = \frac{1}{3}$
	$f^{(2)}(x) = -\frac{2}{9}(x-1)^{-5/3} = -\frac{2}{9(x-1)^{5/3}}$	$f^{(2)}(0) = \frac{2}{9}$
	$f^{(3)}(x) = \frac{10}{27}(x-1)^{-8/3} = \frac{10}{27(x-1)^{8/3}}$	$f^{(3)}(0) = \frac{10}{27}$
	$f^{(4)}(x) = -\frac{80}{81}(x-1)^{-11/3} = -\frac{80}{81(x-1)^{11/3}}$	$f^{(4)}(0) = \frac{80}{81}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= -1 + \frac{\frac{1}{3}}{1!}x + \frac{\frac{2}{9}}{2!}x^2 + \frac{\frac{10}{27}}{3!}x^3 + \frac{\frac{80}{81}}{4!}x^4 \\
 &= -1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{5}{81}x^3 + \frac{10}{243}x^4
 \end{aligned}$$

8.	Derivative	Value at 0
	$f(x) = \sqrt[3]{x+8} = (x+8)^{1/3}$	$f(0) = 2$
	$f^{(1)}(x) = \frac{1}{3}(x+8)^{-2/3} = \frac{1}{3(x+8)^{2/3}}$	$f^{(1)}(0) = \frac{1}{12}$
	$f^{(2)}(x) = -\frac{2}{9}(x+8)^{-5/3} = -\frac{2}{9(x+8)^{5/3}}$	$f^{(2)}(0) = -\frac{1}{144}$
	$f^{(3)}(x) = \frac{10}{27}(x+8)^{-8/3} = \frac{10}{27(x+8)^{8/3}}$	$f^{(3)}(0) = \frac{5}{3456}$
	$f^{(4)}(x) = -\frac{80}{81}(x+8)^{-11/3} = -\frac{80}{81(x+8)^{11/3}}$	$f^{(4)}(0) = -\frac{5}{10,368}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 2 + \frac{\frac{1}{12}}{1!}x + \frac{-\frac{1}{144}}{2!}x^2 + \frac{\frac{5}{3456}}{3!}x^3 + \frac{-\frac{5}{10,368}}{4!}x^4 \\
 &= 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{20,736}x^3 - \frac{5}{248,832}x^4
 \end{aligned}$$

9.

Derivative	Value at 0
$f(x) = \sqrt[4]{x+1} = (x+1)^{1/4}$	$f(0) = 1$
$f^{(1)}(x) = \frac{1}{4}(x+1)^{-3/4} = \frac{1}{4(x+1)^{3/4}}$	$f^{(1)}(0) = \frac{1}{4}$
$f^{(2)}(x) = -\frac{3}{16}(x+1)^{-7/4} = -\frac{3}{16(x+1)^{7/4}}$	$f^{(2)}(0) = -\frac{3}{16}$
$f^{(3)}(x) = \frac{21}{64}(x+1)^{-11/4} = \frac{21}{64(x+1)^{11/4}}$	$f^{(3)}(0) = \frac{21}{64}$
$f^{(4)}(x) = -\frac{231}{256}(x+1)^{-15/4} = -\frac{231}{256(x+1)^{15/4}}$	$f^{(4)}(0) = -\frac{231}{256}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{\frac{1}{4}}{1!}x + \frac{-\frac{3}{16}}{2!}x^2 + \frac{\frac{21}{64}}{3!}x^3 + \frac{-\frac{231}{256}}{4!}x^4 \\
 &= 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 - \frac{77}{2048}x^4
 \end{aligned}$$

10.

Derivative	Value at 0
$f(x) = \sqrt[4]{x+16} = (x+16)^{1/4}$	$f(0) = 2$
$f^{(1)}(x) = \frac{1}{4}(x+16)^{-3/4} = \frac{1}{4(x+16)^{3/4}}$	$f^{(1)}(0) = \frac{1}{32}$
$f^{(2)}(x) = -\frac{3}{16}(x+16)^{-7/4} = -\frac{3}{16(x+16)^{7/4}}$	$f^{(2)}(0) = -\frac{3}{2048}$
$f^{(3)}(x) = \frac{21}{64}(x+16)^{-11/4} = \frac{21}{64(x+16)^{11/4}}$	$f^{(3)}(0) = \frac{21}{131,072}$
$f^{(4)}(x) = -\frac{231}{256}(x+16)^{-15/4} = -\frac{231}{256(x+16)^{15/4}}$	$f^{(4)}(0) = -\frac{231}{8,388,608}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 2 + \frac{\frac{1}{32}}{1!}x + \frac{-\frac{3}{2048}}{2!}x^2 + \frac{\frac{21}{131,072}}{3!}x^3 + \frac{-\frac{231}{8,388,608}}{4!}x^4 \\
 &= 2 + \frac{1}{32}x - \frac{3}{4096}x^2 + \frac{7}{262,144}x^3 - \frac{77}{67,108,864}x^4
 \end{aligned}$$

11.

Derivative	Value at 0
$f(x) = \ln(1 - x)$	$f(0) = 0$
$f^{(1)}(x) = -\frac{1}{1-x} = \frac{1}{x-1} = (x-1)^{-1}$	$f^{(1)}(0) = -1$
$f^{(2)}(x) = -(x-1)^{-2} = -\frac{1}{(x-1)^2}$	$f^{(2)}(0) = -1$
$f^{(3)}(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$	$f^{(3)}(0) = -2$
$f^{(4)}(x) = -6(x-1)^{-4} = -\frac{6}{(x-1)^4}$	$f^{(4)}(0) = -6$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 0 + \frac{-1}{1!}x + \frac{-1}{2!}x^2 + \frac{-2}{3!}x^3 + \frac{-6}{4!}x^4 \\
 &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4
 \end{aligned}$$

12.

Derivative	Value at 0
$f(x) = \ln(1 + 2x)$	$f(0) = 0$
$f^{(1)}(x) = \frac{2}{1+2x}$	$f^{(1)}(0) = 2$
$f^{(2)}(x) = -\frac{4}{(1+2x)^2}$	$f^{(2)}(0) = -4$
$f^{(3)}(x) = \frac{16}{(1+2x)^3}$	$f^{(3)}(0) = 16$
$f^{(4)}(x) = -\frac{96}{(1+2x)^4}$	$f^{(4)}(0) = -96$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 0 + \frac{2}{1!}x + \frac{-4}{2!}x^2 + \frac{16}{3!}x^3 + \frac{-96}{4!}x^4 \\
 &= 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4
 \end{aligned}$$

13.

Derivative	Value at 0
$f(x) = \ln(1 + 2x^2)$	$f(0) = 0$
$f^{(1)}(x) = \frac{4x}{1+2x^2}$	$f^{(1)}(0) = 0$
$f^{(2)}(x) = \frac{4-8x^2}{(1+2x^2)^2}$	$f^{(2)}(0) = 4$
$f^{(3)}(x) = \frac{32x^3-48x}{(1+2x^2)^3}$	$f^{(3)}(0) = 0$
$f^{(4)}(x) = \frac{-192x^4+576x^2-48}{(1+2x^2)^4}$	$f^{(4)}(0) = -48$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 0 + \frac{0}{1!}x + \frac{4}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-48}{4!}x^4 \\
 &= 2x^2 - 2x^4
 \end{aligned}$$

14.

Derivative	Value at 0
$f(x) = \ln(1 - x^3)$	$f(0) = 0$
$f^{(1)}(x) = \frac{-3x^2}{1 - x^3} = \frac{3x^2}{x^3 - 1}$	$f^{(1)}(0) = 0$
$f^{(2)}(x) = \frac{-3x^4 - 6x}{(x^3 - 1)^2}$	$f^{(2)}(0) = 0$
$f^{(3)}(x) = \frac{6x^6 + 42x^3 + 6}{(x^3 - 1)^3}$	$f^{(3)}(0) = -6$
$f^{(4)}(x) = \frac{-18x^8 - 288x^5 - 180x^2}{(x^3 - 1)^4}$	$f^{(4)}(0) = 0$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 0 + \frac{0}{1!}x + \frac{0}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{0}{4!}x^4 \\
 &= -x^3
 \end{aligned}$$

15.

Derivative	Value at 0
$f(x) = xe^{-x}$	$f(0) = 0$
$f^{(1)}(x) = -xe^{-x} + e^{-x}$	$f^{(1)}(0) = 1$
$f^{(2)}(x) = xe^{-x} - 2e^{-x}$	$f^{(2)}(0) = -2$
$f^{(3)}(x) = -xe^{-x} + 3e^{-x}$	$f^{(3)}(0) = 3$
$f^{(4)}(x) = xe^{-x} - 4e^{-x}$	$f^{(4)}(0) = -4$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 0 + \frac{1}{1!}x + \frac{-2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{-4}{4!}x^4 \\
 &= x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4
 \end{aligned}$$

16.

Derivative	Value at 0
$f(x) = x^2e^x$	$f(0) = 0$
$f^{(1)}(x) = x^2e^x + 2xe^x$	$f^{(1)}(0) = 0$
$f^{(2)}(x) = x^2e^x + 4xe^x + 2e^x$	$f^{(2)}(0) = 2$
$f^{(3)}(x) = x^2e^x + 6xe^x + 6e^x$	$f^{(3)}(0) = 6$
$f^{(4)}(x) = x^2e^x + 8xe^x + 12e^x$	$f^{(4)}(0) = 12$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 0 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{12}{4!}x^4 \\
 &= x^2 + x^3 + \frac{1}{2}x^4
 \end{aligned}$$

17.

Derivative	Value at 0
$f(x) = (9 - x)^{3/2}$	$f(0) = 27$
$f^{(1)}(x) = -\frac{3}{2}(9 - x)^{1/2}$	$f^{(1)}(0) = -\frac{9}{2}$
$f^{(2)}(x) = \frac{3}{4}(9 - x)^{-1/2} = \frac{3}{4(9 - x)^{1/2}}$	$f^{(2)}(0) = \frac{1}{4}$
$f^{(3)}(x) = \frac{3}{8}(9 - x)^{-3/2} = \frac{3}{8(9 - x)^{3/2}}$	$f^{(3)}(0) = \frac{1}{72}$
$f^{(4)}(x) = \frac{9}{16}(9 - x)^{-5/2} = \frac{9}{16(9 - x)^{5/2}}$	$f^{(4)}(0) = \frac{1}{432}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 27 + \frac{-\frac{9}{2}}{1!}x + \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{1}{72}}{3!}x^3 + \frac{\frac{1}{432}}{4!}x^4 \\
 &= 27 - \frac{9}{2}x + \frac{1}{8}x^2 + \frac{1}{432}x^3 + \frac{1}{10,368}x^4
 \end{aligned}$$

18.

Derivative	Value at 0
$f(x) = (1 + x)^{3/2}$	$f(0) = 1$
$f^{(1)}(x) = \frac{3}{2}(1 + x)^{1/2}$	$f^{(1)}(0) = \frac{3}{2}$
$f^{(2)}(x) = \frac{3}{4}(1 + x)^{-1/2} = \frac{3}{4(1 + x)^{1/2}}$	$f^{(2)}(0) = \frac{3}{4}$
$f^{(3)}(x) = -\frac{3}{8}(1 + x)^{-3/2} = -\frac{3}{8(1 + x)^{3/2}}$	$f^{(3)}(0) = -\frac{3}{8}$
$f^{(4)}(x) = \frac{9}{16}(1 + x)^{-5/2} = \frac{9}{16(1 + x)^{5/2}}$	$f^{(4)}(0) = \frac{9}{16}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{\frac{3}{2}}{1!}x + \frac{\frac{3}{4}}{2!}x^2 + \frac{-\frac{3}{8}}{3!}x^3 + \frac{\frac{9}{16}}{4!}x^4 \\
 &= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4
 \end{aligned}$$

19.	Derivative	Value at 0
	$f(x) = \frac{1}{1+x} = (1+x)^{-1}$	$f(0) = 1$
	$f^{(1)}(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$	$f^{(1)}(0) = -1$
	$f^{(2)}(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$	$f^{(2)}(0) = 2$
	$f^{(3)}(x) = -6(1+x)^{-4} = -\frac{6}{(1+x)^4}$	$f^{(3)}(0) = -6$
	$f^{(4)}(x) = 24(1+x)^{-5} = \frac{24}{(1+x)^5}$	$f^{(4)}(0) = 24$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{-1}{1!}x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4 \\
 &= 1 - x + x^2 - x^3 + x^4
 \end{aligned}$$

20.	Derivative	Value at 0
	$f(x) = \frac{1}{x-1} = (x-1)^{-1}$	$f(0) = -1$
	$f^{(1)}(x) = -(x-1)^{-2} = -\frac{1}{(x-1)^2}$	$f^{(1)}(0) = -1$
	$f^{(2)}(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$	$f^{(2)}(0) = -2$
	$f^{(3)}(x) = -6(x-1)^{-4} = -\frac{6}{(x-1)^4}$	$f^{(3)}(0) = -6$
	$f^{(4)}(x) = 24(x-1)^{-5} = \frac{24}{(x-1)^5}$	$f^{(4)}(0) = -24$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= -1 + \frac{-1}{1!}x + \frac{-2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{-24}{4!}x^4 \\
 &= -1 - x - x^2 - x^3 - x^4
 \end{aligned}$$

For Exercises 21–34 each approximation can be determined using the Taylor polynomials from Exercises 1–14, respectively.

21. Using the result of Exercise 1, with $f(x) = e^{-2x}$ and $P_4(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4$, we can approximate $e^{-0.04}$ by evaluating $f(0.01) = e^{-2(0.02)} = e^{-0.04}$. Using $P_4(x)$ from Exercise 1 with $x = 0.02$ gives

$$\begin{aligned}
 P_4(0.02) &= 1 - 2(0.02) + 2(0.02)^2 - \frac{4}{3}(0.02)^3 + \frac{2}{3}(0.02)^4 \\
 &\approx 0.96078944.
 \end{aligned}$$

To four decimal places, $P_4(0.02)$ approximates the value of $e^{-0.04}$ as 0.9608.

22. Using the result of Exercise 2, with $f(x) = e^{3x}$ and $P_4(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$, we can approximate $e^{0.06}$ by evaluating $f(0.02) = e^{3(0.02)} = e^{0.06}$. Using $P_4(x)$ from Exercise 2 with $x = 0.02$ gives

$$\begin{aligned} P_4(0.02) &= 1 + 3(0.02) + \frac{9}{2}(0.02)^2 + \frac{9}{2}(0.02)^3 + \frac{27}{8}(0.02)^4 \\ &\approx 1.06183654 \end{aligned}$$

To four decimal places, $P_4(0.02)$ approximates the value of $e^{0.06}$ as 1.0618.

23. Using the result of Exercise 3, with $f(x) = e^{x+1}$ and $P_4(x) = e + ex + \frac{e}{2}x^2 + \frac{e}{6}x^3 + \frac{e}{24}x^4$, we can approximate $e^{1.02}$ by evaluating $f(0.02) = e^{0.02+1} = e^{1.02}$. Using $P_4(x)$ from Exercise 3 with $x = 0.02$ gives

$$\begin{aligned} P_4(0.02) &= e + e(0.02) + \frac{e}{2}(0.02)^2 + \frac{e}{6}(0.02)^3 + \frac{e}{24}(0.02)^4 \\ &\approx 2.773194764. \end{aligned}$$

To four decimal places, $P_4(0.02)$ approximates the value of $e^{1.02}$ as 2.7732.

24. Using the result of Exercise 4, with $f(x) = e^{-x}$ and $P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4$, we can approximate $e^{-0.07}$ by evaluating $f(0.07) = e^{-(0.07)} = e^{-0.07}$. Using $P_4(x)$ from Exercise 4 with $x = 0.07$ gives

$$\begin{aligned} P_4(0.07) &= 1 - 0.07 + \frac{1}{2}(0.07)^2 - \frac{1}{6}(0.07)^3 + \frac{1}{24}(0.07)^4 \\ &\approx 0.9323938338. \end{aligned}$$

To four decimal places, $P_4(0.07)$ approximates the value of $e^{-0.07}$ as 0.9324.

25. Using the result of Exercise 5, with $f(x) = \sqrt{x+9}$ and $P_4(x) = 3 + \frac{1}{6}x - \frac{1}{216}x^2 + \frac{1}{3888}x^3 - \frac{5}{279,936}x^4$, we can approximate $\sqrt{8.92}$ by evaluating $f(-0.08) = \sqrt{-0.08+9} = \sqrt{8.92}$. Using $P_4(x)$ from Exercise 5 with $x = -0.08$ gives

$$\begin{aligned} P_4(-0.08) &= 3 + \frac{1}{6}(-0.08) - \frac{1}{216}(-0.08)^2 + \frac{1}{3888}(-0.08)^3 - \frac{5}{279,936}(-0.08)^4 \\ &\approx 2.986636905. \end{aligned}$$

To four decimal places, $P_4(-0.08)$ approximates the value of $\sqrt{8.92}$ as 2.9866.

26. Using the result of Exercise 6, with $f(x) = \sqrt{x+16}$ and $P_4(x) = 4 + \frac{1}{8}x - \frac{1}{512}x^2 + \frac{1}{16,384}x^3 - \frac{5}{2,097,152}x^4$, we can approximate $\sqrt{16.3}$ by evaluating $f(0.3) = \sqrt{0.3+16} = \sqrt{16.3}$. Using $P_4(x)$ from Exercise 6 with $x = 0.3$ gives

$$\begin{aligned} P_4(0.3) &= 4 + \frac{1}{8}(0.3) - \frac{1}{512}(0.3)^2 + \frac{1}{16,384}(0.3)^3 - \frac{5}{2,097,152}(0.3)^4 \\ &\approx 4.037325847. \end{aligned}$$

To four decimal places, $P_4(0.3)$ approximates the value of $\sqrt{16.3}$ as 4.0373.

27. Using the result of Exercise 7, with $f(x) = \sqrt[3]{x-1}$ and $P_4(x) = -1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{5}{81}x^3 + \frac{10}{243}x^4$, we can approximate $\sqrt[3]{-1.05}$ by evaluating $f(-0.05) = \sqrt[3]{-0.05-1} = \sqrt[3]{-1.05}$. Using $P_4(x)$ from Exercise 7 with $x = -0.05$ gives

$$\begin{aligned} P_4(-0.05) &= -1 + \frac{1}{3}(-0.05) + \frac{1}{9}(-0.05)^2 + \frac{5}{81}(-0.05)^3 + \frac{10}{243}(-0.05)^4 \\ &\approx -1.016396348. \end{aligned}$$

To four decimal places, $P_4(-0.05)$ approximates the value of $\sqrt[3]{-1.05}$ as -1.0164 .

28. Using the result of Exercise 8, with $f(x) = \sqrt[3]{x+8}$ and $P_4(x) = 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{20,736}x^3 - \frac{5}{248,832}x^4$, we can approximate $\sqrt[3]{7.91}$ by evaluating $f(-0.09) = \sqrt[3]{-0.09+8} = \sqrt[3]{7.91}$. Using $P_4(x)$ from Exercise 8 with $x = -0.09$ gives

$$\begin{aligned} P_4(-0.09) &= 2 + \frac{1}{12}(-0.09) - \frac{1}{288}(-0.09)^2 + \frac{5}{20,736}(-0.09)^3 - \frac{5}{248,832}(-0.09)^4 \\ &\approx 1.992471698. \end{aligned}$$

To four decimal places, $P_4(-0.09)$ approximates the value of $\sqrt[3]{7.91}$ as 1.9925 .

29. Using the result of Exercise 9, with $f(x) = \sqrt[4]{x+1}$ and $P_4(x) = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 - \frac{77}{2048}x^4$, we can approximate $\sqrt[4]{1.06}$ by evaluating $f(0.06) = \sqrt[4]{0.06+1} = \sqrt[4]{1.06}$. Using $P_4(x)$ from Exercise 9 with $x = 0.06$ gives

$$\begin{aligned} P_4(0.06) &= 1 + \frac{1}{4}(0.06) - \frac{3}{32}(0.06)^2 + \frac{7}{128}(0.06)^3 - \frac{77}{2048}(0.06)^4 \\ &\approx 1.014673825. \end{aligned}$$

To four decimal places, $P_4(0.06)$ approximates the value of $\sqrt[4]{1.06}$ as 1.0147 .

30. Using the result of Exercise 10, with $f(x) = \sqrt[4]{x+16}$ and $P_4(x) = 2 + \frac{1}{32}x - \frac{3}{4096}x^2 + \frac{7}{262,144}x^3 - \frac{77}{67,108,864}x^4$, we can approximate $\sqrt[4]{15.88}$ by calculating $f(-0.12) = \sqrt[4]{-0.12+16} = \sqrt[4]{15.88}$. Using $P_4(x)$ from Exercise 10 with $x = -0.12$ gives

$$\begin{aligned} P_4(-0.12) &= 2 + \frac{1}{32}(-0.12) - \frac{3}{4096}(-0.12)^2 + \frac{7}{262,144}(-0.12)^3 - \frac{77}{67,108,864}(-0.12)^4 \\ &\approx 1.996239407. \end{aligned}$$

To four decimal places, $P_4(-0.12)$ approximates the value of $\sqrt[4]{15.88}$ as 1.9962 .

31. Using the result of Exercise 11, with $f(x) = \ln(1-x)$ and $P_4(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$, we can approximate $\ln 0.97$ by evaluating $f(0.03) = \ln(1-0.03) = \ln 0.97$. Using $P_4(x)$ from Exercise 11 with $x = 0.03$ gives

$$\begin{aligned} P_4(0.03) &= -(0.03) - \frac{1}{2}(0.03)^2 - \frac{1}{3}(0.03)^3 - \frac{1}{4}(0.03)^4 \\ &\approx -0.0304592025. \end{aligned}$$

To four decimal places, $P_4(0.03)$ approximates the value of $\ln 0.97$ as -0.0305 .

32. Using the result of Exercise 12, with $f(x) = \ln(1 + 2x)$ and $P_4(x) = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4$, we can approximate $\ln 1.06$ by evaluating $f(0.03) = \ln(1 + 2(0.03)) = \ln(1 + 0.06) = \ln 1.06$. Using $P_4(x)$ from Exercise 12 with $x = 0.03$ gives

$$\begin{aligned} P_4(0.03) &= 2(0.03) - 2(0.03)^2 + \frac{8}{3}(0.03)^3 - 4(0.03)^4 \\ &\approx 0.05826876. \end{aligned}$$

To four decimal places, $P_4(0.03)$ approximates the value of $\ln 1.06$ as 0.0583.

33. Using the result of Exercise 13, with $f(x) = \ln(1 + 2x^2)$ and $P_4(x) = 2x^2 - 2x^4$, we can compute $\ln 1.008$ by evaluating $f\left(\frac{\sqrt{10}}{50}\right) = \ln\left[1 + 2\left(\frac{\sqrt{10}}{50}\right)^2\right] = \ln[1 + 2(0.004)] = \ln 1.008$. Using $P_4(x)$ with $x = \frac{\sqrt{10}}{50}$ gives

$$\begin{aligned} P_4\left(\frac{\sqrt{10}}{50}\right) &= 2\left(\frac{\sqrt{10}}{50}\right)^2 - 4\left(\frac{\sqrt{10}}{50}\right)^4 \\ &= 0.007968 \end{aligned}$$

To four decimal places, $P_4\left(\frac{\sqrt{10}}{50}\right)$ approximates $\ln 1.008$ as 0.0080.

34. Using the result of Exercise 14, with $f(x) = \ln(1 - x^3)$ and $P_4(x) = -x^3$, we can approximate $\ln 0.992$ by evaluating $f(0.2) = \ln(1 - 0.2^3) = \ln 0.992$. Using $P_4(x)$ from Exercise 14 with $x = 0.2$ gives

$$\begin{aligned} P_4(.1) &= -(0.2)^3 \\ &\approx -0.008. \end{aligned}$$

To four decimal places, $P_4(0.2)$ approximates the value of $\ln 0.992$ as -0.0080 .

$$35. \quad P_3(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 = 3 + \frac{6}{1}x + \frac{12}{2}x^2 + \frac{24}{6}x^3 = 3 + 6x + 6x^2 + 4x^3$$

$$\begin{aligned} 36. \quad P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + \frac{1}{1}x + \frac{2}{2}x^2 + \frac{6}{6}x^3 + \frac{24}{24}x^4 \\ &= 1 + x + x^2 + x^3 + x^4 \end{aligned}$$

For a polynomial of degree n with $f^{(n)}(0) = n!$ and $0! = 1$,

$$\begin{aligned} P_n(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + \frac{1!}{1!}x + \frac{2!}{2!}x^2 + \cdots + \frac{(n-1)!}{(n-1)!}x^{n-1} + \frac{n!}{n!}x^n \\ &= 1 + x + x^2 + \cdots + x^{n-1} + x^n \end{aligned}$$

37. (a)

Derivative	Value at 0
$f(x) = (a + x)^{\frac{1}{n}}$	$f(0) = a^{\frac{1}{n}}$
$f^{(1)}(x) = \frac{1}{n}(a + x)^{\frac{1}{n}-1}$	$f^{(1)}(0) = \frac{1}{n}a^{\frac{1}{n}-1} = \frac{a^{\frac{1}{n}}}{na}$
$f^{(2)}(x) = \frac{1}{n}\left(\frac{1}{n} - 1\right)(a + x)^{\frac{1}{n}-2}$	$f^{(2)}(0) = \frac{1}{n}\left(\frac{1-n}{n}\right)a^{\frac{1}{n}-2} = \frac{(1-n)a^{\frac{1}{n}}}{n^2a^2}$
$f^{(3)}(x) = \frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)(a + x)^{\frac{1}{n}-3}$	$f^{(3)}(0) = \frac{1}{n}\left(\frac{1-n}{n}\right)\left(\frac{1-2n}{n}\right)a^{\frac{1}{n}-3} = \frac{(1-n)(1-2n)a^{\frac{1}{n}}}{n^3a^3}$
$f^{(4)}(x) = \frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)\left(\frac{1}{n} - 3\right)(a + x)^{\frac{1}{n}-4}$	$f^{(4)}(0) = \frac{1}{n}\left(\frac{1-n}{n}\right)\left(\frac{1-2n}{n}\right)\left(\frac{1-3n}{n}\right)a^{\frac{1}{n}-4}$ $= \frac{(1-n)(1-2n)(1-3n)a^{\frac{1}{n}}}{n^4a^4}$

$$\begin{aligned}
 P_4(x) &= a^{1/n} + \frac{a^{1/n}}{na}x + \frac{(1-n)a^{1/n}}{2!n^2a^2}x^2 + \frac{(1-n)(1-2n)a^{1/n}}{3!n^3a^3}x^3 + \frac{(1-n)(1-2n)(1-3n)a^{1/n}}{4!n^4a^4}x^4 \\
 &= a^{1/n} + \frac{xa^{1/n}}{na} + \frac{x^2a^{1/n}(1-n)}{2!n^2a^2} + \frac{x^3a^{1/n}(1-n)(1-2n)}{3!n^3a^3} + \frac{x^4a^{1/n}(1-n)(1-2n)(1-3n)}{4!n^4a^4} \\
 &= a^{1/n} + \frac{xa^{1/n}}{na} \left[1 + \frac{x(1-n)}{2!na} + \frac{x^2(1-n)(1-2n)}{3!n^2a^2} + \frac{x^3(1-n)(1-2n)(1-3n)}{4!n^3a^3} \right]
 \end{aligned}$$

For values of x that are small compared with a , the terms after 1 in the brackets get closer and closer to zero. For example, consider the term $\frac{x(1-n)}{2!na}$ where $x = \frac{1}{10}a$. Then $\frac{x(1-n)}{2!na} = \frac{1-n}{20n}$. For even the smallest value of n , $n = 2$, the value of this term is $-\frac{1}{40}$. This term, and the terms which follow will have little impact on our approximation. Thus the value of the expression is approximately 1, so $P_4(x) \approx a^{1/n} + \frac{xa^{1/n}}{na}$. Thus, if x is small compared with a , the Taylor polynomial $P_4(x)$ approximates $(a + x)^{1/n}$ as $a^{1/n} + \frac{xa^{1/n}}{na}$.

- (b) Using the result of part (a), we can approximate $\sqrt[3]{66}$ by evaluating $f(2)$ given $f(x) = (64 + x)^{1/3}$ since $f(2) = (64 + 2)^{1/3} = 66^{1/3} = \sqrt[3]{66}$. Using $P_4(x)$ from part (a) with $x = 2$, $a = 64$, and $n = 3$ gives

$$\begin{aligned}
 (64 + 2)^{1/3} &= 64^{1/3} + \frac{2 \cdot 64^{1/3}}{3 \cdot 64} \\
 &= 4 + \frac{8}{192} \\
 &= 4.041\bar{6}
 \end{aligned}$$

To five decimal places, $P_4(x) \approx (64 + x)^{1/3}$ approximates the value of $\sqrt[3]{66}$ as 4.04167. To five decimal places, a calculator gives an approximation of 4.04124.

38. (a)

Derivative	Value at 0
$f(r) = (1 + r)^D$	$f(0) = 1$
$f^{(1)}(r) = D(1 + r)^{D-1}$	$f^{(1)}(0) = D$

$$P_1(r) = 1 + \frac{D}{1!}r = 1 + rD$$

So the Taylor polynomial of degree 1 for the function $f(r) = (1 + r)^D$ is $P_1(r) = 1 + rD$. If r is small, then $(1 + r)^D \approx 1 + rD$, and so $V(1 + r)^D \approx V(1 + rD)$. Therefore, for small r , the two formulas for S give approximately equal values.

- (b)** Using $S \approx V(1 + rD)$ with $V = 1000$, $r = 0.1$, and $D = 3.2$ gives $S \approx 1000[1 + (0.1)(3.2)] = 1320$. Using $S \approx V(1 + r)^D$ with $V = 1000$, $r = 0.1$, and $D = 3.2$ gives $S \approx 1000(1 + 0.1)^{3.2} \approx 1357$. The two approximations for S are 1320 and 1357 which are relatively close.

39. (a)

Derivative	Value at 0
$f(N) = e^{\lambda N}$	$f(0) = 1$
$f^{(1)}(N) = \lambda e^{\lambda N}$	$f^{(1)}(0) = \lambda$
$f^{(2)}(N) = \lambda^2 e^{\lambda N}$	$f^{(2)}(0) = \lambda^2$

$$\begin{aligned} P_2(N) &= f(0) + \frac{f^{(1)}(0)}{1!}N + \frac{f^{(2)}(0)}{2!}N^2 \\ &= 1 + \frac{\lambda}{1!}N + \frac{\lambda^2}{2!}N^2 = 1 + \lambda N + \frac{\lambda^2}{2}N^2 \end{aligned}$$

The Taylor polynomial of degree 2 at $N = 0$ for $e^{\lambda N}$ is $P_2(N) = 1 + \lambda N + \frac{\lambda^2}{2}N^2$.

- (b)** Substituting $P_2(N) = 1 + \lambda N + \frac{\lambda^2}{2}N^2$ for $e^{\lambda N}$ in the original equation gives:

$$\begin{aligned} \frac{1 + \lambda N + \frac{\lambda^2}{2}N^2}{\lambda} - N &= \frac{1}{\lambda} + k \\ \left(1 + \lambda N + \frac{\lambda^2}{2}N^2\right) - \lambda N &= 1 + \lambda k \\ \frac{\lambda^2}{2}N^2 &= \lambda k \\ N^2 &= \frac{2k}{\lambda} \\ N &= \pm \sqrt{\frac{2k}{\lambda}} \end{aligned}$$

Since N is a positive quantity, use the positive square root. So, $N = \sqrt{\frac{2k}{\lambda}}$.

40.

Derivative	Value at 0
$P(x) = \frac{20 + x^2}{50 + x}$	$P(0) = 0.4$
$P^{(1)}(x) = \frac{x^2 + 100x - 20}{(50 + x)^2}$	$^{(1)}P(0) = -0.008$
$P^{(2)}(x) = \frac{(50 + x)^2(2x + 100) - (x^2 + 100x - 20)[2(50 + x)]}{(50 + x)^4}$	$^{(2)}P(0) = 0.04032$

$$P_2(x) = P(0) + \frac{P^{(1)}(0)}{1!}x + \frac{P^{(2)}(0)}{2!}x^2 = 0.4 + \frac{-0.008}{1}x + \frac{0.04032}{2}x^2 = 0.4 - 0.008x + 0.02016x^2$$

$$P_2(0.3) = 0.4 - 0.008(0.3) + (0.02016)(0.3)^2 = 0.3994144$$

The Taylor polynomial $P_2(0.3)$ approximates $P(0.3)$ as 0.3994144. The approximate profit when 300 tons of apples are sold is 0.399 thousand dollars, or \$399.

Finding $P(0.3)$ by direct substitution gives:

$$P(0.3) = \frac{20 + (0.3)^2}{50 + (0.3)} \approx 0.3994035785$$

Direct substitution also estimates the profit as 0.399 thousand dollars, or \$399.

41.

Derivative	Value at 0
$P(x) = \ln(100 + 3x)$	$P(0) = 4.605$
$P^{(1)}(x) = \frac{3}{100 + 3x}$	$P^{(1)}(0) = 0.03$
$P^{(2)}(x) = -\frac{9}{(100 + 3x)^2}$	$P^{(2)}(0) = -0.0009$

$$\begin{aligned} P_2(x) &= P(0) + \frac{P^{(1)}(0)}{1!}x + \frac{P^{(2)}(0)}{2!}x^2 \\ &= 4.605 + \frac{0.03}{1}x + \frac{-0.0009}{2}x^2 \\ &= 4.605 + 0.03x - 0.00045x^2 \end{aligned}$$

$$\begin{aligned} P_2(0.6) &= 4.605 + 0.03(0.6) - 0.00045(0.6)^2 \\ &= 4.622838 \end{aligned}$$

The Taylor polynomial $P_2(0.6)$ approximates $P(0.6)$ as 4.622838. The approximate profit is 4.623 thousand dollars, or \$4623.

Finding $P(0.6)$ by direct substitution gives:

$$\begin{aligned} P(0.6) &= \ln 100 + 3(0.6) \\ &= \ln 101.8 \\ &= 4.623010104 \end{aligned}$$

Direct substitution also estimates the profit as 4.623 thousand dollars, or \$4623.

42.	Derivative	Value at 0
	$C(x) = e^{-x/50}$	$C(0) = 1$
	$C^{(1)}(x) = -\frac{1}{50}e^{-x/50}$	$C^{(1)}(0) = -\frac{1}{50}$
	$C^{(2)}(x) = \frac{1}{2500}e^{-x/50}$	$C^{(2)}(0) = \frac{1}{2500}$

$$P_2(x) = C(0) + \frac{C^{(1)}(0)}{1!}x + \frac{C^{(2)}(0)}{2!}x^2 = 1 + \frac{-\frac{1}{50}}{1!}x + \frac{\frac{1}{2500}}{2!}x^2 = 1 - \frac{1}{50}x + \frac{1}{5000}x^2$$

$$P_2(5) = 1 - \frac{1}{50}(5) + \frac{1}{5000}(5)^2 = 0.905$$

The Taylor polynomial $P_2(5)$ approximates $C(5)$ as 0.905. The approximate cost is 0.905 dollar, or about \$0.91.

Finding $C(5)$ by direct substitution gives:

$$C(5) = e^{-5/50} \approx 0.904837418$$

Direct substitution estimates the profit as 0.905 dollar, or about \$0.90.

43.	Derivative	Value at 0
	$R(x) = 500 \ln\left(4 + \frac{x}{50}\right)$	$R(0) = 500(1.386) = 693$
	$R^{(1)}(x) = \frac{500}{200 + x}$	$R^{(1)}(0) = 2.5$
	$R^{(2)}(x) = -\frac{500}{(200 + x)^2}$	$R^{(2)}(0) = -0.0125$

$$P_2(x) = R(0) + \frac{R^{(1)}(0)}{1!}x + \frac{R^{(2)}(0)}{2!}x^2 = 693 + \frac{2.5}{1}x + \frac{-0.0125}{2}x^2 = 693 + 2.5x - 0.00625x^2$$

$$P_2(10) = 693 + 2.5(10) - 0.00625(10)^2 = 717.375$$

The Taylor polynomial $P_2(10)$ approximates $R(10)$ as 717.375. Thus, the approximate revenue is \$717.

Finding $R(10)$ by direct substitution gives:

$$R(10) = 500 \ln\left(4 + \frac{10}{50}\right) \approx 717.5422626$$

Direct substitution estimates the revenue as \$718.

44.	Derivative	Value at 0
	$A(x) = \frac{6x}{1 + 10x}$	$A(0) = 0$
	$A^{(1)}(x) = \frac{6}{(1 + 10x)^2}$	$A^{(1)}(0) = 6$
	$A^{(2)}(x) = -\frac{120}{(1 + 10x)^3}$	$A^{(2)}(0) = -120$

$$P_2(x) = A(0) + \frac{A^{(1)}(0)}{1!}x + \frac{A^{(2)}(0)}{2!}x^2 = 0 + \frac{6}{1}x + \frac{-120}{2}x^2 = 6x - 60x^2$$

$$P_2(0.05) = 6(0.05) - 60(0.05)^2 = 0.15$$

The Taylor polynomial $P_2(0.05)$ approximates $A(0.05)$ as 0.15. The approximate amount of the drug in the bloodstream after 0.05 minutes is 0.15 milliliters.

Finding $A(0.05)$ by direct substitution gives:

$$A(0.05) = \frac{6(0.05)}{1 + 10(0.05)} = 0.2$$

Direct substitution indicates that 0.2 milliliter of the drug remain in the bloodstream after 0.05 minutes.

45.

Derivative	Value at 0
$P(k) = 1 - e^{-2k}$	$P(0) = 1 - e^0 = 0$
$P^{(1)}(k) = 2e^{-2k}$	$P^{(1)}(0) = 2$

$$P_1(k) = P(0) + \frac{P^{(1)}(0)}{1!}k = 0 + \frac{2}{1!}k = 2k$$

So, if k is small, that is, close to 0, then

$$P(k) \approx P_1(k) = 2k.$$

- 46. (a)** The Taylor polynomial of degree 1 for $\sqrt{1+x}$ was found in Example 2.

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

Therefore,

$$V = k_1 \left(\sqrt{z^2 + R^2} - z \right) = k_1 \left(\sqrt{z^2 \left(1 + \frac{R^2}{z^2} \right)} - z \right) = k_1 \left(z \sqrt{1 + \frac{R^2}{z^2}} - z \right).$$

Now use the Taylor polynomial to approximate the radical expression, letting x be replaced by R^2/z^2 .

$$V \approx k_1 \left(z \left[1 + \frac{1}{2} \left(\frac{R^2}{z^2} \right) \right] - z \right) = k_1 \left(z + \frac{R^2 z}{2z^2} - z \right) = k_1 \left(\frac{R^2}{2z} \right) = \frac{k_1 R^2}{2} \left(\frac{1}{z} \right)$$

If z is much larger than R , the expression is dominated more by the z -factor and less so by the R^2 . Let $k_1 R^2/2 = k_2$. Then

$$V \approx \frac{k_2}{z}.$$

(b)
$$V = k_1 \left(\sqrt{z^2 + R^2} - z \right) = k_1 \left(\sqrt{R^2 \left(1 + \frac{z^2}{R^2} \right)} - z \right) = k_1 \left(R \sqrt{1 + \frac{z^2}{R^2}} - z \right)$$

As before, use the Taylor polynomial of degree 1 for $\sqrt{1+x}$ to approximate the radical expression. This time let x be replaced by z^2/R^2 .

$$V \approx k_1 \left(R \left[1 + \frac{1}{2} \left(\frac{z^2}{R^2} \right) \right] - z \right) = k_1 \left(R + \frac{Rz^2}{2R^2} - z \right) = k_1 \left(R - z + \frac{z^2}{2R} \right)$$

This is the desired result.

12.4 Infinite Series

Your Turn 1

The sequence is $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{4} = \frac{5}{4}$$

$$S_3 = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$$

$$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{205}{144}$$

$$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{5269}{3600}$$

Your Turn 2

- (a) This is a geometric series with $a = a_1 = 2$ and $r = 1/3$. Since r is in $(-1, 1)$, the series converges and has sum

$$\begin{aligned}\frac{a}{1-r} &= \frac{2}{1-(1/3)} \\ &= \frac{2}{2/3} \\ &= 3.\end{aligned}$$

3. $2 + 6 + 18 + 54 + \dots$ is geometric series with $a = a_1 = 2$ and $r = 3$.

- (b) This is a geometric series with $a = a_1 = 4$ and $r = -1/4$. Since r is in $(-1, 1)$, the series converges and has sum

$$\begin{aligned}\frac{a}{1-r} &= \frac{5}{1-(-1/4)} \\ &= \frac{5}{5/4} \\ &= 4.\end{aligned}$$

- (c) This is a geometric series with $a = a_1 = 2$ and $r = 1.01$. Since r is not in $(-1, 1)$, the series diverges.

Your Turn 3

- (a) Solve using algebra.

Let t be the number of hours since 2:00 pm. Between 8:00 am and 2:00 pm, Turtle has traveled $(6)(60)(15) = 5400$ feet.

Turtle's distance from the starting line at any number of hours t after 2:00 pm is

$$d_1 = 5400 + (t)(60)(15) = 5400 + 900t \text{ feet.}$$

Rabbit's distance from the starting line at time t is $d_2 = (t)(60)(45) = 2700t$ feet.

$$\begin{aligned}d_1 &= d_2 \\ 5400 + 900t &= 2700t \\ 5400 &= 1800t \\ t &= 3\end{aligned}$$

Rabbit catches up with Turtle 3 hours after 2 pm, that is, at 5 pm.

- (b) Solve using a geometric series.

At 2 pm Turtle has traveled 5400 ft. Traveling at 45 feet per minute, it will take Rabbit $5400/45 = 120$ minutes to make up this distance. During this time, Turtle will have traveled a further $(120)(15) = 1800$ feet, and it will take Rabbit $1800/45 = 40$ minutes to make up this distance. The total number of minutes it takes Rabbit to make up all these distances is $120 + 40 + 40/3 + \dots$. This is a geometric series with $a = a_1 = 120$ and $r = 1/3$. Since r is in $(-1, 1)$, the series converges and has sum

$$\begin{aligned}\frac{a}{1-r} &= \frac{120}{1-(1/3)} \\ &= \frac{120}{2/3} \\ &= 180,\end{aligned}$$

that is, 180 minutes or 3 hours, as found in (a).

12.4 Exercises

1. $20 + 10 + 5 + \frac{5}{2} + \dots$ is a geometric series

with $a = a_1 = 20$ and $r = \frac{1}{2}$. Since r is in

$(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{20}{1-\frac{1}{2}} = \frac{20}{\frac{1}{2}} = 40.$$

2. $1 + 0.8 + 0.64 + 0.512 + \dots$ is a geometric series with $a = a_1 = 1$ and $r = 0.8$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{1}{1-0.8} = \frac{1}{0.2} = 5.$$

3. $2 + 6 + 18 + 54 + \cdots$ is a geometric series with $a = a_1 = 2$ and $r = 3$. Since $r > 1$, the series diverges.

4. $3 + 6 + 12 + 24 + \cdots$ is a geometric series with $a = a_1 = 3$ and $r = 2$. Since $r > 1$, the series diverges.

5. $27 + 9 + 3 + 1 + \cdots$ is a geometric series with $a = a_1 = 27$ and $r = \frac{1}{3}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{27}{1-\frac{1}{3}} = \frac{27}{\frac{2}{3}} = \frac{81}{2}.$$

6. $64 + 16 + 4 + 1 + \cdots$ is a geometric series with $a = a_1 = 64$ and $r = \frac{1}{4}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{64}{1-\frac{1}{4}} = \frac{64}{\frac{3}{4}} = \frac{256}{3}.$$

7. $100 + 10 + 1 + \cdots$ is a geometric series with $a = a_1 = 100$ and $r = \frac{1}{10}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{100}{1-\frac{1}{10}} = \frac{100}{\frac{9}{10}} = \frac{1000}{9}.$$

8. $44 + 22 + 11 + \cdots$ is a geometric series with $a = a_1 = 44$ and $r = \frac{1}{2}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{44}{1-\frac{1}{2}} = \frac{44}{\frac{1}{2}} = 88.$$

9. $\frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \cdots$ is a geometric series with $a = a_1 = \frac{5}{4}$ and $r = \frac{1}{2}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{\frac{5}{4}}{1-\frac{1}{2}} = \frac{\frac{5}{4}}{\frac{1}{2}} = \frac{5}{2}.$$

10. $\frac{4}{5} + \frac{2}{5} + \frac{1}{5} + \cdots$ is a geometric series with $a = a_1 = \frac{4}{5}$ and $r = \frac{1}{2}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{\frac{4}{5}}{1-\frac{1}{2}} = \frac{\frac{4}{5}}{\frac{1}{2}} = \frac{8}{5}.$$

11. $\frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \frac{8}{81} + \cdots$ is a geometric series with $a = a_1 = \frac{1}{3}$ and $r = -\frac{2}{3}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{\frac{1}{3}}{1+\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{5}{3}} = \frac{1}{5}.$$

12. $1 + \frac{1}{1.01} + \frac{1}{(1.01)^2} + \cdots$ is a geometric series with $a = a_1 = 1$ and $r = \frac{1}{1.01}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{1.01}} = \frac{1}{\frac{0.01}{1.01}} = 101.$$

13. $e - 1 + \frac{1}{e} - \frac{1}{e^2} + \cdots$ is a geometric series with $a = a_1 = e$ and $r = -\frac{1}{e}$. Since r is in $(-1, 1)$, the series converges and the sum

$$\begin{aligned} \frac{a}{1-r} &= \frac{e}{1-\left(-\frac{1}{e}\right)} = \frac{e}{1+\frac{1}{e}} \\ &= \frac{e}{\frac{e+1}{e}} = \frac{e^2}{e+1}. \end{aligned}$$

14. $e + e^2 + e^3 + e^4 + \cdots$ is a geometric series with $a = a_1 = e$ and $r = e$. Since $r > 1$, the series diverges.

$$\begin{aligned}
 15. \quad S_1 &= a_1 = \frac{1}{1} = 1 \\
 S_2 &= a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2} \\
 S_3 &= a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\
 S_4 &= a_1 + a_2 + a_3 + a_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \\
 S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad S_1 &= a_1 = \frac{1}{1+1} = \frac{1}{2} \\
 S_2 &= a_1 + a_2 = \frac{1}{1+1} + \frac{1}{2+1} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \\
 S_3 &= a_1 + a_2 + a_3 = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} \\
 S_4 &= a_1 + a_2 + a_3 + a_4 = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} \\
 S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{4+1} + \frac{1}{5+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{29}{20}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad S_1 &= a_1 = \frac{1}{2(1)+5} = \frac{1}{7} \\
 S_2 &= a_1 + a_2 = \frac{1}{2(1)+5} + \frac{1}{2(2)+5} = \frac{1}{7} + \frac{1}{9} = \frac{16}{63} \\
 S_3 &= a_1 + a_2 + a_3 = \frac{1}{2(1)+5} + \frac{1}{2(2)+5} + \frac{1}{2(3)+5} = \frac{1}{7} + \frac{1}{9} + \frac{1}{11} = \frac{239}{693} \\
 S_4 &= a_1 + a_2 + a_3 + a_4 \\
 &= \frac{1}{2(1)+5} + \frac{1}{2(2)+5} + \frac{1}{2(3)+5} + \frac{1}{2(4)+5} = \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} = \frac{3800}{9009} \\
 S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\
 &= \frac{1}{2(1)+5} + \frac{1}{2(2)+5} + \frac{1}{2(3)+5} + \frac{1}{2(4)+5} + \frac{1}{2(5)+5} \\
 &= \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} = \frac{22,003}{45,045}
 \end{aligned}$$

$$18. \quad S_1 = a_1 = \frac{1}{3(1) - 1} = \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{3(1) - 1} + \frac{1}{3(2) - 1} + \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{3(1) - 1} + \frac{1}{3(2) - 1} + \frac{1}{3(3) - 1} + \frac{1}{2} + \frac{1}{5} + \frac{1}{8} = \frac{33}{40}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{3(1) - 1} + \frac{1}{3(2) - 1} + \frac{1}{3(3) - 1} + \frac{1}{3(4) - 1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} = \frac{403}{440}$$

$$\begin{aligned} S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{3(1) - 1} + \frac{1}{3(2) - 1} + \frac{1}{3(3) - 1} + \frac{1}{3(4) - 1} + \frac{1}{3(5) - 1} \\ &= \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} = \frac{3041}{3080} \end{aligned}$$

$$19. \quad S_1 = a_1 = \frac{1}{(1+1)(1+2)} = \frac{1}{6}$$

$$S_2 = a_1 + a_2 = \frac{1}{(1+1)(1+2)} + \frac{1}{(2+1)(2+2)} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

$$\begin{aligned} S_3 &= a_1 + a_2 + a_3 \\ &= \frac{1}{(1+1)(1+2)} + \frac{1}{(2+1)(2+2)} + \frac{1}{(3+1)(3+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{3}{10} \end{aligned}$$

$$\begin{aligned} S_4 &= a_1 + a_2 + a_3 + a_4 = \frac{1}{(1+1)(1+2)} + \frac{1}{(2+1)(2+2)} + \frac{1}{(3+1)(3+2)} + \frac{1}{(4+1)(4+2)} \\ &= \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= \frac{1}{(1+1)(1+2)} + \frac{1}{(2+1)(2+2)} + \frac{1}{(3+1)(3+2)} + \frac{1}{(4+1)(4+2)} + \frac{1}{(5+1)(5+2)} \\ &= \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{5}{14} \end{aligned}$$

$$20. \quad S_1 = a_1 = \frac{1}{(1+3)[2(1)+1]} = \frac{1}{12}$$

$$S_2 = a_1 + a_2 = \frac{1}{(1+3)[2(1)+1]} + \frac{1}{(2+3)[2(2)+1]} = \frac{1}{12} + \frac{1}{25} = \frac{37}{300}$$

$$\begin{aligned} S_3 &= a_1 + a_2 + a_3 = \frac{1}{(1+3)[2(1)+1]} + \frac{1}{(2+3)[2(2)+1]} + \frac{1}{(3+3)[2(3)+1]} \\ &= \frac{1}{12} + \frac{1}{25} + \frac{1}{42} = \frac{103}{700} \end{aligned}$$

$$\begin{aligned} S_4 &= a_1 + a_2 + a_3 + a_4 = \frac{1}{(1+3)[2(1)+1]} + \frac{1}{(2+3)[2(2)+1]} + \frac{1}{(3+3)[2(3)+1]} + \frac{1}{(4+3)[2(4)+1]} \\ &= \frac{1}{12} + \frac{1}{25} + \frac{1}{42} + \frac{1}{63} = \frac{1027}{6300} \end{aligned}$$

$$\begin{aligned} S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= \frac{1}{(1+3)[2(1)+1]} + \frac{1}{(2+3)[2(2)+1]} + \frac{1}{(3+3)[2(3)+1]} + \frac{1}{(4+3)[2(4)+1]} + \frac{1}{(5+3)[2(5)+1]} \\ &= \frac{1}{12} + \frac{1}{25} + \frac{1}{42} + \frac{1}{63} + \frac{1}{88} = \frac{24,169}{138,600} \end{aligned}$$

21. The infinite geometric series

$$0.2 + 0.2\left(\frac{1}{10}\right) + 0.2\left(\frac{1}{10}\right)^2 + 0.2\left(\frac{1}{10}\right)^3 + \dots$$

is a geometric series with $a = a_1 = 0.2$ and $r = 1/10$. Since r is in $(-1, 1)$, the series converges and has sum

$$\begin{aligned}\frac{a}{1-r} &= \frac{0.2}{1-(1/10)} \\ &= \frac{0.2}{0.9} = 2/9.\end{aligned}$$

22. The infinite geometric series

$$0.18 + 0.18\left(\frac{1}{100}\right) + 0.18\left(\frac{1}{100}\right)^2 + 0.18\left(\frac{1}{100}\right)^3 + \dots$$

is a geometric series with $a = a_1 = 0.18$ and $r = 1/100$. Since r is in $(-1, 1)$, the series converges and has sum

$$\begin{aligned}\frac{a}{1-r} &= \frac{0.18}{1-(1/100)} \\ &= \frac{0.18}{0.99} = 18/99 = 2/11.\end{aligned}$$

23. (a) It follows from Viète's formula that

$$\begin{aligned}\frac{2}{\pi} &\approx \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \\ &= \frac{\sqrt{2}(\sqrt{2+\sqrt{2}})(\sqrt{2+\sqrt{2+\sqrt{2}}})}{8} \\ &\approx 0.641\end{aligned}$$

Thus, $\pi \approx \frac{2}{0.641} \approx 3.12$.

It follows from Leibniz's formula that

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \approx 0.724.$$

Thus, $\pi \approx 4(0.724) \approx 2.90$.

Therefore, approximating π by multiplying the first three terms of Viète's formula together is more accurate than approximating π by adding the first four terms of Leibniz's formula together.

- (b)** Using the graphing calculator with $Y_1 = 4 * \text{sum}\left(\text{seq}\left(\frac{(-1)^{N-1}}{2N-1}, N, 1, X\right)\right)$ and the table function gives the following values for Y_1 for $35 \leq X \leq 41$.

X	Y ₁
35	3.1702
36	3.1138
37	3.1686
38	3.1415
39	3.1672
40	3.1166
41	3.166

Thus, 38 terms of the second formula must be added together to produce the same accuracy as the product of the first three terms of the first formula.

24. (a) With $a = a_1 = 1000$ and $r = 0.1$, the production manager should order

$$\frac{a}{1-r} = \frac{1000}{1-0.1} = \frac{1000}{0.9} = \frac{10,000}{9} \approx 1111 \text{ units.}$$

- (b) $0.9x = 1000$

$$x = \frac{1000}{0.9} = \frac{10,000}{9} \approx 1111 \text{ units}$$

25. (a) The total expenditure is the sum of an infinite geometric series with $a = a_1 = 200$ and $r = 0.9$. The sum of this convergent series is

$$\begin{aligned} \frac{a}{1-r} &= \frac{200}{1-0.9} \\ &= \frac{200}{0.1} \\ &= 2000. \end{aligned}$$

The total expenditure, including the government's original \$200 payout, will be \$2000.

- (b) Since the original payout was \$200, the multiplier is $\$2000/\$200 = 10$.

26. (a) Consider the finite sequence

$$C(1+r)^{-1}, C(1+r)^{-2}, C(1+r)^{-3}, \dots, C(1+r)^{-n}.$$

This is a finite geometric sequence having first term $a = a_1 = C(1+r)^{-1}$ and common ratio $w = (1+r)^{-1}$.

Thus,

$$\begin{aligned} P = S_n &= \frac{a(w^n - 1)}{w - 1} \\ &= \frac{C(1+r)^{-1}[(1+r)^{-n} - 1]}{(1+r)^{-1} - 1} \\ &= C \frac{(1+r)^{-1}[(1+r)^{-n} - 1]}{(1+r)^{-1}[1 - (1+r)]} \\ &= C \frac{(1+r)^{-n} - 1}{-r} \\ &= C \frac{(1+r)^{-n} - 1}{-r} \cdot \frac{(1+r)^n}{(1+r)^n} \\ &= C \frac{(1+r)^n - 1}{r(1+r)^n}. \end{aligned}$$

- (b) The present value over an infinite amount of time is given by the infinite geometric series

$$P = C(1+r)^{-1} + C(1+r)^{-2} + C(1+r)^{-3} + \dots$$

where $a = a_1 = C(1+r)^{-1}$ and the common ratio $w = (1+r)^{-1}$. This series converges to

$$\frac{a}{1-w} = \frac{C(1+r)^{-1}}{1-(1+r)^{-1}} = \frac{C(1+r)^{-1}}{(1+r)^{-1}[(1+r)-1]} = \frac{C}{r}$$

Here, $P = \frac{C}{r}$.

27. Let S_n be the total medical malpractice payments the company pays after n years. If each year's payments are 20% less than those of the previous year, then they are 80% of the previous year's payments. Thus,

$$\begin{aligned} S_0 &= 60 \\ S_1 &= 60 + 60(0.8) \\ S_2 &= 60 + 60(0.8) + 60(0.8)^2 \\ &\vdots \\ S_n &= 60 + 60(0.8) + \cdots + 60(0.8)^n. \end{aligned}$$

The total medical malpractice payments that the company pays in all years is, therefore,

$$S = 60 + 60(0.8) + \cdots + 60(0.8)^n + \cdots$$

This is a geometric series with $a = 60$ and $r = 0.8$.

$$S = \frac{a}{1-r} = \frac{60}{1-0.8} = \frac{60}{0.2} = 300$$

The correct answer choice is **d**.

28. Let $p_0 = p$, the probability that the policyholder files no claims during the period. Then, because $p_{n+1} = \left(\frac{1}{5}\right)p_n$, we have that

$$\begin{aligned} p_0 &= p \\ p_1 &= \frac{1}{5}p_0 = \frac{1}{5}p \\ p_2 &= \frac{1}{5}p_1 = \frac{1}{5} \cdot \frac{1}{5}p = \left(\frac{1}{5}\right)^2 p \\ &\vdots \\ p_n &= \frac{1}{5}p_{n-1} = \frac{1}{5} \cdot \left(\frac{1}{5}\right)^{n-1} p = \left(\frac{1}{5}\right)^n p. \end{aligned}$$

The total probability is the sum of the probabilities of the policyholder filing no claim, 1 claim, 2 claims, ..., n claims, ..., and so on.

$$S = p + p_1 + p_2 + \cdots + p_n + \cdots = p + p\left(\frac{1}{5}\right) + p\left(\frac{1}{5}\right)^2 + \cdots + p\left(\frac{1}{5}\right)^n + \cdots$$

This is a geometric series with $a = p$ and $r = \frac{1}{5}$.

$$S = \frac{a}{1-r} = \frac{p}{1-\frac{1}{5}} = \frac{p}{\frac{4}{5}} = \frac{5}{4}p$$

And, since the sum of all probabilities must equal 1,

$$\begin{aligned} \frac{5}{4}p &= 1 \\ p &= \frac{4}{5}. \end{aligned}$$

The probability that a policyholder files more than one claim is

$$p_{n>1} = 1 - (p_0 + p_1) = 1 - \left(p + \frac{1}{5}p\right) = 1 - \frac{6}{5}p = 1 - \frac{6}{5}\left(\frac{4}{5}\right) = 1 - \frac{24}{25} = 0.04.$$

The correct answer choice is **a**.

- 29.** The height that the ball returns to after the first bounce is given by $10\left(\frac{3}{4}\right)$, after the second bounce $10\left(\frac{3}{4}\right)^2$, after the third bounce $10\left(\frac{3}{4}\right)^3$, and so on. Thus, the distance that the ball travels before it comes to rest is given by

$$10 + 2(10)\left(\frac{3}{4}\right) + 2(10)\left(\frac{3}{4}\right)^2 + 2(10)\left(\frac{3}{4}\right)^3 + \dots$$

$2(10)\left(\frac{3}{4}\right) + 2(10)\left(\frac{3}{4}\right)^2 + 2(10)\left(\frac{3}{4}\right)^3 + \dots$ is an infinite geometric series with $a_1 = a = 2(10)\left(\frac{3}{4}\right) = 15$ and $r = \frac{3}{4}$. This series converges to

$$\frac{a}{1-r} = \frac{15}{1-\frac{3}{4}} = \frac{15}{\frac{1}{4}} = 60.$$

So, the total distance traveled by the ball is $10 + 60 = 70$ meters.

- 30.** The number of times the wheel rotates in the second minute is given by $400\left(\frac{3}{4}\right)$, in the third minute $400\left(\frac{3}{4}\right)^2$, in the fourth minute $400\left(\frac{3}{4}\right)^3$, and so on. Thus, the total number of rotations of the wheel before coming to a complete stop is given by

$$400 + 400\left(\frac{3}{4}\right) + 400\left(\frac{3}{4}\right)^2 + 400\left(\frac{3}{4}\right)^3 + \dots$$

This is an infinite geometric series with $a = a_1 = 400$ and $r = \frac{3}{4}$. This series converges to

$$\frac{a}{1-r} = \frac{400}{1-\frac{3}{4}} = \frac{400}{\frac{1}{4}} = 1600.$$

The wheel makes 1600 rotations before coming to a complete stop.

- 31.** On its second swing, the pendulum bob will swing through an arc $40(0.8)$ centimeters long, on its third swing $40(0.8)^2$ centimeters long, on its fourth swing $40(0.8)^3$ centimeters long, and so on. Thus, the distance it will swing altogether before coming to a complete stop is given by

$$40 + 40(0.8) + 40(0.8)^2 + 40(0.8)^3 + \dots$$

This is an infinite geometric series with $a_1 = a = 40$ and $r = 0.8$. This series converges to

$$\frac{a}{1-r} = \frac{40}{1-0.8} = \frac{40}{0.2} = 200.$$

So, the total distance altogether before coming to a complete stop is 200 centimeters.

- 32.** The second triangle has sides 1 meter in length, the third triangle has sides $\frac{1}{2}$ meter in length, the fourth triangle has sides $\frac{1}{4}$ meter in length, and so on. Thus, the total perimeter of all the triangles given by

$$3(2) + 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \dots$$

is an infinite geometric series with $a = a_1 = 3(2) = 6$ and $r = \frac{1}{2}$. This series converges to

$$\frac{a}{1-r} = \frac{6}{1-\frac{1}{2}} = \frac{6}{\frac{1}{2}} = 12.$$

So, the total perimeter of all the triangles is 12 meters.

- 33.** The first triangle has sides 2 meters in length, the second triangle has sides 1 meter in length, the third triangle has sides $\frac{1}{2}$ meter in length, the fourth triangle has sides $\frac{1}{4}$ meter in length, and so on. Thus, the height of the first triangle is $\sqrt{3}$ meters, the height of the second triangle is $\frac{\sqrt{3}}{2}$ meter, the height of the third triangle is $\frac{\sqrt{3}}{4}$ meter, the height of the fourth triangle is $\frac{\sqrt{3}}{8}$ meter, and so on. The total area of the triangles, disregarding the overlaps, is given by

$$\frac{1}{2}(2)(\sqrt{3}) + \frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{8}\right) + \cdots = \sqrt{3} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{64} + \cdots$$

This is an infinite geometric series with $a = a_1 = \sqrt{3}$ and $r = \frac{1}{4}$.

This series converges to $\frac{a}{1-r} = \frac{\sqrt{3}}{1-\frac{1}{4}} = \frac{\sqrt{3}}{\frac{3}{4}} = \frac{4\sqrt{3}}{3}$.

The total area of the triangles, disregarding the overlaps, is given by $\frac{4\sqrt{3}}{3}$ square meters.

- 34. (a)** Solve using algebra.

Let t be the number of hours since 2:00 pm. Between noon and 2 pm the first train has traveled $(2)(100) = 200$ miles. Its distance from the station at any time t is $d_1 = 200 + 100t$. At time t , the second train has traveled $d_2 = 125t$ miles.

$$\begin{aligned} d_1 &= d_2 \\ 200 + 100t &= 125t \\ 200 &= 25t \\ t &= 8 \end{aligned}$$

The trains are the same distance from the station 8 hours after 2 pm, that is, at 10 pm.

- (b)** Solve using a geometric series.

At 2 pm the first train has traveled 200 miles, and it takes the second train $200/125$ hours to make up this distance. During this time the first train travels an additional $(100)(200/125)$ miles, which takes the second train $(100)(200/125)/125$ hours, to make up. The total of the second train's catch-up times is a geometric series with $a = a_1 = 200/125$ and $r = 100/125$.

The sum of this series is

$$\frac{a}{1-r} = \frac{200/125}{1-(100/125)} = \frac{8/5}{1/5} = 8,$$

or 8 hours, as found in (a). Again the two trains meet 8 hours after 2 pm, at 10 pm.

- 35. (a)** The tortoise's starting point is 10 meters. Since Achilles runs 10 meters per second, it takes him 1 second to reach the tortoise's starting point. The tortoise has traveled 1

meter in this 1 second since the tortoise's rate is 1 meter per second. After Achilles reaches the tortoise's starting point, the tortoise has traveled 1 meter. Since Achilles travels 10 meters per second, it takes Achilles $\frac{1}{10}$ second to reach this new point.

The tortoise has traveled $\frac{1}{10}$ meter in this $\frac{1}{10}$ second since the tortoise's rate is 1 meter per second. The time that it takes Achilles to reach the tortoise is given by

$$1 + \frac{1}{10} + \frac{1}{100} + \cdots$$

This is an infinite geometric series with $a = a_1 = 1$ and $r = \frac{1}{10}$. This series

converges to $\frac{a}{1-r} = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$.

It takes Achilles $\frac{10}{9}$ sec to catch the tortoise

- (b)** Using distance = rate $\times$ time, Achilles' distance is given by $10t$ meters in t seconds and the tortoise's distance is given by $t + 10$ meters in t seconds. The distances are equal when

$$\begin{aligned} 10t &= t + 10 \\ t &= \frac{10}{9} \end{aligned}$$

It takes Achilles $\frac{10}{9}$ seconds to catch the tortoise.

- (c)** The error is the assumption that an infinite series must have an infinite sum.

- 36.** Since the cyclists travel at 10 miles per hour and start 20 miles apart, they will meet in one hour. Since the fly flies at 15 miles per hour for 1 hour, it must travel 15 miles.

Now find the fly's total travel by summing each piece of its trip. It leaves the first bike and will meet the second bike when $15t = 20 - 10t$ or $t = 4/5$ hours. The fly has traveled $15(4/5) = 12$ miles and the second bike has traveled 8 miles toward it, so they meet at a point 12 miles from the first bike's original starting point. During this same $4/5$ hour, the first bike has also traveled 8 miles, so when the fly turns around, the first bike is 4 miles away. Now repeat the calculation: $15t = 4 - 10t$ or $t = 4/25$ hours. During this time the fly travels $15(4/25)$ miles. The fly's travel segments form a geometric series with first term $a = a_1 = 12$ and $r = 1/5$. This series has sum

$$\begin{aligned}\frac{a}{1-r} &= \frac{12}{1-(1/5)} \\ &= \frac{12}{4/5} = 15,\end{aligned}$$

or 15 miles, as found using the argument based on the total travel time.

12.5 Taylor Series

Your Turn 1

- (a) Use the Taylor series for $\ln(1+x)$ with property (2).

$$\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots \\ -7\ln(1+x) &= -7x + \frac{7x^2}{2} - \frac{7x^3}{3} + \frac{7x^4}{4} + \cdots + \frac{7(-1)^n x^n}{n} + \cdots\end{aligned}$$

for all x in $(-1, 1]$

- (b) Use the Taylor series for $\frac{1}{1-x}$ with property (3).

$$\begin{aligned}\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \\ \frac{x^2}{1-x} &= x^2 + x^3 + x^4 + x^5 + \cdots + x^n + \cdots\end{aligned}$$

for all x in $(-1, 1)$

Your Turn 2

- (a) Use the Taylor series for $\ln(1+x)$ together with composition.

$$\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots \\ \ln(1+2x^2) &= (2x^2) - \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{3} - \frac{(2x^2)^4}{4} + \cdots + \frac{(-1)^{n+1}(2x^2)^n}{n} + \cdots \\ &= 2x^2 - 2x^4 + \frac{8x^6}{3} - 4x^8 + \cdots + \frac{(-1)^{n+1}2^n x^{2n}}{n} + \cdots\end{aligned}$$

The interval of convergence of the original series is $-1 < x \leq 1$.

If $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$, then $-1 < 0 \leq 2x^2 \leq 1$, so the interval of convergence is now $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

(b) Use the series for $\frac{1}{1-x}$ together with composition.

$$\begin{aligned}
 g(x) &= \frac{3}{4-x^2} = \frac{3/4}{1-\left(\frac{x}{2}\right)^2} \\
 &= \frac{3}{4} f\left(\frac{x}{2}\right) \text{ where } f(x) = \frac{1}{1-x} \\
 \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \\
 \frac{3/4}{1-\left(\frac{x}{2}\right)^2} &= \frac{3}{4} + \frac{3}{4}\left(\frac{x}{2}\right)^2 + \frac{3}{4}\left[\left(\frac{x}{2}\right)^2\right]^2 + \frac{3}{4}\left[\left(\frac{x}{2}\right)^2\right]^3 + \cdots + \frac{3}{4}\left[\left(\frac{x}{2}\right)^2\right]^n + \cdots \\
 &= \frac{3}{4} + \frac{3x^2}{16} + \frac{3x^4}{64} + \frac{3x^6}{256} + \cdots + \frac{3x^{2n}}{4^{n+1}} + \cdots
 \end{aligned}$$

The interval of convergence of the original series is $-1 < x < 1$.

When $-2 < x < 2$, $-1 < 0 \leq \left(\frac{x}{2}\right)^2 < 1$ so the interval of convergence of the new series is $(-2, 2)$.

Your Turn 3

For an interest rate of 3.5%, the doubling time is $\frac{\ln 2}{\ln(1+0.035)} \approx 20.149$. Since this interest rate is less than 5%, we can use the rule of 70 to estimate this doubling time; the estimate is $\frac{70}{100(0.035)} = 20$ years.

For an interest rate of 8%, the doubling time is $\frac{\ln 2}{\ln(1+0.08)} \approx 9.006$. Since this interest rate is more than 5%, we can use the rule of 72 to estimate this doubling time; the estimate is $\frac{72}{100(0.08)} = 9$ years.

12.5 Exercises

1. Use the Taylor series for $\frac{1}{1-x}$ with property (2).

$$\begin{aligned}
 \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \\
 \frac{6}{1-x} &= 6 + 6x + 6x^2 + 6x^3 + \cdots + 6x^n + \cdots
 \end{aligned}$$

for all x in $(-1, 1)$

2. Use the Taylor series for $\frac{1}{1-x}$ with property (2).

$$\begin{aligned}
 \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \\
 \frac{-3}{1-x} &= -3 - 3x - 3x^2 - 3x^3 - \cdots - 3x^n - \cdots
 \end{aligned}$$

for all x in $(-1, 1)$

3. Use the Taylor series for e^x with property (3).

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$x^2 e^x = x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots + \frac{x^{n+2}}{n!} + \cdots$$

for x in $(-\infty, \infty)$

4. Use the Taylor series for e^x with property (3).

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$x^5 e^x = x^5 + \frac{x^6}{1!} + \frac{x^7}{2!} + \frac{x^8}{3!} + \cdots + \frac{x^{n+5}}{n!} + \cdots$$

for x in $(-\infty, \infty)$

5. This function most nearly matches $\frac{1}{1-x}$. To get 1 in the denominator, instead of 2, divide the numerator and denominator by 2.

$$\frac{5}{2-x} = \frac{\frac{5}{2}}{1-\frac{x}{2}}$$

Thus, we can find the Taylor series for $\frac{\frac{5}{2}}{1-\frac{x}{2}}$ by starting with the Taylor series for $\frac{1}{1-x}$, multiplying each term by $\frac{5}{2}$, and replacing each x with $\frac{x}{2}$.

$$\begin{aligned} \frac{5}{2-x} &= \frac{\frac{5}{2}}{1-\frac{x}{2}} \\ &= \frac{5}{2} \cdot 1 + \frac{5}{2} \left(\frac{x}{2} \right) + \frac{5}{2} \left(\frac{x}{2} \right)^2 + \frac{5}{2} \left(\frac{x}{2} \right)^3 + \cdots + \frac{5}{2} \left(\frac{x}{2} \right)^n + \cdots \\ &= \frac{5}{2} + \frac{5x}{4} + \frac{5x^2}{8} + \frac{5x^3}{16} + \cdots + \frac{5x^n}{2^{n+1}} + \cdots \end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $\frac{x}{2}$ gives

$$-1 < \frac{x}{2} < 1 \text{ or } -2 < x < 2.$$

The interval of convergence of the new series is $(-2, 2)$.

6. This function most nearly matches $\frac{1}{1-x}$. To get 1 in the denominator, instead of 4, divide the numerator and denominator by 4.

$$\frac{-3}{4-x} = \frac{-\frac{3}{4}}{1-\frac{x}{4}}$$

Thus, we can find the Taylor series for $\frac{-\frac{3}{4}}{1-\frac{x}{4}}$ by starting with the Taylor series for $\frac{1}{1-x}$, multiplying each term by $-\frac{3}{4}$, and replacing each x with $\frac{x}{4}$.

$$\begin{aligned}
\frac{-3}{4-x} &= \frac{-\frac{3}{4}}{1-\frac{x}{4}} \\
&= \left(-\frac{3}{4}\right) \cdot 1 + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right) + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right)^2 + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right)^3 + \cdots + \left(-\frac{3}{4}\right)\left(\frac{x}{4}\right)^n + \cdots \\
&= -\frac{3}{4} - \frac{3x}{16} - \frac{3x^2}{64} - \frac{3x^3}{256} - \cdots - \frac{3x^n}{4^{n+1}} - \cdots
\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $\frac{x}{4}$ gives

$$-1 < \frac{x}{4} < 1 \text{ or } -4 < x < 4.$$

The interval of convergence of the new series is $(-4, 4)$.

$$7. \quad \frac{8x}{1+3x} = x \cdot \frac{8}{1-(-3x)}$$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 8, and replace x with $-3x$. Also, use property (3) with $k = 1$.

$$\begin{aligned}
\frac{8x}{1+3x} &= x \cdot \frac{8}{1-(-3x)} \\
&= x \cdot 8 \cdot 1 + x \cdot 8(-3x) + x \cdot 8(-3x)^2 + x \cdot 8(-3x)^3 + \cdots + x \cdot 8(-3x)^n + \cdots \\
&= 8x - 24x^2 + 72x^3 - 216x^4 + \cdots + (-1)^n \cdot 8 \cdot 3^n x^{n+1} + \cdots
\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-3x$ gives

$$-1 < -3x < 1 \quad \text{or} \quad \frac{1}{3} > x > -\frac{1}{3}.$$

The interval of convergence of the new series is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

$$8. \quad \frac{7x}{1+2x} = x \cdot \frac{7}{1-(-2x)}$$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 7, and replace x with $-2x$. Also, use property (3) with $k = 1$.

$$\begin{aligned}
\frac{7x}{1+2x} &= x \cdot \frac{7}{1-(-2x)} \\
&= x \cdot 7 \cdot 1 + x \cdot 7(-2x) + x \cdot 7(-2x)^2 + x \cdot 7(-2x)^3 + \cdots + x \cdot 7(-2x)^n + \cdots \\
&= 7x - 14x^2 + 28x^3 - 56x^4 + \cdots + (-1)^n \cdot 7 \cdot 2^n x^{n+1} + \cdots
\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-2x$ gives

$$-1 < -2x < 1 \quad \text{or} \quad \frac{1}{2} > x > -\frac{1}{2}.$$

The interval of convergence of the new series is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

$$9. \quad \frac{x^2}{4-x} = x^2 \cdot \frac{\frac{1}{4}}{1-\frac{x}{4}}$$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by $\frac{1}{4}$, and replace x with $\frac{x}{4}$. Also, use property (3) with $k = 2$.

$$\begin{aligned} \frac{x^2}{4-x} &= x^2 \cdot \frac{\frac{1}{4}}{1-\frac{x}{4}} \\ &= x^2 \cdot \frac{1}{4} \cdot 1 + x^2 \cdot \frac{1}{4} \left(\frac{x}{4} \right) + x^2 \cdot \frac{1}{4} \left(\frac{x}{4} \right)^2 + x^2 \cdot \frac{1}{4} \left(\frac{x}{4} \right)^3 + \cdots + x^2 \cdot \frac{1}{4} \left(\frac{x}{4} \right)^n + \cdots \\ &= \frac{x^2}{4} + \frac{x^3}{16} + \frac{x^4}{64} + \frac{x^5}{256} + \cdots + \frac{x^{n+2}}{4^{n+1}} + \cdots \end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $\frac{x}{4}$ gives

$$-1 < \frac{x}{4} < 1 \text{ or } -4 < x < 4.$$

The interval of convergence of the new series is $(-4, 4)$.

$$10. \quad \frac{9x^4}{1-x} = x^4 \cdot \frac{9}{1-x}$$

Use the Taylor series for $\frac{1}{1-x}$, and multiply each term by 9. Also, use property (3) with $k = 4$.

$$\begin{aligned} \frac{9x^4}{1-x} &= x^4 \cdot \frac{9}{1-x} \\ &= x^4 \cdot 9 \cdot 1 + x^4 \cdot 9(x) + x^4 \cdot 9(x)^2 + x^4 \cdot 9(x)^3 + \cdots + x^4 \cdot 9(x)^n + \cdots \\ &= 9x^4 + 9x^5 + 9x^6 + 9x^7 + \cdots + 9x^{n+4} + \cdots \end{aligned}$$

The Taylor series for $\frac{9x^4}{1-x}$ has the same interval of convergence, $(-1, 1)$, as the Taylor series for $\frac{1}{1-x}$.

11. We find the Taylor series of $\ln(1+4x)$ by starting with the Taylor series for $\ln(1+x)$ and replacing each x with $4x$.

$$\begin{aligned} \ln(1+4x) &= 4x - \frac{(4x)^2}{2} + \frac{(4x)^3}{3} - \frac{(4x)^4}{4} + \cdots + \frac{(-1)^n(4x)^{n+1}}{n+1} + \cdots \\ &= 4x - 8x^2 + \frac{64}{3}x^3 - 64x^4 + \cdots + \frac{(-1)^n 4^{n+1} x^{n+1}}{n+1} + \cdots \end{aligned}$$

The Taylor series for $\ln(1+x)$ is valid when $-1 < x \leq 1$. Replacing x with $4x$ gives

$$-1 < 4x \leq 1 \text{ or } -\frac{1}{4} < x \leq \frac{1}{4}.$$

The interval of convergence of the new series is $\left(-\frac{1}{4}, \frac{1}{4}\right]$.

12. We find the Taylor series of $\ln\left(1 - \frac{x}{2}\right)$ by starting with the Taylor series for $\ln(1 + x)$ and replacing each x with $-\frac{x}{2}$.

$$\begin{aligned}\ln\left(1 - \frac{x}{2}\right) &= \left(-\frac{x}{2}\right) - \frac{\left(-\frac{x}{2}\right)^2}{2} + \frac{\left(-\frac{x}{2}\right)^3}{3} - \frac{\left(-\frac{x}{2}\right)^4}{4} + \cdots + \frac{(-1)^n \left(-\frac{x}{2}\right)^{n+1}}{n+1} + \cdots \\ &= -\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} - \cdots - \frac{x^{n+1}}{(n+1)2^{n+1}} - \cdots\end{aligned}$$

The Taylor series for $\ln(1 + x)$ is valid when $-1 < x \leq 1$. Replacing x with $-\frac{x}{2}$ gives

$$-1 < -\frac{x}{2} \leq 1 \quad \text{or} \quad 2 > x \geq -2.$$

The interval of convergence of the new series is $[-2, 2)$.

13. We find the Taylor series for e^{4x^2} by starting with the Taylor series for e^x and replacing each x with $4x^2$.

$$\begin{aligned}e^{4x^2} &= 1 + (4x^2) + \frac{1}{2!}(4x^2)^2 + \frac{1}{3!}(4x^2)^3 + \cdots + \frac{1}{n!}(4x^2)^n + \cdots \\ &= 1 + 4x^2 + 8x^4 + \frac{32}{3}x^6 + \cdots + \frac{4^n x^{2n}}{n!} + \cdots\end{aligned}$$

The Taylor series for e^{4x^2} has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

14. We find the Taylor series for e^{-3x^2} by starting with the Taylor series for e^x and replacing each x with $-3x^2$.

$$\begin{aligned}e^{-3x^2} &= 1 + (-3x^2) + \frac{1}{2!}(-3x^2)^2 + \frac{1}{3!}(-3x^2)^3 + \cdots + \frac{1}{n!}(-3x^2)^n + \cdots \\ &= 1 - 3x^2 + \frac{9}{2}x^4 - \frac{9}{2}x^6 + \cdots + \frac{(-1)^n 3^n x^{2n}}{n!} + \cdots\end{aligned}$$

The Taylor series for e^{-3x^2} has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

15. Use the Taylor series for e^x and replace x with $-x$. Also, use property (3) with $k = 3$.

$$\begin{aligned}x^3 e^{-x} &= x^3 \cdot 1 + x^3(-x) + x^3 \cdot \frac{1}{2!}(-x)^2 + x^3 \cdot \frac{1}{3!}(-x)^3 + \cdots + x^3 \cdot \frac{1}{n!}(-x)^n + \cdots \\ &= x^3 - x^4 + \frac{1}{2}x^5 - \frac{1}{6}x^6 + \cdots + \frac{(-1)^n x^{n+3}}{n!} + \cdots\end{aligned}$$

The Taylor series for $x^3 e^{-x}$ has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

16. $f(x) = x^4 e^{2x}$

Use the Taylor series for e^x and property (3) with $k = 4$.

$$\begin{aligned} x^4 e^{2x} &= x^4 \cdot 1 + x^4 \cdot (2x) + x^4 \cdot \frac{1}{2!}(2x)^2 + x^4 \cdot \frac{1}{3!}(2x)^3 + \cdots + x^4 \cdot \frac{1}{n!}(2x)^n + \cdots \\ &= x^4 + 2x^5 + 2x^6 + \frac{4}{3}x^7 + \cdots + \frac{2^n x^{n+4}}{n!} + \cdots \end{aligned}$$

The Taylor series for $x^4 e^{2x}$ has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

17. $\frac{2}{1+x^2} = \frac{2}{1-(-x^2)}$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 2, and replace x with $-x^2$.

$$\begin{aligned} \frac{2}{1+x^2} &= \frac{2}{1-(-x^2)} \\ &= 2 \cdot 1 + 2(-x^2) + 2(-x^2)^2 + 2(-x^2)^3 + \cdots + 2(-x^2)^n + \cdots \\ &= 2 - 2x^2 + 2x^4 - 2x^6 + \cdots + (-1)^n \cdot 2 \cdot x^{2n} + \cdots \end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-x^2$ gives

$$-1 < -x^2 < 1 \text{ or } 1 > x^2 > -1.$$

The inequality is satisfied by any x in the interval $(-1, 1)$.

18. $\frac{6}{3+x^2} = \frac{2}{1-\left(-\frac{x^2}{3}\right)}$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 2, and replace x with $-\frac{x^2}{3}$.

$$\begin{aligned} \frac{6}{3+x^2} &= \frac{2}{1-\left(-\frac{x^2}{3}\right)} \\ &= 2 \cdot 1 + 2\left(-\frac{x^2}{3}\right) + 2\left(-\frac{x^2}{3}\right)^2 + 2\left(-\frac{x^2}{3}\right)^3 + \cdots + 2\left(-\frac{x^2}{3}\right)^n + \cdots \\ &= 2 - \frac{2}{3}x^2 + \frac{2}{9}x^4 - \frac{2}{27}x^6 + \cdots + \frac{(-1)^n \cdot 2 \cdot x^{2n}}{3^n} + \cdots \end{aligned}$$

The Taylor for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-\frac{x^2}{3}$ gives

$$-1 < -\frac{x^2}{3} < 1 \text{ or } 3 > x^2 > -3.$$

The inequality is satisfied by any x in the interval $(-\sqrt{3}, \sqrt{3})$.

$$19. \quad \frac{e^x + e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

We find the Taylor series for $\frac{1}{2}e^x$ by starting with the Taylor series for e^x and multiplying each term by $\frac{1}{2}$.

$$\begin{aligned} \frac{1}{2}e^x &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot x + \frac{1}{2} \cdot \frac{1}{2!}x^2 + \frac{1}{2} \cdot \frac{1}{3!}x^3 + \cdots + \frac{1}{2} \cdot \frac{1}{n!}x^n + \cdots \\ &= \frac{1}{2} + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{12}x^3 + \cdots + \frac{1}{2n!}x^n + \cdots \end{aligned}$$

We find the Taylor series for $\frac{1}{2}e^{-x}$ by starting with the Taylor series for e^x , multiplying each term by $\frac{1}{2}$, and replacing x with $-x$.

$$\begin{aligned} \frac{1}{2}e^{-x} &= \frac{1}{2} \cdot 1 + \frac{1}{2}(-x) + \frac{1}{2} \cdot \frac{1}{2!}(-x)^2 + \frac{1}{2} \cdot \frac{1}{3!}(-x)^3 + \cdots + \frac{1}{2} \cdot \frac{1}{n!}(-x)^n + \cdots \\ &= \frac{1}{2} - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{12}x^3 + \cdots + \frac{(-1)^n}{2n!}x^n + \cdots \end{aligned}$$

Use property (1) with $f(x) = \frac{1}{2}e^x$ and $g(x) = \frac{1}{2}e^{-x}$.

$$\begin{aligned} \frac{e^x + e^{-x}}{2} &= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\ &= \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\right)x + \left(\frac{1}{4} + \frac{1}{4}\right)x^2 + \left(\frac{1}{12} - \frac{1}{12}\right)x^3 + \cdots + \left(\frac{1}{2n!} + \frac{(-1)^n}{2n!}\right)x^n + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots + \frac{1 + (-1)^n}{2n!}x^n + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots + \frac{1}{(2n)!}x^{2n} + \cdots \end{aligned}$$

The new series has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

$$20. \quad \frac{e^x - e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

We find the Taylor series for $\frac{1}{2}e^x$ by starting with the Taylor series for e^x and multiplying each term by $\frac{1}{2}$.

$$\begin{aligned} \frac{1}{2}e^x &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot x + \frac{1}{2} \cdot \frac{1}{2!}x^2 + \frac{1}{2} \cdot \frac{1}{3!}x^3 + \cdots + \frac{1}{2} \cdot \frac{1}{n!}x^n + \cdots \\ &= \frac{1}{2} + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{12}x^3 + \cdots + \frac{1}{2n!}x^n + \cdots \end{aligned}$$

We find the Taylor series for $-\frac{1}{2}e^{-x}$ by starting with the Taylor series for e^x , multiplying each term by $-\frac{1}{2}$, and replacing x with $-x$.

$$\begin{aligned} -\frac{1}{2}e^{-x} &= -\frac{1}{2} \cdot 1 + \left(-\frac{1}{2}\right)(-x) + \left(-\frac{1}{2}\right)\frac{1}{2!}(-x)^2 + \left(-\frac{1}{2}\right)\frac{1}{3!}(-x)^3 + \cdots + \left(-\frac{1}{2}\right)\frac{1}{n!}(-x)^n + \cdots \\ &= -\frac{1}{2} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 + \cdots + \frac{(-1)^{n+1}}{2n!}x^n + \cdots \end{aligned}$$

Use property (1) with $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$.

$$\begin{aligned}
& \frac{e^x - e^{-x}}{2} \\
&= \frac{1}{2}e^x - \frac{1}{2}e^{-x} \\
&= \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{2}\right)x + \left(\frac{1}{4} - \frac{1}{4}\right)x^2 + \left(\frac{1}{12} + \frac{1}{12}\right)x^3 + \cdots + \left(\frac{1}{2n!} + \frac{(-1)^{n+1}}{2n!}\right)x^n + \cdots \\
&= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots + \frac{1 + (-1)^{n+1}}{2n!}x^n + \cdots \\
&= x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + \cdots
\end{aligned}$$

The new series has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

21. Use the Taylor series for $\ln(1+x)$ and replace x with $2x^4$.

$$\begin{aligned}
\ln(1+2x^4) &= 2x^4 - \frac{(2x^4)^2}{2} + \frac{(2x^4)^3}{3} - \frac{(2x^4)^4}{4} + \cdots + \frac{(-1)^n(2x^4)^{n+1}}{n+1} + \cdots \\
&= 2x^4 - 2x^8 + \frac{8}{3}x^{12} - 4x^{16} + \cdots + \frac{(-1)^n \cdot 2^{n+1}x^{4n+4}}{n+1} + \cdots
\end{aligned}$$

The Taylor series for $\ln(1+x)$ is valid when $-1 < x \leq 1$. Replacing x with $2x^4$ gives

$$-1 < 2x^4 \leq 1 \text{ or } -\frac{1}{2} < x^4 \leq \frac{1}{2}.$$

The inequality is satisfied by any x in the interval $\left[-\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right]$.

22. $\ln(1-5x^2) = \ln[1+(-5x^2)]$

Use the Taylor series for $\ln(1+x)$ and replace x with $-5x^2$.

$$\begin{aligned}
\ln(1-5x^2) &= (-5x^2) - \frac{(-5x^2)^2}{2} + \frac{(-5x^2)^3}{3} - \frac{(-5x^2)^4}{4} + \cdots + \frac{(-1)^n(-5x^2)^{n+1}}{n+1} + \cdots \\
&= -5x^2 - \frac{25}{2}x^4 - \frac{125}{3}x^6 - \frac{625}{4}x^8 - \cdots - \frac{5^{n+1}x^{2n+2}}{n+1} - \cdots
\end{aligned}$$

The Taylor series for $\ln(1+x)$ is valid when $-1 < x \leq 1$. Replacing x with $-5x^2$ gives

$$-1 < -5x^2 \leq 1 \text{ or } \frac{1}{5} > x^2 \geq -\frac{1}{5}.$$

The inequality is satisfied by any x in the interval $\left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

23. $\frac{1+x}{1-x} = \frac{1}{1-x} + \frac{x}{1-x}$

The Taylor series for $\frac{1}{1-x}$ is given as

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

We find the Taylor series for $\frac{x}{1-x}$ by starting with the Taylor series for $\frac{1}{1-x}$ and using property (3) with $k = 1$.

$$\begin{aligned}\frac{x}{1-x} &= x \cdot \frac{1}{1-x} \\ &= x \cdot 1 + x \cdot x + x \cdot x^2 + x \cdot x^3 + \cdots + x \cdot x^n + \cdots \\ &= x + x^2 + x^3 + x^4 + \cdots + x^{n+1} + \cdots\end{aligned}$$

Use property (1) with $f(x) = \frac{1}{1-x}$ and $g(x) = \frac{x}{1-x}$.

$$\begin{aligned}\frac{1+x}{1-x} &= \frac{1}{1-x} + \frac{x}{1-x} \\ &= 1 + (1+x) + (1+x)x + (1+x)x^2 + (1+x)x^3 + \cdots + (1+x)x^n + \cdots \\ &= 1 + 2x + 2x^2 + 2x^3 + \cdots + 2x^n + \cdots\end{aligned}$$

24. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

The Taylor series for $\ln(1+x)$ is given as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^{n+1}}{n+1} + \cdots$$

We find the Taylor series for $-\ln(1-x)$ by starting with the Taylor series for $\ln(1+x)$ multiplying each term by -1 , and replacing x with $-x$.

$$\begin{aligned}-\ln(1-x) &= -1(-x) - (-1)\frac{(-x)^2}{2} + (-1)\frac{(-x)^3}{3} - (-1)\frac{(-x)^4}{4} + \cdots + (-1)\frac{(-1)^n (-x)^{n+1}}{n+1} + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^{n+1}}{n+1} + \cdots\end{aligned}$$

Use property (1) with $f(x) = \ln(1+x)$ and $g(x) = -\ln(1-x)$.

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= (1+x) + \left(-\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{3} + \frac{1}{3}\right)x^3 + \left(-\frac{1}{4} + \frac{1}{4}\right)x^4 + \cdots + \left(\frac{(-1)^n}{n+1} + \frac{1}{n+1}\right)x^{n+1} + \cdots \\ &= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots + \frac{(-1)^n + 1}{n+1}x^{n+1} + \cdots \\ &= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots + \frac{2x^{2n+1}}{2n+1} + \cdots\end{aligned}$$

25. The Taylor series for e^x is given as

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{n!}x^n + \cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots + \frac{x^n}{n!} + \cdots \\ &= 1 + x + \frac{x^2}{2} \left[1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \cdots + \frac{x^{n-2}}{\frac{n!}{2}} + \cdots \right] \end{aligned}$$

Compare the series $\frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \cdots + \frac{x^{n-2}}{\frac{n!}{2}} + \cdots$, for $n \geq 3$, with the series

$\frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{48} + \cdots + \frac{x^{n-2}}{3 \cdot 4^{n-3}} + \cdots$, for $n \geq 3$. The second series is an infinite geometric series that is larger than or equal to the first series term by term. The geometric series has $a = \frac{x}{3}$ and $r = \frac{x}{4}$, and sums to $\frac{4x}{12-3x}$. Thus, for $x > 0$,

$$1 + x + \frac{x^2}{2} < 1 + x + \frac{x^2}{2} \left[1 + \frac{x}{3} + \frac{x^2}{12} + \frac{x^3}{60} + \cdots + \frac{x^{n-2}}{\frac{n!}{2}} + \cdots \right] < 1 + x + \frac{x^2}{2} \left[1 + \frac{4x}{12-3x} \right],$$

or
$$1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} \left[1 + \frac{4x}{12-3x} \right].$$

For values of x sufficiently close to 0, $\frac{4x}{12-3x}$ approaches 0, and $1 + x + \frac{x^2}{2} \left[1 + \frac{4x}{12-3x} \right] \approx 1 + x + \frac{x^2}{2}$.

Thus, $e^x \approx 1 + x + \frac{x^2}{2}$.

For $x < 0$, a similar argument can be made.

26. Use the Taylor series for e^x and replace x with $-x$ to find the Taylor series for e^{-x} .

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{1}{2!}(-x)^2 + \frac{1}{3!}(-x)^3 + \frac{1}{4!}(-x)^4 + \frac{1}{5!}(-x)^5 + \cdots + \frac{1}{n!}(-x)^n + \cdots \\ &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots \\ &= 1 - x + \frac{x^2}{2} \left[1 - \frac{x}{3} + \frac{x^2}{12} - \frac{x^3}{60} + \cdots + \frac{(-1)^n x^{n-2}}{\frac{n!}{2}} + \cdots \right] \\ &= 1 - x + \frac{x^2}{2} \left[1 - \frac{x}{3} \left(1 - \frac{x}{4} \right) - \frac{x^3}{60} \left(1 - \frac{x}{6} \right) - \cdots - \frac{x^{n-2}}{\frac{n!}{2}} \left(1 - \frac{x}{n+1} \right) - \cdots \right] \end{aligned}$$

Compare the series $-\frac{x}{3} \left(1 - \frac{x}{4} \right) - \frac{x^3}{60} \left(1 - \frac{x}{6} \right) - \cdots - \frac{x^{n-2}}{\frac{n!}{2}} \left(1 - \frac{x}{n+1} \right) - \cdots$ with the series

$-\frac{x}{3} \left(1 - \frac{x}{4} \right) - \frac{x^3}{48} \left(1 - \frac{x}{4} \right) - \cdots - \frac{x^{n-2}}{3 \cdot 4^{n-3}} \left(1 - \frac{x}{4} \right) - \cdots$. The second series is an infinite geometric series that is less than or equal to the first series term by term. The geometric series has $a = -\frac{x}{3} \left(1 - \frac{x}{4} \right)$ and $r = \frac{x^2}{16}$, and sums to $-\frac{16x-4x^2}{48-3x^2}$. Thus, for $0 < x < 1$,

$$1 - x + \frac{x^2}{2} \left[1 - \frac{16x - 4x^2}{48 - 3x^2} \right] < 1 - x + \frac{x^2}{2} \left[1 - \frac{x}{3} \left(1 - \frac{x}{4} \right) - \frac{x^3}{60} \left(1 - \frac{x}{6} \right) - \cdots - \frac{x^{n-2}}{\frac{n!}{2}} \left(1 - \frac{x}{n+1} \right) - \cdots \right]$$

$$< 1 - x + \frac{x^2}{2},$$

or

$$1 - x + \frac{x^2}{2} \left[1 - \frac{16x - 4x^2}{48 - 3x^2} \right] < e^{-x} < 1 - x + \frac{x^2}{2}.$$

For values of x sufficiently close to 0, $\frac{16x-4x^2}{48-3x^2}$ approaches 0, and

$$1 - x + \frac{x^2}{2} \left[1 - \frac{16x - 4x^2}{48 - 3x^2} \right] \approx 1 - x + \frac{x^2}{2}.$$

Thus, $e^{-x} \approx 1 - x + \frac{x^2}{2}$.

For $x < 0$, a similar argument can be made.

27. The Taylor series for e^x is given as

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{n!}x^n + \cdots$$

For $x > 0$, $\frac{1}{n!}x^n > 0$ when $n \geq 2$. Thus,

$$\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{n!}x^n + \cdots > 0$$

since each term on the left-hand side is positive. Adding $1 + x$ to both sides we have

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{n!}x^n + \cdots > 1 + x$$

or $e^x > 1 + x$.

For $x = 0$, $e^x = e^0 = 1 = 1 + 0 = 1 + x$.

For $-1 \leq x < 0$, when b is an even positive integer, $\frac{x^b}{b} > 0$. Also, when $a \geq 3$, $\frac{|x|}{a} < 1$, or $1 + \frac{x}{a} > 0$. Thus, $\frac{x^b}{b} \left(1 + \frac{x}{a} \right) > 0$ for b an even positive integer and $a \geq 3$, since both factors are positive. Thus

$$\frac{x^2}{2!} \left(1 + \frac{x}{3} \right) + \frac{x^4}{4!} \left(1 + \frac{x}{5} \right) + \frac{x^6}{6!} \left(1 + \frac{x}{7} \right) + \cdots + \frac{x^{2n}}{(2n)!} \left(1 + \frac{x}{2n+1} \right) + \cdots > 0,$$

since each term has been shown to be positive. Consider the Taylor series for e^x ,

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{n!}x^n + \cdots \\ &= 1 + x + \frac{x^2}{2!} \left(1 + \frac{x}{3} \right) + \frac{x^4}{4!} \left(1 + \frac{x}{5} \right) + \cdots + \frac{x^{2n}}{(2n)!} \left(1 + \frac{x}{2n+1} \right) + \cdots \end{aligned}$$

We have already shown the series to be positive from the third term on, so $e^x > 1 + x$.

For $x < -1$, $1 + x < 0$ and $e^x > 0$. Thus $e^x > 1 + x$.

28. The Taylor series for e^{-x} is found from the Taylor series for e^x by replacing x with $-x$.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots$$

For $x \leq 0$, all the terms after the first two are greater than or equal to 0, so $e^{-x} \geq 1 - x$.

For $0 < x \leq 1$, take the terms in pairs beginning with the third term. For example,

$$\frac{x^2}{2!} - \frac{x^3}{3!} = \frac{x^2}{2!} \left(1 - \frac{x}{3}\right). \text{ If } 0 < x \leq 1, \text{ then } 0 < \frac{x}{3} \leq \frac{1}{3}, \text{ so } 1 - \frac{x}{3} > 0 \text{ and } \frac{x^2}{2!} \left(1 - \frac{x}{3}\right) > 0.$$

A similar argument works for every pair of terms of the form $\frac{x^{2n}}{(2n)!} - \frac{x^{2n+1}}{(2n+1)!} = \frac{x^{2n}}{(2n)!} \left(1 - \frac{x}{2n+1}\right)$.

Thus $e^{-x} \geq 1 - x$.

Finally, if $x > 1$, e^{-x} is positive and $1 - x$ is negative, so again $e^{-x} \geq 1 - x$.

29. The area is given by

$$\int_0^{\frac{1}{3}} e^{x^2} dx.$$

Find the Taylor series for e^{x^2} by using the Taylor series for e^x and replacing x with x^2 .

$$e^{x^2} = 1 + (x^2) + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \frac{1}{4!}(x^2)^4 + \cdots + \frac{1}{n!}(x^2)^n + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned} \int_0^{\frac{1}{3}} e^{x^2} dx &\approx \int_0^{\frac{1}{3}} \left[1 + x^2 + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \frac{1}{4!}(x^2)^4 \right] dx \\ &= \int_0^{\frac{1}{3}} \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 \right) dx \\ &= \left(x + \frac{1}{3}x^3 + \frac{1}{10}x^5 + \frac{1}{42}x^7 + \frac{1}{216}x^9 \right) \Big|_0^{\frac{1}{3}} \\ &= \frac{1}{3} + \frac{1}{81} + \frac{1}{2430} + \frac{1}{91,854} + \frac{1}{4,251,528} - 0 \\ &\approx 0.3461. \end{aligned}$$

30. The area is given by

$$\int_0^{\frac{1}{2}} \frac{1}{1-x^3} dx.$$

Find the Taylor series for $\frac{1}{1-x^3}$ by using the Taylor series for $\frac{1}{1-x}$ and replacing x with x^3 .

$$\begin{aligned} \frac{1}{1-x^3} &= 1 + (x^3) + (x^3)^2 + (x^3)^3 + (x^3)^4 + \cdots + (x^3)^n + \cdots \\ &= 1 + x^3 + x^6 + x^9 + x^{12} + \cdots + x^{3n} + \cdots \end{aligned}$$

Using the first five terms of this series gives

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{1}{1-x^3} dx &\approx \int_0^{\frac{1}{2}} (1 + x^3 + x^6 + x^9 + x^{12}) dx = \left(x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + \frac{1}{10}x^{10} + \frac{1}{13}x^{13} \right) \bigg|_0^{\frac{1}{2}} \\ &= \frac{1}{2} + \frac{1}{64} + \frac{1}{896} + \frac{1}{10,240} + \frac{1}{106,496} - 0 \approx 0.5168.\end{aligned}$$

31. The area is given by

$$\int_{\frac{1}{4}}^{\frac{1}{3}} \frac{1}{1-\sqrt{x}} dx.$$

Find the Taylor series for $\frac{1}{1-\sqrt{x}}$ by using the Taylor series for $\frac{1}{1-x}$ and replacing x with $\sqrt{x}$.

$$\frac{1}{1-\sqrt{x}} = 1 + \sqrt{x} + \sqrt{x}^2 + \sqrt{x}^3 + \cdots + \sqrt{x}^n + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned}\int_{\frac{1}{4}}^{\frac{1}{3}} \frac{1}{1-\sqrt{x}} dx &\approx \int_{\frac{1}{4}}^{\frac{1}{3}} (1 + \sqrt{x} + x + x^{3/2} + x^2) dx = \left(x + \frac{2}{3}x^{3/2} + \frac{x^2}{2} + \frac{2}{5}x^{5/2} + \frac{x^3}{3} \right) \bigg|_{\frac{1}{4}}^{\frac{1}{3}} \\ &= \left(\frac{1}{3} + \frac{2}{3}\left(\frac{1}{3}\right)^{3/2} + \frac{1}{18} + \frac{2}{5}\left(\frac{1}{3}\right)^{5/2} + \frac{1}{81} \right) - \left(\frac{1}{4} + \frac{1}{12} + \frac{1}{32} + \frac{1}{80} + \frac{1}{192} \right) \\ &\approx 0.1729.\end{aligned}$$

32. The area is given by

$$\int_0^1 e^{\sqrt{x}} dx.$$

Find the Taylor series for $e^{\sqrt{x}}$ by using the Taylor series for e^x and replacing x with $\sqrt{x}$.

$$e^{\sqrt{x}} = 1 + \sqrt{x} + \frac{\sqrt{x}^2}{2!} + \frac{\sqrt{x}^3}{3!} + \frac{\sqrt{x}^4}{4!} + \cdots + \frac{\sqrt{x}^n}{n!} + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned}\int_0^1 e^{\sqrt{x}} dx &\approx \int_0^1 \left(1 + \sqrt{x} + \frac{x}{2} + \frac{x^{3/2}}{6} + \frac{x^2}{24} \right) dx = \left(x + \frac{2}{3}x^{3/2} + \frac{x^2}{4} + \frac{x^{5/2}}{15} + \frac{x^3}{72} \right) \bigg|_0^1 \\ &= 1 + \frac{2}{3} + \frac{1}{4} + \frac{1}{15} + \frac{1}{72} - 0 \approx 1.9972\end{aligned}$$

33. The area is given by

$$\int_0^{0.4} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_0^{0.4} e^{-x^2/2} dx.$$

Find the Taylor series for $e^{-x^2/2}$ by using the Taylor series for e^x and replacing x with $-\frac{x^2}{2}$.

$$e^{-x^2/2} = 1 - \frac{1}{2}x^2 + \frac{1}{2!2^2}x^4 - \frac{1}{3!2^3}x^6 + \frac{1}{4!2^4}x^8 + \cdots + \frac{(-1)^n}{n!2^n}x^{2n} + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_0^{0.4} e^{-x^2/2} dx &\approx \frac{1}{\sqrt{2\pi}} \int_0^{0.4} \left(1 - \frac{1}{2}x^2 + \frac{1}{2!2^2}x^4 - \frac{1}{3!2^3}x^6 + \frac{1}{4!2^4}x^8 \right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{0.4} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 \right) dx \\
&= \frac{1}{\sqrt{2\pi}} \left(x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \frac{1}{336}x^7 + \frac{1}{3456}x^9 \right) \Big|_0^{0.4} \\
&= \frac{1}{\sqrt{2\pi}} \left(0.4 - \frac{1}{6}(0.4)^3 + \frac{1}{40}(0.4)^5 - \frac{1}{336}(0.4)^7 + \frac{1}{3456}(0.4)^9 - 0 \right) \\
&\approx \frac{1}{\sqrt{2(3.1416)}} (0.389585) \\
&\approx 0.1554.
\end{aligned}$$

34. The area is given by

$$\int_0^{0.6} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_0^{0.6} e^{-x^2/2} dx.$$

Find the Taylor series for $e^{-x^2/2}$ by using the Taylor series for e^x and replacing x with $-\frac{x^2}{2}$.

$$e^{-x^2/2} = 1 - \frac{1}{2}x^2 + \frac{1}{2!2^2}x^4 - \frac{1}{3!2^3}x^6 + \frac{1}{4!2^4}x^8 + \cdots + \frac{(-1)^n}{n!2^n}x^{2n} + \cdots$$

Using the first five terms of this series gives

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_0^{0.6} e^{-x^2/2} dx &\approx \frac{1}{\sqrt{2\pi}} \int_0^{0.6} \left(1 - \frac{1}{2}x^2 + \frac{1}{2!2^2}x^4 - \frac{1}{3!2^3}x^6 + \frac{1}{4!2^4}x^8 \right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{0.6} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 \right) dx \\
&= \frac{1}{\sqrt{2\pi}} \left(x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \frac{1}{336}x^7 + \frac{1}{3456}x^9 \right) \Big|_0^{0.6} \\
&= \frac{1}{\sqrt{2\pi}} \left(0.6 - \frac{1}{6}(0.6)^3 + \frac{1}{40}(0.6)^5 - \frac{1}{336}(0.6)^7 + \frac{1}{3456}(0.6)^9 - 0 \right) \\
&\approx \frac{1}{\sqrt{2(3.1416)}} (0.565864) \\
&\approx 0.2257.
\end{aligned}$$

35. The doubling time n for a quantity that increases at an annual rate r is given by

$$n = \frac{\ln 2}{\ln(1+r)} = \frac{\ln 2}{\ln 0.0475} \approx 14.94.$$

It will take about 14.94 years.

According to the Rule of 70, the doubling time is given by

$$\begin{aligned}
\text{Doubling time} &\approx \frac{70}{100r} = \frac{70}{100(0.0475)} \\
&= 14.74.
\end{aligned}$$

It will take about 14.74 years, a difference of 0.2 years, or about 10 weeks.

36. The doubling time n for a quantity that increases at an annual rate r is given by

$$n = \frac{\ln 2}{\ln(1+r)} = \frac{\ln 2}{\ln 1.06} \approx 11.896.$$

It will take about 11.9 years.

According to the Rule of 72, the doubling time is given by

$$\begin{aligned} \text{Doubling time} &\approx \frac{72}{100r} = \frac{72}{100(0.06)} \\ &= 12. \end{aligned}$$

It will take about 12 years, a difference of 0.1 years, or about 2 weeks.

$$\begin{aligned} 37. \quad (a) \quad \sum_{x=0}^{\infty} f(x) &= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \cdots + \frac{\lambda^n e^{-\lambda}}{n!} + \cdots \\ &= e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \cdots + \frac{\lambda^n e^{-\lambda}}{n!} + \cdots \\ &= e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots + \frac{\lambda^n}{n!} + \cdots \right) \\ &= e^{-\lambda} \cdot e^{\lambda} \\ &= e^0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} (b) \quad \sum_{x=0}^{\infty} x f(x) &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\ &= 0 \frac{\lambda^0 e^{-\lambda}}{0!} + 1 \frac{\lambda^1 e^{-\lambda}}{1!} + 2 \frac{\lambda^2 e^{-\lambda}}{2!} + 3 \frac{\lambda^3 e^{-\lambda}}{3!} + \cdots + n \frac{\lambda^n e^{-\lambda}}{n!} + \cdots \\ &= 0 + \lambda e^{-\lambda} + \lambda^2 e^{-\lambda} + \frac{\lambda^3 e^{-\lambda}}{2!} + \cdots + \frac{\lambda^n e^{-\lambda}}{(n-1)!} + \cdots \\ &= \lambda e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots + \frac{\lambda^{n-1}}{(n-1)!} + \cdots \right) \\ &= \lambda e^{-\lambda} \cdot e^{\lambda} \\ &= \lambda e^0 \\ &= \lambda \end{aligned}$$

- (c) Since 6.14 is the expected value from part b we have $\lambda = 6.14$.

$$\begin{aligned} P(x < 4) &= P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) \\ &= \frac{6.14^0 e^{-6.14}}{0!} + \frac{6.14^1 e^{-6.14}}{1!} + \frac{6.14^2 e^{-6.14}}{2!} + \frac{6.14^3 e^{-6.14}}{3!} \\ &= e^{-6.14} \left(1 + 6.14 + \frac{37.6996}{2} + \frac{231.475544}{6} \right) \\ &\approx 0.1391 \end{aligned}$$

38. (a)

$$\begin{aligned}\sum_{x=1}^{\infty} f(x) &= \sum_{x=1}^{\infty} (1-p)^{x-1} p = (1-p)^{1-1} p + (1-p)^{2-1} p + (1-p)^{3-1} p + (1-p)^{4-1} p + \cdots + (1-p)^{n-1} p + \cdots \\ &= p + p(1-p) + p(1-p)^2 + p(1-p)^3 + \cdots + p(1-p)^{n-1} + \cdots\end{aligned}$$

This is an infinite geometric series with $a = p$ and $r = (1-p)$. Thus the series sums to

$$\frac{a}{1-r} = \frac{p}{1-(1-p)} = \frac{p}{1-1+p} = \frac{p}{p} = 1.$$

(b) Let $g(z) = p \sum_{x=1}^{\infty} z^x = p(z + z^2 + z^3 + z^4 + \cdots + z^n + \cdots)$.

Then $g'(z) = p(1 + 2z + 3z^2 + 4z^3 + \cdots + nz^{n-1} + \cdots)$

and $g'(1-p) = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \cdots + n(1-p)^{n-1} + \cdots]$

$$= p[1(1-p)^{1-1} + 2(1-p)^{2-1} + 3(1-p)^{3-1} + 4(1-p)^{4-1} + \cdots + n(1-p)^{n-1} + \cdots]$$

$$= p \sum_{x=1}^{\infty} x(1-p)^{x-1} = \sum_{x=1}^{\infty} x(1-p)^{x-1} p = \sum_{x=1}^{\infty} x f(x).$$

Using the Taylor series for $\frac{1}{1-x}$, write $g(z)$ as

$$\begin{aligned}g(z) &= p \sum_{x=1}^{\infty} z^x = p(z + z^2 + z^3 + z^4 + \cdots + z^n + \cdots) = pz(1 + z + z^2 + z^3 + \cdots + z^{n-1} + \cdots) \\ &= pz \cdot \frac{1}{1-z} = \frac{zp}{1-z}.\end{aligned}$$

Then $g'(z) = \frac{(1-z)p - zp(-1)}{(1-z)^2} = \frac{p}{(1-z)^2}$, and $g'(1-p) = \frac{p}{[1-(1-p)]^2} = \frac{p}{p^2} = \frac{1}{p}$.

Thus, $\sum_{x=1}^{\infty} x f(x) = g'(1-p) = \frac{1}{p}$.

(c) The probability p of meeting a foreign-born player is $p = \frac{231}{833} = \frac{33}{119}$. The expected value is $\frac{1}{p} = \frac{833}{231} \approx 3.606$.

(d) $P(\text{meeting in first three trials}) = f(1) + f(2) + f(3)$

$$\begin{aligned}&= \left(1 - \frac{33}{119}\right)^{1-1} \left(\frac{33}{119}\right) + \left(1 - \frac{33}{119}\right)^{2-1} \left(\frac{33}{119}\right) + \left(1 - \frac{33}{119}\right)^{3-1} \left(\frac{33}{119}\right) \\ &\approx 0.6226.\end{aligned}$$

39. (a) The probability, p , of popping a 6 is $p = \frac{1}{6}$. From Exercise 31b, the expected value is given by

$$\sum_{x=1}^{\infty} x f(x) = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6.$$

(b) For $x \geq 4$, find

$$\begin{aligned}
\sum_{x=4}^{\infty} f(x) &= 1 - \sum_{x=1}^3 f(x) \\
&= 1 - \sum_{x=1}^3 (1-p)^{x-1} p \\
&= 1 - \sum_{x=1}^3 \left(1 - \frac{1}{6}\right)^{x-1} \left(\frac{1}{6}\right) \\
&= 1 - \left[\left(1 - \frac{1}{6}\right)^{1-1} \left(\frac{1}{6}\right) + \left(1 - \frac{1}{6}\right)^{2-1} \left(\frac{1}{6}\right) + \left(1 - \frac{1}{6}\right)^{3-1} \left(\frac{1}{6}\right) \right] \\
&\approx 1 - 0.4213 \\
&= 0.5787.
\end{aligned}$$

12.6 Newton's Method

Your Turn 1

$$f(x) = 2x^3 - 5x^2 + 6x - 10$$

First note that $f(1) = -7$ and $f(3) = 17$, so f has a zero on $[1, 3]$. Use $c_1 = 2$ as a first guess.

For Newton's method we need to know f' , which is $f'(x) = 6x^2 - 10x + 6$.

$$\begin{aligned}
c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \\
&= 2 - \frac{-2}{10} \\
&= 2.2
\end{aligned}$$

$$\begin{aligned}
c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \\
&= 2.2 - \frac{0.296}{13.04} \\
&\approx 2.177301
\end{aligned}$$

$$\begin{aligned}
c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} \\
&\approx 2.177301 - \frac{0.004207}{12.670828} \\
&\approx 2.176969
\end{aligned}$$

To three decimal places the root is 2.177. In fact c_4 is correct to six decimal places.

Your Turn 2

We find $\sqrt[3]{15}$ by finding a zero of the function

$f(x) = x^3 - 15$. $f'(x) = 3x^2$. Use 2.5 as the initial guess c_1 .

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} \\ &= 2.5 - \frac{0.625}{18.75} \\ &\approx 2.466667 \end{aligned}$$

$$\begin{aligned} c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} \\ &= 2.466667 - \frac{0.008302}{18.25334} \\ &\approx 2.466212 \end{aligned}$$

$\sqrt[3]{15} \approx 2.466$. In fact, c_3 is correct to six decimal places.

12.6 Exercises

1. $f(x) = 5x^2 - 3x - 3$
 $f'(x) = 10x - 3$

$f(1) = -1 < 0$ and $f(2) = 11 > 0$ so a solution exists in $(1, 2)$.

Let $c_1 = 1$.

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-1}{7} = 1.1429 \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} = 1.1429 - \frac{0.1020}{8.4286} = 1.1308 \\ c_4 &= c_3 - \frac{f(c_3)}{f'(c_3)} = 1.1308 - \frac{0.0007}{8.3075} = 1.1307 \end{aligned}$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 1.13$.

2. $f(x) = 2x^2 - 8x + 3$
 $f'(x) = 4x - 8$

$f(3) = -3 < 0$ and $f(4) = 3 > 0$ so a solution exists in $(3, 4)$.

Let $c_1 = 3$.

$$\begin{aligned} c_2 &= c_1 - \frac{f(c_1)}{f'(c_1)} = 3 - \frac{-3}{4} = 3.75 \\ c_3 &= c_2 - \frac{f(c_2)}{f'(c_2)} = 3.75 - \frac{1.125}{7} = 3.5893 \end{aligned}$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 3.5893 - \frac{0.0517}{6.3571} = 3.5812$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 3.5812 - \frac{0.0001}{6.3246} = 3.5811$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 3.58$.

3. $f(x) = 2x^3 - 6x^2 - x + 2$
 $f'(x) = 6x^2 - 12x - 1$

$f(3) = -1 < 0$ and $f(4) = 30 > 0$ so a solution exists in $(3, 4)$.

Let $c_1 = 3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 3 - \frac{-1}{17} = 3.0588$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 3.0588 - \frac{0.0419}{18.4325} = 3.0565$$

Subsequent approximations will agree with c_2 and c_3 to the nearest hundredth. Thus, $x = 3.06$.

4. $f(x) = -x^3 + 4x^2 - 5x + 4$
 $f'(x) = -3x^2 + 8x - 5$

$f(1) = 2 > 0$ and $f(3) = -2 < 0$ so a solution exists in $(2, 3)$.

Let $c_1 = 2$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{2}{-1} = 4$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 4 - \frac{-16}{-21} = 3.2381$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 3.2381 - \frac{-4.2017}{-10.5510} = 2.8399$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 2.8399 - \frac{-0.8430}{-6.4756} = 2.7097$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = 2.7097 - \frac{-0.0744}{-5.3497} = 2.6958$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus, $x = 2.70$.

5. $f(x) = -3x^3 + 5x^2 + 3x + 2$
 $f'(x) = -9x^2 + 10x + 3$

$f(2) = 4 > 0$ and $f(3) = -25 < 0$ so a solution exists in $(2, 3)$.

Let $c_1 = 2$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{4}{-13} = 2.3077$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 2.3077 - \frac{-1.3182}{-21.8521} = 2.2474$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 2.2474 - \frac{-0.0567}{-19.9824} = 2.2445$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = -2.2445 - \frac{-0.0001}{-19.8960} = 2.2445$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 2.24$.

6. $f(x) = 4x^3 - 5x^2 - 6x + 6$

$$f'(x) = 12x^2 - 10x - 6$$

$f(1) = -1 < 0$ and $f(2) = 6 > 0$ so a solution exists in $(1, 2)$.

Let $c_1 = 1$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-1}{-4} = 0.75$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 0.75 - \frac{0.375}{-6.75} = 0.8056$$

The approximations clearly are not leading to a solution in $(1, 2)$. (There is another solution to the equation in $[0, 1]$, $x = 0.81$.) Therefore, start again, this time letting $c_1 = 2$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{6}{22} = 1.7273$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.7273 - \frac{1.3324}{12.5298} = 1.6210$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.6210 - \frac{0.1734}{9.3217} = 1.6024$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 1.6024 - \frac{0.0050}{8.7882} = 1.6018$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 1.60$.

7. $f(x) = 2x^4 - 2x^3 - 3x^2 - 5x - 8$

$$f'(x) = 8x^3 - 6x^2 - 6x - 5$$

In the interval $[-2, -1]$: $f(-2) = 38 > 0$ and $f(-1) = -2 < 0$ so a solution exists in $(-2, -1)$.

Let $c_1 = -2$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = -2 - \frac{38}{-81} = -1.5309$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -1.5309 - \frac{10.7834}{-38.5773} = -1.2513$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -1.2513 - \frac{2.3817}{-22.5622} = -1.1458$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = -1.1458 - \frac{0.2457}{-18.0356} = -1.1322$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = -1.1322 - \frac{0.0036}{-17.5069} = -1.1319$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus, $x = -1.13$ is a solution in $[-2, -1]$.

In the interval $[2, 3]$: $f(2) = -14 < 0$ and

$f(3) = 58 > 0$ so a solution exists in $(2, 3)$.

Let $c_1 = 2$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{-14}{23} = 2.6087$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 2.6087 - \frac{15.6588}{80.5396} = 2.4143$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 2.4143 - \frac{2.2460}{58.1188} = 2.3756$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 2.3756 - \frac{0.0774}{54.1413} = 2.3742$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = 2.3742 - \frac{0.0001}{53.9972} = 2.3742$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus, $x = 2.37$ is a solution in $[2, 3]$.

8. $f(x) = 3x^4 + 4x^3 - 6x^2 - 2x - 12$

$$f'(x) = 12x^3 + 12x^2 - 12x - 2$$

In the interval $[-3, -2]$: $f(-3) = 75 > 0$ and $f(-2) = -16 < 0$ so a solution exists in $(-3, -2)$.

Let $c_1 = -3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = -3 - \frac{75}{-182} = -2.5879$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -2.5879 - \frac{18.2141}{-98.5428} = -2.4030$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -2.4030 - \frac{2.6873}{-65.2105} = -2.3633$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = -2.3633 - \frac{0.0005}{-65.0118} = -2.3633$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = -2.36$ is a solution in $[-3, -2]$.

In the interval $[1, 2]$: $f(1) = -13 < 0$ and $f(2) = 40 > 0$ so a solution exists in $(1, 2)$.

Let $c_1 = 1$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-13}{10} = 2.3$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 2.3 - \frac{84.2803}{179.884} = 1.8315$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.8315 - \frac{22.5408}{89.9975} = 1.5810$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 1.5810 - \frac{4.3913}{56.4444} = 1.5032$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = 1.5032 - \frac{0.3400}{47.8367} = 1.4961$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus, $x = 1.50$ is a solution in $[1, 2]$.

9. $f(x) = 4x^{1/3} - 2x^2 + 4$
 $f'(x) = \frac{4}{3}x^{-2/3} - 4x$
 $f(-3) = -19.7690 < 0$ and $f(0) = 4 > 0$
 so a solution exists in $(-3, 0)$.
 Let $c_1 = -3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = -3 - \frac{-19.7690}{12.6410} = -1.4361$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -1.4361 - \frac{-4.6378}{6.7920} = -0.7533$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -0.7533 - \frac{-0.7744}{4.6236} = -0.5858$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = -0.5858 - \frac{-0.0332}{4.2477} = -0.5780$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = -0.5780 - \frac{-0.0001}{4.2335} = -0.5780$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus, $x = -0.58$.

10. $f(x) = 4x^{1/3} - 2x^2 + 4$
 $f'(x) = \frac{4}{3}x^{-2/3} - 4x$
 $f(0) = 4 > 0$ and $f(3) = -8.2310 < 0$ so a solution exists in $(0, 3)$. Notice that $f'(0)$ is undefined. Therefore, instead of choosing 0 for c_1 , let $c_1 = 3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 3 - \frac{-8.2310}{-11.3590} = 2.2754$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 2.2754 - \frac{-1.0938}{-8.3309} = 2.1441$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 2.1441 - \frac{-0.0364}{-7.7745} = 2.1394$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 2.14$.

11. $f(x) = e^x + x - 2$
 $f'(x) = e^x + 1$
 $f(0) = -1 < 0$ and $f(3) = 21.085537$ so a solution exists in $(0, 3)$.

Let $c_1 = 0$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0 - \frac{-1}{2} = 0.5$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 0.5 - \frac{0.14872}{2.6487} = 0.4439$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 0.4439 - \frac{0.00267}{2.5588} = 0.4429$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 0.44$.

12. $f(x) = e^{2x} + 3x - 4$
 $f'(x) = 2e^{2x} + 3$
 $f(0) = -3 < 0$ and $f(3) = 408.42879 > 0$
 so a solution exists in $(0, 3)$.

Let $c_1 = 0$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0 - \frac{-3}{5} = 0.6$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 0.6 - \frac{1.1201}{9.6402} = 0.4838$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 0.4838 - \frac{0.08302}{8.2632} = 0.4738$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 0.4738 - \frac{9.1 \cdot 10^{-4}}{8.159} = 0.4737$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 0.47$.

13. $f(x) = x^2 e^{-x} + x^2 - 2$

$$f'(x) = -x^2 e^{-x} + 2xe^{-x} + 2x$$

$f(0) = -2$ and $f(3) = 7.4480836$ so a solution exists in $(0, 3)$.

Let $c_1 = 0$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0 - \frac{-2}{0}$$

Since c_2 is undefined, let $c_1 = 1$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-0.6321}{2.3679} = 1.2670$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.2670 - \frac{0.05746}{2.7956} = 1.2464$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.2464 - \frac{2.1 \cdot 10^{-4}}{2.7629} = 1.2463$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 1.25$.

14. $f(x) = x^2 e^{-x} + x^2 - 2$

$$f'(x) = -x^2 e^{-x} + 2xe^{-x} + 2x$$

$f(-3) = 187.76983 > 0$ and $f(0) = -2 < 0$ so a solution exists in $(-3, 0)$.

Let $c_1 = -3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = -3 - \frac{187.77}{-307.3} = -2.3890$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -2.3890 - \frac{65.932}{-119.1} = -1.8354$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -1.8354 - \frac{22.482}{-47.79} = -1.3650$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = -1.3650 - \frac{7.1591}{-20.72} = -1.0195$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = -1.0195 - \frac{1.9203}{-10.57} = -0.08378$$

$$c_7 = c_6 - \frac{f(c_6)}{f'(c_6)} = -0.8378 - \frac{0.32421}{-7.171} = -0.7926$$

$$c_8 = c_7 - \frac{f(c_7)}{f'(c_7)} = -0.7926 - \frac{0.01602}{-6.475} = -0.7901$$

Subsequent approximations will agree with c_7 and c_8 to the nearest hundredth. Thus, $x = -0.79$.

15. $f(x) = \ln x + x - 2$

$$f'(x) = \frac{1}{x} + 1$$

$f(1) = -1 < 0$ and $f(4) = 3.3862944 > 0$ so a solution exists in $(1, 4)$.

Let $c_1 = 1$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-1}{2} = 1.5$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.5 - \frac{-0.0945}{1.6667} = 1.5567$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.5567 - \frac{-7 \cdot 10^{-4}}{1.6424} = 1.5571$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 1.56$.

16. $f(x) = 2 \ln x + x - 3$

$$f'(x) = \frac{2}{x} + 1$$

$f(1) = -2 < 0$ and $f(4) = 3.7725887 > 0$ so a solution exists in $(1, 4)$.

Let $c_1 = 1$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{-2}{3} = 1.6667$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.6667 - \frac{-0.3116}{2.2} = 1.8083$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.8083 - \frac{-0.0069}{2.106} = 1.8116$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 1.81$.

17. $\sqrt{2}$ is a solution of $x^2 - 2 = 0$.

$$f(x) = x^2 - 2$$

$$f'(x) = 2x$$

Since $1 < \sqrt{2} < 2$, let $c_1 = 1$.

$$c_2 = 1 - \frac{-1}{2} = 1.5$$

$$c_3 = 1.5 - \frac{0.25}{3} = 1.417$$

$$c_4 = 1.417 - \frac{0.00789}{2.834} = 1.414$$

$$c_5 = 1.414 - \frac{-6 \cdot 10^{-4}}{2.828} = 1.414$$

Since $c_4 = c_5 = 1.414$, to the nearest thousandth, $\sqrt{2} = 1.414$.

18. $\sqrt{3}$ is a solution of $x^2 - 3 = 0$.

$$f(x) = x^2 - 3$$

$$f'(x) = 2x$$

Since $1 < \sqrt{3} < 2$, let $c_1 = 1$.

$$c_2 = 1 - \frac{-2}{2} = 2$$

$$c_3 = 2 - \frac{1}{4} = 1.75$$

$$c_4 = 1.75 - \frac{0.0625}{3.5} = 1.732$$

$$c_5 = 1.732 - \frac{-2 \cdot 10^{-4}}{3.464} = 1.732$$

Since $c_4 = c_5 = 1.732$, to the nearest thousandth, $\sqrt{3} = 1.732$.

19. $\sqrt{11}$ is a solution of $x^2 - 11 = 0$.

$$f(x) = x^2 - 11$$

$$f'(x) = 2x$$

Since $3 < \sqrt{11} < 4$, let $c_1 = 3$.

$$c_2 = 3 - \frac{-2}{6} = 3.333$$

$$c_3 = 3.333 - \frac{0.10889}{6.666} = 3.317$$

$$c_4 = 3.317 - \frac{0.00249}{6.634} = 3.317$$

Since $c_3 = c_4 = 3.317$, to the nearest thousandth, $\sqrt{11} = 3.317$.

20. $\sqrt{15}$ is a solution of $x^2 - 15 = 0$.

$$f(x) = x^2 - 15$$

$$f'(x) = 2x$$

Since $3 < \sqrt{15} < 4$, let $c_1 = 3$.

$$c_2 = 3 - \frac{-6}{6} = 4$$

$$c_3 = 4 - \frac{1}{8} = 3.875$$

$$c_4 = 3.875 - \frac{0.01563}{7.75} = 3.873$$

$$c_5 = 3.873 - \frac{1.3 \cdot 10^{-4}}{7.746} = 3.873$$

Since $c_4 = c_5 = 3.873$, to the nearest thousandth, $\sqrt{15} = 3.873$.

21. $\sqrt{250}$ is a solution of $x^2 - 250 = 0$.

$$f(x) = x^2 - 250$$

$$f'(x) = 2x$$

Since $15 < \sqrt{250} < 16$, let $c_1 = 15$.

$$c_2 = 15 - \frac{-25}{30} = 15.833$$

$$c_3 = 15.833 - \frac{0.68389}{31.666} = 15.811$$

$$c_4 = 15.811 - \frac{-0.0123}{31.622} = 15.811$$

Since $c_3 = c_4 = 15.811$, to the nearest thousandth, $\sqrt{250} = 15.811$.

22. $\sqrt{300}$ is a solution of $x^2 - 300 = 0$.

$$f(x) = x^2 - 300$$

$$f'(x) = 2x$$

Since $17 < \sqrt{300} < 18$, let $c_1 = 17$.

$$c_2 = 17 - \frac{-11}{34} = 17.324$$

$$c_3 = 17.324 - \frac{0.12098}{34.648} = 17.321$$

$$c_4 = 17.321 - \frac{0.01704}{34.642} = 17.321$$

Since $c_3 = c_4 = 17.321$, to the nearest thousandth, $\sqrt{300} = 17.321$.

23. $\sqrt[3]{9}$ is a solution of $x^3 - 9 = 0$.

$$f(x) = x^3 - 9$$

$$f'(x) = 3x^2$$

Since $2 < \sqrt[3]{9} < 3$, let $c_1 = 2$.

$$c_2 = 2 - \frac{-1}{12} = 2.083$$

$$c_3 = 2.083 - \frac{0.03791}{13.017} = 2.080$$

$$c_4 = 2.080 - \frac{-0.0011}{12.979} = 2.080$$

Since $c_3 = c_4 = 2.080$, to the nearest thousandth, $\sqrt[3]{9} = 2.080$.

24. $\sqrt[3]{15}$ is a solution of $x^3 - 15 = 0$.

$$f(x) = x^3 - 15$$

$$f'(x) = 3x^2$$

Since $2 < \sqrt[3]{15} < 3$, let $c_1 = 2$.

$$c_2 = 2 - \frac{-7}{12} = 2.583$$

$$c_3 = 2.583 - \frac{2.2335}{20.016} = 2.471$$

$$c_4 = 2.471 - \frac{0.08753}{18.318} = 2.466$$

$$c_5 = 2.466 - \frac{-0.0039}{18.243} = 2.466$$

Since $c_4 = c_5 = 2.466$, to the nearest thousandth, $\sqrt[3]{15} = 2.466$.

25. $\sqrt[3]{100}$ is a solution of $x^3 - 100 = 0$.

$$f(x) = x^3 - 100$$

$$f'(x) = 3x^2$$

Since $4 < \sqrt[3]{100} < 5$, let $c_1 = 4$.

$$c_2 = 4 - \frac{-36}{48} = 4.75$$

$$c_3 = 4.75 - \frac{7.1719}{67.688} = 4.644$$

$$c_4 = 4.644 - \frac{0.15592}{64.7} = 4.642$$

$$c_5 = 4.642 - \frac{0.02658}{64.644} = 4.642$$

Since $c_4 = c_5 = 4.642$, to the nearest thousandth, $\sqrt[3]{100} = 4.642$.

26. $\sqrt[3]{121}$ is a solution of $x^3 - 121 = 0$.

$$f(x) = x^3 - 121$$

$$f'(x) = 3x^2$$

Since $4 < \sqrt[3]{121} < 5$, let $c_1 = 4$.

$$c_2 = 4 - \frac{-57}{48} = 5.188$$

$$c_3 = 5.188 - \frac{18.637}{80.746} = 4.957$$

$$c_4 = 4.957 - \frac{0.80266}{73.716} = 4.946$$

$$c_5 = 4.946 - \frac{-0.0064}{73.389} = 4.946$$

Since $c_4 = c_5 = 4.946$, to the nearest thousandth, $\sqrt[3]{121} = 4.946$.

27. $f(x) = x^3 - 3x^2 - 18x + 4$
 $f'(x) = 3x^2 - 6x - 18$

To find critical points, solve

$$f'(x) = 3x^2 - 6x - 18 = 0.$$

$$f''(x) = 6x - 6$$

$f'(-2) = 6 > 0$ and $f'(-1) = -9 < 0$ so a solution exists in $(-2, -1)$.

Let $c_1 = -2$.

$$c_2 = -2 - \frac{6}{-18} = -1.67$$

$$c_3 = -1.67 - \frac{0.3867}{-16.02} = -1.65$$

$$c_4 = -1.65 - \frac{0.0675}{-15.9} = -1.65$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = -1.65$. Since $f''(-1.65) = -15.9 < 0$, the graph has a relative maximum at $x = -1.65$.

$f'(3) = -9 < 0$ and $f'(4) = 6 > 0$ so a solution exists in $(3, 4)$.

Let $c_1 = 3$.

$$c_2 = 3 - \frac{-9}{12} = 3.75$$

$$c_3 = 3.75 - \frac{1.6875}{16.5} = 3.65$$

$$c_4 = 3.65 - \frac{0.0675}{15.9} = 3.65$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 3.65$. Since $f''(3.65) = 15.9 > 0$, the graph has a relative minimum at $x = 3.65$.

$$28. \quad f(x) = x^3 + 9x^2 - 6x + 4$$

$$f'(x) = 3x^2 + 18x - 6$$

To find critical points, solve

$$f'(x) = 3x^2 + 18x - 6 = 0.$$

$$f''(x) = 6x + 18$$

$f'(-7) = 15 > 0$ and $f'(-6) = -6 < 0$ so a solution exists in $(-7, -6)$.

Let $c_1 = -7$.

$$c_2 = -7 - \frac{15}{-24} = -6.38$$

$$c_3 = -6.38 - \frac{1.2732}{-20.28} = -6.32$$

$$c_4 = -6.32 - \frac{0.0672}{-19.92} = -6.32$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = -6.32$. Since $f''(-6.32) = -19.92 < 0$, the graph has a relative maximum at $x = -6.32$.

$f'(0) = -6 < 0$ and $f'(1) = 15 > 0$ so a solution exists in $(0, 1)$.

Let $c_1 = 0$.

$$c_2 = 0 - \frac{-6}{18} = 0.33$$

$$c_3 = 0.33 - \frac{0.2667}{19.98} = 0.32$$

$$c_4 = 0.32 - \frac{0.0672}{19.92} = 0.32$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 0.32$. Since $f''(0.32) = 19.92 > 0$, the graph has a relative minimum at $x = 0.32$.

$$29. \quad f(x) = x^4 - 3x^3 + 6x - 1$$

$$f'(x) = 4x^3 - 9x^2 + 6$$

To find critical points, solve

$$f'(x) = 4x^3 - 9x^2 + 6 = 0.$$

$$f''(x) = 12x^2 - 18x$$

$f'(-1) = -7 < 0$ and $f'(0) = 6 > 0$ so a solution exists in $(-1, 0)$.

Let $c_1 = -1$.

$$c_2 = -1 - \frac{-7}{30} = -0.77$$

$$c_3 = -0.77 - \frac{-1.162}{20.975} = -0.71$$

$$c_4 = -0.71 - \frac{0.03146}{18.829} = -0.71$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = -0.71$. Since $f''(-0.71) = 18.829 > 0$, the graph has a relative minimum at $x = -0.71$.

$f'(1) = 1 > 0$ and $f'(1.5) = -0.75 < 0$ so a solution exists in $(1, 1.5)$.

Let $c_1 = 1$.

$$c_2 = 1 - \frac{1}{-6} = 1.17$$

$$c_3 = 1.17 - \frac{0.08635}{-4.633} = 1.19$$

$$c_4 = 1.19 - \frac{-0.00043}{-4.427} = 1.19$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 1.19$. Since $f''(1.19) = -4.427 < 0$, the graph has a relative maximum at $x = 1.19$.

$f'(1.5) = -0.75 < 0$ and $f'(2) = 2 > 0$ so a solution exists in $(1.5, 2)$.

Let $c_1 = 1.5$.

$$c_2 = 1.5 - \frac{-0.75}{0}$$

Since c_2 is undefined, let $c_1 = 1.6$.

$$c_2 = 1.6 - \frac{-0.656}{1.92} = 1.94$$

$$c_3 = 1.94 - \frac{1.3331}{10.243} = 1.81$$

$$c_4 = 1.81 - \frac{0.23406}{6.7332} = 1.78$$

$$c_5 = 1.78 - \frac{0.04341}{5.9808} = 1.77$$

$$c_6 = 1.77 - \frac{-0.0152}{5.7348} = 1.77$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus, $x = 1.77$.
 Since $f''(1.77) = 5.7348 > 0$, the graph has a relative minimum at $x = 1.77$.

$$30. \quad f(x) = x^4 + 2x^3 - 5x + 2$$

$$f'(x) = 4x^3 + 6x^2 - 5$$

To find critical points, solve

$$f'(x) = 4x^3 + 6x^2 - 5 = 0.$$

$$f''(x) = 12x^2 + 12x$$

$f'(0) = -5 < 0$ and $f'(1) = 5 > 0$ so a solution exists in $(0, 1)$.

Let $c_1 = 0$.

$$c_2 = 0 - \frac{-5}{0}$$

Since c_2 is undefined, let $c_1 = 0.5$.

$$c_2 = 0.5 - \frac{-3}{9} = 0.83$$

$$c_3 = 0.83 - \frac{1.4205}{18.227} = 0.75$$

$$c_4 = 0.75 - \frac{0.0625}{15.75} = 0.75$$

Subsequent approximations will agree with c_3 and c_4 to the nearest hundredth. Thus, $x = 0.75$.
 Since $f''(0.75) = 15.75 > 0$, the graph has a relative minimum at $x = 0.75$.

$$31. \quad f(x) = (x - 1)^{1/3}$$

$$f'(x) = \frac{1}{3}(x - 1)^{-2/3}$$

$f(0) = -1 < 0$ and $f(2) = 1 > 0$ so a solution exists in $(0, 2)$.

Let $c_1 = 0$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 0 - \frac{-1}{0.3333} = 3$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 3 - \frac{1.2599}{0.2100} = -3$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -3 - \frac{-1.5874}{0.1323} = 9$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 9 - \frac{2}{0.0833} = -15$$

Not only are successive approximations alternating in sign but they are moving further and further apart. Thus, the approximations are not approaching any specific value. The method fails in this case because the derivative, $f'(x) = \frac{1}{3}(x - 1)^{-2/3}$, is undefined at $x = 1$; the function has a vertical tangent line there.

32. Break-even occurs when $R(x) = C(x)$, so solve $R(x) - C(x) = 0$.

$$f(x) = R(x) - C(x) = 10x^{2/3} - (2x - 9)$$

$$= 10x^{2/3} - 2x + 9$$

$$f'(x) = \frac{20}{3}x^{-1/3} - 2$$

Graphing the function $f(x)$ on a graphing calculator will help locate a value for c_1 . With a little trial-and-error, you should see that $f(x)$ intersects the x -axis around 138.

$$f(138) = 0.0460 > 0 \text{ and}$$

$$f(139) = -0.6655 < 0 \text{ so a solution exists in } (138, 139). \text{ Let } c_1 = 138.$$

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 138 - \frac{0.0460}{-0.7099} = 138.06$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 138.06 - \frac{0.0034}{-0.7101} = 138.06$$

Subsequent approximations will agree with c_2 and c_3 to the nearest hundredth. Thus, $x = 138.06$.
 The break-even point for the product is 138.06 units.

33. The process should be used at least until the savings produced is equal to the increased costs incurred, costs incurred, or when $S(x) = C(x)$. Therefore, solve $S(x) - C(x)$.

$$\begin{aligned} f(x) &= S(x) - C(x) \\ &= (x^3 + 5x^2 + 9) - (x^2 + 40x + 20) \\ &= x^3 + 4x^2 - 40x - 11 \\ f'(x) &= 3x^2 + 8x - 40 \\ f'(x) &= 3x^2 + 8x - 40 \end{aligned}$$

$f(4) = -43 < 0$ and $f(5) = 14 > 0$ so a solution exists in $(4, 5)$.

Let $c_1 = 4$.

$$\begin{aligned} c_2 &= 4 - \frac{-43}{40} = 5.08 \\ c_3 &= 5.08 - \frac{20.122}{78.059} = 4.82 \\ c_4 &= 4.82 - \frac{1.1098}{68.257} = 4.80 \\ c_5 &= 4.80 - \frac{-0.248}{67.52} = 4.80 \end{aligned}$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus, $x = 4.80$. The process should be used for at least 4.80 years.

34. (a) $f(i) = \frac{1 - (1+i)^{-n}}{i} - \frac{P}{M}$

$$\begin{aligned} f'(i) &= \frac{i[n(1+i)^{-n-1}] - [1 - (1+i)^{-n}]}{i^2} \\ &= \frac{ni(1+i)^{-n-1} - 1 + (1+i)^{-n}}{i^2} \\ &= \frac{-1 + ni(1+i)^{-n-1} + (1+i)^{-n}}{i^2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{f(i)}{f'(i)} &= \left[\frac{1 - (1+i)^{-n}}{i} - \frac{P}{M} \right] \div \left[\frac{-1 + ni(1+i)^{-n-1} + (1+i)^{-n}}{i^2} \right] \\ &= \frac{M - M(1+i)^{-n} - Pi}{Mi} \cdot \frac{i^2}{-1 + ni(1+i)^{-n-1} + (1+i)^{-n}} \\ &= \frac{Mi - Mi(1+i)^{-n} - Pi^2}{M[-1 + ni(1+i)^{-n-1} + (1+i)^{-n}]} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad i_2 &= 0.01 - \frac{197(0.01) - 197(0.01)(1+0.01)^{-24} - 4000(0.01)^2}{197[-1 + (24)(0.01)(1+0.01)^{-24-1} + (1+0.01)^{-24}]} = 0.01 - \frac{0.018494729}{-4.982020913} \\ &= 0.013712295 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad i_3 &= 0.01371229 - \frac{197(0.01371229) - 197(0.01371229)(1 + 0.01371229)^{-24} - 4000(0.01371229)^2}{197[-1 + (24)(0.01371229)(1 + 0.01371229)^{-24-1} + (1 + 0.01371229)^{-24}]} \\
 &= 0.01371229 - \frac{0.0010602245}{-8.803254706} \\
 &= 0.01383273
 \end{aligned}$$

$$\begin{aligned}
 35. \quad i_2 &= 0.02 - \frac{57(0.02) - 57(0.02)(1 + 0.02)^{-12} - 600(0.02)^2}{57[-1 + (12)(0.02)(1 + 0.02)]^{-12-1} + (1 + 0.02)^{-12}} \\
 &= 0.02 - \frac{0.0011177798}{-1.480804049} \\
 &= 0.02075485
 \end{aligned}$$

$$\begin{aligned}
 i_3 &= 0.02075485 - \frac{57(0.02075485) - 57(0.02075485)(1 + 0.02075485)^{-12} - 600(0.02075485)^2}{57[-1 + (12)(0.02075485)(1 + 0.02075485)^{-12-1} + (1 + 0.02075485)^{-12}]} \\
 &= 0.02075485 - \frac{4.0638916 \cdot 10^{-6}}{-1.583924743} \\
 &= 0.02075742
 \end{aligned}$$

$$\begin{aligned}
 36. \quad i_2 &= 0.01 - \frac{337(0.01) - 337(0.01)(1 + 0.01)^{-60} - 15,000(0.01)^2}{337[-1 + (60)(0.01)(1 + 0.01)^{-60-1} + (1 + 0.01)^{-60}]} \\
 &= 0.01 - \frac{0.0149847943}{-41.29955632} \\
 &= 0.01036283
 \end{aligned}$$

$$\begin{aligned}
 i_3 &= 0.01036283 - \frac{337(0.01036283) - 337(0.01036283)(1 + 0.01036283)^{-60} - 15,000(0.01036283)^2}{337[-1 + (60)(0.01036283)(1 + 0.01036283)^{-60-1} + (1 + 0.01036283)^{-60}]} \\
 &= 0.01036283 - \frac{1.128666375 \cdot 10^{-4}}{-43.73087288} \\
 &= 0.01036541
 \end{aligned}$$

12.7 L'Hospital's Rule

Your Turn 1

Find $\lim_{x \rightarrow 4} \frac{3x - 12}{\sqrt[3]{x + 4} - 2}$.

First note that $\lim_{x \rightarrow 4} 3x - 12 = 0$ and

$\lim_{x \rightarrow 4} \sqrt[3]{x + 4} - 2 = 0$, so the conditions of l'Hospital's

rule are satisfied. Differentiate the numerator and denominator.

For $f(x) = 3x - 12$, $f'(x) = 3$.

For $g(x) = \sqrt[3]{x + 4} - 2$, $g'(x) = \frac{1}{3}(x + 4)^{-2/3}$.

$$\begin{aligned}
 \lim_{x \rightarrow 4} \frac{f'(x)}{g'(x)} &= \lim_{x \rightarrow 4} \frac{3}{(1/3)(x + 4)^{-2/3}} \\
 &= \frac{3}{(1/3)(8)^{-2/3}} \\
 &= 36
 \end{aligned}$$

By l'Hospital's rule, $\lim_{x \rightarrow 4} \frac{3x - 12}{\sqrt[3]{x + 4} - 2} = 36$.

Your Turn 2

Find $\lim_{x \rightarrow 1} \frac{e^{x-1} - 1}{x^2 - 2x + 1}$.

First note that $\lim_{x \rightarrow 1} e^{x-1} - 1 = 0$ and $\lim_{x \rightarrow 1} x^2 - 2x + 1$

$= 0$, so the conditions of l'Hospital's rule are satisfied.

Differentiate the numerator and denominator.

For $f(x) = e^{x-1} - 1$, $f'(x) = e^{x-1}$.

For $g(x) = x^2 - 2x + 1$, $g'(x) = 2x - 2$.

$$\lim_{x \rightarrow 1} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 1} \frac{e^{x-1}}{2x - 2}$$

The numerator has limit 1 but the denominator has limit 0, so the limit of the quotient does not exist.

By l'Hospital's rule, $\lim_{x \rightarrow 1} \frac{e^{x-1} - 1}{x^2 - 2x + 1}$ does not exist.

Your Turn 3

Find $\lim_{x \rightarrow 0} \frac{x^3}{\ln(x+1)}$.

First note that $\lim_{x \rightarrow 0} x^3 = 0$ and $\lim_{x \rightarrow 0} \ln(x+1) = 0$, so

the conditions of l'Hospital's rule are satisfied. Differentiate the numerator and denominator.

For $f(x) = x^3$, $f'(x) = 3x^2$.

For $g(x) = \ln(x+1)$, $g'(x) = \frac{1}{x+1}$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} &= \lim_{x \rightarrow 0} \frac{3x^2}{\frac{1}{x+1}} \\ &= \lim_{x \rightarrow 0} [3x^2(x+1)] \\ &= 0 \end{aligned}$$

By l'Hospital's rule, $\lim_{x \rightarrow 0} \frac{x^3}{\ln(x+1)} = 0$.

Your Turn 4

Find $\lim_{x \rightarrow 0} \frac{e^{3x} - \frac{9}{2}x^2 - 3x - 1}{x^3}$.

First note that $\lim_{x \rightarrow 0} e^{3x} - \frac{9}{2}x^2 - 3x - 1 = 1 - 1 = 0$,

and $\lim_{x \rightarrow 0} x^3 = 0$, so the conditions of l'Hospital's rule are satisfied. Differentiate the numerator and denominator.

For $f(x) = e^{3x} - \frac{9}{2}x^2 - 3x - 1$,

$$f'(x) = 3e^{3x} - 9x - 3.$$

For $g(x) = x^3$, $g'(x) = 3x^2$.

$$\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0} \frac{3e^{3x} - 9x - 3}{3x^2}$$

Both the numerator and denominator have limit equal to 0, so we apply l'Hospital's rule again with

$$f(x) = 3e^{3x} - 9x - 3 \quad \text{and} \quad g(x) = 3x^2.$$

$$f'(x) = 9e^{3x} - 9$$

$$g'(x) = 6x$$

We see that both f' and g' still have limit 0 as $x \rightarrow 0$, so we need one more application of l'Hospital's rule, now with $f(x) = 9e^{3x} - 9$ and $g(x) = 6x$.

$$f'(x) = 27e^{3x}$$

$$g'(x) = 6$$

$$\begin{aligned} \text{Thus } \lim_{x \rightarrow 0} \frac{e^{3x} - \frac{9}{2}x^2 - 3x - 1}{x^3} &= \lim_{x \rightarrow 0} \frac{27e^{3x}}{6} \\ &= \frac{27}{6} \\ &= \frac{9}{2}. \end{aligned}$$

Your Turn 5

Find $\lim_{x \rightarrow 0^+} x^2 \ln(3x)$.

Write this as

$$\lim_{x \rightarrow 0^+} \frac{\ln(3x)}{\frac{1}{x^2}}.$$

Now both numerator and denominator become infinite in magnitude as $x \rightarrow 0$, so we can apply l'Hospital's rule, differentiating the numerator and the denominator.

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln(3x)}{\frac{1}{x^2}} &= \lim_{x \rightarrow 0^+} \frac{3/x}{-2/x^3} \\ &= \lim_{x \rightarrow 0^+} -\frac{3}{2}x^2 = 0 \end{aligned}$$

Thus $\lim_{x \rightarrow 0^+} x^2 \ln(3x) = 0$.

Your Turn 6

Find $\lim_{x \rightarrow \infty} \frac{\ln x}{e^x}$ and $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2}$.

Each expression is a quotient in which both the numerator and denominator tend to infinity as $x \rightarrow \infty$, so we can apply l'Hospital's rule.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln x}{e^x} &= \lim_{x \rightarrow \infty} \frac{1/x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{xe^x} = 0. \\ \lim_{x \rightarrow \infty} \frac{\ln x}{x^2} &= \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0. \end{aligned}$$

12.7 Exercises

1. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 1} \frac{3x^2 + 2x - 1}{2x - 1} = \frac{3(1)^2 + 2(1) - 1}{2(1) - 1} = 4.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 1} \frac{x^3 + x^2 - x - 1}{x^2 - x} = 4.$$

2. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 3} \frac{3x^2 + 2x - 11}{2x - 3} = \frac{3(3)^2 + 2(3) - 11}{2(3) - 3} = \frac{22}{3}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 3} \frac{x^3 + x^2 - 11x - 3}{x^2 - 3x} = \frac{22}{3}.$$

3. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{5x^4 - 6x^2 + 8x}{40x^4 - 4x + 5} = \frac{5(0)^4 - 6(0)^2 + 8(0)}{40(0)^4 - 4(0) + 5} = 0.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{x^5 - 2x^3 + 4x^2}{8x^5 - 2x^2 + 5x} = 0.$$

4. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{48x^5 + 12x^3 - 9}{63x^6 - 8x^3 + 3x^2} \text{ which does not exist.}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{8x^6 + 3x^4 - 9x}{9x^7 - 2x^4 + x^3} \text{ does not exist.}$$

5. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 2} \frac{\frac{1}{x-1}}{1} = \frac{1}{2-1} = 1.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 2} \frac{\ln(x-1)}{x-2} = 1.$$

6. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{\frac{1}{x+1}}{1} = \frac{1}{0+1} = 1.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1.$$

7. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{e^x}{4x^3} \text{ which does not exist.}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x^4} \text{ does not exist.}$$

8. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{2e^{2x}}{10x - 1} = \frac{2e^{2(0)}}{10(0) - 1} = -2.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{5x^2 - x} = -2.$$

9. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{e^x + xe^x}{e^x} = \frac{e^0 + 0 \cdot e^0}{e^0} = 1.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{xe^x}{e^x - 1} = 1.$$

10. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{e^{-x} - xe^{-x}}{4e^{2x}} = \frac{e^0 - 0 \cdot e^0}{4e^{2(0)}} = \frac{1}{4}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{xe^{-x}}{2e^{2x} - 2} = \frac{1}{4}.$$

11. $\lim_{x \rightarrow 0} e^x = 1$ and l'Hospital's rule does not apply.

However,

$$\lim_{x \rightarrow 0} \frac{e^x}{2x^3 + 9x^2 - 11x} \text{ does not exist.}$$

12. $\lim_{x \rightarrow 0} e^x = 1$ and l'Hospital's rule does not apply.

However

$$\lim_{x \rightarrow 0} \frac{e^x}{8x^5 - 3x^4} \text{ does not exist.}$$

13. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{\frac{1}{2(2+x)^{1/2}}}{1} = \frac{1}{2(2)^{1/2}} = \frac{1}{2\sqrt{2}} \text{ or } \frac{\sqrt{2}}{4}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \frac{1}{2\sqrt{2}} \text{ or } \frac{\sqrt{2}}{4}.$$

14. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{\frac{1}{2(9+x)^{1/2}}}{1} = \frac{1}{2(9+0)^{1/2}} = \frac{1}{6}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\sqrt{9+x} - 3}{x} = \frac{1}{6}.$$

15. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 4} \frac{\frac{1}{2x^{1/2}}}{1} = \frac{1}{2 \cdot 4^{1/2}} = \frac{1}{4}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{1}{4}.$$

16. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 9} \frac{\frac{1}{2x^{1/2}}}{1} = \frac{1}{2 \cdot 9^{1/2}} = \frac{1}{6}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \frac{1}{6}.$$

17. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 8} \frac{\frac{1}{3x^{2/3}}}{1} = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{12}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 8} \frac{\sqrt[3]{x} - 2}{x - 8} = \frac{1}{12}.$$

18. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 27} \frac{\frac{1}{3x^{2/3}}}{1} = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 27} \frac{\sqrt[3]{x} - 3}{x - 27} = \frac{1}{27}.$$

19. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 1} \frac{9x^8 + 24x^7 + 20x^4}{1} = \frac{9 + 24 + 20}{1} = 53.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 1} \frac{x^9 + 3x^8 + 4x^5 - 8}{x - 1} = 53.$$

20. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 2} \frac{7x^6 - 30x^5 + 25x^4}{1} = 448 - 960 + 400 = -112.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 2} \frac{x^7 - 5x^6 + 5x^5 + 32}{x - 2} = -112.$$

21. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{1} = e^0 - e^0 = 0.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{x} = 0.$$

22. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{e^x + 1}{-e^{-x} - 1} = \frac{e^0 + 1}{-e^0 - 1} = \frac{2}{-2} = -1.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{e^x - 1 + x}{e^{-x} - 1 - x} = -1.$$

23. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\frac{x}{(x^2+7)^{1/2}}}{2x} &= \lim_{x \rightarrow 3} \frac{1}{2(x^2+7)^{1/2}} \\ &= \frac{1}{2(3^2+7)^{1/2}} = \frac{1}{8}. \end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 3} \frac{\sqrt{x^2+7}-4}{x^2-9} = \frac{1}{8}.$$

24. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\begin{aligned} \lim_{x \rightarrow 5} \frac{\frac{x}{(x^2+11)^{1/2}}}{2x} &= \lim_{x \rightarrow 5} \frac{1}{2(x^2+11)^{1/2}} \\ &= \frac{1}{2(5^2+11)^{1/2}} = \frac{1}{12}. \end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 5} \frac{\sqrt{x^2+11}-6}{x^2-25} = \frac{1}{12}.$$

25. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\lim_{x \rightarrow 0} \frac{\frac{1}{3} - \frac{1}{3(1+x)^{2/3}}}{2x} = \frac{\frac{1}{3} - \frac{1}{3(1+0)^{2/3}}}{2 \cdot 0} = \frac{0}{0}.$$

Using l'Hospital's rule a second time, gives

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{2}{9(1+x)^{5/3}}}{2} &= \lim_{x \rightarrow 0} \frac{1}{(1+x)^{5/3}} \\ &= \frac{1}{9(1+0)^{5/3}} = \frac{1}{9}. \end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{1 + \frac{1}{3}x - (1+x)^{1/3}}{x^2} = \frac{1}{9}.$$

26. Find $\lim_{x \rightarrow 0} \frac{2e^{5x} - 25x^2 - 10x - 2}{5x^3}.$

$$\lim_{x \rightarrow 0} 2e^{5x} - 25x^2 - 10x - 2 = 2 - 2 = 0, \text{ and}$$

$$\lim_{x \rightarrow 0} 5x^3 = 0, \text{ so l'Hospital's rule applies.}$$

Differentiate the numerator and denominator.

$$\text{For } f(x) = 2e^{5x} - 25x^2 - 10x - 2,$$

$$f'(x) = 10e^{5x} - 50x - 10.$$

$$\text{For } g(x) = 5x^3, g'(x) = 15x^2.$$

Both derivatives still have limit equal to 0, so we apply l'Hospital's rule again with

$$f(x) = 10e^{5x} - 50x - 10 \text{ and } g(x) = 15x^2.$$

$$f'(x) = 50e^{5x} - 50$$

$$g'(x) = 30x$$

We see that both f' and g' still have limit 0 as $x \rightarrow 0$, so we need one more application of l'Hospital's rule, now with $f(x) = 50e^{5x} - 50$ and $g(x) = 30x$.

$$f'(x) = 250e^{5x}$$

$$g'(x) = 30$$

Thus

$$\begin{aligned} \lim_{x \rightarrow 0} \lim_{x \rightarrow 0} \frac{2e^{5x} - 25x^2 - 10x - 2}{5x^3} &= \lim_{x \rightarrow 0} \frac{250e^{5x}}{30} \\ &= \frac{250}{30} = \frac{25}{3}. \end{aligned}$$

27. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{1}{2(1+x)^{1/2}} + \frac{1}{2(1-x)^{1/2}}}{1} \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{2(1+x)^{1/2}} + \frac{1}{2(1-x)^{1/2}} \right) \\ &= \frac{1}{2(1+0)^{1/2}} + \frac{1}{2(1-0)^{1/2}} = 1. \end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = 1.$$

28. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{-1}{2(3-x)^{1/2}} - \frac{-1}{2(3+x)^{1/2}}}{1} \\ &= \lim_{x \rightarrow 0} \left(\frac{-1}{2(3-x)^{1/2}} - \frac{1}{2(3+x)^{1/2}} \right) \\ &= \frac{-1}{2 \cdot 3^{1/2}} - \frac{1}{2 \cdot 3^{1/2}} = -\frac{1}{\sqrt{3}} \text{ or } -\frac{\sqrt{3}}{3}. \end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\sqrt{3-x} - \sqrt{3+x}}{x} = -\frac{1}{\sqrt{3}} \text{ or } -\frac{\sqrt{3}}{3}.$$

29. $\lim_{x \rightarrow 0} \sqrt{x^2 - 5x + 4} = 2$ and l'Hospital's rule does not apply. However,

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 - 5x + 4}}{x} \text{ does not exist.}$$

30. $\lim_{x \rightarrow 1} \sqrt{x^2 + 5x + 9} = \sqrt{15}$ and l'Hospital's rule does not apply. However

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 5x + 9}}{x - 1} \text{ does not exist.}$$

31. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(x+1) + \frac{5+x}{x+1}}{e^x} \\ &= \lim_{x \rightarrow 0} \frac{(x+1)\ln(x+1) + 5+x}{e^x(x+1)} \\ &= \frac{(0+1)\ln(0+1) + 5+0}{e^0(0+1)} = 5. \end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{(5+x)\ln(x+1)}{e^x - 1} = 5.$$

32. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{-\ln(1-x) - \frac{7-x}{1-x}}{-e^{-x}} \\ &= \lim_{x \rightarrow 0} \frac{-(1-x)\ln(1-x) - 7+x}{-e^{-x}(1-x)} \\ &= \frac{-(1-0)\ln(1-0) - 7+0}{-e^0(1-0)} = 7 \end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{(7-x)\ln(1-x)}{e^{-x} - 1} = 7.$$

33. We have a limit of the form $0 \times \infty$. Rewrite the expression.

$$x^2(\ln x)^2 = \frac{(\ln x)^2}{\frac{1}{x^2}}$$

Now both the numerator and denominator become infinite and l'Hospital's rule applies to the limit of the form ∞/∞ . Differentiate the numerator and denominator.

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{\frac{1}{x^2}} &= \lim_{x \rightarrow 0^+} \frac{\frac{2(\ln x)}{x}}{\frac{-2}{x^3}} \\ &= \lim_{x \rightarrow 0^+} -x^2 \ln x\end{aligned}$$

This problem is similar to what we started with so we handle it in the same manner.

$$\begin{aligned}\lim_{x \rightarrow 0^+} -x^2 \ln x &= \lim_{x \rightarrow 0^+} \frac{-\ln x}{\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} \frac{-1}{\frac{-2}{x^3}} \\ &= \lim_{x \rightarrow 0^+} \frac{x^2}{2} = 0\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow 0^+} x^2(\ln x)^2 = 0.$$

34. We have a limit of the form $0 \times \infty$. Rewrite the expression.

$$xe^{1/x} = \frac{e^{1/x}}{\frac{1}{x}}$$

Now both the numerator and denominator become infinite and l'Hospital's rule applies to the limit of the form ∞/∞ . Differentiate the numerator and the denominator.

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{e^{1/x}}{\frac{1}{x}} &= \lim_{x \rightarrow 0^+} \frac{e^{1/x} \left(\frac{-1}{x^2} \right)}{\frac{-1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} e^{1/x} = \infty\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow 0^+} xe^{1/x} = \infty.$$

35. We have a limit of the form $0 \times \infty$. Rewrite the expression.

$$x \ln(e^x - 1) = \frac{(\ln(e^x - 1))}{\frac{1}{x}}$$

Now both the numerator and denominator become infinite and l'Hospital's rule applies to the limit of the form ∞/∞ . Differentiate the numerator and denominator.

$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{e^x}{(e^x - 1)}}{\frac{-1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{-x^2 e^x}{e^x - 1}$$

Notice that $\frac{e^x}{e^x - 1} = 1 + \frac{1}{e^x - 1}$.

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{-x^2 e^x}{e^x - 1} &= \lim_{x \rightarrow 0^+} -x^2 \cdot \left(1 + \frac{1}{e^x - 1} \right) \\ &= 0 \cdot 1 = 0\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow 0^+} x \ln(e^x - 1) = 0.$$

36. The limit in the numerator is ∞ , as in the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} &= \lim_{x \rightarrow \infty} \frac{\frac{2(\ln x)}{x}}{1} \\ &= \lim_{x \rightarrow \infty} \frac{2(\ln x)}{x}\end{aligned}$$

This problem is similar to what we started with so we handle it in the same manner.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2(\ln x)}{x} &= \lim_{x \rightarrow \infty} \frac{\frac{2}{x}}{1} \\ &= \lim_{x \rightarrow \infty} \frac{2}{x} = 0\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} = 0.$$

37. The limit in the numerator is ∞ , as in the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(\ln x)} &= \lim_{x \rightarrow \infty} \frac{\frac{1}{(2\sqrt{x})}}{\frac{1}{(x \ln x)}} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x} \ln x}{2} = \infty.\end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(\ln x)} = \infty \text{ (does not exist).}$$

38. The limit in the numerator is ∞ , as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{x^3} &= \lim_{x \rightarrow \infty} \frac{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}}{3x^2} \\ &= \lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{6x^{5/2}}\end{aligned}$$

This problem is similar to what we started with so we handle it in the same manner. In fact, perform the same steps five times.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{6x^{5/2}} &= \lim_{x \rightarrow \infty} \frac{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}}{15x^{3/2}} = \lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{30x^2} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}}{60x} = \lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{120x^{3/2}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}}{180x^{1/2}} = \lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{360x} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}}{360} = \lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{720x} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}}{720} = \lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{1440\sqrt{x}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}}{\frac{1440}{(2\sqrt{x})}} = \lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{1440} = \infty\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow \infty} \frac{e^{\sqrt{x}}}{x^3} = \infty \text{ (does not exist).}$$

39. The limit in the numerator is ∞ , as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\ln(e^x + 1)}{5x} &= \lim_{x \rightarrow 0^+} \frac{\frac{e^x}{(e^x + 1)}}{5} \\ &= \lim_{x \rightarrow 0^+} \frac{e^x}{5(e^x + 1)}\end{aligned}$$

$$\text{Notice that } \frac{e^x}{e^x + 1} = 1 - \frac{1}{e^x + 1}.$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{e^x}{5(e^x + 1)} &= \frac{1}{5} \cdot \lim_{x \rightarrow 0^+} \frac{e^x}{e^x + 1} \\ &= \frac{1}{5} \cdot \lim_{x \rightarrow 0^+} \left(1 - \frac{1}{e^x + 1} \right) \\ &= \frac{1}{5} \cdot 1 = \frac{1}{5}\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow 0^+} \frac{\ln(e^x + 1)}{5x} = \frac{1}{5}.$$

40. The limit in the numerator is ∞ , as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\ln(4e^{\sqrt{x}} - 1)}{3\sqrt{x}} &= \lim_{x \rightarrow 0^+} \frac{\frac{2e^{\sqrt{x}}}{\sqrt{x}(4e^{\sqrt{x}} - 1)}}{\frac{3}{(2\sqrt{x})}} \\ &= \lim_{x \rightarrow 0^+} \frac{4e^{\sqrt{x}}}{3(4e^{\sqrt{x}} - 1)}\end{aligned}$$

$$\text{Notice that } \frac{4e^{\sqrt{x}}}{4e^{\sqrt{x}} - 1} = 1 + \frac{1}{4e^{\sqrt{x}} - 1}.$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{4e^{\sqrt{x}}}{3(4e^{\sqrt{x}} - 1)} &= \frac{1}{3} \cdot \lim_{x \rightarrow 0^+} \frac{4e^{\sqrt{x}}}{(4e^{\sqrt{x}} - 1)} \\ &= \frac{1}{3} \cdot \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{4e^{\sqrt{x}} - 1} \right) \\ &= \frac{1}{3} \cdot 1 = \frac{1}{3}\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow 0^+} \frac{\ln(4e^{\sqrt{x}} - 1)}{3\sqrt{x}} = \frac{1}{3}.$$

41. We have a limit of the form $\infty \cdot 0$. Rewrite the expression as a quotient.

$$x^5 e^{-0.001x} = \frac{x^5}{e^{0.001x}}$$

The limit in the numerator is ∞ , as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\lim_{x \rightarrow \infty} \frac{x^5}{e^{0.001x}} = \lim_{x \rightarrow \infty} \frac{5x^4}{0.001e^{0.001x}}$$

This problem is similar to what we started with so we handle it in the same manner. In fact, continue the process four more times.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{5x^4}{0.001e^{0.001x}} &= \lim_{x \rightarrow \infty} \frac{20x^3}{(0.001)^2 e^{0.001x}} \\ &= \lim_{x \rightarrow \infty} \frac{60x^2}{(0.001)^3 e^{0.001x}} \\ &= \lim_{x \rightarrow \infty} \frac{120x}{(0.001)^4 e^{0.001x}} \\ &= \lim_{x \rightarrow \infty} \frac{120}{(0.001)^5 e^{0.001x}} \\ &= 0 \end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow \infty} x^5 e^{-0.001x} = 0.$$

42. The limit in the numerator is ∞ , as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^3 + 1}{x^2 \ln x} &= \lim_{x \rightarrow \infty} \frac{3x^2}{2x \ln x + x} \\ &= \lim_{x \rightarrow \infty} \frac{x(3x)}{x(2 \ln x + 1)} \\ &= \lim_{x \rightarrow \infty} \frac{3x}{2 \ln x + 1} \end{aligned}$$

This problem is similar to what we started with so we handle it in the same manner.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x}{2 \ln x + 1} &= \lim_{x \rightarrow \infty} \frac{3}{\frac{2}{x}} \\ &= \lim_{x \rightarrow \infty} \frac{3x}{2} = \infty \end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow \infty} \frac{x^3 + 1}{x^2 \ln x} = \infty \text{ (does not exist).}$$

43. Find $\lim_{x \rightarrow 0} \left(\frac{e^x}{x^2} - \frac{1}{x^2} - \frac{1}{x} \right)$.

Rewrite over a common denominator.

$$\lim_{x \rightarrow 0} \left(\frac{e^x - 1 - x}{x^2} \right)$$

The numerator and denominator have limit 0, so l'Hospital's rule applies. Differentiate the numerator and denominator.

$$\lim_{x \rightarrow 0} \left(\frac{e^x - 1 - x}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{2x} \right)$$

The numerator and denominator still have limit 0, so differentiate the numerator and denominator again.

$$\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{2x} \right) = \lim_{x \rightarrow 0} \left(\frac{e^x}{2} \right) = \frac{1}{2}$$

Thus

$$\lim_{x \rightarrow 0} \left(\frac{e^x}{x^2} - \frac{1}{x^2} - \frac{1}{x} \right) = \frac{1}{2}.$$

44. Find $\lim_{x \rightarrow 0} \left(\frac{12e^x}{x^3} - \frac{12}{x^3} - \frac{12}{x^2} - \frac{6}{x} \right)$.

Rewrite over a common denominator.

$$\lim_{x \rightarrow 0} \left(\frac{12e^x - 12 - 12x - 6x^2}{x^3} \right)$$

The numerator and denominator have limit 0, so l'Hospital's rule applies. Differentiate the numerator and denominator.

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{12e^x - 12 - 12x - 6x^2}{x^3} \right) \\ = \lim_{x \rightarrow 0} \left(\frac{12e^x - 12 - 12x}{3x^2} \right) \end{aligned}$$

The numerator and denominator still have limit 0, so differentiate the numerator and denominator again.

$$\lim_{x \rightarrow 0} \left(\frac{12e^x - 12 - 12x}{3x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{12e^x - 12}{6x} \right)$$

The numerator and denominator still have limit 0, so differentiate the numerator and denominator again.

$$\lim_{x \rightarrow 0} \left(\frac{12e^x - 12}{6x} \right) = \lim_{x \rightarrow 0} \left(\frac{12e^x}{6} \right) = \frac{12}{6} = 2$$

Thus

$$\lim_{x \rightarrow 0} \left(\frac{12e^x}{x^3} - \frac{12}{x^3} - \frac{12}{x^2} - \frac{6}{x} \right) = 2.$$

45. Find $\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

Rewrite over a common denominator: $\lim_{x \rightarrow 1} \left(\frac{x \ln x - x + 1}{(x-1) \ln(x)} \right)$.

The numerator and denominator have limit 0, so l'Hospital's rule applies. Differentiate the numerator and denominator.

$$\lim_{x \rightarrow 1} \left(\frac{x \ln x - x + 1}{(x-1) \ln(x)} \right) = \lim_{x \rightarrow 1} \left(\frac{\ln x}{\frac{x-1}{x} + \ln x} \right) = \lim_{x \rightarrow 1} \left(\frac{x \ln x}{x-1+x \ln x} \right)$$

The numerator and denominator still have limit 0, so differentiate the numerator and denominator again.

$$\lim_{x \rightarrow 1} \left(\frac{x \ln x}{x-1+x \ln x} \right) = \lim_{x \rightarrow 1} \left(\frac{1+\ln x}{1+1+\ln x} \right) = \frac{1}{2}$$

Thus

$$\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \frac{1}{2}.$$

46. Find $\lim_{x \rightarrow 0} \left(\frac{2}{x} - \frac{\ln(1+2x)}{x^2} \right)$.

Rewrite over a common denominator: $\lim_{x \rightarrow 0} \left(\frac{2x - \ln(1+2x)}{x^2} \right)$

The numerator and denominator have limit 0, so l'Hospital's rule applies. Differentiate the numerator and denominator.

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{2x - \ln(1+2x)}{x^2} \right) &= \lim_{x \rightarrow 0} \left(\frac{2 - \frac{2}{1+2x}}{2x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{2x}{x(1+2x)} \right) = \lim_{x \rightarrow 0} \left(\frac{2}{1+2x} \right) = 2 \end{aligned}$$

Thus

$$\lim_{x \rightarrow 0} \left(\frac{2}{x} - \frac{\ln(1+2x)}{x^2} \right) = 2.$$

47. $\lim_{x \rightarrow 0} x^2 + 3 \neq 0$, so l'Hospital's rule does not apply.

48. First, verify that the form is one for which l'Hospital's rule applies.

$$\begin{aligned}\lim_{x \rightarrow a} \left(\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x} \right) &= \sqrt{2a^4 - a^4} - a\sqrt[3]{a^3} = a^2 - a^2 = 0 \\ \lim_{x \rightarrow a} \left(a - \sqrt[4]{ax^3} \right) &= a - \sqrt[4]{a^4} = a - a = 0\end{aligned}$$

The limit in the numerator is 0, as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\begin{aligned}\lim_{x \rightarrow a} \frac{(\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x})}{a - \sqrt[4]{ax^3}} &= \lim_{x \rightarrow a} \frac{\frac{a^3 - 2x^3}{\sqrt{2a^3x - x^4}} - \frac{a^{5/3}}{3\sqrt[3]{x}}}{\frac{-3ax^2}{4(ax^3)^{3/4}}} \\ &= \frac{4(a^4)^{3/4}(a^{5/3}\sqrt{a^4} + 6a^{11/3} - 3a^{11/3})}{9a(a^{8/3})\sqrt{a^4}} \\ &= \frac{4a^3(a^{11/3} + 6a^{11/3} - 3a^{11/3})}{9a^{11/3}a^2} \\ &= \frac{4a^3(4a^{11/3})}{9a^{17/3}} \\ &= \frac{16a}{9}\end{aligned}$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow a} \frac{\sqrt{2a^3 - x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}} = \frac{16a}{9}.$$

Chapter 12 Review Problems

1. True
2. False: The amounts are a constant arithmetic sequence.
3. True
4. True
5. False: In general the fifth derivatives will be different at 0.
6. True
7. False: It converges as long as $-1 < r < 1$.
8. True
9. True
10. False: It converges only for x in $(-1, 1]$.
11. False: Newton's method may fail to converge by alternating between two values, neither of which is a zero.
12. False: The rule applies to limits of quotients, not derivatives of quotients.
13.
$$\begin{aligned} a_4 &= a_1 \cdot r^3 \\ &= 5(-2)^3 \\ &= -40 \\ a_n &= 5(-2)^{n-1} \\ S_5 &= \frac{a_1(r^5 - 1)}{r - 1} \\ &= \frac{5[(-2)^5 - 1]}{(-2) - 1} \\ &= \frac{5(-33)}{-3} \\ &= 55 \end{aligned}$$
14.
$$\begin{aligned} a_4 &= a_1 \cdot r^3 \\ &= 128\left(\frac{1}{2}\right)^3 \\ &= 16 \end{aligned}$$

$$\begin{aligned} a_n &= 128\left(\frac{1}{2}\right)^{n-1} \\ S_5 &= \frac{a_1(r^5 - 1)}{r - 1} \\ &= \frac{128[(1/2)^5 - 1]}{(1/2) - 1} \\ &= \frac{128\left(-\frac{31}{32}\right)}{-\frac{1}{2}} \\ &= (8)(31) \\ &= 248 \end{aligned}$$

$$\begin{aligned} 15. \quad a_4 &= a_1 \cdot r^3 \\ &= 27\left(\frac{1}{3}\right)^3 \\ &= 1 \\ a_n &= 27\left(\frac{1}{3}\right)^{n-1} \\ S_5 &= \frac{a_1(r^5 - 1)}{r - 1} \\ &= \frac{27[(1/3)^5 - 1]}{(1/3) - 1} \\ &= \frac{27\left(-\frac{242}{243}\right)}{-\frac{2}{3}} \\ &= \frac{242}{6} \\ &= \frac{121}{3} \end{aligned}$$

$$\begin{aligned} 16. \quad a_4 &= a_1 \cdot r^3 \\ &= 2(-5)^3 \\ &= -250 \\ a_n &= 2(-5)^{n-1} \\ S_5 &= \frac{a_1(r^5 - 1)}{r - 1} \\ &= \frac{2[(-5)^5 - 1]}{(-5) - 1} \\ &= \frac{2(-3126)}{-6} \\ &= 1042 \end{aligned}$$

17.

Derivative	Value at 0
$f(x) = e^{2-x}$	$f(0) = e^2$
$f^{(1)}(x) = -e^{2-x}$	$f^{(1)}(0) = -e^2$
$f^{(2)}(x) = e^{2-x}$	$f^{(2)}(0) = e^2$
$f^{(3)}(x) = -e^{2-x}$	$f^{(3)}(0) = -e^2$
$f^{(4)}(x) = e^{2-x}$	$f^{(4)}(0) = e^2$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= e^2 + \frac{-e^2}{1!}x + \frac{e^2}{2!}x^2 + \frac{-e^2}{3!}x^3 + \frac{e^2}{4!}x^4 \\
 &= e^2 - e^2x + \frac{e^2}{2}x^2 - \frac{e^2}{6}x^3 + \frac{e^2}{24}x^4
 \end{aligned}$$

18.

Derivative	Value at 0
$f(x) = 5e^{2x}$	$f(0) = 5$
$f^{(1)}(x) = 10e^{2x}$	$f^{(1)}(0) = 10$
$f^{(2)}(x) = 20e^{2x}$	$f^{(2)}(0) = 20$
$f^{(3)}(x) = 40e^{2x}$	$f^{(3)}(0) = 40$
$f^{(4)}(x) = 80e^{2x}$	$f^{(4)}(0) = 80$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 5 + \frac{10}{1!}x + \frac{20}{2}x^2 + \frac{40}{6}x^3 + \frac{80}{24}x^4 = 5 + 10x + 10x^2 + \frac{20}{3}x^3 + \frac{10}{3}x^4
 \end{aligned}$$

19.

Derivative	Value at 0
$f(x) = \sqrt{x+1} = (x+1)^{1/2}$	$f(0) = 1$
$f^{(1)}(x) = \frac{1}{2}(x+1)^{-1/2} = \frac{1}{2(x+1)^{1/2}}$	$f^{(1)}(0) = \frac{1}{2}$
$f^{(2)}(x) = -\frac{1}{4}(x+1)^{-3/2} = -\frac{1}{4(x+1)^{3/2}}$	$f^{(2)}(0) = -\frac{1}{4}$
$f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} = \frac{3}{8(x+1)^{5/2}}$	$f^{(3)}(0) = \frac{3}{8}$
$f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2} = -\frac{15}{16(x+1)^{7/2}}$	$f^{(4)}(0) = -\frac{15}{16}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{\frac{1}{2}}{1}x + \frac{-\frac{1}{4}}{2}x^2 + \frac{\frac{3}{8}}{6}x^3 + \frac{-\frac{15}{16}}{24}x^4 \\
 &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4
 \end{aligned}$$

20.

Derivative	Value at 0
$f(x) = \sqrt[3]{x+27} = (x+27)^{1/3}$	$f(0) = 3$
$f^{(1)}(x) = \frac{1}{3}(x+27)^{-2/3} = \frac{1}{3(x+27)^{2/3}}$	$f^{(1)}(0) = \frac{1}{27}$
$f^{(2)}(x) = -\frac{2}{9}(x+27)^{-5/3} = -\frac{2}{9(x+27)^{5/3}}$	$f^{(2)}(0) = -\frac{2}{2187}$
$f^{(3)}(x) = \frac{10}{27}(x+27)^{-8/3} = \frac{10}{27(x+27)^{8/3}}$	$f^{(3)}(0) = \frac{10}{177,147}$
$f^{(4)}(x) = -\frac{80}{81}(x+27)^{-11/3} = -\frac{80}{81(x+27)^{11/3}}$	$f^{(4)}(0) = -\frac{80}{14,348,907}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 3 + \frac{\frac{1}{27}}{1}x + \frac{-\frac{2}{2187}}{2}x^2 + \frac{\frac{10}{177,147}}{6}x^3 + \frac{-\frac{80}{14,348,907}}{24}x^4 \\
 &= 3 + \frac{1}{27}x - \frac{1}{2187}x^2 + \frac{5}{531,441}x^3 - \frac{10}{43,046,721}x^4
 \end{aligned}$$

21.

Derivative	Value at 0
$f(x) = \ln(2-x)$	$f(0) = \ln 2$
$f^{(1)}(x) = -\frac{1}{2-x} = -(2-x)^{-1}$	$f^{(1)}(0) = -\frac{1}{2}$
$f^{(2)}(x) = -(2-x)^{-2} = -\frac{1}{(2-x)^2}$	$f^{(2)}(0) = -\frac{1}{4}$
$f^{(3)}(x) = -2(2-x)^{-3} = -\frac{2}{(2-x)^3}$	$f^{(3)}(0) = -\frac{1}{4}$
$f^{(4)}(x) = -6(2-x)^{-4} = -\frac{6}{(2-x)^4}$	$f^{(4)}(0) = -\frac{3}{8}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= \ln 2 + \frac{-\frac{1}{2}}{1}x + \frac{-\frac{1}{4}}{2}x^2 + \frac{-\frac{1}{4}}{6}x^3 + \frac{-\frac{3}{8}}{24}x^4 \\
 &= \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3 - \frac{1}{64}x^4
 \end{aligned}$$

22.

Derivative	Value at 0
$f(x) = \ln(3+2x)$	$f(0) = \ln 3$
$f^{(1)}(x) = \frac{2}{3+2x} = 2(3+2x)^{-1}$	$f^{(1)}(0) = \frac{2}{3}$
$f^{(2)}(x) = -4(3+2x)^{-2} = -\frac{4}{(3+2x)^2}$	$f^{(2)}(0) = -\frac{4}{9}$
$f^{(3)}(x) = 16(3+2x)^{-3} = \frac{16}{(3+2x)^3}$	$f^{(3)}(0) = \frac{16}{27}$
$f^{(4)}(x) = -96(3+2x)^{-4} = -\frac{96}{(3+2x)^4}$	$f^{(4)}(0) = -\frac{32}{27}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= \ln 3 + \frac{\frac{2}{3}}{1}x + \frac{-\frac{4}{9}}{2}x^2 + \frac{\frac{16}{27}}{6}x^3 + \frac{-\frac{32}{27}}{24}x^4 \\
 &= \ln 3 + \frac{2}{3}x - \frac{2}{9}x^2 + \frac{8}{81}x^3 - \frac{4}{81}x^4
 \end{aligned}$$

23.

Derivative	Value at 0
$f(x) = (1+x)^{2/3}$	$f(0) = 1$
$f^{(1)}(x) = \frac{2}{3}(1+x)^{-1/3} = \frac{2}{3(1+x)^{1/3}}$	$f^{(1)}(0) = \frac{2}{3}$
$f^{(2)}(x) = -\frac{2}{9}(1+x)^{-4/3} = -\frac{2}{9(1+x)^{4/3}}$	$f^{(2)}(0) = -\frac{2}{9}$
$f^{(3)}(x) = \frac{8}{27}(1+x)^{-7/3} = \frac{8}{27(1+x)^{7/3}}$	$f^{(3)}(0) = \frac{8}{27}$
$f^{(4)}(x) = -\frac{56}{81}(1+x)^{-10/3} = -\frac{56}{81(1+x)^{10/3}}$	$f^{(4)}(0) = -\frac{56}{81}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 1 + \frac{\frac{2}{3}}{1}x + \frac{-\frac{2}{9}}{2}x^2 + \frac{\frac{8}{27}}{6}x^3 + \frac{-\frac{56}{81}}{24}x^4 \\
 &= 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 - \frac{7}{243}x^4
 \end{aligned}$$

24.

Derivative	Value at 0
$f(x) = (4+x)^{3/2}$	$f(0) = 8$
$f^{(1)}(x) = \frac{3}{2}(4+x)^{1/2}$	$f^{(1)}(0) = 3$
$f^{(2)}(x) = \frac{3}{4}(4+x)^{-1/2}$	$f^{(2)}(0) = \frac{3}{8}$
$f^{(3)}(x) = -\frac{3}{8}(4+x)^{-3/2}$	$f^{(3)}(0) = -\frac{3}{64}$
$f^{(4)}(x) = \frac{9}{16}(4+x)^{-5/2}$	$f^{(4)}(0) = \frac{9}{512}$

$$\begin{aligned}
 P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
 &= 8 + \frac{3}{1}x + \frac{\frac{3}{8}}{2}x^2 + \frac{-\frac{3}{64}}{6}x^3 + \frac{\frac{9}{512}}{24}x^4 \\
 &= 8 + 3x + \frac{3}{16}x^2 - \frac{1}{128}x^3 + \frac{3}{4096}x^4
 \end{aligned}$$

25. Using the result of Exercise 17, with $f(x) = e^{2-x}$ and $P_4(x) = e^2 - e^2x + \frac{e^2}{2}x^2 - \frac{e^2}{6}x^3 + \frac{e^2}{24}x^4$, we can approximate $e^{1.93}$ by evaluating $f(0.07) = e^{2-0.07} = e^{1.93}$. Using $P_4(x)$ from Exercise 17 with $x = 0.07$ gives

$$\begin{aligned} P_4(0.07) &= e^2 - e^2(0.07) + \frac{e^2}{2}(0.07)^2 - \frac{e^2}{6}(0.07)^3 + \frac{e^2}{24}(0.07)^4 \\ &\approx 6.88951034388. \end{aligned}$$

To four decimal places, $P_4(0.07)$ approximates the value of $e^{1.93}$ as 6.8895.

26. Using the result of Exercise 18, with $f(x) = 5e^{2x}$ and $P_4(x) = 5 + 10x + 10x^2 + \frac{20}{3}x^3 + \frac{10}{3}x^4$, we can approximate $5e^{0.04}$ by evaluating $f(0.02) = 5e^{2(0.02)} = 5e^{0.04}$. Using $P_4(x)$ from Exercise 18 with $x = 0.02$ gives

$$\begin{aligned} P_4(0.02) &= 5 + 10(0.02) + 10(0.02)^2 + \frac{20}{3}(0.02)^3 + \frac{10}{3}(0.02)^4 \\ &\approx 5.20405386667. \end{aligned}$$

To four decimal places, $P_4(0.02)$ approximates the value of $5e^{0.04}$ as 5.2041.

27. Using the result of Exercise 19, with $f(x) = \sqrt{x+1}$ and $P_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$, we can approximate $\sqrt{1.03}$ by evaluating $f(0.03) = \sqrt{0.03+1} = \sqrt{1.03}$. Using $P_4(x)$ from Exercise 19 with $x = 0.03$ gives

$$\begin{aligned} P_4(0.03) &= 1 + \frac{1}{2}(0.03) - \frac{1}{8}(0.03)^2 + \frac{1}{16}(0.03)^3 - \frac{5}{128}(0.03)^4 \\ &\approx 1.01488915586. \end{aligned}$$

To four decimal places, $P_4(0.03)$ approximates the value of $\sqrt{1.03}$ as 1.0149.

28. Using the result of Exercise 20, with $f(x) = \sqrt[3]{x+27}$ and $P_4(x) = 3 + \frac{1}{27}x - \frac{1}{2187}x^2 + \frac{5}{531,441}x^3 - \frac{10}{43,046,721}x^4$, we can approximate $\sqrt[3]{26.94}$ by evaluating $f(-0.06) = \sqrt[3]{-0.06+27} = \sqrt[3]{26.94}$. Using $P_4(x)$ from Exercise 20 with $x = -0.06$ gives

$$\begin{aligned} P_4(-0.06) &= 3 + \frac{1}{27}(-0.06) - \frac{1}{2187}(-0.06)^2 + \frac{5}{531,441}(-0.06)^3 - \frac{10}{43,046,721}(-0.06)^4 \\ &\approx 2.99777612965. \end{aligned}$$

To four decimal places, $P_4(-0.06)$ approximates the value of $\sqrt[3]{26.94}$ as 2.9978.

29. Using the result of Exercise 21, with $f(x) = \ln(2-x)$ and $P_4(x) = \ln 2 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{24}x^3 - \frac{1}{64}x^4$, we can approximate $\ln 2.05$ by evaluating $f(-0.05) = \ln(2 - (-0.05)) = \ln 2.05$. Using $P_4(x)$ from Exercise 21 with $x = -0.05$ gives

$$\begin{aligned} P_4(-0.05) &= \ln 2 - \frac{1}{2}(-0.05) - \frac{1}{8}(-0.05)^2 - \frac{1}{24}(-0.05)^3 - \frac{1}{64}(-0.05)^4 \\ &\approx 0.717842610677 \text{ (using } \ln 2 = 0.69315). \end{aligned}$$

To four decimal places, $P_4(-0.05)$ approximates the value of $\ln 2.05$ as 0.7178.

30. Using the result of Exercise 22, with $f(x) = \ln(3 + 2x)$ and $P_4(x) = \ln 3 + \frac{2}{3}x - \frac{2}{9}x^2 + \frac{8}{81}x^3 - \frac{4}{81}x^4$, we can approximate $\ln 3.06$ by evaluating $f(0.03) = f(3 + 2(0.03)) = f(3 + 0.06) = f(3.06)$. Using $P_4(x)$ from Exercise 22 with $x = 0.03$ gives

$$\begin{aligned} P_4(0.03) &= \ln 3 + \frac{2}{3}(0.03) - \frac{2}{9}(0.03)^2 + \frac{8}{81}(0.03)^3 - \frac{4}{81}(0.03)^4 \\ &= 1.118412627 \text{ (using } \ln 3 = 1.1184). \end{aligned}$$

To four decimal places, $P_4(0.03)$ approximates the value of $\ln 3.06$ as 1.1184.

31. Using the result of Exercise 23, with $f(x) = (1 + x)^{2/3}$ and $P_4(x) = 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 - \frac{7}{243}x^4$, we can approximate $(0.92)^{2/3}$ by evaluating $f(-0.08) = (1 + (-0.08))^{2/3} = (0.92)^{2/3}$. Using $P_4(x)$ from Exercise 23 with $x = -0.08$ gives

$$\begin{aligned} P_4(-0.08) &= 1 + \frac{2}{3}(-0.08) - \frac{1}{9}(-0.08)^2 + \frac{4}{81}(-0.08)^3 - \frac{7}{243}(-0.08)^4 \\ &\approx 0.945929091687. \end{aligned}$$

To four decimal places, $P_4(-0.08)$ approximates the value of $(0.92)^{2/3}$ as 0.9459.

32. Using the result of Exercise 24, with $f(x) = (4 + x)^{3/2}$ and $P_4(x) = 8 + 3x + \frac{3}{16}x^2 - \frac{1}{128}x^3 + \frac{3}{4096}x^4$, we can approximate $4.02^{3/2}$ by evaluating $f(0.02) = (4 + 0.02)^{3/2} = 4.02^{3/2}$. Using $P_4(x)$ from Exercise 24 with $x = 0.02$ gives

$$\begin{aligned} P_4(0.02) &= 8 + 3(0.02) + \frac{3}{16}(0.02)^2 - \frac{1}{128}(0.02)^3 + \frac{3}{4096}(0.02)^4 \\ &\approx 8.06007493762. \end{aligned}$$

To four decimal places, $P_4(0.02)$ approximates the value of $4.02^{3/2}$ as 8.0601.

33. $9 - 6 + 4 - \frac{8}{3} + \cdots$ is a geometric series with $a = a_1 = 9$ and $r = -\frac{2}{3}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{9}{1-(-\frac{2}{3})} = \frac{9}{1+\frac{2}{3}} = \frac{9}{\frac{5}{3}} = \frac{27}{5}.$$

34. $2 + 1.4 + .98 + .686 + \cdots$ is a geometric series with $a = a_1 = 2$ and $r = .7$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{2}{1-.7} = \frac{2}{.3} = \frac{20}{3}.$$

35. $3 + 9 + 27 + 81 + \cdots$ is a geometric series with $a = a_1 = 3$ and $r = 3$. Since $r > 1$, the series diverges.

36. $4 + 4.8 + 5.76 + 6.912 + \cdots$ is a geometric series with $a = a_1 = 4$ and $r = 1.2$. Since $r > 1$, the series diverges.

37. $\frac{2}{5} - \frac{2}{25} + \frac{2}{125} - \frac{2}{625} + \cdots$ is a geometric series with $a = a_1 = \frac{2}{5}$ and $r = -\frac{1}{5}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{\frac{2}{5}}{1 - \left(-\frac{1}{5}\right)} = \frac{\frac{2}{5}}{1 + \frac{1}{5}} = \frac{\frac{2}{5}}{\frac{6}{5}} = \frac{1}{3}.$$

38. $36 + 3 + \frac{1}{4} + \frac{1}{48} + \cdots$ is a geometric series with $a = a_1 = 36$ and $r = \frac{1}{12}$. Since r is in $(-1, 1)$, the series converges and has sum

$$\frac{a}{1-r} = \frac{36}{1 - \frac{1}{12}} = \frac{36}{\frac{11}{12}} = \frac{432}{11}.$$

39. $S_1 = a_1 = \frac{1}{2(1) - 1} = 1$

$$S_2 = a_1 + a_2 = \frac{1}{2(1) - 1} + \frac{1}{2(2) - 1} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{2(1) - 1} + \frac{1}{2(2) - 1} + \frac{1}{2(3) - 1} = 1 + \frac{1}{3} + \frac{1}{5} = \frac{23}{15}$$

$$\begin{aligned} S_4 &= a_1 + a_2 + a_3 + a_4 = \frac{1}{2(1) - 1} + \frac{1}{2(2) - 1} + \frac{1}{2(3) - 1} + \frac{1}{2(4) - 1} \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{176}{105} \end{aligned}$$

$$\begin{aligned} S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{2(1) - 1} + \frac{1}{2(2) - 1} + \frac{1}{2(3) - 1} + \frac{1}{2(4) - 1} + \frac{1}{2(5) - 1} \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} = \frac{563}{315} \end{aligned}$$

40. $S_1 = a_1 = \frac{1}{(1+2)(1+3)} = \frac{1}{12}$

$$S_2 = a_1 + a_2 = \frac{1}{(1+2)(1+3)} + \frac{1}{(2+2)(2+3)} = \frac{1}{12} + \frac{1}{20} = \frac{2}{15}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{(1+2)(1+3)} + \frac{1}{(2+2)(2+3)} + \frac{1}{(3+2)(3+3)} = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{1}{6}$$

$$\begin{aligned} S_4 &= a_1 + a_2 + a_3 + a_4 = \frac{1}{(1+2)(1+3)} + \frac{1}{(2+2)(2+3)} + \frac{1}{(3+2)(3+3)} + \frac{1}{(4+2)(4+3)} \\ &= \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{4}{21} \end{aligned}$$

$$\begin{aligned} S_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= \frac{1}{(1+2)(1+3)} + \frac{1}{(2+2)(2+3)} + \frac{1}{(3+2)(3+3)} + \frac{1}{(4+2)(4+3)} + \frac{1}{(5+2)(5+3)} \\ &= \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} = \frac{5}{24} \end{aligned}$$

41. This function most nearly matches $\frac{1}{1-x}$. To get 1 in the denominator, instead of 3, divide the numerator and denominator by 3.

$$\frac{4}{3-x} = \frac{\frac{4}{3}}{1-\frac{x}{3}}$$

Thus, we can find the Taylor series for $\frac{\frac{4}{3}}{1-\frac{x}{3}}$ by starting with the Taylor series for $\frac{1}{1-x}$, multiplying each term by $\frac{4}{3}$, and replacing x with $\frac{x}{3}$.

$$\begin{aligned}\frac{4}{3-x} &= \frac{\frac{4}{3}}{1-\frac{x}{3}} \\ &= \frac{4}{3} \cdot 1 + \frac{4}{3} \left(\frac{x}{3} \right) + \frac{4}{3} \left(\frac{x}{3} \right)^2 + \frac{4}{3} \left(\frac{x}{3} \right)^3 + \cdots + \frac{4}{3} \left(\frac{x}{3} \right)^n + \cdots \\ &= \frac{4}{3} + \frac{4x}{9} + \frac{4x^2}{27} + \frac{4x^3}{81} + \cdots + \frac{4x^n}{3^{n+1}} + \cdots\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $\frac{x}{3}$ gives

$$-1 < \frac{x}{3} < 1 \text{ or } -3 < x < 3.$$

The interval of convergence of the new series is $(-3, 3)$.

42.
$$\frac{2x}{1+3x} = x \cdot \frac{2}{1-(-3x)}$$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by 2, and replace x with $-3x$. Also, use property (3) with $k = 1$.

$$\begin{aligned}\frac{2x}{1+3x} &= x \cdot \frac{2}{1-(-3x)} = x \cdot 2 \cdot 1 + x \cdot 2(-3x) + x \cdot 2(-3x)^2 + x \cdot 2(-3x)^3 + \cdots + x \cdot 2(-3x)^n + \cdots \\ &= 2x - 6x^2 + 18x^3 - 54x^4 + \cdots + (-1)^n \cdot 2 \cdot 3^n \cdot x^{n+1} + \cdots\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-3x$ gives

$$-1 < -3x < 1 \text{ or } \frac{1}{3} > x > -\frac{1}{3}.$$

The interval of convergence of the new series is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

43.
$$\frac{x^2}{x+1} = x^2 \cdot \frac{1}{1-(-x)}$$

Use the Taylor series for $\frac{1}{1-x}$ and replace x with $-x$. Also, use property (3) with $k = 2$.

$$\begin{aligned}\frac{x^2}{x+1} &= x^2 \cdot \frac{1}{1-(-x)} \\ &= x^2 \cdot 1 + x^2(-x) + x^2(-x)^2 + x^2(-x)^3 + \cdots + x^2(-x)^n + \cdots \\ &= x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^{n+2} + \cdots\end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $-x$ gives

$$-1 < -x < 1 \quad \text{or} \quad 1 > x > -1.$$

The interval of convergence of the new series is $(-1, 1)$.

$$44. \quad \frac{3x^3}{2-x} = x^3 \cdot \frac{\frac{3}{2}}{1-\frac{x}{2}}$$

Use the Taylor series for $\frac{1}{1-x}$, multiply each term by $\frac{3}{2}$, and replace x with $\frac{x}{2}$. Also, use property (3) with $k = 3$.

$$\begin{aligned} \frac{3x^3}{2-x} &= x^3 \cdot \frac{\frac{3}{2}}{1-\frac{x}{2}} = x^3 \cdot \frac{3}{2} \cdot 1 + x^3 \cdot \frac{3}{2} \left(\frac{x}{2} \right) + x^3 \cdot \frac{3}{2} \left(\frac{x}{2} \right)^2 + x^3 \cdot \frac{3}{2} \left(\frac{x}{2} \right)^3 + \cdots + x^3 \cdot \frac{3}{2} \left(\frac{x}{2} \right)^n + \cdots \\ &= \frac{3x^3}{2} + \frac{3x^4}{4} + \frac{3x^5}{8} + \frac{3x^6}{16} + \cdots + \frac{3x^{n+3}}{2^{n+1}} + \cdots \end{aligned}$$

The Taylor series for $\frac{1}{1-x}$ is valid when $-1 < x < 1$. Replacing x with $\frac{x}{2}$ gives

$$-1 < \frac{x}{2} < 1 \quad \text{or} \quad -2 < x < 2.$$

The interval of convergence of the new series is $(-2, 2)$.

45. We find the Taylor series for $\ln(1-2x)$ by starting with the Taylor series for $\ln(1+x)$ and replacing each x with $-2x$.

$$\begin{aligned} \ln(1-2x) &= -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \cdots + \frac{(-1)^n(-2x)^{n+1}}{n+1} + \cdots \\ &= -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - \cdots - \frac{2^{n+1}x^{n+1}}{n+1} - \cdots \end{aligned}$$

The Taylor series for $\ln(1+x)$ is valid when $-1 < x \leq 1$. Replacing x with $-2x$ gives

$$-1 < -2x \leq 1 \quad \text{or} \quad \frac{1}{2} > x \geq -\frac{1}{2}.$$

The interval of convergence of the new series is $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

46. We find the Taylor series for $\ln\left(1 + \frac{1}{3}x\right)$ by starting with the Taylor series for $\ln(1+x)$ and replacing each x with $\frac{1}{3}x$.

$$\begin{aligned} \ln\left(1 + \frac{1}{3}x\right) &= \frac{1}{3}x - \frac{\left(\frac{1}{3}x\right)^2}{2} + \frac{\left(\frac{1}{3}x\right)^3}{3} - \frac{\left(\frac{1}{3}x\right)^4}{4} + \cdots + \frac{(-1)^n\left(\frac{1}{3}x\right)^{n+1}}{n+1} + \cdots \\ &= \frac{1}{3}x - \frac{1}{18}x^2 + \frac{1}{81}x^3 - \frac{1}{324}x^4 + \cdots + \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)} + \cdots \end{aligned}$$

The Taylor series for $\ln(1+x)$ is valid when $-1 < x \leq 1$. Replacing x with $\frac{1}{3}x$ gives

$$-1 < \frac{1}{3}x \leq 1 \quad \text{or} \quad -3 < x \leq 3.$$

The interval of convergence of the new series is $(-3, 3]$.

47. We find the Taylor series for e^{-2x^2} by starting with the Taylor series for e^x and replacing each x with $-2x^2$.

$$\begin{aligned} e^{-2x^2} &= 1 + (-2x^2) + \frac{1}{2!}(-2x^2)^2 + \frac{1}{3!}(-2x^2)^3 + \cdots + \frac{1}{n!}(-2x^2)^n + \cdots \\ &= 1 - 2x^2 + 2x^4 - \frac{4}{3}x^6 + \cdots + \frac{(-1)^n \cdot 2^n x^{2n}}{n!} + \cdots \end{aligned}$$

The Taylor series for e^{-2x^2} has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

48. We find the Taylor series for e^{-5x} by starting with the Taylor series for e^x and replacing each x with $-5x$.

$$\begin{aligned} e^{-5x} &= 1 + (-5x) + \frac{1}{2!}(-5x)^2 + \frac{1}{3!}(-5x)^3 + \cdots + \frac{1}{n!}(-5x)^n + \cdots \\ &= 1 - 5x + \frac{25}{2}x^2 - \frac{125}{6}x^3 + \cdots + \frac{(-1)^n 5^n x^n}{n!} + \cdots \end{aligned}$$

The Taylor series for e^{-5x} has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

49. Use the Taylor series for e^x , multiply each term by 2, and replace x with $-3x$. Also, use property (3) with $k = 3$.

$$\begin{aligned} 2x^3 e^{-3x} &= 2x^3 \cdot 1 + 2x^3(-3x) + 2x^3 \cdot \frac{1}{2!}(-3x)^2 + 2x^3 \cdot \frac{1}{3!}(-3x)^3 + \cdots + 2x^3 \cdot \frac{1}{n!}(-3x)^n + \cdots \\ &= 2x^3 - 6x^4 + 9x^5 - 9x^6 + \cdots + \frac{(-1)^n \cdot 2 \cdot 3^n x^{n+3}}{n!} + \cdots \end{aligned}$$

The Taylor series for $2x^3 e^{-3x}$ has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

50. Use the Taylor series for e^x and replace x with $-x$. Also, use property (3) with $k = 6$.

$$\begin{aligned} x^6 e^{-x} &= x^6 \cdot 1 + x^6 \cdot (-x) + x^6 \cdot \frac{1}{2!}(-x)^2 + x^6 \cdot \frac{1}{3!}(-x)^3 + \cdots + x^6 \cdot \frac{1}{n!}(-x)^n + \cdots \\ &= x^6 - x^7 + \frac{x^8}{2} - \frac{x^9}{6} + \cdots + \frac{(-1)^n x^{n+6}}{n!} + \cdots \end{aligned}$$

The Taylor series for $x^6 e^{-x}$ has the same interval of convergence, $(-\infty, \infty)$, as the Taylor series for e^x .

51. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x \rightarrow 2} \frac{3x^2 - 2x - 1}{2x} = \frac{3(2)^2 - 2(2) - 1}{2(2)} = \frac{7}{4}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 2} \frac{x^3 - x^2 - x - 2}{x^2 - 4} = \frac{7}{4}.$$

52. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x \rightarrow 0} \frac{3x^2 - 8x + 6}{3} = \frac{3(0)^2 - 8(0) + 6}{3} = 2.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{x^3 - 4x^2 + 6x}{3x} = 2.$$

53. $\lim_{x \rightarrow -5} x^3 - 3x^2 + 4x - 1 = -221$ and l'Hospital's rule does not apply. However,

$$\lim_{x \rightarrow -5} \frac{x^3 - 3x^2 + 4x - 1}{x^2 - 25} \text{ does not exist.}$$

54. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x \rightarrow 0} \frac{\frac{3}{3x+1}}{1} = \frac{3}{3(0) + 1} = 3.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\ln(3x + 1)}{x} = 3.$$

55. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x \rightarrow 0} \frac{5e^x}{3x^2 - 16x + 7} = \frac{5e^0}{3(0)^2 - 16(0) + 7} = \frac{5}{7}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{5e^x - 5}{x^3 - 8x^2 + 7x} = \frac{5}{7}.$$

56. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x \rightarrow 0} \frac{\frac{1}{2(5+x)^{1/2}}}{1} = \frac{1}{2(5+0)^{1/2}} = \frac{1}{2\sqrt{5}} \text{ or } \frac{\sqrt{5}}{10}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\sqrt{5+x} - \sqrt{5}}{x} = \frac{1}{2\sqrt{5}} \text{ or } \frac{\sqrt{5}}{10}.$$

57. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{-e^{2x} - 2xe^{2x}}{2e^{2x}} &= \frac{-e^{2(0)} - 2(0)e^{2(0)}}{2e^{2(0)}} \\ &= -\frac{1}{2}.\end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{-xe^{2x}}{e^{2x} - 1} = -\frac{1}{2}.$$

58. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\lim_{x \rightarrow 16} \frac{\frac{1}{2x^{1/2}}}{1} = \frac{1}{2(16)^{1/2}} = \frac{1}{8}.$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 16} \frac{\sqrt{x} - 4}{x - 16} = \frac{1}{8}.$$

59. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2 - \frac{1}{2}(1+x)^{-1/2}}{3x^2} &= \lim_{x \rightarrow 0} \frac{4(1+x)^{1/2} - 1}{3x^2(2(1+x)^{1/2})} \\ &= \frac{3}{0} \text{ which does not exist.}\end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{1 + 2x - (1+x)^{1/2}}{x^3} \text{ does not exist.}$$

60. The limit in the numerator is 0, as is the limit in the denominator, so that l'Hospital's rule applies. Taking derivatives separately in the numerator and denominator gives,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\frac{1}{2(5+x)^{1/2}} + \frac{1}{2(5-x)^{1/2}}}{2} &= \frac{1}{2} \lim_{x \rightarrow 0} \left(\frac{1}{2(5+x)^{1/2}} + \frac{1}{2(5-x)^{1/2}} \right) \\ &= \frac{1}{2} \left[\frac{1}{2(5+0)^{1/2}} + \frac{1}{2(5-0)^{1/2}} \right] = \frac{1}{2\sqrt{5}} \text{ or } \frac{\sqrt{5}}{10}\end{aligned}$$

By l'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\sqrt{5+x} - \sqrt{5-x}}{2x} = \frac{1}{2\sqrt{5}} \text{ or } \frac{\sqrt{5}}{10}.$$

61. We have a limit of the form $\infty \cdot 0$. Rewrite the expression.

$$x^2 e^{-\sqrt{x}} = \frac{x^2}{e^{\sqrt{x}}}$$

Now both the numerator and denominator become infinite and l'Hospital's rule applies to the limit of the form ∞/∞ . Differentiate the numerator and the denominator.

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2x}{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}} = \lim_{x \rightarrow \infty} \frac{4x^{3/2}}{e^{\sqrt{x}}}$$

This problem is similar to what we started with so we handle it in the same manner. In fact, continue the process three more times.

$$\lim_{x \rightarrow \infty} \frac{4x^{3/2}}{e^{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{6x^{1/2}}{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}} = \lim_{x \rightarrow \infty} \frac{12x}{e^{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{12}{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}} = \lim_{x \rightarrow \infty} \frac{24\sqrt{x}}{e^{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{\frac{12}{\sqrt{x}}}{\frac{e^{\sqrt{x}}}{(2\sqrt{x})}} = \lim_{x \rightarrow \infty} \frac{24}{e^{\sqrt{x}}} = 0$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow \infty} x^2 e^{-\sqrt{x}} = 0 \text{ (does not exist)}$$

62. The limit in the numerator is ∞ , as is the limit in the denominator, so l'Hospital's rule applies. Taking derivatives separately in the numerator and the denominator gives

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(x^3 + 1)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{(2\sqrt{x})}}{\frac{3x^2}{(x^3 + 1)}} = \lim_{x \rightarrow \infty} \frac{x^3 + 1}{6x^{5/2}}$$

This problem is similar to what we started with so we handle it in the same manner. In fact, continue the process three more times.

$$\lim_{x \rightarrow \infty} \frac{x^3 + 1}{6x^{5/2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{15x^{3/2}} = \lim_{x \rightarrow \infty} \frac{6x}{\frac{45x^{1/2}}{2}} = \lim_{x \rightarrow \infty} \frac{6}{\frac{45}{(4x^{1/2})}} = \lim_{x \rightarrow \infty} \frac{8x^{1/2}}{15} = \infty$$

Therefore, by l'Hospital's rule,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln(x^3 + 1)} = \infty.$$

63.
$$\lim_{x \rightarrow 0} \left(\frac{e^{3x}}{x^2} - \frac{1}{x^2} - \frac{3}{x} \right).$$

Rewrite over a common denominator.

$$\lim_{x \rightarrow 0} \left(\frac{e^{3x} - 1 - 3x}{x^2} \right)$$

The numerator and denominator have limit 0, so l'Hospital's rule applies. Differentiate the numerator and denominator.

$$\lim_{x \rightarrow 0} \left(\frac{e^{3x} - 1 - 3x}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{3e^{3x} - 3}{2x} \right)$$

The numerator and denominator still have limit 0, so differentiate the numerator and denominator again.

$$\lim_{x \rightarrow 0} \left(\frac{3e^{3x} - 3}{2x} \right) = \lim_{x \rightarrow 0} \left(\frac{9e^{3x}}{2} \right) = \frac{9}{2}$$

Thus

$$\lim_{x \rightarrow 0} \left(\frac{e^{3x}}{x^2} - \frac{1}{x^2} - \frac{3}{x} \right) = \frac{9}{2}.$$

64. Find $\lim_{x \rightarrow 0} \left(\frac{2}{x^3} + \frac{2}{x^2} + \frac{1}{x} - \frac{2e^x}{x^3} \right)$.

Rewrite over a common denominator.

$$\lim_{x \rightarrow 0} \left(\frac{2 + 2x + x^2 - 2e^x}{x^3} \right)$$

The numerator and denominator have limit 0, so l'Hospital's rule applies. Differentiate the numerator and denominator.

$$\lim_{x \rightarrow 0} \left(\frac{2 + 2x + x^2 - 2e^x}{x^3} \right) = \lim_{x \rightarrow 0} \left(\frac{2 + 2x - 2e^x}{3x^2} \right)$$

The numerator and denominator still have limit 0, so differentiate the numerator and denominator again.

$$\lim_{x \rightarrow 0} \left(\frac{2 + 2x - 2e^x}{3x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{2 - 2e^x}{6x} \right)$$

The numerator and denominator still have limit 0, so differentiate the numerator and denominator again.

$$\lim_{x \rightarrow 0} \left(\frac{2 - 2e^x}{6x} \right) = \lim_{x \rightarrow 0} \left(\frac{-2e^x}{6} \right) = \frac{-2}{6} = -\frac{1}{3}$$

Thus

$$\lim_{x \rightarrow 0} \left(\frac{2}{x^3} + \frac{2}{x^2} + \frac{1}{x} - \frac{2e^x}{x^3} \right) = -\frac{1}{3}.$$

65. Find $\lim_{x \rightarrow 0} \left(\frac{\ln(1 - 4x)}{x^2} + \frac{4}{x} \right)$.

Rewrite over a common denominator.

$$\lim_{x \rightarrow 0} \left(\frac{\ln(1 - 4x) + 4x}{x^2} \right)$$

The numerator and denominator have limit 0, so l'Hospital's rule applies. Differentiate the numerator and denominator.

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{\ln(1 - 4x) + 4x}{x^2} \right) &= \lim_{x \rightarrow 0} \left(\frac{\frac{-4}{1-4x} + 4}{2x} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{-16x}{(1 - 4x)(2x)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{-8}{1 - 4x} \right) = -8 \end{aligned}$$

Thus

$$\lim_{x \rightarrow 0} \left(\frac{\ln(1 - 4x)}{x^2} + \frac{4}{x} \right) = -8.$$

66. Find $\lim_{x \rightarrow 0} \left(\frac{1}{x^3} + \frac{1}{x^2} \right)$.

Rewrite over a common denominator.

$$\lim_{x \rightarrow 0} \left(\frac{x + 1}{x^3} \right)$$

The numerator has limit 1 and the denominator has limit 0, so this limit does not exist.

67. $f(x) = x^3 - 8x^2 + 18x - 12$

$$f'(x) = 3x^2 - 16x + 18$$

$f(4) = -4 < 0$ and $f(5) = 3 > 0$ so a solution exists in $(4, 5)$.

Let $c_1 = 4$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 4 - \frac{-4}{2} = 6$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 6 - \frac{24}{30} = 5.2$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 5.2 - \frac{5.888}{15.92} = 4.8302$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 4.8302 - \frac{0.98953}{10.709} = 4.7378$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = 4.7378 - \frac{0.05462}{9.5354} = 4.7321$$

$$c_7 = c_6 - \frac{f(c_6)}{f'(c_6)} = 4.7321 - \frac{4.7 \cdot 10^{-4}}{9.4647} = 4.7321$$

Subsequent approximations will agree with c_6 and c_7 to the nearest hundredth. Thus,
 $x = 4.73$.

68. $f(x) = 3x^3 - 4x^2 - 4x - 7$

$$f'(x) = 9x^2 - 8x - 4$$

$f(2) = -7 < 0$ and $f(3) = 26 > 0$ so a solution exists in $(2, 3)$.

Let $c_1 = 2$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{-7}{16} = 2.4375$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 2.4375 - \frac{2.9309}{29.973} = 2.3397$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 2.3397 - \frac{0.16835}{26.55} = 2.3334$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 2.3334 - \frac{0.00176}{26.336} = 2.3333$$

Subsequent approximations will agree with c_4 and c_5 to the nearest hundredth. Thus,
 $x = 2.33$.

69. $f(x) = x^4 + 3x^3 - 4x^2 - 21x - 21$

$$f'(x) = 4x^3 + 9x^2 - 8x - 21$$

$f(2) = -39 < 0$ and $f(3) = 42 > 0$ so a solution exists in $(2, 3)$.

Let $c_1 = 2$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{-39}{31} = 3.2581$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 3.2581 - \frac{84.558}{186.81} = 2.8055$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 2.8055 - \frac{16.796}{115.72} = 2.6604$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 2.6604 - \frac{1.4037}{96.735} = 2.6459$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = 2.6459 - \frac{0.01411}{94.933} = 2.6458$$

Subsequent approximations will agree with c_5 and c_6 to the nearest hundredth. Thus,
 $x = 2.65$.

70. $f(x) = x^4 + x^3 - 14x^2 - 15x - 15$

$$f'(x) = 4x^3 + 3x^2 - 28x - 15$$

$f(3) = -78 < 0$ and $f(4) = 21 > 0$ so a solution exists in $(3, 4)$.

Let $c_1 = 3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 3 - \frac{-78}{36} = 5.1667$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 5.1667 - \frac{384.31}{472.11} = 4.3527$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 4.3527 - \frac{95.883}{249.83} = 3.9689$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 3.9689 - \frac{15.586}{171.202} = 3.8779$$

$$c_6 = c_5 - \frac{f(c_5)}{f'(c_5)} = 3.8779 - \frac{0.75897}{154.8} = 3.8730$$

$$c_7 = c_6 - \frac{f(c_6)}{f'(c_6)} = 3.8730 - \frac{0.00256}{153.94} = 3.8730$$

Subsequent approximations will agree with c_6 and c_7 to the nearest hundredth. Thus,
 $x = 3.87$.

71. $\sqrt{37.6}$ is a solution of $x^2 - 37.6 = 0$.

$$f(x) = x^2 - 37.6$$

$$f'(x) = 2x$$

Since $6 < \sqrt{37.6} < 7$, let $c_1 = 6$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 6 - \frac{-1.6}{12} = 6.1333$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 6.1333 - \frac{0.01737}{12.2666} = 6.1319$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 6.1319 - \frac{0.0002}{12.2638} = 6.1319$$

Since $c_4 = c_3 = 6.1319$, to the nearest thousandth, $\sqrt{37.6} = 6.132$.

72. $\sqrt{51.7}$ is a solution of $x^2 - 51.7 = 0$.

$$f(x) = x^2 - 51.7$$

$$f'(x) = 2x$$

Since $7 < \sqrt{51.7} < 8$, let $c_1 = 7$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 7 - \frac{-2.7}{14} = 7.1929$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 7.1929 - \frac{0.03781}{14.3858} = 7.1903$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 7.1903 - \frac{0.0004}{14.3806} = 7.1903$$

Since $c_4 = c_3 = 7.1903$, to the nearest thousandth, $\sqrt{51.7} = 7.190$.

73. $\sqrt[3]{94.7}$ is a solution of $x^3 - 94.7 = 0$.

$$f(x) = x^3 - 94.7$$

$$f'(x) = 3x^2$$

Since $4 < \sqrt[3]{94.7} < 5$, let $c_1 = 4$

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 4 - \frac{-30.7}{48} = 4.6396$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 4.6396 - \frac{5.1715}{64.5777} = 4.5595$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 4.5595 - \frac{0.08763}{62.3671} = 4.5581$$

$$c_5 = c_4 - \frac{f(c_4)}{f'(c_4)} = 4.5581 - \frac{0.0003}{62.3288} = 4.5581$$

Since $c_3 = c_4 = 4.5581$, to the nearest thousandth, $\sqrt[3]{94.7} = 4.558$.

74. $\sqrt[4]{102.6}$ is a solution of $x^4 - 102.6 = 0$.

$$f(x) = x^4 - 102.6$$

$$f'(x) = 4x^3$$

Since $3 < \sqrt[4]{102.6} < 4$, let $c_1 = 3$.

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 3 - \frac{-21.6}{108} = 3.2$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 3.2 - \frac{2.2576}{131.072} = 3.1828$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 3.1828 - \frac{0.02127}{128.9698} = 3.1826$$

Rounded to the nearest thousandth, c_3 , c_4 , and subsequent approximations agree. Thus, $\sqrt[4]{102.6} = 3.183$.

75. The yearly incomes produced by the mine form a geometric sequence with $r = 118\% = 1.18$ and $a_1 = 750,000$. To determine the total amount produced in 8 years, use the formula to find S_n with $n = 8$, $r = 1.18$, and $a = a_1 = 750,000$.

$$\begin{aligned} S_8 &= \frac{750,000[(1.18)^8 - 1]}{1.18 - 1} \\ &= \frac{750,000(3.75886 - 1)}{0.18} \approx 11,495,247 \end{aligned}$$

The total amount of income produced by the mine in five years is \$11,495,247.

76. This ordinary annuity will amount to \$5000 in 4 years at 8% compounded semiannually. Thus, $S = 5000$,
 $n = 4 \cdot 2 = 8$, and $i = \frac{8\%}{2} = 4\% = 0.04$,
 so

$$5000 = R \cdot s_{\overline{8}|0.04}$$

$$R = \frac{5000}{s_{\overline{8}|0.04}} = \frac{5000}{9.21423} \approx 542.64$$

or \$542.64.

77. The payments form an ordinary annuity with $R = 491$, $n = 9 \cdot 4 = 36$, and
 $i = \frac{9.4\%}{4} = 2.35\% = 0.0235$. The amount of this annuity is

$$S = 491 \left[\frac{(1.0235)^{36} - 1}{0.0235} \right]$$

The number in brackets, $s_{\overline{36}|0.0235}$, is 55.64299673, so that

$$S = 491(55.64299673) \approx 27,320.71.$$

or \$27,320.71

78. The payments form an ordinary annuity with $R = 1526.38$, $n = 5 \cdot 2 = 10$, and $i = \frac{7.6\%}{2} = 3.8\% = 0.038$. The amount of this annuity is

$$S = 1526.38 \left[\frac{(1.038)^{10} - 1}{0.038} \right]$$

The number in brackets, $s_{\overline{10}|0.038}$, is 11.89534558, so that

$$S = 1526.38(11.89534558) \approx 18,156.82$$

or \$18,156.82.

79. \$20,000 is the present value of an annuity of R dollars, with 9 periods, and $i = 8.9\% = 0.089$ per period.

$$P = R \cdot a_{\overline{n}|i}$$

$$20,000 = R \cdot a_{\overline{9}|0.089}$$

$$\begin{aligned} R &= \frac{20,000}{a_{\overline{9}|0.089}} = \frac{20,000}{6.019696915} \\ &\approx 3322.43 \end{aligned}$$

Each payment is \$3322.43.

80. \$49,275 is the present value of an annuity of R dollars, with 48 period, and $i = \frac{12.2\%}{12} = 1.01\overline{6}\%$
 $= 0.0101\overline{6}$ per period.

$$\begin{aligned} P &= R \cdot a_{\overline{n}|i} \\ 49,275 &= R \cdot a_{\overline{48}|0.0101\overline{6}} \\ R &= \frac{49,275}{a_{\overline{48}|0.0101\overline{6}}} \\ &= \frac{49,275}{37.83272858} \\ &\approx 1302.44 \end{aligned}$$

Each payment is \$1302.44.

81. The present value, P , is 156,890,
 $i = \frac{0.0774}{12} = 0.00645$, and $n = 12 \cdot 25 = 300$.

$$\begin{aligned} 156,890 &= R \cdot a_{\overline{300}|0.00645} \\ &= R \left[\frac{1 - (1 + 0.00645)^{-300}}{0.00645} \right] \\ &= R \left[\frac{1 - 0.1453244675}{0.00645} \right] \\ &= R \left[\frac{0.8546755325}{0.00645} \right] \\ R &\approx 1184.01 \end{aligned}$$

Monthly payments of \$1184.01 will be required to amortize the loan.

82. The present value, P , is 177,110, $i = \frac{0.0845}{12}$
 $= 0.007041\overline{6}$, and $n = 12 \cdot 30 = 360$.

$$\begin{aligned} 177,110 &= R \cdot a_{\overline{360}|0.007041\overline{6}} \\ &= R \left[\frac{1 - (1 + 0.007041\overline{6})^{-360}}{0.007041\overline{6}} \right] \\ &= R \left[\frac{1 - 0.0799689882}{0.007041\overline{6}} \right] \\ &= R \left[\frac{0.9200310118}{0.007041\overline{6}} \right] \\ R &\approx 1355.55 \end{aligned}$$

Monthly payments of \$1355.55 will be required to amortize the loan.

83. The doubling time n for a quantity that increases at an annual rate r is given by

$$n = \frac{\ln 2}{\ln(1 + r)} = \frac{\ln 2}{\ln 1.0325} \approx 21.67.$$

It will take about 21.67 years.

According to the Rule of 70, the doubling time is given by

$$\begin{aligned} \text{Doubling time} &\approx \frac{70}{100r} \\ &= \frac{70}{100(0.0325)} \\ &= 21.54. \end{aligned}$$

It will take about 21.54 years, a difference of 0.13 year, or about 7 weeks.

84. The doubling time n for a quantity that increases at an annual rate r is given by

$$n = \frac{\ln 2}{\ln(1 + r)} = \frac{\ln 2}{\ln 1.09} \approx 8.04.$$

It will take about 8.04 years.

According to the Rule of 72, the doubling time is given by

$$\text{Doubling time} \approx \frac{70}{100r} = \frac{72}{100(0.09)} = 8.$$

It will take about 8 years, a difference of 0.04 year, or about 2 weeks.

85. 2 hours amounts to six doubling times, so after two hours the number of bacteria will be $1000(2^6)$
 $= 64,000$ bacteria.

86. Since the number of crimes decreases by 8% each year, each year it will be 0.92 times the number reported the previous year. Since number of crimes was 22,700 in the year before the neighborhood program started, the number of crimes n years after program began will be $22,700(0.92)^n$. The number after five years will be $22,700(0.92)^5 \approx 14,961$.

Extended Application: Living Assistance and Subsidized Housing

1. Solve the following system:

$$S = 0.3R$$

$$0.2(100 + S) = R$$

Substitute for S in the second equation.

$$0.2(100 + 0.3R) = R$$

$$20 + 0.06R = R$$

$$R = \frac{20}{0.94} \approx 21.28$$

Then $S = 0.3R$

$$= 0.3 \left(\frac{20}{0.94} \right) \\ \approx 6.38$$

Thus the rent will be $300 + R = \$321.28$, and the stipend will be $1000 + 100 + S = \$1106.38$, as found using infinite series.

2. Under this scenario, the sequence of stipends will be

$$1000 - 50 - 50(0.3)(0.2) - 50[(0.3)(0.2)]^2 - \dots$$

After the first term, the amount subtracted is a geometric series with $a_1 = 50$ and $r = 0.06$.

Its sum is $\frac{50}{1 - 0.06} = \frac{50}{0.94} \approx 53.19$. Thus the final stipend will be $1000 - 53.19 = 946.81$, or \$946.81.

The rents will be $300 - 10 - 10(0.06)^2 - 10(0.06)^2 - \dots$

After the first term, the amount subtracted is a geometric series with $a_1 = 10$ and $r = 0.06$.

Its sum is $\frac{10}{1 - 0.06} = \frac{10}{0.94} \approx 10.64$. Thus the final rent will be $300 - 10.64 = 289.36$, or \$289.36.

3. The distances proposed by Eastville, measured from Eastville toward Westville, are $6 + \frac{3}{2} + \frac{1.5}{2} + \dots$. Which we can write as $6 \left(1 + \frac{1}{4} + \frac{1}{16} + \dots \right)$. In parentheses is a geometric series with sum $\frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$, so the total changes proposed by Eastville would locate the fire station $6/(3/4) = 8$ miles from Eastville.

The distances proposed by Westville, measured from Westville toward Eastville, are half of the corresponding changes proposed by Westville (since Westville chose first), so the series of distances will be $3 + \frac{3}{4} + \frac{1.5}{4} + \dots$ or

$\left(1 + \frac{1}{4} + \frac{1}{16} + \dots \right)$ with a sum of $3/(3/4) = 4$ miles from Westville, the same location as found by summing Eastville's proposals.

4. Under this scenario, Eastville's proposed changes from the series $4 + \frac{4}{3} + \frac{4}{9} + \dots$ which has sum

$$\frac{4}{1 - \frac{1}{3}} = 6, \text{ and Westville's changes form the}$$

same series, so in this case the chosen location for the pool is half way between the two towns.

5. The initial \$1000 doesn't affect our answers to this questions, so we will look at the terms after the first term, which are a geometric series with $a_1 = 100$ and $r = 0.06$. The sum of the first n term is

$$S_n = \frac{100(0.06^n - 1)}{0.06 - 1} = \frac{100}{0.94}(1 - 0.06^n).$$

$$S_2 = 106$$

$$S_3 = 106.36$$

$$S_4 = 106.3816$$

Thus, including the first term of 1000, 3 terms give the value to the nearest dollar, four terms to the nearest dime and 5 terms to the nearest penny.

6. Experimentation shows that you need to sum about 50 terms to get within 0.01 of $\ln 2$.

THE TRIGONOMETRIC FUNCTIONS

13.1 Definitions of the Trigonometric Functions

Your Turn 1

- (a) Since $1^\circ = \pi/180$ radians,

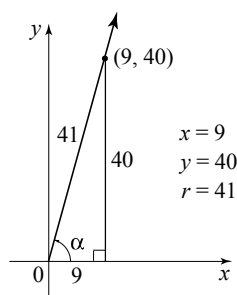
$$\begin{aligned} 210^\circ &= 210 \left(\frac{\pi}{180} \right) \text{ radians} \\ &= \frac{7\pi}{6} \text{ radians.} \end{aligned}$$

- (b) Since 1 radian $= 180^\circ/\pi$,

$$\begin{aligned} \frac{3\pi}{4} \text{ radians} &= \frac{3\pi}{4} \left(\frac{180^\circ}{\pi} \right) \\ &= 3(45^\circ) \\ &= 135^\circ \end{aligned}$$

Your Turn 2

Sketch the triangle formed by joining the origin, the point (9, 40), and the point (9, 0) on the x -axis.



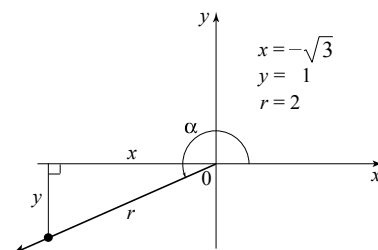
To find r , use $r = \sqrt{x^2 + y^2}$. Here $x = 9$, and $y = 40$, so

$$\begin{aligned} r &= \sqrt{9^2 + 40^2} \\ &= \sqrt{1681} \\ &= 41. \end{aligned}$$

$$\begin{aligned} \sin \alpha &= \frac{y}{r} = \frac{40}{41} & \tan \alpha &= \frac{y}{x} = \frac{40}{9} \\ \sec \alpha &= \frac{r}{x} = \frac{41}{9} & \cos \alpha &= \frac{x}{r} = \frac{9}{41} \\ \cot \alpha &= \frac{x}{y} = \frac{9}{40} & \csc \alpha &= \frac{r}{y} = \frac{41}{40} \end{aligned}$$

Your Turn 3

Sketch the angle and label the sides of an appropriate right triangle.



The triangle formed in the third quadrant is a 30° - 60° - 90° triangle, so we label the sides as in Example 4, except that here in the third quadrant both x and y are negative.

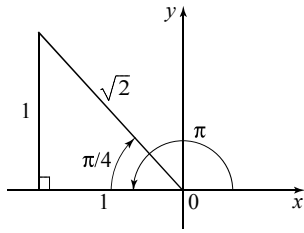
$$\begin{aligned} \sin \frac{7\pi}{6} &= \frac{-1}{2} = -\frac{1}{2} & \tan \frac{7\pi}{6} &= \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \sec \frac{7\pi}{6} &= \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3} & \cos \frac{7\pi}{6} &= \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \\ \cot \frac{7\pi}{6} &= \frac{-\sqrt{3}}{-1} = \sqrt{3} & \csc \frac{7\pi}{6} &= \frac{2}{-1} = -2 \end{aligned}$$

Your Turn 4

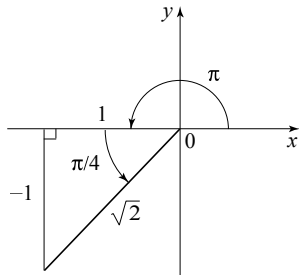
- (a) $\cos 6^\circ \approx 0.9945$
 (b) $\sec 4 \approx -1.5299$

Your Turn 5

The cosine function is negative in quadrants II and III. In quadrant II we draw a triangle with an angle whose cosine is $-\sqrt{2}/2$, which we recognize as a 45° - 45° - 90° degree triangle (Figure (a)). Thus the angle corresponding to the terminal side has measure $\pi - (\pi/4) = 3\pi/4$. Drawing the same triangle in quadrant III (Figure (b)) we see that the angle corresponding to the terminal side is $\pi + (\pi/4) = 5\pi/4$. Thus the equation $\cos x = -\sqrt{2}/2$ has two solutions between 0 and 2π , namely $3\pi/4$ and $5\pi/4$.



(a)



(b)

13.1 Exercises

$$1. \quad 60^\circ = 60 \left(\frac{\pi}{180} \right) = \frac{\pi}{3}$$

$$2. \quad 90^\circ = 90 \left(\frac{\pi}{180} \right) = \frac{\pi}{2}$$

$$3. \quad 150^\circ = 150 \left(\frac{\pi}{180} \right) = \frac{5\pi}{6}$$

$$4. \quad 135^\circ = 135 \left(\frac{\pi}{180} \right) = \frac{3\pi}{4}$$

$$5. \quad 270^\circ = 270 \left(\frac{\pi}{180} \right) = \frac{3\pi}{2}$$

$$6. \quad 320^\circ = 320 \left(\frac{\pi}{180} \right) = \frac{16\pi}{9}$$

$$7. \quad 495^\circ = 495 \left(\frac{\pi}{180} \right) = \frac{11\pi}{4}$$

$$8. \quad 510^\circ = 510 \left(\frac{\pi}{180} \right) = \frac{17\pi}{6}$$

$$9. \quad \frac{5\pi}{4} = \frac{5\pi}{4} \left(\frac{180^\circ}{\pi} \right) = 225^\circ$$

$$10. \quad \frac{2\pi}{3} = \frac{2\pi}{3} \left(\frac{180^\circ}{\pi} \right) = 120^\circ$$

$$11. \quad -\frac{13\pi}{6} = -\frac{13\pi}{6} \left(\frac{180^\circ}{\pi} \right) = -390^\circ$$

$$12. \quad -\frac{\pi}{4} = -\frac{\pi}{4} \left(\frac{180^\circ}{\pi} \right) = -45^\circ$$

$$13. \quad \frac{8\pi}{5} = \frac{8\pi}{5} \left(\frac{180^\circ}{\pi} \right) = 288^\circ$$

$$14. \quad \frac{5\pi}{9} = \frac{5\pi}{9} \left(\frac{180^\circ}{\pi} \right) = 100^\circ$$

$$15. \quad \frac{7\pi}{12} = \frac{7\pi}{12} \left(\frac{180^\circ}{\pi} \right) = 105^\circ$$

$$16. \quad 5\pi = 5\pi \left(\frac{180^\circ}{\pi} \right) = 900^\circ$$

17. Let α = the angle with terminal side through $(-3, 4)$. Then $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (4)^2} \\ = \sqrt{25} = 5.$$

$$\sin \alpha = \frac{y}{r} = \frac{4}{5} \quad \cot \alpha = \frac{x}{y} = -\frac{3}{4}$$

$$\cos \alpha = \frac{x}{r} = -\frac{3}{5} \quad \sec \alpha = \frac{r}{x} = -\frac{5}{3}$$

$$\tan \alpha = \frac{y}{x} = -\frac{4}{3} \quad \csc \alpha = \frac{r}{y} = \frac{5}{4}$$

18. Let α = the angle with terminal side through $(-12, -5)$. Then $x = -12$, $y = -5$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{144 + 25} \\ = \sqrt{169} = 13.$$

$$\sin \alpha = \frac{y}{r} = -\frac{5}{13} \quad \cot \alpha = \frac{x}{y} = \frac{12}{5}$$

$$\cos \alpha = \frac{x}{r} = -\frac{12}{13} \quad \sec \alpha = \frac{r}{x} = -\frac{13}{12}$$

$$\tan \alpha = \frac{y}{x} = \frac{5}{12} \quad \csc \alpha = \frac{r}{y} = -\frac{13}{5}$$

19. Let α = the angle with terminal side through $(7, -24)$. Then $x = 7$, $y = -24$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{49 + 576} \\ = \sqrt{625} = 25.$$

$$\begin{aligned} \sin \alpha &= \frac{y}{r} = -\frac{24}{25} & \cot \alpha &= \frac{x}{y} = -\frac{7}{24} \\ \cos \alpha &= \frac{x}{r} = \frac{7}{25} & \sec \alpha &= \frac{r}{x} = \frac{25}{7} \\ \tan \alpha &= \frac{y}{x} = -\frac{24}{7} & \csc \alpha &= \frac{r}{y} = -\frac{25}{24} \end{aligned}$$

20. Let α = the angle with terminal side through $(20, 15)$. Then $x = 20$, $y = 15$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{400 + 225} \\ = \sqrt{625} = 25.$$

$$\begin{aligned} \sin \alpha &= \frac{y}{r} = \frac{3}{5} & \cot \alpha &= \frac{x}{y} = \frac{4}{3} \\ \cos \alpha &= \frac{x}{r} = \frac{4}{5} & \sec \alpha &= \frac{r}{x} = \frac{5}{4} \\ \tan \alpha &= \frac{y}{x} = \frac{3}{4} & \csc \alpha &= \frac{r}{y} = \frac{5}{3} \end{aligned}$$

21. In quadrant I, all six trigonometric functions are positive, so their sign is +.
22. In quadrant II, $x < 0$ and $y > 0$.

Furthermore, $r > 0$.

$$\begin{aligned} \sin \theta &= \frac{y}{r} > 0, \text{ so the sign is } +. \\ \cos \theta &= \frac{x}{r} < 0, \text{ so the sign is } -. \\ \tan \theta &= \frac{y}{x} < 0, \text{ so the sign is } -. \\ \cot \theta &= \frac{x}{y} < 0, \text{ so the sign is } -. \\ \sec \theta &= \frac{r}{x} < 0, \text{ so the sign is } -. \\ \csc \theta &= \frac{r}{y} > 0, \text{ so the sign is } +. \end{aligned}$$

23. In quadrant III, $x < 0$ and $y < 0$.

Furthermore, $r > 0$.

$$\begin{aligned} \sin \theta &= \frac{y}{r} < 0, \text{ so the sign is } -. \\ \cos \theta &= \frac{x}{r} < 0, \text{ so the sign is } -. \\ \tan \theta &= \frac{y}{x} > 0, \text{ so the sign is } +. \\ \cot \theta &= \frac{x}{y} > 0, \text{ so the sign is } +. \\ \sec \theta &= \frac{r}{x} < 0, \text{ so the sign is } -. \\ \csc \theta &= \frac{r}{y} < 0, \text{ so the sign is } -. \end{aligned}$$

24. In quadrant IV, $x > 0$ and $y < 0$.

Also, $r > 0$.

$$\begin{aligned} \sin \theta &= \frac{y}{r} < 0, \text{ so the sign is } -. \\ \cos \theta &= \frac{x}{r} > 0, \text{ so the sign is } +. \\ \tan \theta &= \frac{y}{x} < 0, \text{ so the sign is } -. \\ \cot \theta &= \frac{x}{y} < 0, \text{ so the sign is } -. \\ \sec \theta &= \frac{r}{x} > 0, \text{ so the sign is } +. \\ \csc \theta &= \frac{r}{y} < 0, \text{ so the sign is } -. \end{aligned}$$

25. When an angle θ of 30° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (\sqrt{3}, 1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2.$$

$$\begin{aligned} \tan \theta &= \frac{y}{x} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \cot \theta &= \frac{x}{y} = \sqrt{3} \\ \csc \theta &= \frac{r}{y} = 2 \end{aligned}$$

26. When an angle θ of 45° is drawn in standard position, $(x, y) = (1, 1)$ is one point on its terminal side. Then

$$\begin{aligned} r &= \sqrt{1+1} = \sqrt{2}. \\ \sin \theta &= \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \cos \theta &= \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sec \theta &= \frac{r}{x} = \sqrt{2} \\ \csc \theta &= \frac{r}{y} = \sqrt{2} \end{aligned}$$

27. When an angle θ of 60° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, \sqrt{3})$. Then

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{1+3} = 2. \\ \sin \theta &= \frac{y}{r} = \frac{\sqrt{3}}{2} \\ \cot \theta &= \frac{x}{y} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \csc \theta &= \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \end{aligned}$$

28. When an angle θ of 120° is drawn in standard position, $(x, y) = (-1, \sqrt{3})$ is one point on its terminal side. Then

$$\begin{aligned} r &= \sqrt{1+3} = 2. \\ \cos \theta &= \frac{x}{r} = -\frac{1}{2} \\ \cot \theta &= \frac{x}{y} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \\ \sec \theta &= \frac{r}{x} = -2 \end{aligned}$$

29. When an angle θ of 135° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-1, 1)$. Then

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}. \\ \tan \theta &= \frac{y}{x} = -1 \\ \cot \theta &= \frac{x}{y} = -1 \end{aligned}$$

30. When an angle θ of 150° is drawn in standard position, $(x, y) = (-\sqrt{3}, 1)$ is one point on its terminal side. Then

$$\begin{aligned} r &= \sqrt{3+1} = 2. \\ \sin \theta &= \frac{y}{r} = \frac{1}{2} \\ \cot \theta &= \frac{x}{y} = -\sqrt{3} \\ \sec \theta &= \frac{r}{x} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3} \end{aligned}$$

31. When an angle θ of 210° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-\sqrt{3}, -1)$. Then

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{3+1} = 2. \\ \cos \theta &= \frac{x}{r} = -\frac{\sqrt{3}}{2} \\ \sec \theta &= \frac{r}{x} = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3} \end{aligned}$$

32. When an angle θ of 240° is drawn in standard position, $(x, y) = (-1, -\sqrt{3})$ is one point on its terminal side.

$$\begin{aligned} \tan \theta &= \frac{y}{x} = \sqrt{3} \\ \cot \theta &= \frac{x}{y} = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \end{aligned}$$

33. When an angle of $\frac{\pi}{3}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, \sqrt{3})$. Then

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{1+3} = 2. \\ \sin \frac{\pi}{3} &= \frac{y}{r} = \frac{\sqrt{3}}{2} \end{aligned}$$

34. When an angle of $\frac{\pi}{6}$ is drawn in standard position, $(x, y) = (\sqrt{3}, 1)$ is one point on its terminal side. Then

$$\begin{aligned} r &= \sqrt{3+1} = 2. \\ \cos \frac{\pi}{6} &= \frac{x}{r} = \frac{\sqrt{3}}{2} \end{aligned}$$

35. When an angle of $\frac{\pi}{4}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, 1)$.

$$\tan \frac{\pi}{4} = \frac{y}{x} = 1$$

36. When an angle of $\frac{\pi}{3}$ is drawn in standard position, $(x, y) = (1, \sqrt{3})$ is one point on its terminal side.

$$\cot \frac{\pi}{3} = \frac{x}{y} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

37. When an angle of $\frac{\pi}{6}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (\sqrt{3}, 1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2.$$

$$\csc \frac{\pi}{6} = \frac{r}{y} = \frac{2}{1} = 2$$

38. When an angle of $\frac{3\pi}{2}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (0, -1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{0 + 1} = 1.$$

$$\sin \frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1$$

39. When an angle of 3π is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-1, 0)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1} = 1.$$

$$\cos 3\pi = \frac{x}{r} = -1$$

40. When an angle of π is drawn in standard position, $(x, y) = (-1, 0)$ is one point on its terminal side. Then

$$r = \sqrt{1 + 0} = 1.$$

$$\sec \pi = \frac{r}{x} = -1$$

41. When an angle of $\frac{7\pi}{4}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, -1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 1} = \sqrt{2}.$$

$$\sin \frac{7\pi}{4} = \frac{y}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

42. When an angle of $\frac{5\pi}{2}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (0, 1)$. Then

$$\tan \frac{5\pi}{2} = \frac{y}{x} = \frac{1}{0} \text{ undefined.}$$

43. When an angle of $\frac{5\pi}{4}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-1, -1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 1} = \sqrt{2}.$$

$$\sec \frac{5\pi}{4} = \frac{r}{x} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$$

44. When an angle of 5π is drawn in standard position, $(x, y) = (-1, 0)$ is one point on its terminal side. Then

$$r = \sqrt{1 + 0} = 1.$$

$$\cos 5\pi = \frac{x}{r} = -1$$

45. When an angle of $-\frac{3\pi}{4}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-1, -1)$. Then

$$\cot \left(-\frac{3\pi}{4} \right) = \frac{y}{x} = \frac{-1}{-1} = 1$$

46. When an angle of $-\frac{5\pi}{6}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-\sqrt{3}, -1)$. Then

$$\tan \left(-\frac{5\pi}{6} \right) = \frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

47. When an angle of $-\frac{7\pi}{6}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-\sqrt{3}, 1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2.$$

$$\sin\left(-\frac{7\pi}{6}\right) = \frac{y}{r} = \frac{1}{2}$$

48. When an angle of $-\frac{\pi}{6}$ is drawn in standard position, $(x, y) = (\sqrt{3}, -1)$ is one point on its terminal side. Then

$$r = \sqrt{3 + 1} = 2.$$

$$\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

49. The cosine function is positive in quadrants I and IV. We know that $\cos(\pi/3) = 1/2$, so the solution in quadrant I is $\pi/3$. The solution in quadrant IV is $2\pi - (\pi/3) = 5\pi/3$. The two solutions of $\cos x = 1/2$ between 0 and 2π are $\pi/3$ and $5\pi/3$.

50. The sine function is negative in quadrants III and IV. We know that $\sin(\pi/6) = 1/2$. The solution in quadrant III is $\pi + (\pi/6) = 7\pi/6$. The solution in quadrant IV is $2\pi - (\pi/6) = 11\pi/6$. The two solutions of $\sin x = -1/2$ between 0 and 2π are $7\pi/6$ and $11\pi/6$.

51. The tangent function is negative in quadrants II and IV. We know that $\tan(\pi/4) = 1$. The solution in quadrant II is $\pi - (\pi/4) = 3\pi/4$. The solution in quadrant IV is $2\pi - (\pi/4) = 7\pi/4$. The two solutions of $\tan x = -1$ between 0 and 2π are $3\pi/4$ and $7\pi/4$.

52. The tangent function is positive in quadrants I and III. We know that $\tan(\pi/3) = \sqrt{3}$, so the solution in quadrant I is $\pi/3$. The solution in quadrant III is $\pi + (\pi/3) = 4\pi/3$. The two solutions of $\tan x = \sqrt{3}$ between 0 and 2π are $\pi/3$ and $4\pi/3$.

53. The secant function is negative in quadrants II and III. We know that $\sec(\pi/6) = 2/\sqrt{3}$, so the solution in quadrant II is $\pi - (\pi/6) = 5\pi/6$. The solution in quadrant III is $\pi + (\pi/6) = 7\pi/6$. The two solutions of $\sec x = -2/\sqrt{3}$ between 0 and 2π are $5\pi/6$ and $7\pi/6$.

54. The secant function is positive in quadrants I and IV. We know that $\sec(\pi/4) = \sqrt{2}$, so the solution in quadrant I is $\pi/4$. The solution in quadrant IV is $2\pi - (\pi/4) = 7\pi/4$. The two solutions of $\sec x = \sqrt{2}$ between 0 and 2π are $\pi/4$ and $7\pi/4$.

55. $\sin 39^\circ \approx 0.6293$

56. $\cos 67^\circ \approx 0.3907$

57. $\tan 123^\circ \approx -1.5399$

58. $\tan 54^\circ \approx 1.3764$

59. $\sin 0.3638 \approx 0.3558$

60. $\tan 1.0123 \approx 1.6004$

61. $\cos 1.2353 \approx 0.3292$

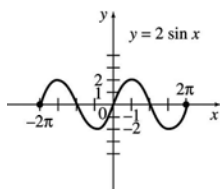
62. $\sin 1.5359 \approx 0.9994$

63. $f(x) = \cos(3x)$ is of the form $f(x) = a \cos(bx)$ where $a = 1$ and $b = 3$. Thus, $a = 1$ and $T = \frac{2\pi}{b} = \frac{2\pi}{3}$.

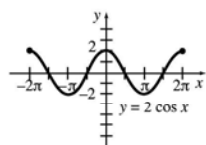
64. $g(t) = 2 \sin\left(\frac{\pi}{4}t + 2\right)$ is of the form $g(t) = a \sin(bt + c)$ where $a = 1$, $b = \frac{\pi}{4}$, and $c = 2$. Thus, $a = 2$ and $T = \frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{4}} = 8$.

65. $s(x) = 3\sin(880\pi t - 7)$ is of the form $s(x) = a\sin(bt + c)$ where $a = 3$, $b = 880\pi$, and $c = -7$. Thus, $a = 3$ and $T = \frac{2\pi}{b} = \frac{2\pi}{880\pi} = \frac{1}{440}$.

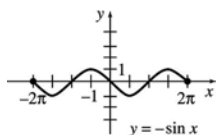
66. The graph of $y = 2\sin x$ is similar to the graph of $y = \sin x$ except that it has twice the amplitude. (That is, its height is twice as great.)



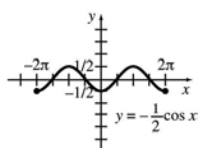
67. The graph of $y = 2\cos x$ is similar to the graph of $y = \cos x$ except that it has twice the amplitude. (That is, its height is twice as great.)



68. The graph of $y = -\sin x$ is similar to the graph of $y = \sin x$ except that it is reflected about the x -axis.

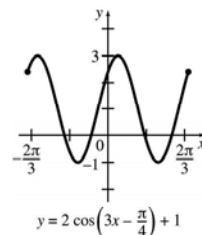


69. The graph of $y = -\frac{1}{2}\cos x$ is similar to the graph of $y = \cos x$ except that it has half the amplitude and is reflected about the x -axis.



70. $y = 2\cos\left(3x - \frac{\pi}{4}\right) + 1$ has amplitude $a = 2$, period $T = \frac{2\pi}{b} = \frac{2\pi}{3}$, phase shift $\frac{-\pi/4}{3} = -\frac{\pi}{12}$, and vertical shift $d = 1$. Thus, the graph of $y = 2\cos\left(3x - \frac{\pi}{4}\right) + 1$ is similar to the graph of $f(x) = \cos x$ except that it has 2 times the amplitude, a third of the period, and is shifted up 1 unit vertically. Also, $y = 2\cos\left(3x - \frac{\pi}{4}\right) + 1$

is shifted $\frac{\pi}{12}$ units to the right relative to the graph of $g(x) = \cos x$.

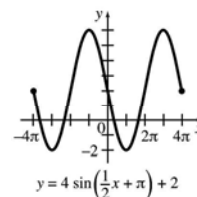


71. $y = 4\sin\left(\frac{1}{2}x + \pi\right) + 2$ has amplitude $a = 4$, period $T = \frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$, phase shift

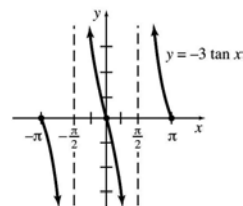
$$\frac{c}{b} = \frac{\pi}{\frac{1}{2}} = 2\pi, \text{ and vertical shift } d = 2. \text{ Thus,}$$

the graph of $y = 4\sin\left(\frac{1}{2}x + \pi\right) + 2$ is similar to the graph of $f(x) = \sin x$ except that it has 4 times the amplitude, twice the period, and is shifted up 2 units vertically. Also,

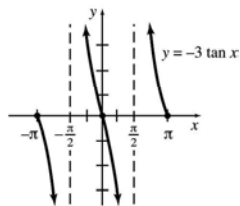
$y = \sin\left(\frac{1}{2}x + \pi\right) + 2$ is shifted 2π units to the left relative to the graph of $g(x) = \sin\left(\frac{1}{2}x\right)$.



72. The graph of $y = \frac{1}{2}\tan x$ is similar to the graph of $y = \tan x$ except that the y -values of points on the graph are one-half the y -values of points on the graph of $y = \tan x$.



73. The graph of $y = -3\tan x$ is similar to the graph of $y = \tan x$ except that it is reflected about the x -axis and each ordinate value is three times larger in absolute value. Note that the points $\left(-\frac{\pi}{4}, 3\right)$ and $\left(\frac{\pi}{4}, -3\right)$ lie on the graph.



74. (a) Since the three angles θ are equal and their sum is 180° , each angle θ is 60° .

- (b) The base angle on the left is still 60° . The bisector is perpendicular to the base, so the other base angle is 90° . The angle formed by bisecting the original vertex angle θ is 30° .

- (c) Two sides of the triangle on the left are given in the diagram: the hypotenuse is 2, and the base is half of the original base of 2, or 1. The Pythagorean Theorem gives the length of the remaining side (the vertical bisector) as $\sqrt{2^2 - 1^2} = \sqrt{3}$.

75. (a) The Pythagorean Theorem gives the length of the hypotenuse as $\sqrt{1^2 + 1^2} = \sqrt{2}$.

- (b) The angle at the lower left is a right angle with measure 90° . Since the sum of all three angles is 180° , the measures of the remaining two angles sum to 90° . Since these two angles are the base angles of an isosceles triangle they are equal, and thus each has measure 45° .

76. $S(t) = 500 + 500 \cos \frac{\pi}{6} t$

- (a) November corresponds to $t = 0$.
Therefore,

$$\begin{aligned} S(0) &= 500 + 500 \left[\cos \left(\frac{\pi}{6} \right) (0) \right] \\ &= 500 + 500 \cos 0 \\ &= 1000 \text{ snowblowers.} \end{aligned}$$

- (b) January corresponds to $t = 2$.
Therefore,

$$\begin{aligned} S(2) &= 500 + 500 \left[\cos \left(\frac{\pi}{6} \right) (2) \right] \\ &= 500 + 500 \cos \frac{\pi}{3} \\ &= 500 + 500 \left(\frac{1}{2} \right) \\ &= 750 \text{ snowblowers.} \end{aligned}$$

- (c) February corresponds to $t = 3$.

Therefore,

$$\begin{aligned} S(3) &= 500 + 500 \left[\cos \left(\frac{\pi}{6} \right) (3) \right] \\ &= 500 + 500 \cos \frac{\pi}{2} \\ &= 500 + 500(0) \\ &= 500 \text{ snowblowers.} \end{aligned}$$

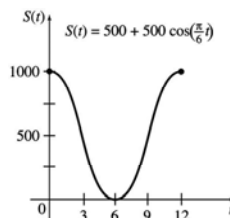
- (d) May corresponds to $t = 6$.
Therefore,

$$\begin{aligned} S(6) &= 500 + 500 \left[\cos \left(\frac{\pi}{6} \right) (6) \right] \\ &= 500 + 500 \cos \pi \\ &= 500 + 500(-1) \\ &= 0 \text{ snowblowers.} \end{aligned}$$

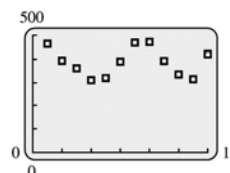
- (e) August corresponds to $t = 9$.
Therefore,

$$\begin{aligned} S(9) &= 500 + 500 \left[\cos \left(\frac{\pi}{6} \right) (9) \right] \\ &= 500 + 500 \cos \frac{3\pi}{2} \\ &= 500 + 500(0) \\ &= 500 \text{ snowblowers.} \end{aligned}$$

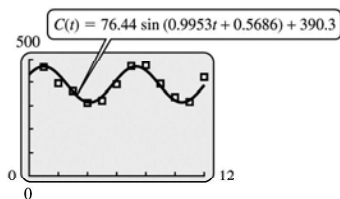
- (f) Use the ordered pairs obtained in parts (a)-(e) to plot the graph.



77. (a) It is reasonable to assume that electrical consumption is periodic across the seasons of the year, with higher use in summer and winter and lower use in spring and fall.



(b) $C(t) = 76.44 \sin(0.9953t + 0.5686) + 390.3$



(c) $T = \frac{2\pi}{b} \approx 6.3$

The period is about 6.3 months, which seems reasonable since it is about half a year.

- (d) $C(9) \approx 383$. The estimated September consumption is 383 trillion BTUs, a bit less than the actual value given in the table.

78. (a) Since the amplitude is 2 and the period is 0.350,

$$\begin{aligned} 0.350 &= \frac{2\pi}{b} \\ \frac{1}{0.350} &= \frac{b}{2\pi} \\ \frac{2\pi}{0.350} &= b. \end{aligned}$$

Therefore, the equation is $y = 2 \sin(2\pi t / 0.350)$, where t is the time in seconds.

(b)
$$\begin{aligned} 2 &= 2 \sin\left(\frac{2\pi t}{0.350}\right) \\ 1 &= \sin\left(\frac{2\pi t}{0.350}\right) \\ \frac{\pi}{2} &= \frac{2\pi t}{0.350} \\ \frac{0.350\pi}{4\pi} &= t \\ 0.0875 &= t \end{aligned}$$

The image reaches its maximum amplitude after 0.0875 seconds.

(c) $t = 2$

$$\begin{aligned} y &= 2 \sin\left(\frac{2\pi(2)}{0.350}\right) \\ &\approx -1.95. \end{aligned}$$

The position of the object after 2 seconds is -1.95° .

79. (a) The period is $\frac{2\pi}{\left(\frac{\pi}{14.77}\right)} = 29.54$

There is a lunar cycle every 29.54 days.

- (b) $y = 100 + 1.8 \cos\left[\frac{(t-6)\pi}{14.77}\right]$ reaches a maximum value when $\cos\left[\frac{(t-6)\pi}{14.77}\right] = 1$ which occurs when

$$\begin{aligned} t - 6 &= 0 \\ t &= 6 \end{aligned}$$

Six days from October 11, 2011, is October 17, 2011.

$$\begin{aligned} y &= 100 + 1.8 \cos\left[\frac{(6-6)\pi}{14.77}\right] \\ &= 101.8 \end{aligned}$$

There is a percent increase of 1.8 percent.

- (c) On October 28, $t = 17$.

$$\begin{aligned} y &= 100 + 1.8 \cos\left[\frac{(17-6)\pi}{14.77}\right] \\ &\approx 98.75 \end{aligned}$$

The formula predicts that the number of consultations was 98.75% of the daily mean.

80. $P(t) = 7(1 - \cos 2\pi t)(t + 10) + 100e^{0.2t}$

- (a) Since January 1 of the base year corresponds to $t = 0$, the pollution level is

$$\begin{aligned} P(0) &= 7(1 - \cos 0)(0 + 10) + 100e^0 \\ &= 7(0)(10) + 100 = 100. \end{aligned}$$

- (b) Since July 1 of the base year corresponds to $t = 0.5$, the pollution level is

$$\begin{aligned} P(0.5) &= 7(1 - \cos \pi)(0.5 + 10) + 100e^{0.1} \\ &= 7(2)(10.5) + 100e^{0.1} = 258. \end{aligned}$$

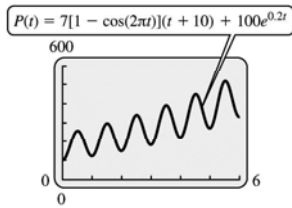
- (c) Since January 1 of the following year corresponds to $t = 1$, the pollution level is

$$\begin{aligned} P(1) &= 7(1 - \cos 2\pi)(1 + 10) + 100e^{0.2} \\ &\approx 122. \end{aligned}$$

- (d) Since July 1 of the following year corresponds to $t = 1.5$, the pollution level is

$$\begin{aligned} P(1.5) &= 7(1 - \cos 3\pi)(1.5 + 10) + 100e^{0.3} \\ &= 7(2)(11.5) + 100e^{0.3} \approx 296. \end{aligned}$$

81.

82. Solving $\frac{c_1}{c_2} = \frac{\sin \theta_1}{\sin \theta_2}$ for c_2 gives

$$c_2 = \frac{c_1 \sin \theta_2}{\sin \theta_1}.$$

When $\theta_1 = 39^\circ$, $\theta_2 = 28^\circ$, and $c_1 = 3 \times 10^8$,

$$\begin{aligned} c_2 &= \frac{(3 \times 10^8)(\sin 28^\circ)}{\sin 39^\circ} \\ &\approx 2.2 \times 10^8 \text{ m/sec.} \end{aligned}$$

83. Solving $\frac{c_1}{c_2} = \frac{\sin \theta_1}{\sin \theta_2}$ for c_2 gives

$$c_2 = \frac{c_1 \sin \theta_2}{\sin \theta_1}.$$

 $c_1 = 3 \cdot 10^8$, $\theta_1 = 46^\circ$, and $\theta_2 = 31^\circ$ so

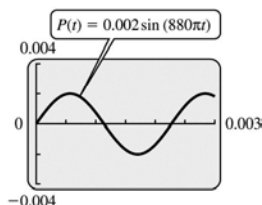
$$\begin{aligned} c_2 &= \frac{3 \cdot 10^8 (\sin 31^\circ)}{\sin 46^\circ} \\ &= 214,796,150 \\ &\approx 2.1 \times 10^8 \text{ m/sec.} \end{aligned}$$

84. Since each square represents 30° and the sine wave repeats itself every 8 squares, the period of the sine wave is

$$8 \times 30^\circ = 240^\circ.$$

85. On the horizontal scale, one whole period clearly spans four square, so $4 \cdot 30^\circ = 120^\circ$ is the period.

86. (a)

(b) Since the sine function is zero for multiples of π , we can determine the values (s) of t where $P = 0$ by setting $880\pi t = n\pi$, where n is an integer, and solving for t . After some algebraic manipulations,

$$t = \frac{n}{880}$$

and $P = 0$ when $n = \dots, -2, -1, 0, 1, 2, \dots$. However, only values of $n = 0, n = 1$, or $n = 2$ produce values of t that lie in the interval $[0, 0.003]$. Thus, $P = 0$ when $t = 0$, $t = \frac{1}{880} \approx 0.0011$, and $t = \frac{2}{880} \approx 0.0023$. These values check with the graph in part (a).(c) The period is $T = \frac{2\pi}{880\pi} = \frac{1}{440}$. Therefore, the frequency is 440 cycles per second.

87. $T(t) = 60 - 30 \cos\left(\frac{t}{2}\right)$

(a) $t = 1$ represents February, so the maximum afternoon temperature in February is

$$T(0) = 60 - 30 \cos \frac{1}{2} \approx 34^\circ\text{F}.$$

(b) $t = 3$ represents April, so the maximum afternoon temperature in April is

$$T(3) = 60 - 30 \cos \frac{3}{2} \approx 58^\circ\text{F}.$$

(c) $t = 8$ represents September, so the maximum afternoon temperature in September is

$$T(8) = 60 - 30 \cos 4 \approx 80^\circ\text{F}.$$

(d) $t = 6$ represents July, so the maximum afternoon temperature in July is

$$T(6) = 60 - 30 \cos 3 \approx 90^\circ\text{F}.$$

(e) $t = 11$ represents December, so the maximum afternoon temperature in December is

$$T(11) = 60 - 30 \cos \frac{11}{2} \approx 39^\circ\text{F}.$$

88. $T(x) = 37 \sin \left[\frac{2\pi}{365}(t - 101) \right] + 25$

(a) $T(74) = 37 \sin \left[\frac{2\pi}{365}(-27) \right] + 25$
 $\approx 8^\circ\text{F}$

(b) $T(121) = 37 \sin \left[\frac{2\pi}{365}(20) \right] + 25$
 $\approx 37^\circ\text{F}$

(c) $T(250) = 37 \sin \left[\frac{2\pi}{365}(149) \right] + 25$
 $\approx 45^\circ\text{F}$

(d) $T(325) = 37 \sin \left[\frac{2\pi}{365}(224) \right] + 25$
 $\approx 1^\circ\text{F}$

- (e) The maximum and minimum values of the sine function are 1 and -1 , respectively. Thus, the maximum value of T is

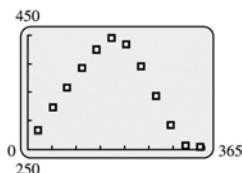
$$37(1) + 25 = 62^\circ\text{F}$$

and the minimum value of T is

$$37(-1) + 25 = -12^\circ\text{F}.$$

- (f) The period is $\frac{2\pi}{\frac{2\pi}{365}} = 365$.

89. (a)



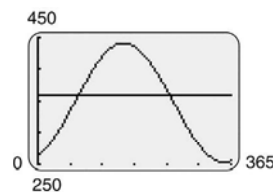
Yes; because of the cyclical nature of the days of the year, it is reasonable to assume that the times of the sunset are periodic.

- (b) The function $s(t)$, derived by a TI-84 Plus using the sine regression function under the STATCALC menu, is given by

$$s(t) = 94.0872 \sin(0.0166t - 1.2213) + 347.4158.$$

(c) $s(60) = 94.0872 \sin[0.0166(60) - 1.2213] + 347.4158$
 $= 326 \text{ minutes}$
 $= 5:26 \text{ P.M.}$
 $s(120) = 94.0872 \sin[0.0166(120) - 1.2213] + 347.4158$
 $= 413 \text{ minutes} + 60 \text{ minutes}$
 $(\text{daylight savings})$
 $= 7:53 \text{ P.M.}$
 $s(240) = 94.0872 \sin[0.0166(240) - 1.2213] + 347.4158$
 $= 382 \text{ minutes} + 60 \text{ minutes}$
 $(\text{daylight savings})$
 $= 7:22 \text{ P.M.}$

- (d) The following graph shows $s(t)$ and $y = 360$ (corresponding to a sunset at 6:00 P.M.). These graphs first intersect on day 82. However because of daylight savings time, to find the second value we find where the graphs of $s(t)$ and $y = 360 - 60 = 300$ intersect. These graphs intersect on day 295. Thus, the sun sets at approximately 6:00 P.M. on the 82nd and 295th days of the year.



90. $w_1 = \frac{u^2(\tan \theta)}{L + u(\tan \theta)}$ and

$$w_2 = \frac{u^2(\tan \theta)}{L - u(\tan \theta)}$$

If $\theta = \frac{1}{30}^\circ$, $L = 0.00625$, and $u = 6$, then

$$w_1 = \frac{6^2 \left(\tan \frac{1}{30}^\circ \right)}{0.00625 + 6 \left(\tan \frac{1}{30}^\circ \right)} \approx 2.2 \text{ and}$$

$$w_2 = \frac{6^2 \left(\tan \frac{1}{30}^\circ \right)}{0.00625 - 6 \left(\tan \frac{1}{30}^\circ \right)} \approx 7.6.$$

The near and far limits of the depth of field when the object being photographed is 6 m from the camera, are 2.2 m and 7.6 m, respectively.

91. Let h = the height of the building.

$$\tan 42.8^\circ = \frac{h}{65}$$

$$h = 65 \tan 42.8^\circ \approx 60.2$$

The height of the building is approximately 60.2 meters.

92. Let x be the distance to the opposite side of the canyon. Then

$$\tan 27^\circ = \frac{105}{x}$$

$$x = \frac{105}{\tan 27^\circ} \approx 206$$

The distance to the opposite side of the canyon is approximately 206 ft.

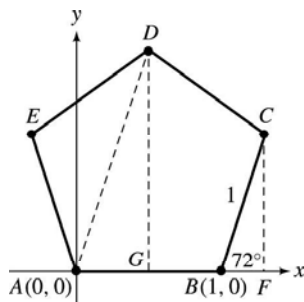
93. Let θ = the average angle with the horizontal.

$$\tan \theta = \frac{26}{5280}$$

Using the TAN^{-1} key on the calculator,

$$\theta = \text{TAN}^{-1}\left(\frac{26}{5280}\right) \approx 0.28^\circ.$$

94. Label the vertices as shown. Since the five exterior angles have equal measure and the sum of their measures is 360° , the measure of each exterior angle is 72° .



The coordinates of C can be found by solving the right triangle BCF . Those coordinates are $(1 + \cos 72^\circ, \sin 72^\circ)$, or about $(1.309, 0.951)$.

The y -coordinate of D can be found by solving the right triangle ADG , noting that the measure of angle DAG is also 72° . That gives the coordinates of D to be $(0.5, 0.5 \tan 72^\circ)$, or about $(0.5, 1.539)$.

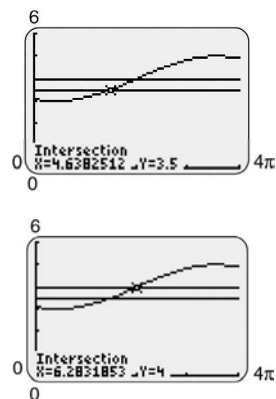
Similarly, the coordinate of E are $(-\cos 72^\circ, \sin 72^\circ)$, or about $(-0.309, 0.951)$. The x -coordinate of D is half-way between the x -coordinates of E and C , namely at 0.5.

95. We need to find the values of t for which

$$3.5 \leq h(t) \leq 4.$$

The following graphs show where

$h(t) = \sin\left(\frac{t}{\pi} - 2\right) + 4$ intersects the horizontal lines $y = 3.5$ and $y = 4$.



Thus, $3.5 \leq h(t) \leq 4$ when t is in the interval $[4.6, 6.3]$, to the nearest tenth.

13.2 Derivatives of Trigonometric Functions

Your Turn 1

Find the derivative of $y = 5 \sin(3x^4)$.

By the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= 5 \cos(3x^4) \cdot D_x(3x^4) \\ &= 60x^3 \cos(3x^4). \end{aligned}$$

Your Turn 2

Find the derivative of $y = 2 \sin^3(\sqrt{x})$.

This derivative requires two applications of the chain rule.

$$\begin{aligned} \frac{dy}{dx} &= 6 \sin^2(\sqrt{x}) \cdot D_x[\sin(\sqrt{x})] \\ &= 6 \sin^2(\sqrt{x}) \cdot \cos(\sqrt{x}) \cdot D_x(\sqrt{x}) \\ &= 6 \sin^2(\sqrt{x}) \cdot \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{3 \sin^2(\sqrt{x}) \cdot \cos(\sqrt{x})}{\sqrt{x}} \end{aligned}$$

Your Turn 3

Find the derivative of $y = x \cos(x^2)$.

By the product rule and two applications of the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= 1 \cdot \cos(x^2) + x \cdot D_x[\cos(x^2)] \\ &= \cos(x^2) + x \cdot [-\sin(x^2)] \cdot D_x(x^2) \\ &= \cos(x^2) + x \cdot [-\sin(x^2)] \cdot (2x) \\ &= -2x^2 \sin(x^2) + \cos(x^2).\end{aligned}$$

Your Turn 4

Find the derivative of $y = x \tan^2(x)$.

By the product rule and two applications of the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= 1 \cdot \tan^2 x + x \cdot D_x(\tan^2 x) \\ &= 1 \cdot \tan^2 x + x \cdot (2 \tan x) \cdot D_x(\tan x) \\ &= 1 \cdot \tan^2 x + x \cdot (2 \tan x) \cdot \sec^2 x \\ &= 2x \tan x \sec^2 x + \tan^2 x.\end{aligned}$$

Your Turn 5

Find the derivative of $y = x \sec^2(\sqrt{x})$.

By two applications of the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= [2 \sec(\sqrt{x})] \cdot D_x[\sec(\sqrt{x})] \\ &= [2 \sec(\sqrt{x})] \cdot [\sec(\sqrt{x}) \tan(\sqrt{x})] \cdot D_x(\sqrt{x}) \\ &= [2 \sec(\sqrt{x})] \cdot [\sec(\sqrt{x}) \tan(\sqrt{x})] \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\sec^2(\sqrt{x}) \tan(\sqrt{x})}{\sqrt{x}}.\end{aligned}$$

Your Turn 6

Find the derivative of $f(x) = \sin(\cos x)$ when $x = \pi/2$.

Use the chain rule.

$$\begin{aligned}f'(x) &= \cos(\cos x) \cdot (-\sin x) \\ f'\left(\frac{\pi}{2}\right) &= \cos\left(\cos \frac{\pi}{2}\right) \cdot \left(-\sin \frac{\pi}{2}\right) \\ &= (\cos 0) \cdot (-1) \\ &= (1)(-1) \\ &= -1\end{aligned}$$

13.2 Exercises

1. $y = \frac{1}{2} \sin 8x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2} (\cos 8x) \cdot D_x(8x) \\ &= \frac{1}{2} (\cos 8x) \cdot 8 \\ &= 4 \cos 8x\end{aligned}$$

2. $y = -\cos 2x + \cos \frac{\pi}{6}$

$$\begin{aligned}\frac{dy}{dx} &= (\sin 2x) \cdot D_x(2x) + 0 \\ &= (\sin 2x) \cdot 2 \\ &= 2 \sin 2x\end{aligned}$$

3. $y = 12 \tan(9x + 1)$

$$\begin{aligned}\frac{dy}{dx} &= [12 \sec^2(9x + 1)] \cdot D_x(9x + 1) \\ &= [12 \sec^2(9x + 1)] \cdot 9 \\ &= 108 \sec^2(9x + 1)\end{aligned}$$

4. $y = -4 \cos(7x^2 - 4)$

$$\begin{aligned}\frac{dy}{dx} &= [4 \sin(7x^2 - 4)] \cdot D_x(7x^2 - 4) \\ &= [4 \sin(7x^2 - 4)] \cdot 14x \\ &= 56x \sin(7x^2 - 4)\end{aligned}$$

5. $y = \cos^4 x$

$$\begin{aligned}\frac{dy}{dx} &= [4(\cos x)^3] D_x(\cos x) \\ &= (4 \cos^3 x)(-\sin x) \\ &= -4 \sin x \cos^3 x\end{aligned}$$

6. $y = -9 \sin^5 x$

$$\begin{aligned}\frac{dy}{dx} &= -45(\sin x)^4 \cdot D_x(\sin x) \\ &= -45 \sin^4 x \cos x\end{aligned}$$

7. $y = \tan^8 x$

$$\begin{aligned}\frac{dy}{dx} &= 8(\tan x)^7 \cdot D_x(\tan x) \\ &= 8 \tan^7 x \sec^2 x\end{aligned}$$

8. $y = 3 \cot^5 x$
 $\frac{dy}{dx} = 15(\cot x)^4 \cdot D_x(\cot x)$
 $= -15 \cot^4 x \csc^2 x$
9. $y = -6x \cdot \sin 2x$
 $\frac{dy}{dx} = -6x \cdot D_x(\sin 2x) + \sin 2x \cdot D_x(-6x)$
 $= -6x(\cos 2x) \cdot D_x(2x) + (\sin 2x)(-6)$
 $= -6x(\cos 2x) \cdot 2 - 6 \sin 2x$
 $= -12x \cos 2x - 6 \sin 2x$
10. $y = 2x \cdot \sec 4x$
 $\frac{dy}{dx} = 2x \cdot D_x(\sec 4x) + \sec 4x \cdot D_x(2x)$
 $= 2x(\sec 4x \tan 4x) \cdot D_x(4x) + (\sec 4x)(2)$
 $= 2x(\sec 4x \tan 4x) \cdot 4 + 2 \sec 4x$
 $= 8x \sec 4x \tan 4x + 2 \sec 4x$
11. $y = \frac{\csc x}{x}$
 $\frac{dy}{dx} = \frac{x \cdot D_x(\csc x) - (\csc x) \cdot D_x x}{x^2}$
 $= \frac{-x \csc x \cot x - \csc x}{x^2}$
 $= \frac{-(x \csc x \cot x + \csc x)}{x^2}$
12. $y = \frac{\tan x}{x-1}$
 $\frac{dy}{dx} = \frac{(x-1) \cdot D_x(\tan x) - (\tan x) D_x(x-1)}{(x-1)^2}$
 $= \frac{(x-1) \sec^2 x - \tan x}{(x-1)^2}$
13. $y = \sin e^{4x}$
 $\frac{dy}{dx} = \cos e^{4x} \cdot D_x(e^{4x})$
 $= (\cos e^{4x}) \cdot e^{4x} \cdot D_x(4x)$
 $= (\cos e^{4x}) \cdot e^{4x} \cdot 4$
 $= 4e^{4x} \cos e^{4x}$
14. $y = \cos 4e^{2x}$
 $\frac{dy}{dx} = (-\sin 4e^{2x}) \cdot D_x(4e^{2x})$
 $= -8e^{2x} \sin 4e^{2x}$
15. $y = e^{\cos x}$
 $\frac{dy}{dx} = e^{\cos x} \cdot D_x(\cos x)$
 $= e^{\cos x} \cdot (-\sin x)$
 $= (-\sin x)e^{\cos x}$
16. $y = -2e^{\cot x}$
 $\frac{dy}{dx} = -2e^{\cot x} \cdot D_x(\cot x)$
 $= -2e^{\cot x} \cdot (-\csc^2 x)$
 $= (2 \csc^2 x)e^{\cot x}$
17. $y = \sin(\ln 3x^4)$
 $\frac{dy}{dx} = [\cos(\ln 3x^4)] \cdot D_x(\ln 3x^4)$
 $= \cos(\ln 3x^4) \cdot \frac{D_x(3x^4)}{3x^4}$
 $= \cos(\ln 3x^4) \cdot \frac{12x^3}{3x^4}$
 $= \cos(\ln 3x^4) \cdot \frac{4}{x}$
 $= \frac{4}{x} \cos(\ln 3x^4)$
18. $y = \cos(\ln |2x^3|)$
 $\frac{dy}{dx} = [-\sin(\ln |2x^3|)] \cdot D_x(\ln |2x^3|)$
 $= -\sin(\ln |2x^3|) \cdot \frac{D_x(2x^3)}{2x^3}$
 $= -\frac{3}{x} \sin(\ln |2x^3|)$
19. $y = \ln |\sin x^2|$
 $\frac{dy}{dx} = \frac{D_x(\sin x^2)}{\sin x^2} = \frac{(\cos x^2) \cdot D_x(x^2)}{\sin x^2}$
 $= \frac{(\cos x^2) \cdot 2x}{\sin x^2} = \frac{2x \cos x^2}{\sin x^2}$
or $2x \cot x^2$

$$\begin{aligned}
 20. \quad y &= \ln |\tan^2 x| \\
 \frac{dy}{dx} &= \frac{1}{\tan^2 x} \cdot D_x(\tan^2 x) \\
 &= \frac{1}{\tan^2 x} \cdot (2 \tan x) \cdot D_x(\tan x) \\
 &= \frac{2 \sec^2 x}{\tan x}
 \end{aligned}$$

$$\begin{aligned}
 21. \quad y &= \frac{2 \sin x}{3 - 2 \sin x} \\
 \frac{dy}{dx} &= \frac{(3 - 2 \sin x) D_x(2 \sin x) - (2 \sin x) \cdot D_x(3 - 2 \sin x)}{(3 - 2 \sin x)^2} \\
 &= \frac{(3 - 2 \sin x) \cdot 2 D_x(\sin x) - (2 \sin x) \cdot [-2 D_x(\sin x)]}{(3 - 2 \sin x)^2} \\
 &= \frac{6 \cos x - 4 \sin x \cos x + 4 \sin x \cos x}{(3 - 2 \sin x)^2} \\
 &= \frac{6 \cos x}{(3 - 2 \sin x)^2}
 \end{aligned}$$

$$\begin{aligned}
 22. \quad y &= \frac{3 \cos x}{5 - \cos x} \\
 \frac{dy}{dx} &= \frac{(5 - \cos x) \cdot D(3 \cos x) - (3 \cos x) \cdot D_x(5 - \cos x)}{(5 - \cos x)^2} \\
 &= \frac{(5 - \cos x)(-3 \sin x) - (3 \cos x) \cdot (\sin x)}{(5 - \cos x)^2} \\
 &= \frac{-15 \sin x}{(5 - \cos x)^2}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad y &= \sqrt{\frac{\sin x}{\sin 3x}} = \left(\frac{\sin x}{\sin 3x} \right)^{1/2} \\
 \frac{dy}{dx} &= \frac{1}{2} \left(\frac{\sin x}{\sin 3x} \right)^{-1/2} \cdot D_x \left(\frac{\sin x}{\sin 3x} \right) \\
 &= \frac{1}{2} \left(\frac{\sin 3x}{\sin x} \right)^{-1/2} \cdot \left[\frac{(\sin 3x) \cdot D_x(\sin x) - (\sin x) \cdot D_x(\sin 3x)}{(\sin 3x)^2} \right] \\
 &= \frac{1}{2} \left(\frac{\sin 3x}{\sin x} \right)^{-1/2} \cdot \left[\frac{(\sin 3x)(\cos x) - (\sin x)(\cos 3x) \cdot D_x(3x)}{\sin^2 3x} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1(\sin 3x)^{1/2}}{2(\sin x)^{1/2}} \cdot \frac{\sin 3x \cos x - 3 \sin x \cos 3x}{\sin^2 3x} \\
 &= \frac{(\sin 3x)^{1/2}(\sin 3x \cos x - 3 \sin x \cos 3x)}{2(\sin x)^{1/2}(\sin^2 3x)} \\
 &= \frac{\sqrt{\sin 3x} [\sin 3x \cos x - 3 \sin x \cos 3x]}{2\sqrt{\sin x}(\sin^2 3x)}
 \end{aligned}$$

$$\begin{aligned}
 24. \quad y &= \sqrt{\frac{\cos 4x}{\cos x}} = \left(\frac{\cos 4x}{\cos x} \right)^{1/2} \\
 \frac{dy}{dx} &= \frac{1}{2} \left(\frac{\cos 4x}{\cos x} \right)^{-1/2} \cdot D_x \left(\frac{\cos 4x}{\cos x} \right) \\
 &= \frac{1}{2} \left(\frac{\cos 4x}{\cos x} \right)^{-1/2} \cdot \left[\frac{(\cos x) \cdot D_x(\cos 4x) - (\cos 4x) \cdot D_x(\cos x)}{\cos^2 x} \right] \\
 &= \frac{1}{2} \left(\frac{\cos x}{\cos 4x} \right)^{1/2} \cdot \left(\frac{-4 \cos x \sin 4x + \cos 4x \sin x}{\cos^2 x} \right) \\
 &= \frac{-4 \cos x \sin 4x + \cos 4x \sin x}{2 \cos^{3/2} x \cos^{1/2} 4x}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad y &= 3 \tan \left(\frac{1}{4} x \right) + 4 \cot 2x - 5 \csc x + e^{-2x} \\
 \frac{dy}{dx} &= 3 \tan \left(\frac{1}{4} x \right) + 4 \cot 2x - 5 \csc x + e^{-2x} \\
 &= 3 \sec^2 \left(\frac{1}{4} x \right) \cdot D_x \left(\frac{1}{4} x \right) \\
 &\quad + 4(-\csc^2 2x) \cdot D_x(2x) - 5(-\csc x \cot x) \\
 &\quad + e^{-2x} \cdot D_x(-2x) \\
 &= 3 \sec^2 \left(\frac{1}{4} x \right) \cdot \frac{1}{4} - (4 \csc^2 2x) \cdot 2 \\
 &\quad + 5 \csc x \cot x + e^{-2x} \cdot (-2) \\
 &= \frac{3}{4} \sec^2 \left(\frac{1}{4} x \right) - 8 \csc^2 2x \\
 &\quad + 5 \csc x \cot x - 2e^{-2x}
 \end{aligned}$$

$$\begin{aligned}
 26. \quad y &= (\sin 3x + \cot(x^3))^8 \\
 \frac{dy}{dx} &= 8[\sin 3x + \cot(x^3)]^7 \cdot D_x[\sin 3x + \cot(x^3)] \\
 &= 8[\sin 3x + \cot(x^3)]^7 \cdot [\cos 3x \cdot D_x(3x) - \csc^2(x^3) \cdot D_x(x^3)] \\
 &= 8[\sin 3x + \cot(x^3)]^7 [3 \cos 3x - 3x^2 \csc^2(x^3)]
 \end{aligned}$$

27. $y = \sin x; x = 0$

Let $f(x) = \sin x$.

Then $f'(x) = \cos x$, so

$$f'(0) = \cos 0 = 1.$$

The slope of the tangent line to the graph of $y = \sin x$ at $x = 0$ is 1.

28. $y = \sin x; x = \frac{\pi}{4}$

Let $f(x) = \sin x$.

Then $f'(x) = \cos x$ and

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

The slope of the tangent line at $x = \frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$.

29. $y = \cos x; x = -\frac{5\pi}{6}$

Let $f(x) = \cos x$.

Then $f'(x) = -\sin x$, so

$$f'\left(-\frac{5\pi}{6}\right) = -\sin\left(-\frac{5\pi}{6}\right) = \frac{1}{2}.$$

The slope of the tangent line to the graph of $y = \cos x$ at $x = -\frac{5\pi}{6}$ is $\frac{1}{2}$.

30. $y = \cos x; x = -\frac{\pi}{4}$

Let $f(x) = \cos x$.

Then $f'(x) = -\sin x$ and

$$f'\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

The slope of the tangent line at $x = -\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$.

31. $y = \tan x; x = 0$

Let $f(x) = \tan x$.

Then $f'(x) = \sec^2 x$, so

$$f'(0) = \sec^2 0 = \frac{1}{\cos^2 0} = 1.$$

The slope of the tangent line to the graph of $y = \tan x$ at $x = 0$ is 1.

32. $y = \cot x; x = \frac{\pi}{4}$

Let $f(x) = \cot x$.

Then $f'(x) = -\csc^2 x$, so

$$f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -(\sqrt{2})^2 = -2.$$

The slope of the tangent line to the graph of $y = \cot x$ at $x = \frac{\pi}{4}$ is -2 .

33. Since $\cot x = \frac{\cos x}{\sin x}$, by using the quotient rule,

$$\begin{aligned} D_x(\cot x) &= D_x\left(\frac{\cos x}{\sin x}\right) \\ &= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\csc^2 x. \end{aligned}$$

34. $D_x(\sec x) = D_x\left(\frac{1}{\cos x}\right)$

$$\begin{aligned} &= D_x[(\cos x)^{-1}] \\ &= -1(\cos x)^{-2}(-\sin x) \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\ &= \sec x \tan x \end{aligned}$$

35. Since $\csc x = \frac{1}{\sin x} = (\sin x)^{-1}$,

$$\begin{aligned} D_x(\csc x) &= D_x\left(\frac{1}{\sin x}\right) = D_x(\sin x)^{-1} \\ &= -1(\sin x)^{-2} \cos x \\ &= -\frac{\cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ &= -\csc x \cot x. \end{aligned}$$

37. $R(t) = 120 \cos 2\pi t + 150$

(a) $R'(t) = 120 \cdot (-\sin 2\pi t)(2\pi) + 0$
 $= -240\pi \sin 2\pi t$

(b) Replace t with $\frac{1}{12}$ (for $\frac{1}{12}$ of a year).

$$\begin{aligned} R'\left(\frac{1}{12}\right) &= -240\pi \sin 2\pi\left(\frac{1}{12}\right) \\ &= -240\pi \sin \frac{\pi}{6} \\ &= -240\pi\left(\frac{1}{2}\right) \\ &= -120\pi \end{aligned}$$

$R'(t)$ for August 1 is $-\$120\pi$ per year.

(c) January 1 is 6 months, or $\frac{6}{12} = \frac{1}{2}$ of a year for July 1. Replace t with $\frac{1}{2}$.

$$\begin{aligned} R'\left(\frac{1}{2}\right) &= -240\pi \sin 2\pi\left(\frac{1}{2}\right) \\ &= -240\pi \sin \pi \\ &= -240\pi(0) \\ &= 0 \end{aligned}$$

$R'(t)$ for January 1 is $\$0$ per year.

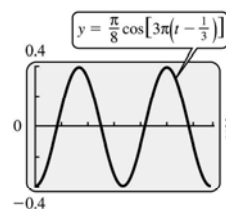
(d) June 1, is $\frac{11}{12}$ of a year from July 1. Replace t with $\frac{11}{12}$.

$$\begin{aligned} R'\left(\frac{11}{12}\right) &= -240\pi \sin 2\pi\left(\frac{11}{12}\right) \\ &= -240\pi \sin \frac{11\pi}{6} \\ &= -240\pi\left(-\frac{1}{2}\right) \\ &= 120\pi \end{aligned}$$

$R'(t)$ for June 1 is $\$120\pi$ per year.

38. $y = \frac{\pi}{8} \cos 3\pi\left(t - \frac{1}{3}\right)$

- (a) The graph should resemble the graph of $y = \cos x$ with following difference: The maximum and minimum values of y are $\frac{\pi}{8}$ and $-\frac{\pi}{8}$. The period of the graph will be $\frac{2\pi}{3\pi} = \frac{2}{3}$ units. The graph will be shifted horizontally $\frac{1}{3}$ units to the right.



(b) velocity $= \frac{dy}{dt}$

$$\begin{aligned} &= D_t\left[\frac{\pi}{8} \cos 3\pi\left(t - \frac{1}{3}\right)\right] \\ &= \frac{\pi}{8} D_t\left[\cos 3\pi\left(t - \frac{1}{3}\right)\right] \\ &= \frac{\pi}{8} \left[-\sin 3\pi\left(t - \frac{1}{3}\right)\right] D_t\left[3\pi\left(t - \frac{1}{3}\right)\right] \\ &= \frac{\pi}{8} \left[-\sin 3\pi\left(t - \frac{1}{3}\right)\right] \cdot 3\pi \\ &= -\frac{3\pi^2}{8} \sin\left[3\pi\left(t - \frac{1}{3}\right)\right] \end{aligned}$$

acceleration $= \frac{d^2y}{dt^2}$

$$\begin{aligned} &= D_t\left[\frac{-3\pi^2 \sin 3\pi\left(t - \frac{1}{3}\right)}{8}\right] \\ &= \frac{-3\pi^2}{8} D_t\left[\sin 3\pi\left(t - \frac{1}{3}\right)\right] \\ &= \frac{-3\pi^2}{8} \left[\cos 3\pi\left(t - \frac{1}{3}\right)\right] D_t\left[3\pi\left(t - \frac{1}{3}\right)\right] \\ &= \frac{-3\pi^2}{8} \left[\cos 3\pi\left(t - \frac{1}{3}\right)\right] \cdot 3\pi \\ &= -\frac{9\pi^3}{8} \cos\left[3\pi\left(t - \frac{1}{3}\right)\right] \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{d^2y}{dt^2} + 9\pi^2 y &= -\frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3} \right) \\
 &\quad + 9\pi^2 \left[\frac{\pi}{8} \cos 3\pi \left(t - \frac{1}{3} \right) \right] \\
 &= -\frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3} \right) \\
 &\quad + \frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3} \right) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad a(1) &= -\frac{9\pi^3}{8} \cos 3\pi \left(t - \frac{1}{3} \right) \\
 &= -\frac{9\pi^3}{8} \cos 2\pi \\
 &= -\frac{9\pi^3}{8} \cdot 1 \\
 &= -\frac{9\pi^3}{8} < 0 \\
 y(1) &= \frac{\pi}{8} \cos 3\pi \left(t - \frac{1}{3} \right) \\
 &= \frac{\pi}{8} \cos 2\pi \\
 &= \frac{\pi}{8} \cdot 1 = \frac{\pi}{8}
 \end{aligned}$$

Therefore, at $t = 1$ second, the force is clockwise and the arm makes an angle of $\frac{\pi}{8}$ radians forward from the vertical. The arm is moving clockwise.

$$\begin{aligned}
 a\left(\frac{4}{3}\right) &= -\frac{9\pi^3}{8} \cos 3\pi \left(\frac{4}{3} - \frac{1}{3} \right) \\
 &= -\frac{9\pi^3}{8} \cos (3\pi) \\
 &= -\frac{9\pi^3}{8} (-1) = \frac{9\pi^3}{8} > 0 \\
 y\left(\frac{4}{3}\right) &= \frac{\pi}{8} \cos 3\pi \left(\frac{4}{3} - \frac{1}{3} \right) \\
 &= \frac{\pi}{8} \cos (3\pi) \\
 &= \frac{\pi}{8} (-1) = -\frac{\pi}{8}
 \end{aligned}$$

Therefore, at $t = \frac{4}{3}$ seconds, the force is counter clockwise and the arm makes an angle of $-\frac{\pi}{8}$ radians from the vertical. The arm is moving counterclockwise.

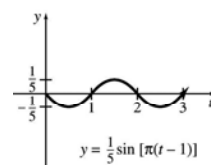
$$\begin{aligned}
 a\left(\frac{5}{3}\right) &= -\frac{9\pi^3}{8} \cos 3\pi \left(\frac{5}{3} - \frac{1}{3} \right) \\
 &= -\frac{9\pi^3}{8} \cos (4\pi) \\
 &= -\frac{9\pi^3}{8} \cdot 1 \\
 &= -\frac{9\pi^3}{8} < 0 \\
 y\left(\frac{5}{3}\right) &= \frac{\pi}{8} \cos 3\pi \left(\frac{5}{3} - \frac{1}{3} \right) \\
 &= \frac{\pi}{8} \cos (4\pi) \\
 &= \frac{\pi}{8} \cdot 1 \\
 &= \frac{\pi}{8}
 \end{aligned}$$

Therefore, at $t = \frac{5}{3}$ seconds, the answer corresponds to $t = 1$ second. So the arm is moving clockwise and makes an angle of $\frac{\pi}{8}$ from the vertical.

39. (a) The following is a table of values for

$$y = \frac{1}{5} \sin \pi(t - 1).$$

t	0	0.5	1.0	1.5	2.0	2.5	2.0
y	0	-0.2	0	0.2	0	-0.2	0



$$\begin{aligned}
 \text{(b)} \quad v(t) &= \frac{dy}{dt} \\
 &= \left[\frac{1}{5} \cos \pi(t - 1) \right] \cdot D_x[\pi(t - 1)] \\
 &= \frac{\pi}{5} \cos \pi(t - 1)
 \end{aligned}$$

$$\begin{aligned}
 a(t) &= v'(t) = \frac{d^2 y}{dt^2} \\
 &= \left[-\frac{\pi}{5} \sin \pi(t-1) \right] \cdot D_x [\pi(t-1)] \\
 &= -\frac{\pi^2}{5} \sin \pi(t-1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{d^2 y}{dt^2} + \pi^2 y &= -\frac{\pi^2}{5} \sin [\pi(t-1)] \\
 &\quad + (\pi^2) \frac{1}{5} \sin \pi(t-1) \\
 &= 0
 \end{aligned}$$

- (d)** Since the constant of proportionality is positive, the force and acceleration are in the same direction.

At $t = 1.5$ sec,

$$\begin{aligned}
 a(t) &= -\frac{\pi^2}{5} \sin \pi(t-1) \\
 a(1.5) &= -\frac{\pi^2}{5} \sin \pi(0.5) \\
 &= -\frac{\pi^2}{5} \sin \frac{\pi}{2} \\
 &= -\frac{\pi^2}{5} \cdot 1 \\
 &= -\frac{\pi^2}{5} < 0,
 \end{aligned}$$

and

$$\begin{aligned}
 y &= \frac{1}{5} \sin \pi(t-1) = \frac{1}{5} \sin \pi(0.5) \\
 &= \frac{1}{5} \sin \frac{\pi}{2} = \frac{1}{5} \cdot 1 = \frac{1}{5}.
 \end{aligned}$$

Thus, at $t = 1.5$, acceleration is negative, the arm is moving clockwise, and the arm is at an angle of $\frac{1}{5}$ radian from vertical.

At $t = 2.5$ sec,

$$\begin{aligned}
 a(2.5) &= -\frac{\pi^2}{5} \sin \pi(1.5) \\
 &= -\frac{\pi^2}{5} \sin \frac{3\pi}{2} \\
 &= -\frac{\pi^2}{5}(-1) = \frac{\pi^2}{5} > 0,
 \end{aligned}$$

and

$$y = \frac{1}{5} \sin \frac{3\pi}{2} = \frac{1}{5}(-1) = -\frac{1}{5}.$$

Thus, at $t = 2.5$, acceleration is positive, the arm is moving counterclockwise, and the arm is at angle of $-\frac{1}{5}$ radian from vertical.

At $t = 3.5$ sec,

$$\begin{aligned}
 a(3.5) &= -\frac{\pi^2}{5} \sin \pi(2.5) \\
 &= -\frac{\pi^2}{5} \sin \frac{5\pi}{2} \\
 &= -\frac{\pi^2}{5} \cdot 1 = -\frac{\pi^2}{5} < 0,
 \end{aligned}$$

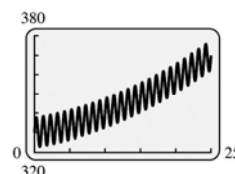
and

$$y = \frac{1}{5} \sin \frac{5\pi}{2} = \frac{1}{5} \cdot 1 = \frac{1}{5}.$$

Thus, at $t = 3.5$, acceleration is negative, the arm is moving clockwise, and the arm is at an angle of $\frac{1}{5}$ radian from vertical.

- 40. (a)** Using the calculator to graph

$C(t) = 0.04t^2 + 0.6t + 330 + 7.5\sin(2\pi t)$ in a $[0, 25]$ by $[320, 380]$ viewing window, gives the following graph.



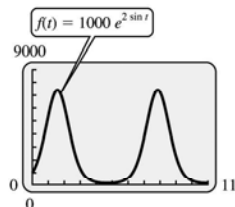
- (b)**
- $$\begin{aligned}
 C(25) &= 0.04(25)^2 + 0.6(25) + 330 \\
 &\quad + 7.5\sin[2\pi(25)] \\
 &= 370 \text{ parts per million} \\
 C(35.5) &= 0.04(35.5)^2 + 0.6(35.5) + 330 \\
 &\quad + 7.5\sin[2\pi(35.5)] \\
 &= 401.71 \text{ parts per million} \\
 C(50.2) &= 0.04(50.2)^2 + 0.6(50.2) + 330 \\
 &\quad + 7.5\sin[2\pi(50.2)] \\
 &\approx 468.05 \text{ parts per million}
 \end{aligned}$$
- (c)** Since
- $$C'(t) = 0.08t + 0.6 + 15\pi \cos(2\pi t),$$
- $C'(50.2)$ is given by

$$\begin{aligned}
 C'(50.2) &= 0.08(50.2) + 0.6 \\
 &\quad + 15\pi \cos[2\pi(50.2)] \\
 &\approx 19.18 \text{ parts per million} \\
 &\quad \text{per year.}
 \end{aligned}$$

The level of carbon dioxide will be increasing at the beginning of 2010 at 19.18 parts per million.

41. (a) $f(t) = 1000e^{2\sin t}$
 $f(0.2) = 1000e^{2\sin(0.2)}$
 ≈ 1488
- (b) $f(t) = 1000e^{2\sin t}$
 $f(1) = 1000e^{2\sin(1)}$
 ≈ 5381
- (c) Since $f'(t) = 2000\cos(t)e^{2\sin t}$, $f'(0)$ is given by
 $f'(0) = 2000\cos(0)e^{2\sin(0)}$
 $= 2000.$
- (d) Since $f'(t) = 2000\cos(t)e^{2\sin t}$, $f'(0.2)$ is given by
 $f'(0.2) = 2000\cos(0.2)e^{2\sin(0.2)}$
 $\approx 2916.$

(e)



- (f) Set $f'(t) = 2000\cos(t)e^{2\sin t}$ equal to zero to find critical numbers.

$$2000\cos(t)e^{2\sin t} = 0$$

$$\cos(t)e^{2\sin t} = 0$$

$$\cos(t) = 0 \quad \text{or} \quad e^{2\sin t} = 0$$

$$t = \frac{\pi}{2} + n\pi \quad e^{2\sin t} \neq 0,$$

for n any integer for all t

We will use the second derivative test.

$$f''(t) = -2000\sin(t)e^{2\sin t}$$

$$+ 4000\cos^2(t)e^{2\sin t}$$

$$= 2000e^{2\sin t}(-\sin t + 2\cos^2 t)$$

$$\text{Since } 2000e^{2\sin t} > 0 \text{ for all } t, f''(t) > 0$$

$$\text{when } -\sin t + 2\cos^2 t > 0, \text{ when}$$

$$\sin t < 2\cos^2 t. \text{ Also, } f''(t) < 0 \text{ when}$$

$$-\sin t + 2\cos^2 t < 0, \text{ when}$$

$$\sin t > 2\cos^2 t. \text{ Since}$$

$$\sin\left(\frac{\pi}{2} + n\pi\right) < 0 = 2\cos^2\left(\frac{\pi}{2} + n\pi\right) \text{ for}$$

n any odd integer, and

$$\sin\left(\frac{\pi}{2} + n\pi\right) > 0 = 2\cos^2\left(\frac{\pi}{2} + n\pi\right)$$

for n any even integer, f has a maximum at

$$t = \frac{\pi}{2} + n\pi \text{ for } n \text{ any even integer and } f$$

$$\text{has a minimum at } t = \frac{\pi}{2} + 2\pi n \text{ for } n \text{ any}$$

odd integer. This is equivalent to f having a

$$\text{maximum for } t = \frac{\pi}{2} + 2\pi n \text{ for } n \text{ any}$$

integer and f having a minimum for

$$t = \frac{3\pi}{2} + 2\pi n \text{ for } n \text{ any integer. To find}$$

the maximum and minimum values we will

$$\text{find } f\left(\frac{\pi}{2}\right) \text{ and } f\left(\frac{3\pi}{2}\right) \text{ respectively.}$$

$$f\left(\frac{\pi}{2}\right) = 1000e^{2\sin\left(\frac{\pi}{2}\right)} \approx 7390$$

$$f\left(\frac{3\pi}{2}\right) = 1000e^{2\sin\left(\frac{3\pi}{2}\right)} \approx 135$$

$$42. \quad s(\theta) = 2.625\cos\theta + 2.625(15 + \cos^2\theta)^{1/2}$$

- (a) The position of the piston when $\theta = \frac{\pi}{2}$ is given by

$$\begin{aligned}
 s\left(\frac{\pi}{2}\right) &= 2.625(0) + 2.625(15 + 0)^{1/2} \\
 &= 2.625\sqrt{15} \approx 10.17 \text{ in.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \frac{ds}{d\theta} &= 2.625(-\sin\theta) + 2.625\left(\frac{1}{2}\right) \\
 &\quad (15 + \cos^2\theta)^{-1/2} \cdot 2\cos\theta(-\sin\theta) \\
 &= -2.625\sin\theta - \frac{2.625\sin\theta\cos\theta}{\sqrt{15 + \cos^2\theta}} \\
 &= -2.625\sin\theta \left(1 + \frac{\cos\theta}{\sqrt{15 + \cos^2\theta}}\right)
 \end{aligned}$$

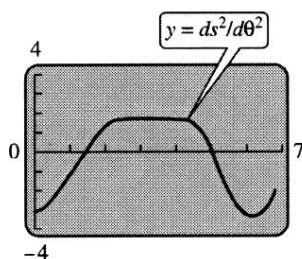
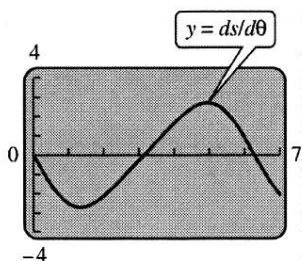
- (c) One way to find the maximum velocity of the piston is to take the second derivative of $s(\theta)$ and then find the values of θ where the second derivative is equal to zero.

Using the second derivative, we can determine which critical numbers maximize $ds/d\theta$.

Because the second derivative is so

complicated, we will employ a graphing calculator to find the critical number(s).

The figures below show the calculator-generated graphs of the functions $ds/d\theta$ and $d^2s/d\theta^2$, respectively.

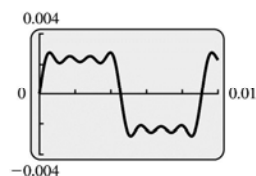


Although there are two critical numbers, the graph of $ds/d\theta$ shows that only one critical number is associated with maximum velocity. This occurs when $\theta \approx 5$. The equation solver on the graphing calculator can be used to find a more precise value. The maximum velocity occurs at 4.944 radians.

43. (a) Using the graphing calculator to graph

$$\begin{aligned} P(t) &= 0.003\sin(220\pi t) \\ &+ \frac{0.003}{3}\sin(660\pi t) \\ &+ \frac{0.003}{5}\sin(1100\pi t) \\ &+ \frac{0.003}{7}\sin(1540\pi t) \end{aligned}$$

in a $[0, 0.01]$ by $[-0.004, 0.004]$ viewing window, gives the following graph.



- (b) Using the graphing calculator to find $P'(x)$ and evaluate $P'(0.002)$ gives .151. However, the slope of the graph of $P(x)$ at $x = 0.002$ appears to be approximately -1 . Thus, we calculate $P'(x)$ by hand and use the

calculator to evaluate $P'(0.002)$.

Since $P'(t) = 0.66\pi\cos(220\pi t) + 0.66\pi\cos(660\pi t) + 0.66\pi\cos(1100\pi t) + 0.66\pi\cos(1540\pi t)$, $P'(0.002)$ is given by $P'(0.002) = 0.66\pi\cos[220\pi(0.002)] + 0.66\pi\cos[660\pi(0.002)] + 0.66\pi\cos[1100\pi(0.002)] + 0.66\pi\cos[1540\pi(0.002)] \approx -1.05$ pounds per square foot per second. At 0.002 seconds the pressure is decreasing at a rate of about 1.05 pounds per square foot per second.

44. (a)

$$\begin{aligned} u(x, t) &= T_0 + A_0 e^{-ax} \cos\left(\frac{\pi}{6}t - ax\right) \\ &= 16 + 11e^{-0.00706x} \cos\left(\frac{\pi}{6}t - 0.00706x\right) \end{aligned}$$

The amplitude of $\mu(x, t)$ is given by $11e^{-0.00706x}$. We need to find where $11e^{-0.00706x} \leq 1$.

$$\begin{aligned} 11e^{-0.00706x} &\leq 1 \\ e^{-0.00706x} &\leq \frac{1}{11} \\ -0.00706x &\leq \ln\left(\frac{1}{11}\right) \end{aligned}$$

$$\begin{aligned} x &\leq \frac{\ln\left(\frac{1}{11}\right)}{-0.00706} \\ &\approx 340 \text{ cm} \end{aligned}$$

The amplitude is at most 1°C at a minimum depth of about 340 centimeters.

- (b) We wish to find x for which $14 \leq u(x, t) \leq 18$. Since

$$-1 \leq \cos\left(\frac{\pi}{6}t - 0.00706x\right) \leq 1, \text{ we have}$$

$$16 - 11e^{-0.00706x} \leq u(x, t) \leq 16 + 11e^{-0.00706x}.$$

For $14 \leq u(x, t) \leq 18$, we need to find where $16 - 11e^{-0.00706x} = 14$ and $16 + 11e^{-0.00706x} = 18$.

These two conditions are equivalent to

$$11e^{-0.00706x} = 2$$

$$e^{-0.00706x} = \frac{2}{11}$$

$$-0.00706x = \ln\left(\frac{2}{11}\right)$$

$$x = \frac{\ln\left(\frac{2}{11}\right)}{-0.00706} \approx 242 \text{ cm}$$

A minimum depth of about 242 centimeters will keep the wine at a temperature between 14°C and 18°C.

- (c) The phase shift will correspond to $\frac{1}{2}$ year or 6 months when

$$\frac{0.00706}{\frac{\pi}{6}} = 6$$

$$0.00706x = \pi$$

$$x = \frac{\pi}{0.00706} \approx 445 \text{ cm}$$

A depth of about 445 centimeters gives a ground temperature prediction of winter when it is summer and vice versa.

- (d) We show that the more general function

$u(x, t) = T_0 + A_0 e^{-ax} \cos(wt - ax)$ satisfies the heat equation. (In our case $w = \pi/6$.)

$$\frac{\partial u}{\partial t} = A_0 e^{-ax} [-\sin(wt - ax)](w)$$

$$= -wA_0 e^{-ax} \sin(wt - ax)$$

$$\frac{\partial u}{\partial x} = -aA_0 e^{-ax} \cos(wt - ax)$$

$$+ A_0 e^{-ax} [-\sin(wt - ax)](-a)$$

$$= -aA_0 e^{-ax} [\cos(wt - ax)$$

$$- \sin(wt - ax)]$$

$$\frac{\partial^2 u}{\partial x^2}$$

$$= a^2 A_0 e^{-ax} [\cos(wt - ax) - \sin(wt - ax)]$$

$$- aA_0 e^{-x} [a \sin(wt - ax) + a \cos(wt - ax)]$$

$$= a^2 A_0 e^{-ax} [\cos(wt - ax) - \sin(wt - ax)]$$

$$- a^2 A_0 e^{-ax} [\sin(wt - ax) + \cos(wt - ax)]$$

$$= -2a^2 A_0 e^{-ax} \sin(wt - ax)$$

Now $a = \sqrt{\frac{w}{2k}}$ implies that

$$a^2 = \frac{w}{2k}$$

$$k = \frac{w}{2a^2}.$$

It follows that

$$\begin{aligned} k \frac{\partial^2 u}{\partial x^2} &= \frac{w}{2a^2} [-2a^2 A_0 e^{-ax} \sin(wt - ax)] \\ &= -wA_0 e^{-ax} \sin(wt - ax) = \frac{\partial u}{\partial t}. \end{aligned}$$

Thus,

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$

$$\begin{aligned} 45. \quad (a) \quad y &= x \tan \alpha - \frac{16x^2}{V^2} \sec^2 \alpha \\ &= 40 \tan\left(\frac{\pi}{4}\right) - \frac{16(40)^2}{44^2} \sec^2\left(\frac{\pi}{4}\right) \\ &= 40(1) - \frac{16(40)^2}{44^2} (2) \approx 13.55 \text{ feet} \end{aligned}$$

$$\begin{aligned} (b) \quad y &= x \tan \alpha - \frac{16x^2}{V^2} \sec^2 \alpha \\ 0 &= x \tan \alpha - \frac{16x^2}{V^2} \sec^2 \alpha \\ 0 &= x \left[\tan \alpha - \frac{16x}{V^2} \sec^2 \alpha \right] \\ 0 &= \tan \alpha - \frac{16x}{V^2} \sec^2 \alpha \quad (\text{for } x \neq 0) \end{aligned}$$

$$\frac{16x}{V^2} \sec^2 \alpha = \tan \alpha$$

$$\begin{aligned} x &= \frac{V^2}{16} \cdot \frac{\tan \alpha}{\sec^2 \alpha} \\ &= \frac{V^2}{16} \cdot \sin \alpha \cos \alpha \\ &= \frac{V^2}{32} \cdot 2 \sin \alpha \cos \alpha \\ &= \frac{V^2}{32} \sin(2\alpha) \end{aligned}$$

$$\begin{aligned} (c) \quad x &= \frac{V^2}{32} \sin(2\alpha) \\ &= \frac{44^2}{32} \sin\left[2\left(\frac{\pi}{3}\right)\right] \\ &\approx 52.39 \text{ feet} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \frac{dx}{d\alpha} &= \frac{V^2}{32} \cos(2\alpha) \cdot D_\alpha(2\alpha) \\ &= \frac{V^2}{16} \cos(2\alpha) \end{aligned}$$

Find critical values.

$$\begin{aligned} \frac{V^2}{16} \cos(2\alpha) &= 0 \\ \cos(2\alpha) &= 0 \end{aligned}$$

$$2\alpha = \frac{\pi}{2} + n\pi, \text{ for } n \text{ any integer}$$

$$\alpha = \frac{\pi}{4} + \frac{n\pi}{2}, \text{ for } n \text{ any integer}$$

Since $0 < \alpha < \frac{\pi}{2}$, $\frac{dx}{d\alpha} = 0$ for $\alpha = \frac{\pi}{4}$.

Furthermore, $\frac{d^2x}{d\alpha^2} = -\frac{V^2}{8} \sin(2\alpha)$ which is

less than zero at $\alpha = \frac{\pi}{4}$. Therefore, x is maximized when $\alpha = \frac{\pi}{4}$.

- (e) Since 60 miles per hour is 88 feet per second, evaluate

$$\begin{aligned} x &= \frac{V^2}{32} \sin(2\alpha) \text{ when } V \\ &= 88 \text{ and } \alpha = \frac{\pi}{4} \\ &= \frac{88^2}{32} \sin\left[2\left(\frac{\pi}{4}\right)\right] = 242 \text{ feet} \end{aligned}$$

46.

$$\text{(a)} \quad s(\theta) = 2.625 \cos \theta + 2.625(15 + \cos^2 \theta)^{1/2}$$

$$\begin{aligned} \frac{ds}{dt} &= \frac{ds}{d\theta} \cdot \frac{d\theta}{dt} \\ &= \left[-2.625 \sin \theta + (15 + \cos^2 \theta)^{-1/2} \cdot D_\theta(15 + \cos^2 \theta) \right] \cdot \frac{d\theta}{dt} \\ &= \left[-2.625 \sin \theta + \frac{1.3125}{\sqrt{15 + \cos^2 \theta}} \cdot (-2 \sin \theta \cos \theta) \right] \cdot \frac{d\theta}{dt} \\ &= -2.625 \sin \theta \left(1 + \frac{\cos \theta}{\sqrt{15 + \cos^2 \theta}} \right) \frac{d\theta}{dt} \end{aligned}$$

- (b) With $\theta = 4.944$ and

$$\frac{d\theta}{dt} = 505,168.1 \frac{\text{radians}}{\text{hour}}, \text{ we have}$$

$$\begin{aligned} \frac{ds}{dt} &= -2.625 \sin(4.944) \\ &\quad \times \left[1 + \frac{\cos(4.944)}{\sqrt{15 + \cos^2(4.944)}} \right] (505,168.1) \\ &\approx 1,367,018.749 \frac{\text{inches}}{\text{hour}} \\ &= 1,367,018.749 \frac{\text{inches}}{\text{hour}} \times \frac{1 \text{ foot}}{12 \text{ inches}} \times \frac{1 \text{ mile}}{5280 \text{ feet}} \\ &\approx 21.6 \text{ miles per hour} \end{aligned}$$

$$47. \quad s(t) = \sin t + 2 \cos t$$

$$v(t) = s'(t) = \cos t - 2 \sin t$$

$$\text{(a)} \quad v(0) = 1 - 2(0) = 1$$

$$\begin{aligned} \text{(b)} \quad v\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} - 2\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \\ &\approx -0.7071 \end{aligned}$$

$$\text{(c)} \quad v\left(\frac{3\pi}{2}\right) = 0 - 2(-1) = 2$$

$$a(t) = v'(t) = -\sin t - 2 \cos t$$

$$\text{(d)} \quad a(0) = 0 - 2(1) = -2$$

$$\begin{aligned} \text{(e)} \quad a\left(\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2} - 2\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2} \\ &\approx -2.1213 \end{aligned}$$

$$\text{(f)} \quad a(\pi) = -0 - (-2) = 2$$

$$48. \quad \text{(a)} \quad \tan \theta = \frac{x}{50}$$

Differentiate both sides with respect to time, t .

$$D_t(\tan \theta) = D_t\left(\frac{x}{50}\right)$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{50} \cdot \frac{dx}{dt}$$

Since the light rotates twice per minute,

$$\frac{d\theta}{dt} = \frac{2(2\pi \text{ radians})}{1 \text{ min}} = 4\pi \text{ radians per minute.}$$

When the light beam and shoreline are at right angles, $\theta = 0$ and $\sec \theta = 1$. Thus,

$$(1)^2(4\pi) = \frac{1}{50} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = 200\pi.$$

The beam is moving along the shoreline at 200π m/min.

- (b) When the beam hits the shoreline 50 m from the point on the shoreline closest to the lighthouse, $\theta = \frac{\pi}{4}$ and $\sec \theta = \sqrt{2}$. Thus,

$$(\sqrt{2})^2(4\pi) = \frac{1}{50} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = 400\pi.$$

The beam is moving along the shoreline at 400π m/min.

49. From the figure, we see that

$$\tan \theta = \frac{x}{60}$$

$$60 \tan \theta = x.$$

Differentiating with respect to time, t , gives

$$60 \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{\frac{dx}{dt}}{60 \sec^2 \theta}.$$

Since

$$\frac{dx}{dt} = 600,$$

$$\frac{d\theta}{dt} = \frac{600}{60 \sec^2 \theta}$$

$$= \frac{10}{\sec^2 \theta}.$$

- (a) When the car is at the point on the road closest to the camera $\theta = 0$. Thus,

$$\frac{d\theta}{dt} = \frac{10}{(\sec 0)^2}$$

$$= \frac{10}{1^2} = 10 \text{ radians/min}$$

$$\frac{d\theta}{dt} = \frac{10 \text{ radians}}{\text{min}} \cdot \frac{1 \text{ rev}}{2\pi \text{ radians}}$$

$$= \frac{5}{\pi} \text{ rev/min}$$

- (b) Six seconds later is $\frac{1}{10}$ of a minute later, and the car has traveled 60 feet. Thus,

$$\tan \theta = \frac{60}{60} \text{ and } \theta = \frac{\pi}{4}.$$

$$\frac{d\theta}{dt} = \frac{10}{\left(\sec \frac{\pi}{4}\right)^2} = \frac{10}{(\sqrt{2})^2}$$

$$= 5 \text{ radians/min}$$

This is one-half the previous value, so

$$\frac{d\theta}{dt} = \frac{1}{2} \cdot \frac{5}{\pi} \text{ rev/min}$$

$$= \frac{5}{2\pi} \text{ rev/min.}$$

50. Let x be the length of the ladder and let y be the distance from the wall to the bottom of the ladder. Then

$$\cos \theta = \frac{y+2}{x} \quad \text{and} \quad \cot \theta = \frac{y}{9}$$

$$x \cos \theta = y + 2 \quad \text{and} \quad y = 9 \cot \theta.$$

Thus, $x \cos \theta = 9 \cot \theta + 2$

$$x = \frac{9 \cot \theta + 2}{\cos \theta}$$

$$x = 9 \csc \theta + 2 \sec \theta.$$

This expression gives the length of the ladder as a function of θ . Find the minimum value of this function.

$$\frac{dx}{d\theta} = -9 \csc \theta \cot \theta + 2 \sec \theta \tan \theta$$

$$= -9 \left(\frac{1}{\sin \theta} \right) \left(\frac{\cos \theta}{\sin \theta} \right) + 2 \left(\frac{1}{\cos \theta} \right) \left(\frac{\sin \theta}{\cos \theta} \right)$$

$$= \frac{-9 \cos \theta}{\sin^2 \theta} + \frac{2 \sin \theta}{\cos^2 \theta}$$

If $\frac{dx}{d\theta} = 0$, then

$$\frac{2 \sin \theta}{\cos^2 \theta} = \frac{9 \cos \theta}{\sin^2 \theta}$$

$$2 \sin^3 \theta = 9 \cos^3 \theta$$

$$\frac{\sin^3 \theta}{\cos^3 \theta} = \frac{9}{2}$$

$$\tan^3 \theta = \frac{9}{2}$$

$$\tan \theta = \sqrt[3]{\frac{9}{2}}$$

$$\theta \approx 1.02619 \text{ radians.}$$

If $\theta < 1.02619$, $\frac{dx}{d\theta} < 0$.

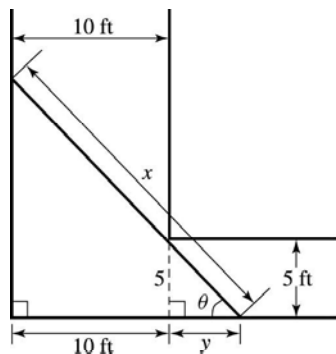
If $\theta > 1.02619$, $\frac{dx}{d\theta} > 0$.

Therefore, there is a minimum when $\theta = 1.02619$. If $\theta = 1.02619$,

$$x \approx 14.383.$$

The minimum length of the ladder is approximately 14.38 ft.

51. Let x represent the length of the ladder.



$$\tan \theta = \frac{5}{y} \quad \text{and} \quad \cos \theta = \frac{10 + y}{x}$$

$$y = 5 \cot \theta \quad \text{and} \quad x = \frac{10 + y}{\cos \theta}$$

Thus,

$$x = \frac{10 + 5 \cot \theta}{\cos \theta} = 10 \sec \theta + 5 \csc \theta$$

$$\frac{dx}{d\theta} = 10 \sec \theta \tan \theta - 5 \csc \theta \cot \theta$$

$$\begin{aligned} &= 10 \left(\frac{1}{\cos \theta} \right) \left(\frac{\sin \theta}{\cos \theta} \right) - 5 \left(\frac{1}{\sin \theta} \right) \left(\frac{\cos \theta}{\sin \theta} \right) \\ &= \frac{10 \sin \theta}{\cos^2 \theta} - \frac{5 \cos \theta}{\sin^2 \theta} \end{aligned}$$

If $\frac{dx}{d\theta} = 0$, then

$$\frac{10 \sin \theta}{\cos^2 \theta} = \frac{5 \cos \theta}{\sin^2 \theta}$$

$$10 \sin^3 \theta = 5 \cos^3 \theta$$

$$\frac{\sin^3 \theta}{\cos^3 \theta} = \frac{1}{2}$$

$$\tan^3 \theta = \frac{1}{2}$$

$$\tan \theta = \sqrt[3]{\frac{1}{2}}$$

$$\theta \approx 0.6709 \text{ radian.}$$

If $\theta < 0.6709$ radian, $\frac{dx}{d\theta} < 0$.

If $\theta > 0.6709$ radian, $\frac{dx}{d\theta} > 0$.

Thus, x will be minimum when $\theta = 0.6709$.

If $\theta = 0.6709$, then

$$x = \frac{10 + 5 \cot 0.6709}{\cos 0.6709} \approx 20.81.$$

The length of the longest possible ladder is approximately 20.81 feet.

13.3 Integrals of Trigonometric Functions

Your Turn 1

- (a) Find $\int \sin\left(\frac{x}{2}\right) dx$.

If $u = x/2$, then $du = (1/2) dx$.

$$\begin{aligned} \int \sin\left(\frac{x}{2}\right) dx &= 2 \int \sin u \left(\frac{1}{2} du\right) \\ &= 2 \int \sin u du \\ &= 2(-\cos u) + C \\ &= -2 \cos\left(\frac{x}{2}\right) + C \end{aligned}$$

- (b) Find $\int 6(\sec^2 x) \sqrt{\tan x} dx$.

If $u = \tan x$, then $du = \sec^2 x dx$.

$$\begin{aligned} \int 6(\sec^2 x) \sqrt{\tan x} dx &= 6 \int \sqrt{\tan x} (\sec^2 x dx) \\ &= 6 \int \sqrt{u} du \\ &= 6 \left(\frac{2}{3} u^{3/2} \right) + C \\ &= 4u^{3/2} + C \\ &= 4(\tan x)^{3/2} + C \end{aligned}$$

Your Turn 2

Find $\int \frac{\tan(\sqrt{x})}{\sqrt{x}} dx$.

If $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}} dx$.

$$\begin{aligned}\int \frac{\tan(\sqrt{x})}{\sqrt{x}} dx &= 2 \int \tan(\sqrt{x}) \left(\frac{1}{2\sqrt{x}} dx \right) \\ &= 2 \int \tan u du \\ &= -2 \ln |\cos u| + C \\ &= -2 \ln |\cos(\sqrt{x})| + C\end{aligned}$$

Your Turn 3

Find $\int x \cos(3x) dx$.

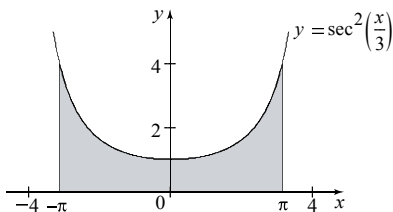
Let $u = x$ and $dv = \cos(3x) dx$.

Then $du = dx$ and $v = \frac{1}{3} \sin 3x$.

$$\begin{aligned}\int x \cos(3x) dx &= \int u dv \\ &= uv - \int v du \\ &= \frac{x}{3} \sin(3x) - \frac{1}{3} \int \sin(3x) dx \\ &= \frac{x}{3} \sin(3x) - \frac{1}{3} \left(-\frac{1}{3} \cos(3x) \right) + C \\ &= \frac{x}{3} \sin(3x) + \frac{1}{9} \cos(3x) + C\end{aligned}$$

Your Turn 4

Find the area under the curve $y = \sec^2(x/3)$ between $x = -\pi$ and $x = \pi$.



$$\begin{aligned}\int_{-\pi}^{\pi} \sec^2\left(\frac{x}{3}\right) dx &= 3 \int_{-\pi}^{\pi} \sec^2\left(\frac{x}{3}\right) \left(\frac{1}{3} dx\right) \\ &= 3 \tan\left(\frac{x}{3}\right) \Big|_{-\pi}^{\pi} \\ &= 3 \left(\tan\left(\frac{\pi}{3}\right) - \tan\left(-\frac{\pi}{3}\right) \right) \\ &= 3[\sqrt{3} - (-\sqrt{3})] \\ &= 6\sqrt{3}\end{aligned}$$

13.3 Exercises

1. $\int \cos 3x dx$

Let $u = 3x$, so $du = 3 dx$ or $\frac{1}{3} du = dx$.

$$\begin{aligned}\int \cos 3x dx &= \int \cos u \cdot \frac{1}{3} du \\ &= \frac{1}{3} \int \cos u du \\ &= \frac{1}{3} \sin u + C \\ &= \frac{1}{3} \sin 3x + C\end{aligned}$$

2. $\int \sin 5x dx$

Let $u = 5x$, so $du = 5 dx$ or $\frac{1}{5} du = dx$.

$$\begin{aligned}\int \sin 5x dx &= \int \sin u \cdot \frac{1}{5} du \\ &= \frac{1}{5} \int \sin u du \\ &= -\frac{1}{5} \cos u + C \\ &= -\frac{1}{5} \cos 5x + C\end{aligned}$$

3. $\int (3 \cos x - 4 \sin x) dx$

$$\begin{aligned}&= \int 3 \cos x dx - \int 4 \sin x dx \\ &= 3 \int \cos x dx - 4 \int \sin x dx \\ &= 3 \sin x + 4 \cos x + C\end{aligned}$$

$$\begin{aligned}
4. \quad & \int (9 \sin x + 8 \cos x) dx \\
&= \int 9 \sin x \, dx + \int 8 \cos x \, dx \\
&= 9 \int \sin x \, dx + 8 \int \cos x \, dx \\
&= -9 \cos x + 8 \sin x + C
\end{aligned}$$

$$5. \quad \int x \sin x^2 \, dx$$

Let $u = x^2$.

Then $du = 2x \, dx$

$$\frac{1}{2} = du = x \, dx.$$

$$\begin{aligned}
\int x \sin x^2 \, dx &= \int \sin u \cdot \frac{1}{2} du \\
&= \frac{1}{2} \int \sin u \, du \\
&= -\frac{1}{2} \cos u + C \\
&= -\frac{1}{2} \cos x^2 + C
\end{aligned}$$

$$6. \quad \int 2x \cos x^2 \, dx$$

Let $u = x^2$, so that $du = 2x \, dx$.

$$\begin{aligned}
\int 2x \cos x^2 \, dx &= \int \cos u \, du \\
&= \sin u + C \\
&= \sin x^2 + C
\end{aligned}$$

$$7. \quad -\int 3 \sec^2 3x \, dx$$

Let $u = 3x$, so $du = 3 \, dx$ or $\frac{1}{3} du = dx$.

$$\begin{aligned}
-\int 3 \sec^2 3x \, dx &= -\int 3 \sec^2 u \cdot \frac{1}{3} du \\
&= -\int \sec^2 u \, du \\
&= -\tan u + C \\
&= -\tan 3x + C
\end{aligned}$$

$$8. \quad -\int 2 \csc^2 8x \, dx$$

Let $u = 8x$, so that $du = 8 \, dx$.

$$\begin{aligned}
-\int 2 \csc^2 8x \, dx &= -\frac{1}{4} \int \csc^2 8x (8 \, dx) \\
&= \frac{1}{4} \int (-\csc^2 u) du \\
&= \frac{1}{4} \cot u + C \\
&= \frac{1}{4} \cot 8x + C
\end{aligned}$$

$$9. \quad \int \sin^7 x \cos x \, dx$$

Let $u = \sin x$.

Then $du = \cos x \, dx$.

$$\begin{aligned}
\int \sin^7 x (\cos x) \, dx &= \int u^7 \, du \\
&= \frac{1}{8} u^8 + C \\
&= \frac{1}{8} \sin^8 x + C
\end{aligned}$$

$$10. \quad \int \sin^4 x \cos x \, dx$$

Let $u = \sin x$, so $du = \cos x \, dx$.

$$\begin{aligned}
\int \sin^4 x \cos x \, dx &= \int u^4 \, du \\
&= \frac{1}{5} u^5 + C \\
&= \frac{1}{5} \sin^5 x + C
\end{aligned}$$

$$11. \quad \int 3\sqrt{\cos x} (\sin x) \, dx$$

Let $u = \cos x$, so $du = -\sin x \, dx$ or $-du = \sin x \, dx$.

$$\begin{aligned}
&\int 3\sqrt{\cos x} (\sin x) \, dx \\
&= \int 3\sqrt{u} (-du) \\
&= -3 \int u^{1/2} du \\
&= -3 \left(\frac{2}{3} u^{3/2} \right) + C \\
&= -2u^{3/2} + C \\
&= -2(\cos x)^{3/2} + C
\end{aligned}$$

$$12. \quad \int \frac{\cos x}{\sqrt{\sin x}} dx$$

Let $u = \sin x$, so that $du = \cos x dx$.

$$\begin{aligned} \int \frac{\cos x}{\sqrt{\sin x}} dx &= \int \sin^{-1/2} x \cos x dx \\ &= \int u^{-1/2} du = 2u^{1/2} + C \\ &= 2 \sin^{1/2} x + C \end{aligned}$$

$$13. \quad \int \frac{\sin x}{1 + \cos x} dx$$

Let $u = 1 + \cos x$.

Then $du = -\sin x$
 $-du = -\sin x dx$.

$$\begin{aligned} \int \frac{\sin x}{1 + \cos x} dx &= \int \frac{1}{u} (-du) = -\int \frac{1}{u} du \\ &= -\ln |u| + C \\ &= -\ln |1 + \cos x| + C \end{aligned}$$

$$14. \quad \int \frac{\cos x}{1 - \sin x} dx$$

Let $u = 1 - \sin x$, that $du = -\cos x dx$.

$$\begin{aligned} \int \frac{\cos x}{1 - \sin x} dx &= -\int (1 - \sin x)^{-1} (-\cos x) dx \\ &= -\int u^{-1} du \\ &= -\ln |u| + C \\ &= -\ln |1 - \sin x| + C \end{aligned}$$

$$15. \quad \int 2x^7 \cos x^8 dx$$

Let $u = x^8$, so $du = 8x^7 dx$ or
 $\frac{1}{8} du = x^7 dx$.

$$\begin{aligned} \int 2x^7 \cos x^8 dx &= \int 2(\cos u) \cdot \frac{1}{8} du \\ &= \frac{1}{4} \int \cos u du \\ &= \frac{1}{4} \sin u + C \\ &= \frac{1}{4} \sin x^8 + C \end{aligned}$$

$$16. \quad \int (x + 2)^4 \sin(x + 2)^5 dx$$

Let $u = (x + 2)^5$, so that

$$du = 5(x + 2)^4 dx.$$

$$\begin{aligned} \int (x + 2)^4 \sin(x + 2)^5 dx &= \frac{1}{5} \int \sin(x + 2)^5 \cdot 5(x + 2)^4 dx \\ &= \frac{1}{5} \int \sin u du \\ &= -\frac{1}{5} \int \cos u + C \\ &= -\frac{1}{5} \cos(x + 2)^5 + C \end{aligned}$$

$$17. \quad \int \tan \frac{1}{3} x dx$$

Let $u = \frac{1}{3}x$, so $du = \frac{1}{3} dx$ or $3 du = dx$.

$$\begin{aligned} \int \tan \frac{1}{3} x dx &= \int (\tan u) \cdot 3 du \\ &= 3 \int \tan u du \\ &= -3 \ln |\cos u| + C \\ &= -3 \ln \left| \cos \frac{1}{3} x \right| + C \end{aligned}$$

$$18. \quad \int \cot \left(-\frac{3x}{8} \right) dx$$

Let $u = -\frac{3x}{8}$, so that $du = -\frac{3}{8} dx$. Then

$$\begin{aligned} \int \cot \left(-\frac{3x}{8} \right) dx &= -\frac{8}{3} \int \cot \left(-\frac{3x}{8} \right) \left(-\frac{3}{8} dx \right) \\ &= -\frac{8}{3} \int \cot u du \\ &= -\frac{8}{3} \ln |\sin u| + C \\ &= -\frac{8}{3} \ln \left| \sin \left(-\frac{3x}{8} \right) \right| + C \end{aligned}$$

$$19. \quad \int x^5 \cot x^6 dx$$

Let $u = x^6$, so $du = 6x^5 dx$ or $\frac{1}{6} du = x^5 dx$.

$$\begin{aligned}
 \int x^5 \cot x^6 dx &= \int (\cot u) \cdot \frac{1}{6} du \\
 &= \frac{1}{6} \int \cot u du \\
 &= \frac{1}{6} \ln |\sin u| + C \\
 &= \frac{1}{6} \ln |\sin x^6| + C
 \end{aligned}$$

20. $\int \frac{x}{4} \tan \left(\frac{x}{4} \right)^2 dx$

Let $u = \left(\frac{x}{4} \right)^2$, so that

$$du = 2 \left(\frac{x}{4} \right) \left(\frac{1}{4} dx \right) = \frac{x}{8} dx.$$

$$\begin{aligned}
 \int \frac{x}{4} \tan \left(\frac{x}{4} \right)^2 dx &= 2 \int \frac{x}{8} \tan \left(\frac{x}{4} \right)^2 dx \\
 &= 2 \int \tan u du \\
 &= -2 \ln |\cos u| + C \\
 &= -2 \ln \left| \cos \left(\frac{x}{4} \right)^2 \right| + C
 \end{aligned}$$

21. $\int e^x \sin e^x dx$

Let $u = e^x$.

Then $du = e^x dx$.

$$\begin{aligned}
 \int e^x \sin e^x dx &= \int \sin u du \\
 &= -\cos u + C \\
 &= -\cos e^x + C
 \end{aligned}$$

22. $\int e^{-x} \tan e^{-x} dx$

Let $u = e^{-x}$, so that $du = -e^{-x} dx$.

$$\begin{aligned}
 \int e^{-x} \tan e^{-x} dx &= -\int \tan u du \\
 &= \ln |\cos u| + C \\
 &= \ln |\cos e^{-x}| + C
 \end{aligned}$$

23. $\int e^x \csc e^x \cot e^x dx$

Let $u = e^x$, so that $du = e^x dx$.

$$\begin{aligned}
 \int e^x \csc e^x \cot e^x dx &= \int \csc u \cot u du \\
 &= \int \csc u \cot u du \\
 &= -\csc u + C \\
 &= -\csc e^x + C
 \end{aligned}$$

24. $\int x^4 \sec x^5 \tan x^5 dx$

Let $u = x^5$, so that $du = 5x^4 dx$

$$\begin{aligned}
 \int x^4 \sec x^5 \tan x^5 dx &= \frac{1}{5} \int \sec u \tan u du \\
 &= \frac{1}{5} \int \sec u \tan u du \\
 &= \frac{1}{5} \int \sec u du + C \\
 &= \frac{1}{5} \ln |\sec u + \tan u| + C \\
 &= \frac{1}{5} \ln |\sec x^5 + \tan x^5| + C
 \end{aligned}$$

25. $\int -6x \cos 5x dx$

Let $u = -6x$ and $dv = \cos 5x dx$.

Then $du = -6 dx$ and $v = \frac{1}{5} \sin 5x$.

$$\begin{aligned}
 \int -6x \cos 5x dx &= (-6x) \left(\frac{1}{5} \sin 5x \right) - \int \left(\frac{1}{5} \sin 5x \right) (-6 dx) \\
 &= -\frac{6}{5} x \sin 5x + \frac{6}{5} \int \sin 5x dx \\
 &= -\frac{6}{5} x \sin 5x + \frac{6}{5} \cdot \frac{1}{5} (-\cos 5x) + C \\
 &= -\frac{6}{5} x \sin 5x - \frac{6}{25} \cos 5x + C
 \end{aligned}$$

26. $\int 7x \sin 5x dx$

Let $u = 7x$ and $dv = \sin 5x dx$.

Then $du = 7 dx$ and $v = -\frac{1}{5} \cos 5x$.

$$\begin{aligned}
& \int 7x \sin 5x \, dx \\
&= 7x \left(-\frac{1}{5} \cos 5x \right) - \int \left(-\frac{1}{5} \cos 5x \right) \cdot 7 \, dx \\
&= -\frac{7}{5} x \cos 5x + \frac{7}{5} \int \cos 5x \, du \\
&= -\frac{7}{5} x \cos 5x + \frac{7}{5} \cdot \frac{1}{5} \sin 5x + C \\
&= -\frac{7}{5} x \cos 5x + \frac{7}{25} \sin 5x + C
\end{aligned}$$

27. $\int 4x \sin x \, dx$

Let $u = 4x$ and $dv = \sin x \, dx$.

Then $du = 4 \, dx$ and $v = -\cos x$.

$$\begin{aligned}
& \int 4x \sin x \, dx \\
&= 4x(-\cos x) - \int (-\cos x) \cdot 4 \, dx \\
&= -4x \cos x + 4 \int \cos x \, du \\
&= -4x \cos x + 4 \sin x + C
\end{aligned}$$

28. $\int -11x \cos x \, dx$

Let $u = -11x$ and $dv = \cos x \, dx$.

Then $du = -11 \, dx$ and $v = \sin x$.

$$\begin{aligned}
& \int -11x \cos x \, dx \\
&= -11x \sin x - \int (\sin x)(-11 \, dx) \\
&= -11x \sin x + 11 \int \sin x \, dx \\
&= -11x \sin x - 11 \cos x + C
\end{aligned}$$

29. $\int -6x^2 \cos 8x \, dx$

Let $u = -6x^2$ and $dv = \cos 8x \, dx$.

Then $du = -12x \, dx$ and $v = \frac{1}{8} \sin 8x$.

$$\begin{aligned}
& \int -6x^2 \cos 8x \, dx \\
&= (-6x^2) \left(\frac{1}{8} \sin 8x \right) \\
&\quad - \int \left(\frac{1}{8} \sin 8x \right) (-12x \, dx) \\
&= -\frac{3}{4} x^2 \sin 8x + \frac{3}{2} \int x \sin 8x \, dx
\end{aligned}$$

In $\int x \sin 8x \, dx$, let

$$u = x \quad \text{and} \quad dv = \sin 8x \, dx.$$

Then $du = dx$ and $v = -\frac{1}{8} \cos 8x$.

$$\begin{aligned}
& \int -6x^2 \cos 8x \, dx \\
&= -\frac{3}{4} x^2 \sin 8x \\
&\quad + \frac{3}{2} \left[-\frac{1}{8} x \cos 8x - \int \left(-\frac{1}{8} \cos 8x \right) dx \right] \\
&= -\frac{3}{4} x^2 \sin 8x - \frac{3}{16} x \cos 8x \\
&\quad + \frac{3}{16} \int \cos 8x \, dx \\
&= -\frac{3}{4} x^2 \sin 8x - \frac{3}{16} x \cos 8x \\
&\quad + \frac{3}{16} \cdot \frac{1}{8} \sin 8x + C \\
&= -\frac{3}{4} x^2 \sin 8x - \frac{3}{16} x \cos 8x \\
&\quad + \frac{3}{128} \sin 8x + C
\end{aligned}$$

30. $\int 10x^2 \sin \frac{1}{2} x \, dx$

Let $u = 10x^2$ and $dv = \sin \frac{1}{2} x \, dx$

Then $du = 20x \, dx$ and $v = -2 \cos \frac{1}{2} x$.

$$\begin{aligned}
& \int 10x^2 \sin \frac{1}{2} x \, dx \\
&= (10x^2) \left(-2 \cos \frac{1}{2} x \right) \\
&\quad - \int \left(-2 \cos \frac{1}{2} x \right) (20x \, dx) \\
&= -20x^2 \cos \frac{1}{2} x + 40 \int x \cos \frac{1}{2} x \, dx
\end{aligned}$$

Let $u = x$ and $dv = \cos \frac{1}{2} x \, dx$.

Then $du = dx$ and $v = 2 \sin \frac{1}{2}x$.

$$\begin{aligned} \int 10x^2 \sin \frac{1}{2}x \, dx &= -20x^2 \cos \frac{1}{2}x \\ &\quad + 40 \left(2x \sin \frac{1}{2}x - \int 2 \sin \frac{1}{2}x \, dx \right) \\ &= -20x^2 \cos \frac{1}{2}x + 80x \sin \frac{1}{2}x \\ &\quad + 160 \cos \frac{1}{2}x + C \end{aligned}$$

$$\begin{aligned} 31. \quad \int_0^{\pi/4} \sin x \, dx &= -\cos x \Big|_0^{\pi/4} \\ &= -\cos \frac{\pi}{4} - (-\cos 0) \\ &= -\frac{\sqrt{2}}{2} + 1 \\ &= 1 - \frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} 32. \quad \int_{-\pi/2}^0 \cos x \, dx &= \sin x \Big|_{-\pi/2}^0 \\ &= \sin 0 - \sin \left(-\frac{\pi}{2} \right) \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} 33. \quad \int_0^{\pi/6} \tan x \, dx &= -\ln |\cos x| \Big|_0^{\pi/6} \\ &= -\ln \left| \cos \frac{\pi}{6} \right| - (-\ln |\cos 0|) \\ &= -\ln \frac{\sqrt{3}}{2} + \ln 1 \\ &= -\ln \frac{\sqrt{3}}{2} + 0 \\ &= -\ln \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} 34. \quad \int_{\pi/4}^{\pi/2} \cot x \, dx &= \ln |\sin x| \Big|_{\pi/4}^{\pi/2} \\ &= \ln \left| \sin \frac{\pi}{2} \right| - \ln \left| \sin \frac{\pi}{4} \right| \\ &= \ln 1 - \ln \frac{\sqrt{2}}{2} \\ &= -\ln \frac{\sqrt{2}}{2} \\ \text{or } \ln \left(\frac{\sqrt{2}}{2} \right)^{-1} &= \ln \sqrt{2} = \frac{1}{2} \ln 2 \end{aligned}$$

$$\begin{aligned} 35. \quad \int_{\pi/2}^{2\pi/3} \cos x \, dx &= \sin x \Big|_{\pi/2}^{2\pi/3} \\ &= \sin \frac{2\pi}{3} - \sin \frac{\pi}{2} \\ &= \frac{\sqrt{3}}{2} - 1 \end{aligned}$$

$$\begin{aligned} 36. \quad \int_{\pi/6}^{\pi/4} \sin x \, dx &= -\cos x \Big|_{\pi/6}^{\pi/4} \\ &= -\cos \frac{\pi}{4} - \left(-\cos \frac{\pi}{6} \right) \\ &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3} - \sqrt{2}}{2} \end{aligned}$$

37. Use the fnInt function on the graphing calculator to enter $\text{fnInt}(e^{(-x)} \sin(x), x, 0, b)$, for successively larger values of b , which returns a value of 0.5 for sufficiently large enough b . Thus, an estimate of $\int_0^{\infty} e^{-x} \sin x \, dx$ is 0.5.

38. Use the fnInt function on the graphing calculator to enter $\text{fnInt}(e^{(-x)} \cos x, x, 0, b)$, for successively larger values of b , which returns a value of 0.5 for sufficiently large enough b . Thus, an estimate of $\int_0^{\infty} e^{-x} \cos x \, dx$ is 0.5.

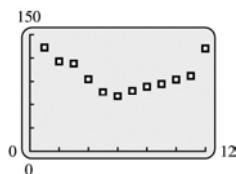
$$39. \quad S(t) = 500 + 500 \cos \left(\frac{\pi}{6}t \right)$$

Total sales over a year's time are approximated by the area under the graph of S during any twelve-month period due to periodicity of S .

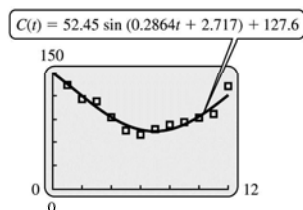
Total sales

$$\begin{aligned}
&\approx \int_0^{12} S(t) dt \\
&= \int_0^{12} \left[500 + 500 \cos \left(\frac{\pi}{6} t \right) \right] dt \\
&= \int_0^{12} 500 dt + 500 \int_0^{12} \cos \left(\frac{\pi}{6} t \right) dt \\
&= 500t \Big|_0^{12} + 500 \left[\frac{6}{\pi} \sin \left(\frac{\pi}{6} t \right) \right]_0^{12} \\
&= 6000 + \frac{3000}{\pi} (\sin 2\pi - \sin 0) \\
&= 6000 + \frac{3000}{\pi} (0 - 0) = 6000
\end{aligned}$$

40. (a) It is not clear whether petroleum consumption is likely to be periodic or not.



- (b) $C(t) = 52.45 \sin(0.2864t + 2.717) + 127.552$



- (c) $C(9) \approx 83.7$, so our estimate for September is 83.7 trillion BTUs; the actual value is 86.7 trillion BTUs.
- (d) Using the numerical differentiation function on a graphing calculator we find that $C'(9) \approx 8.26$, so in September the rate of consumption is changing at about 8.26 trillion BTUs per month.
- (e) Integrating $C(t)$ from 0 to 12 using a calculator gives the estimate 1182 trillion BTUs; the sum of the values in the table is 1171 trillion BTUs.
- (f) The function in the model has a period of
- $$\begin{aligned}
T &= \frac{2\pi}{b} \\
&= \frac{2\pi}{0.2864} \\
&\approx 21.9 \text{ months.}
\end{aligned}$$

One might expect a period of 12 months, but longer-term trends might make this annual period hard to see.

41. $T(t) = 50 + 50 \cos \left(\frac{\pi}{6} t \right)$

Since T is periodic, the number of animals passing the checkpoint is equal to the area under the curve for any 12-month period. Let t vary from 0 to 12.

$$\begin{aligned}
\text{Total} &= \int_0^{12} \left[50 + 50 \cos \left(\frac{\pi}{6} t \right) \right] dt \\
&= \int_0^{12} 50 dt \\
&\quad + \frac{6}{\pi} \int_0^{12} 50 \cos \left(\frac{\pi}{6} t \right) \left(\frac{\pi}{6} dt \right) \\
&= 50t \Big|_0^{12} + \frac{300}{\pi} \sin \left(\frac{\pi}{6} t \right) \Big|_0^{12} \\
&= (600 - 0) + \frac{300}{\pi} (0 - 0) \\
&= 600 \text{ (in hundreds)}
\end{aligned}$$

The total number of animals is 60,000.

42. $V(t) = 170 \sin(120\pi t)$

$$\text{Root mean square} = \sqrt{\frac{\int_0^T V^2(t) dt}{T}}$$

(a) The period is $T = \frac{2\pi}{120\pi} = \frac{1}{60}$ sec.

(b)

$$\begin{aligned}
&\int_0^T V^2(t) dt \\
&= \int_0^{1/60} [170 \sin(120\pi t)]^2 dt \\
&= 170^2 \int_0^{1/60} \sin^2(120\pi t) dt \\
&= 170^2 \int_0^{1/60} \frac{1}{2} [1 - \cos(240\pi t)] dt \\
&= \frac{170^2}{2} \int_0^{1/60} [1 - \cos(240\pi t)] dt \\
&= \frac{170^2}{2} \left[t - \frac{1}{240\pi} \sin(240\pi t) \right] \Big|_0^{1/60}
\end{aligned}$$

$$= \frac{170^2}{2} \left[\frac{1}{60} - \frac{1}{240\pi} \sin \left(240\pi \frac{1}{60} \right) \right] \\ - \frac{170^2}{2} \cdot 0 \\ = \frac{170^2}{120}$$

Root mean square

$$= \sqrt{\frac{\frac{170^2}{120}}{\frac{1}{60}}} = \sqrt{\frac{170^2}{2}} \approx 120.21$$

Thus, 120 volts is the root mean square value for $V(t)$.

43. The total amount of daylight is given by

$$\int_0^{365} N(t) dt = \int_0^{365} [183.549 \sin(0.0172t \\ - 1.329) + 728.124] dt \\ = \left[-\frac{183.549}{0.0172} \cos(0.0172t \\ - 1.329) + 728.124t \right]_0^{365} \\ \approx 265,819.0192 \text{ minutes} \\ \approx 4430 \text{ hours.}$$

The result is relatively close to the actual value.

44. The amount of water in the sink is

$$\int_0^k \cos t \, dt = \sin t \Big|_0^k \\ = \sin k - \sin 0 \\ = \sin k \text{ ("sink").}$$

45. The amount of water in the tank is

$$\int_0^k \sec^2 t \, dt = \tan t \Big|_0^k \\ = \tan k - \tan 0 \\ = \tan k \text{ ("tank").}$$

46. $C(t) = \int -\sin t = \cos t + C$

When $t = 0$, $\cos 0 + C = 1$

$$1 + C = 1$$

$$C = 0.$$

Then a formula for cost is $C(t) = \cos t$ ("cost").

Chapter 13 Review Exercises

- False; The period of cosine is 2π .
- True
- False; There's no reason to suppose that the Dow is periodic.
- False; $\cos(a + b) = \cos a \cos b - \sin a \sin b$
- True
- False; $D_x \tan(x^2) = 2x \sec^2(x^2)$
- True
- False; The secant function has no absolute maximum or minimum
- False; Since the sine function is negative on half of this interval, the true area is $\int_0^{2\pi} |\sin x| dx$.
- False; Use substitution, with $u = 5 + \sin x$ and $du = \cos x \, dx$.
- The exact value for the trigonometric functions can be determined for any integer multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.
- $90^\circ = 90 \left(\frac{\pi}{180} \right) = \frac{90\pi}{180} = \frac{\pi}{2}$
- $160^\circ = 160 \left(\frac{\pi}{180} \right) = \frac{8\pi}{9}$
- $225^\circ = 225 \left(\frac{\pi}{180} \right) = \frac{5\pi}{4}$
- $270^\circ = 270 \left(\frac{\pi}{180} \right) = \frac{3\pi}{2}$
- $360^\circ = 2\pi$
- $405^\circ = 405 \left(\frac{\pi}{180} \right) = \frac{9\pi}{4}$

$$21. \quad 5\pi = 5\pi \left(\frac{180^\circ}{\pi} \right) = 900^\circ$$

$$22. \quad \frac{3\pi}{4} = \frac{3\pi}{4} \left(\frac{180^\circ}{\pi} \right) = 135^\circ$$

$$23. \quad \frac{9\pi}{20} = \frac{9\pi}{20} \left(\frac{180^\circ}{\pi} \right) = 81^\circ$$

$$24. \quad \frac{3\pi}{10} = \frac{3\pi}{10} \left(\frac{180^\circ}{\pi} \right) = 54^\circ$$

$$25. \quad \frac{13\pi}{20} = \frac{13\pi}{20} \left(\frac{180^\circ}{\pi} \right) = 117^\circ$$

$$26. \quad \frac{13\pi}{15} = \frac{13\pi}{15} \left(\frac{180^\circ}{\pi} \right) = 156^\circ$$

27. When an angle 60° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, \sqrt{3})$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2,$$

so

$$\sin 60^\circ = \frac{y}{r} = \frac{\sqrt{3}}{2}.$$

28. When an angle of 120° is drawn in standard position, $(x, y) = (-1, \sqrt{3})$ is one point on its terminal sides, so

$$\tan 120^\circ = \frac{y}{x} = -\sqrt{3}.$$

29. When an angle of -45° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, -1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 1} = \sqrt{2},$$

so

$$\cos(-45^\circ) = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

30. When an angle of 150° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-\sqrt{3}, 1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2,$$

so

$$\sec 150^\circ = \frac{r}{x} = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3}.$$

31. When an angle of 120° is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (-1, \sqrt{3})$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2,$$

so

$$\csc 120^\circ = \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

32. When an angle of 300° is drawn in standard position, $(x, y) = (1, -\sqrt{3})$ is one point on its terminal side, so

$$\cot 300^\circ = \frac{x}{y} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

33. When an angle of $\frac{\pi}{6}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (\sqrt{3}, 1)$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2,$$

so

$$\sin \frac{\pi}{6} = \frac{y}{r} = \frac{1}{2}.$$

34. When an angle of $\frac{7\pi}{3}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, \sqrt{3})$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2,$$

so

$$\cos \frac{7\pi}{3} = \frac{x}{r} = \frac{1}{2}.$$

35. When an angle of $\frac{5\pi}{3}$ is drawn in standard position, one choice of a point on its terminal side is $(x, y) = (1, -\sqrt{3})$. Then

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2,$$

so

$$\sec \frac{5\pi}{3} = \frac{r}{x} = \frac{2}{1} = 2.$$

36. When an angle of $\frac{7\pi}{3}$ is drawn in standard position, $(x, y) = (1, \sqrt{3})$ is one point on its terminal side. Then

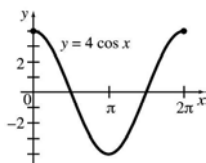
$$r = \sqrt{1 + 3} = 2,$$

so

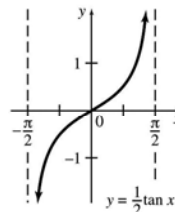
$$\csc \frac{7\pi}{3} = \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

37. $\sin 47^\circ \approx 0.7314$
38. $\cos 72^\circ \approx 0.3090$
39. $\tan 115^\circ \approx -2.1445$
40. $\sin (-123^\circ) \approx -0.8387$
41. $\sin 2.3581 \approx 0.7058$
42. $\cos 0.8215 \approx 0.6811$
43. $\cos 0.5934 \approx 0.8290$
44. $\tan 1.2915 \approx 3.4868$

45. The graph of $y = \cos x$ appears in Figure 17 in Section 1 of this chapter. To get $y = 4 \cos x$, each value of y in $y = \cos x$ must be multiplied by 4. This gives a graph going through $(0, 4)$, $(\pi, -4)$ and $(2\pi, 4)$.



46. The graph of $y = \frac{1}{2} \tan x$ is similar to the graph of $y = \tan x$ except that each ordinate value is multiplied by a factor of $\frac{1}{2}$. Note that the points $(\frac{\pi}{4}, \frac{1}{2})$ and $(-\frac{\pi}{4}, -\frac{1}{2})$ lie on the graph.

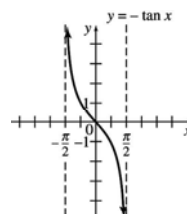


47. The graph of $y = \tan x$ appears in Figure 18 in Section 1 in this chapter. The difference between the graph of $y = \tan x$ and $y = -\tan x$ is that the y -values of points on the graph of $y = -\tan x$ are the opposites of the y -values of the corresponding points on the graph of $y = \tan x$.

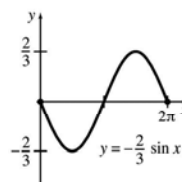
A sample calculation:

When $x = \frac{\pi}{4}$,

$$y = -\tan \frac{\pi}{4} = -1.$$



48. The graph of $y = -\frac{2}{3} \sin x$ is similar to the graph of $y = \sin x$ except that it has two-thirds the amplitude and is reflected about the x -axis.



49. Because the derivative of $y = \sin x$ is $y' = \cos x$, the slope of $y = \sin x$ varies from -1 to 1 .

$$50. \quad y = -4\sin 7x$$

$$\begin{aligned}\frac{dy}{dx} &= -4(\cos 7x) \cdot D_x(7x) \\ &= -4(\cos 7x)7 \\ &= -28 \cos 7x\end{aligned}$$

$$51. \quad y = 2 \tan 5x$$

$$\begin{aligned}\frac{dy}{dx} &= 2 \sec^2 5x \cdot D_x(5x) \\ &= 10 \sec^2 5x\end{aligned}$$

$$52. \quad y = \tan(4x^2 + 3)$$

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(4x^2 + 3) \cdot D_x(4x^2 + 3) \\ &= \sec^2(4x^2 + 3) \cdot (8x) \\ &= 8x \sec^2(4x^2 + 3)\end{aligned}$$

$$53. \quad y = \cot(6 - 3x^2)$$

$$\begin{aligned}\frac{dy}{dx} &= [-\csc^2(6 - 3x^2)] \cdot D_x(6 - 3x^2) \\ &= 6x \csc^2(6 - 3x^2)\end{aligned}$$

$$54. \quad y = 2 \cos^5 x$$

$$\begin{aligned}\frac{dy}{dx} &= 2D_x(\cos x)^5 \\ &= [2 \cdot 5(\cos x)^4] \cdot D_x(\cos x) \\ &= 10(\cos x)^4(-\sin x) \\ &= -10 \sin x \cos^4 x\end{aligned}$$

$$55. \quad y = 2 \sin^4(4x^2)$$

$$\begin{aligned}\frac{dy}{dx} &= [8 \sin^3(4x^2)] \cdot D_x[\sin(4x^2)] \\ &= 8 \sin^3(4x^2) \cdot \cos(4x^2) \cdot D_x(4x^2) \\ &= 64x \sin^3(4x^2) \cos(4x^2)\end{aligned}$$

$$56. \quad y = \cot\left(\frac{1}{2}x^4\right)$$

$$\begin{aligned}\frac{dy}{dx} &= -\csc\left(\frac{1}{2}x^4\right) \cdot D_x\left(\frac{1}{2}x^4\right) \\ &= -\csc^2\left(\frac{1}{2}x^4\right) \cdot 2x^3 \\ &= -2x^3 \csc^2\left(\frac{1}{2}x^4\right)\end{aligned}$$

$$57. \quad y = \cos(1 + x^2)$$

$$\begin{aligned}\frac{dy}{dx} &= [-\sin(1 + x^2)] \cdot D_x(1 + x^2) \\ &= -2x \sin(1 + x^2)\end{aligned}$$

$$58. \quad y = x^2 \csc x$$

$$\begin{aligned}\frac{dy}{dx} &= x^2 D_x(\csc x) + \csc x D_x(x^2) \\ &= x^2(-\csc x \cot x) + \csc x(2x) \\ &= -x^2 \csc x \cot x + 2x \csc x\end{aligned}$$

$$59. \quad y = e^{-2x} \sin x$$

$$\begin{aligned}\frac{dy}{dx} &= e^{-2x} \cdot D_x(\sin x) + \sin x \cdot D_x(e^{-2x}) \\ &= e^{-2x}(\cos x) \\ &\quad + (\sin x)(e^{-2x}) \cdot D_x(-2x) \\ &= e^{-2x}(\cos x) + (\sin x)(e^{-2x})(-2) \\ &= e^{-2x}(\cos x - 2\sin x)\end{aligned}$$

$$60. \quad y = \frac{\sin x - 1}{\sin x + 1}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\sin x + 1)(\cos x) - (\sin x - 1)(\cos x)}{(\sin x + 1)^2} \\ &= \frac{\sin x \cos x + \cos x - \sin x \cos x + \cos x}{(\sin x + 1)^2} \\ &= \frac{2 \cos x}{(\sin x + 1)^2}\end{aligned}$$

$$61.$$

$$\begin{aligned}y &= \frac{\cos^2 x}{1 - \cos x} \\ \frac{dy}{dx} &= \frac{(1 - \cos x)(-2 \cos x \sin x) - (\cos^2 x)(\sin x)}{(1 - \cos x)^2} \\ &= \frac{-2 \cos x \sin x + \cos^2 x \sin x}{(1 - \cos x)^2}\end{aligned}$$

$$\begin{aligned}
 62. \quad y &= \frac{6-x}{\sec x} \\
 \frac{dy}{dx} &= \frac{(\sec x) \cdot D_x(6-x) - (6-x) \cdot D_x(\sec x)}{\sec^2 x} \\
 &= \frac{(\sec x) \cdot (-1) - (6-x) \cdot (\sec x \tan x)}{\sec^2 x} \\
 &= \frac{\sec x[-1 - (6-x)\tan x]}{\sec^2 x} \\
 &= \frac{1 + (6-x)\tan x}{-\sec x}
 \end{aligned}$$

$$\begin{aligned}
 63. \quad y &= \frac{\tan x}{1+x} \\
 \frac{dy}{dx} &= \frac{(1+x)(\sec^2 x) - (\tan x)(1)}{(1+x)^2} \\
 &= \frac{\sec^2 x + x\sec^2 x - \tan x}{(1+x)^2}
 \end{aligned}$$

$$\begin{aligned}
 64. \quad y &= \ln |\cos x| \\
 \frac{dy}{dx} &= \frac{D_x(\cos x)}{\cos x} \\
 &= \frac{-\sin x}{\cos x} = -\tan x
 \end{aligned}$$

$$\begin{aligned}
 65. \quad y &= \ln |5 \sin x| \\
 \frac{dy}{dx} &= \frac{1}{5 \sin x} \cdot D_x(5 \sin x) \\
 &= \frac{\cos x}{\sin x} \\
 &\text{or } \cot x
 \end{aligned}$$

$$\begin{aligned}
 66. \quad &\int \sin 2x \, dx \\
 &\text{Let } u = 2x. \\
 &\text{Then } du = 2dx \\
 &\quad \frac{1}{2} du = dx. \\
 &\int \sin 2x \, dx = \int (\sin u) \left(\frac{1}{2} du \right) \\
 &= \frac{1}{2} \int \sin u \, du \\
 &= \frac{1}{2} (-\cos u) + C \\
 &= -\frac{1}{2} \cos 2x + C
 \end{aligned}$$

$$\begin{aligned}
 67. \quad &\int \cos 5x \, dx \\
 &\text{Let } u = 5x, \text{ so } du = 5dx \text{ or } \frac{1}{5} du = dx.
 \end{aligned}$$

$$\begin{aligned}
 \int \cos 5x \, dx &= \int \cos u \cdot \frac{1}{5} du \\
 &= \frac{1}{5} \int \cos u \, du \\
 &= \frac{1}{5} \sin u + C \\
 &= \frac{1}{5} \sin 5x + C
 \end{aligned}$$

$$\begin{aligned}
 68. \quad &\int \tan 7x \, dx \\
 &\text{Let } u = 7x, \text{ so } du = 7dx \text{ or } \frac{1}{7} du = dx.
 \end{aligned}$$

$$\begin{aligned}
 \int \tan 7x \, dx &= \int (\tan u) \cdot \frac{1}{7} du \\
 &= \frac{1}{7} \int \tan u \, du \\
 &= -\frac{1}{7} \ln |\cos u| + C \\
 &= -\frac{1}{7} \ln |\cos 7x| + C
 \end{aligned}$$

$$\begin{aligned}
 69. \quad &\int \sec^2 5x \, dx \\
 &\text{Let } u = 5x, \text{ so that } du = 5 \, dx. \\
 \int \sec^2 5x \, dx &= \frac{1}{5} \int (\sec^2 5x) (5 \, dx) \\
 &= \frac{1}{5} \int \sec^2 u \, du \\
 &= \frac{1}{5} \tan u + C \\
 &= \frac{1}{5} \tan 5x + C
 \end{aligned}$$

$$\begin{aligned}
 70. \quad \int 8 \sec^2 x \, dx &= 8 \int \sec^2 x \, dx \\
 &= 8 \tan x + C
 \end{aligned}$$

$$\begin{aligned}
 71. \quad \int 4 \csc^2 x \, dx &= -4 \int -\csc^2 x \, dx \\
 &= -4 \cot x + C
 \end{aligned}$$

72. $\int x^2 \sin 4x^3 dx$

Let $u = 4x^3$, so $du = 12x^2 dx$ or $\frac{1}{12} du = x^2 dx$.

$$\begin{aligned}\int x^2 \sin 4x^3 dx &= \int (\sin u) \cdot \frac{1}{12} du \\ &= \frac{1}{12} \int \sin u du \\ &= -\frac{1}{12} \cos u + C \\ &= -\frac{1}{12} \cos 4x^3 + C\end{aligned}$$

73. $\int 5x \sec 2x^2 \tan 2x^2 dx$

Let $u = 2x^2$, so that $du = 4x dx$.

$$\begin{aligned}\int 5x \sec 2x^2 \tan 2x^2 dx &= \frac{5}{4} \int \sec 2x^2 \tan 2x^2 (4x dx) \\ &= \frac{5}{4} \int \sec u \tan u du \\ &= \frac{5}{4} \sec u + C \\ &= \frac{5}{4} \sec 2x^2 + C\end{aligned}$$

74. $\int \sqrt{\cos x} \sin x dx$

Let $u = \cos x$.

Then $du = -\sin x dx$
 $-du = \sin x dx$.

$$\begin{aligned}\int \sqrt{\cos x} \sin x dx &= \int \sqrt{u} (-du) \\ &= -\int u^{1/2} du \\ &= -\frac{2}{3} u^{3/2} + C \\ &= -\frac{2}{3} (\cos x)^{3/2} + C\end{aligned}$$

75. $\int \cos^8 x \sin x dx$

Let $u = \cos x$, so $du = -\sin x dx$.

$$\begin{aligned}\int \cos^8 x \sin x dx &= -\int u^8 du \\ &= -\frac{1}{9} u^9 + C \\ &= -\frac{1}{9} \cos^9 x + C\end{aligned}$$

76. $\int x \tan 11x^2 dx$

Let $u = 11x^2$. Then $du = 22x dx$.

$$\begin{aligned}\int x \tan 11x^2 dx &= \frac{1}{22} \int (\tan 11x^2) \cdot (22x dx) \\ &= \frac{1}{22} \int \tan u du \\ &= \frac{1}{22} (-\ln |\cos u|) + C \\ &= -\frac{1}{22} \ln |\cos 11x^2| + C\end{aligned}$$

77. $\int x^2 \cot 8x^3 dx$

Let $u = 8x^3$, so that $du = 24x^2 dx$.

$$\begin{aligned}\int x^2 \cot 8x^3 dx &= \frac{1}{24} \int (\cot 8x^3) (24x^2 dx) \\ &= \frac{1}{24} \int \cot u du \\ &= \frac{1}{24} \ln |\sin u| + C \\ &= \frac{1}{24} \ln |\sin 8x^3| + C\end{aligned}$$

78. $\int (\sin x)^{3/2} \cos x \, dx$

Let $u = \sin x$, so $du = \cos x \, dx$.

$$\begin{aligned} \int (\sin x)^{3/2} \cos x \, dx &= \int u^{3/2} \, du \\ &= \frac{u^{3/2+1}}{\frac{3}{2}+1} + C \\ &= \frac{u^{5/2}}{\frac{5}{2}} + C \\ &= \frac{2}{5} u^{5/2} + C \\ &= \frac{2}{5} (\sin x)^{5/2} + C \end{aligned}$$

79. $\int (\cos x)^{-4/3} \sin x \, dx$

Let, $u = \cos x$ so that $du = -\sin x \, dx$.

$$\begin{aligned} \int (\cos x)^{-4/3} \sin x \, dx &= -\int (\cos x)^{-4/3} (\sin x) \, dx \\ &= -\int u^{-4/3} \, du \\ &= 3u^{-1/3} + C \\ &= 3(\cos x)^{-1/3} + C \end{aligned}$$

80. $\int \sec^2 5x \tan 5x \, dx$

Let $u = \tan 5x$, so $du = 5 \sec^2 5x \, dx$.

$$\begin{aligned} \int \sec^2 5x \tan 5x \, dx &= \frac{1}{5} \int u \, du \\ &= \frac{1}{5} \cdot \frac{u^2}{2} + C \\ &= \frac{1}{10} u^2 + C \\ &= \frac{1}{10} \tan^2 5x + C \end{aligned}$$

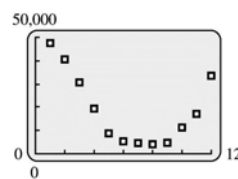
81. $\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2}$
 $= \sin \frac{\pi}{2} - \sin 0$
 $= 1 - 0 = 1$

82. $\int_{-\pi}^{2\pi/3} -\sin x \, dx = \cos x \Big|_{-\pi}^{2\pi/3}$
 $= \cos \frac{2\pi}{3} - \cos(-\pi)$
 $= -\frac{1}{2} - (-1) = \frac{1}{2}$

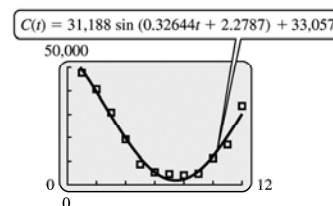
83. $\int_0^{2\pi} (10 + 10 \cos x) \, dx$
 $= \int_0^{2\pi} 10 \, dx + \int_0^{2\pi} 10 \cos x \, dx$
 $= 10(2\pi) + 10 \int_0^{2\pi} \cos x \, dx$
 $= 20\pi + 10(\sin x) \Big|_0^{2\pi}$
 $= 20\pi + 10(\sin 2\pi - \sin 0)$
 $= 20\pi + 10(0 - 0)$
 $= 20\pi$

84. $\int_0^{\pi/3} (3 - 3 \sin x) \, dx$
 $= \int_0^{\pi/3} 3 \, dx - \int_0^{\pi/3} 3 \sin x \, dx$
 $= 3x \Big|_0^{\pi/3} + 3 \cos x \Big|_0^{\pi/3}$
 $= (\pi - 0) + \left(\frac{3}{2} - 3 \right) = \pi - \frac{3}{2}$

85. (a) Residential usage of natural gas is probably periodic.



- (b) $C(t) = 31,188 \sin(0.32644t + 2.2787) + 33,057$.



- (c) Integrating $C(t)$ from 0 to 12 using a calculator gives the estimate 239,389 million cubic feet; the sum of the values in the table is 227,577 million cubic feet. (Note that in carrying out the integration we used the unrounded coefficients as given by sine regression on the calculator. When possible, all intermediate results should be kept in full precision until the final answer is obtained.)

- (d) The function in the model has a period of

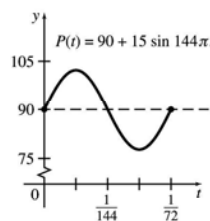
$$\begin{aligned} T &= \frac{2\pi}{b} \\ &= \frac{2\pi}{0.32644} \\ &\approx 19.2 \text{ months.} \end{aligned}$$

86. $P(t) = 90 + 15 \sin 144\pi t$

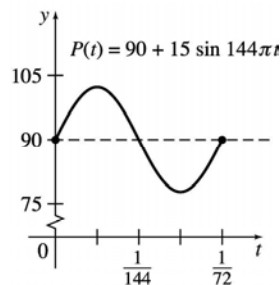
The maximum possible value of $\sin \alpha$ is 1, while the minimum possible value is -1 . Replacing α with $144\pi t$ gives

$$\begin{aligned} -1 &\leq \sin 144\pi t \leq 1, \\ 90 + 15(-1) &\leq P(t) \leq 90 + 15(1) \\ 75 &\leq P(t) \leq 105. \end{aligned}$$

Therefore, the minimum value of $P(t)$ is 75 and the maximum value of $P(t)$ is 105.



87. We draw θ in standard position.



From this diagram, we see that

$$\sin \theta = \frac{s}{L_2}.$$

88. $\sin \theta = \frac{s}{L_2}$

$$L_2 \sin \theta = s$$

$$L_2 = \frac{s}{\sin \theta}$$

89. Using the sketch of θ in the solution to Exercise 87 and the definition of cotangent, we see that

$$\cot \theta = \frac{L_0 - L_1}{s}.$$

90. $\cot \theta = \frac{L_0 - L_1}{s}$

$$s \cot \theta = L_0 - L_1$$

$$L_1 + s \cot \theta = L_0$$

$$L_1 = L_0 - s \cot \theta$$

91. The length of AD is L_1 , and the radius of that section of blood vessel is r_1 , so the general equation

$$k = \frac{L}{r^4}$$

is similar to

$$R_1 = k \frac{L_1}{r_1^4}$$

for that particular segment of the blood vessel.

92. $R_2 = k \cdot \frac{L_2}{r_2^4}$ where R_2 is the resistance along DC .

93. $R = R_1 + R_2$

$$= k \frac{L_1}{r_1^4} + k \frac{L_2}{r_2^4}$$

$$= k \left(\frac{L_1}{r_1^4} + \frac{L_2}{r_2^4} \right)$$

94. $R = k \left(\frac{L_1}{r_1^4} + \frac{L_2}{r_2^4} \right)$

$$= k \left(\frac{L_0 - s \cot \theta}{r_1^4} + \frac{s}{(\sin \theta) r_2^4} \right)$$

$$= \frac{k(L_0 - s \cot \theta)}{r_1^4} + \frac{ks}{r_2^4 \sin \theta}$$

- 95.** Since k, L_1, L_0, s, r_1 , and r_2 are all constants, the only letter left as a variable is θ , so the differentiation in the symbol R' must be differentiation with respect to θ .

$$\begin{aligned}
 R' &= D_\theta R \\
 &= D_\theta \left(k \frac{L_0}{r_1^4} - \frac{sk}{r_1^4} \cot \theta + \frac{s}{r_2^4} \cdot \frac{k}{\sin \theta} \right) \\
 &= k D_\theta \left(\frac{L_0}{r_1^4} - \frac{s}{r_1^4} \cot \theta + \frac{s}{r_2^4} \cdot \frac{1}{\sin \theta} \right) \\
 &= k \left[D_\theta \left(\frac{L_0}{r_1^4} \right) - D_\theta \left(\frac{s}{r_1^4} \cot \theta \right) \right. \\
 &\quad \left. + D_\theta \left(\frac{s}{r_2^4} \cdot \frac{1}{\sin \theta} \right) \right] \\
 &= k \left[0 - \frac{s}{r_1^4} D_\theta (\cot \theta) + \frac{s}{r_2^4} D_\theta \left(\frac{1}{\sin \theta} \right) \right] \\
 &= k \left[-\frac{s}{r_1^4} (-\csc^2 \theta) + \frac{s}{r_2^4} \left(\frac{-\cos \theta}{\sin^2 \theta} \right) \right] \\
 &= k \left(\frac{s}{r_1^4} \frac{1}{\sin^2 \theta} - \frac{s}{r_2^4} \cdot \frac{\cos \theta}{\sin^2 \theta} \right) \\
 &= \frac{ks}{\sin^2 \theta} \left(\frac{1}{r_1^4} - \frac{\cos \theta}{r_2^4} \right) \\
 &= \frac{ks \csc^2 \theta}{r_1^4} - \frac{ks \cos \theta}{r_2^4 \sin^2 \theta}
 \end{aligned}$$

- 96.** Using Exercise 95 gives

$$\frac{ks \csc^2 \theta}{r_1^4} - \frac{ks \cos \theta}{r_2^4 \sin^2 \theta} = 0.$$

- 97.** If the left side of the equation in the solution to Exercise 95 is multiplied by $\frac{\sin^2 \theta}{s}$, we get

$$\begin{aligned}
 &\frac{\sin^2 \theta}{s} \cdot \frac{ks}{\sin^2 \theta} \left(\frac{1}{r_1^4} - \frac{\cos \theta}{r_2^4} \right) \\
 &= k \left(\frac{1}{r_1^4} - \frac{\cos \theta}{r_2^4} \right)
 \end{aligned}$$

This gives the equation

$$\frac{k}{r_1^4} - \frac{k \cos \theta}{r_2^4} = 0.$$

- 98.** Using Exercise 97 gives

$$\begin{aligned}
 k \left(\frac{1}{r_1^4} - \frac{\cos \theta}{r_2^4} \right) &= 0 \\
 \frac{1}{r_1^4} - \frac{\cos \theta}{r_2^4} &= 0 \quad k \neq 0 \\
 \frac{1}{r_1^4} &= \frac{\cos \theta}{r_2^4} \\
 \cos \theta &= \frac{r_2^4}{r_1^4}
 \end{aligned}$$

- 99.** If $r_1 = 1$ and $r_2 = \frac{1}{4}$, then

$$\begin{aligned}
 \cos \theta &= \left(\frac{\frac{1}{4}}{1} \right)^4 = \left(\frac{1}{4} \right)^4 \\
 &= \frac{1}{256} \approx 0.0039,
 \end{aligned}$$

from which we get

$$\theta \approx 90^\circ.$$

- 100.** If $r_1 = 1.4$ and $r_2 = 0.8$, then

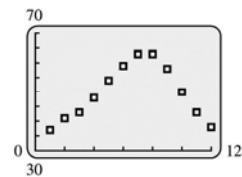
$$\cos \theta = \frac{(0.8)^4}{(1.4)^4} \approx 0.1066.$$

Thus,

$$\theta \approx 84^\circ.$$

- 101.** $s(t) = A \cos(Bt + C)$
 $s'(t) = -A \sin(Bt + C) \cdot D_t(Bt + C)$
 $= -A \sin(Bt + C) \cdot B$
 $= -AB \sin(Bt + C)$
 $s''(t) = -AB \cos(Bt + C) \cdot D_t(Bt + C)$
 $= -AB \cos(Bt + C) \cdot B$
 $= -B^2 A \cos(Bt + C)$
 $= -B^2 s(t)$

- 102. (a)** Enter the data into a graphing calculator and plot.



- (b)** Use the graphing calculator to find a trigonometric function.

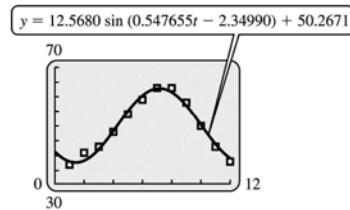
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SinReg
y = a*sin(bx+c)+d
a = 12.5679995
b = .5476551394
c = -2.34990222
d = 50.26714035

```

$$y = 12.5680 \sin(0.54655t - 2.34990) + 50.2671$$

- (c) Graph the function with the data.



(d) $T = \frac{2\pi}{b} \approx 11.4729$

The period is about 11.5 months.

103. (a) $y = x \tan \alpha - \frac{16x^2}{v^2} \sec^2 \alpha + h$

$$= 39 \tan \frac{\pi}{24} - \frac{16(39)^2}{73^2} \sec^2 \frac{\pi}{24} + 9$$

$$\approx 9.5$$

Yes, the ball will make it over the net since the height of the ball is about 9.5 feet when x is 39 feet.

- (b) Entering

$$Y_1 = 39 \tan x - \frac{16(39)^2}{44^2} \sec^2 x + 9 \text{ and}$$

$$Y_2 = \frac{44^2 \sin x \cos x + 44^2 \cos^2 x \sqrt{\tan^2 x + \frac{576}{44^2} \sec^2 x}}{32}$$

into the graphing calculator and using the table function, indicates that the tennis ball will clear the net and travel between 39 and 60 feet for $0.18 \leq x \leq 0.41$ or $0.18 \leq \alpha \leq 0.41$ in radians. In degrees,

$$10.3 \leq \alpha \leq 23.5.$$

X	Y1	Y2
.17	2.754	44.141
.18	3.1103	44.827
.22	4.5223	47.571
.34	8.6526	55.486
.38	10.002	57.9
.41	11.006	59.602
.42	11.339	60.146

- (c) Using Y_2 from part (b) and the graphing calculator, we get

```

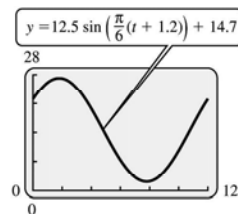
nDeriv(Y2,X,pi/8)
57.01184054

```

Note that $57.01 \frac{\text{feet}}{\text{radian}} \approx 0.995 \frac{\text{feet}}{\text{degree}}$.

The distance the tennis ball travels will increase by approximately 1 foot by increasing the angle of the tennis racket by one degree.

104. (a)



- (b) For February:

$$y = 12.5 \sin \left[\frac{\pi}{6}(2 + 1.2) \right] + 14.7$$

$$= 12.5 \sin \left[\frac{\pi}{6}(3.2) \right] + 14.7$$

$$\approx 27.13 \text{ MCF}$$

For July:

$$y = 12.5 \sin \left[\frac{\pi}{6}(7 + 1.2) \right] + 14.7$$

$$= 12.5 \sin \left[\frac{\pi}{6}(8.2) \right] + 14.7$$

$$\approx 3.28 \text{ MCF}$$

$$(c) \quad y = 12.5 \sin \left[\frac{\pi}{6}(t + 1.2) \right] + 14.7$$

$$\frac{dy}{dt}$$

$$= 12.5 \cos \left[\frac{\pi}{6}(t + 1.2) \right] D_t \left[\left(\frac{\pi}{6}(t + 1.2) \right) \right]$$

$$= 12.5 \cos \left[\frac{\pi}{6}(t + 1.2) \right] \cdot \frac{\pi}{6}$$

$$= \frac{12.5\pi}{6} \cos \left[\frac{\pi}{6}(t + 1.2) \right]$$

At $t = 7$,

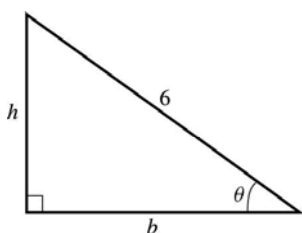
$$\frac{dy}{dt} = \frac{12.5\pi}{6} \cos \left[\frac{\pi}{6}(7 + 1.2) \right]$$

$$= \frac{12.5\pi}{6} \cos \left[\frac{\pi}{6}(8.2) \right]$$

$$\approx -2.66 \text{ MCF/month}$$

In July, the monthly gas usage is decreasing by 2.66 MCF per month.

105. Refer to the figure below.



Let A be the area of the triangle.

$$A = \frac{1}{2}bh$$

In this triangle,

$$\sin \theta = \frac{h}{6} \quad \text{and} \quad \cos \theta = \frac{b}{6}$$

$$h = 6 \sin \theta \quad \text{and} \quad b = 6 \cos \theta.$$

Thus,

$$A = \frac{1}{2}(6 \cos \theta)(6 \sin \theta)$$

$$= 18 \sin \theta \cos \theta$$

$$= 9(2 \sin \theta \cos \theta)$$

$$A = 9 \sin (2\theta)$$

$$\frac{dA}{d\theta} = 9 \cos (2\theta) \cdot 2 = 18 \cos (2\theta).$$

If $\frac{dA}{d\theta} = 0$,

$$18 \cos (2\theta) = 0$$

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4}.$$

If $\theta < \frac{\pi}{4}$, $\frac{dA}{d\theta} > 0$.

If $\theta > \frac{\pi}{4}$, $\frac{dA}{d\theta} < 0$.

Thus, A is maximum when $\theta = \frac{\pi}{4}$ or 45° .

$$106 \quad (a) \quad \frac{d}{dx} \ln |\sec x + \tan x|$$

$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x(\tan x + \sec x)}{(\sec x + \tan x)}$$

$$= \sec x$$

$$(b) \quad \frac{d}{dx} (-\ln |\sec x - \tan x|)$$

$$= -\frac{\sec x \tan x - \sec^2 x}{\sec x - \tan x}$$

$$= -\frac{\sec x(\tan x - \sec x)}{(\sec x - \tan x)}$$

$$= -(-\sec x)$$

$$= \sec x$$

$$(d) \quad D(\theta) = k \int_0^\theta \sec x \, dx$$

$$= k \ln |\sec x + \tan x| \Big|_0^\theta$$

$$= k \ln |\sec \theta + \tan \theta|$$

$$- k \ln |\sec 0 + \tan 0|$$

$$= k \ln |\sec \theta + \tan \theta|$$

$$- k \ln |1 + 0|$$

$$= k \ln |\sec \theta + \tan \theta| - k \ln 1$$

$$= k \ln |\sec \theta + \tan \theta| - 0$$

$$= k \ln |\sec \theta + \tan \theta|$$

Since $D(34^\circ 03') = 7$,

$$7 = k \ln |\sec 34^\circ 03' + \tan 34^\circ 03'|$$

$$k = \frac{7}{\ln |\sec 34^\circ 03' + \tan 34^\circ 03'|}$$

$$\approx 11.0635.$$

$$D(40^\circ 45') \approx 11.0635 \ln |\sec 40^\circ 45' + \tan 40^\circ 45'|$$

$$\approx 8.63$$

New York City should be placed approximately 8.63 inches from the equator.

(e) $D(25^\circ 46')$

$$\approx 11.0635 \ln |\sec 25^\circ 46' + \tan 25^\circ 46'|$$

$$\approx 5.15$$

Miami should be placed approximately 5.15 inches from the equator.

Extended Application: The Shortest Time and the Cheapest Path

1. Since both rays are coming from point A , but meet the interface at two different points, they cannot be parallel. If the two rays were parallel, the alternate interior angle of the labeled right angle would also measure 90° implying that the two angles θ_1 are equal. As Δx becomes closer to zero the rays become more nearly parallel.
2. The segment labeled Δp is only an approximation to the change in length of the ray from point A . The actual change in length would be found by measuring the length of the original ray from point A to the crossing point along the new ray which would leave the change in length left over. As Δx becomes closer to zero the two rays become more nearly parallel and Δp becomes a better approximation.

$$3. \quad \frac{\text{Speed of light in air}}{\text{Speed of light in plastic}} = 1.6$$

$$\text{Speed of light in plastic} = \frac{\text{Speed of light in air}}{1.6}$$

The percentage of the speed of light in 1.6-index plastic is given by $\frac{1}{1.6} = 0.625$ or 62.5%.

4. The cheapest route would be where the route goes almost perpendicularly across the swamp, making x_1 almost zero.

5. If construction over land was actually more expensive than construction over the swamp, we would solve

$$k \frac{7-x}{\sqrt{(7-x)^2 + 5^2}} = \frac{x}{\sqrt{x^2 + 3^2}}$$

where k represents how many more times expensive building over land is than building over the swamp. Letting $k = 2$ in the above equation, we get

$$2 \frac{7-x}{\sqrt{(7-x)^2 + 5^2}} = \frac{x}{\sqrt{x^2 + 3^2}}$$

The solver on the calculator gives $x \approx 4.68$ miles. The angle θ_1 has tangent equal to $4.68/3$ which means that θ_1 is about 57.3° . For the straight line route, θ_1 has tangent equal to $7/8$ which means that θ_1 is about 41.2° . The angle between the north-south line which passes A and the line from A to the point where the road emerges from the swamp is also θ_1 , the larger θ_1 corresponds to the route which lies to the west of the other route. Thus, the route which minimizes cost lies to the west of the straight line route.

6. When you see the sun begin to set, the center of the sun looks to be $\frac{1}{2}(0.53^\circ) = 0.265^\circ$ above the horizon, but is actually $0.265^\circ - 0.57^\circ = -0.305^\circ$ below the horizon. Since the radius of the sun's disk is only 0.265° , this means the sun is already below the horizon.